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LE PRODIGE DE LA MINE

PAR

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53/II

(Communicated at the meeting of May 20, 1950)

1. *Introduction.*

Dans les grands charbonnages de Hollande, de la Campine belge, de la Ruhr et même d'Angleterre, on emploie sur une large échelle la méthode d'exploitation par longues tailles chassantes et foudroyage du toit, avec abatage et chargement mécaniques.

Les figures 1 et 2 donnent une idée d'un tel chantier.

La sécurité du personnel, la bonne tenue des travaux, dépendent du soutènement du toit. Le premier souci du mineur est de chercher à prevenir les éboulements. Pour placer l'ingénieur des mines de pair avec ses collègues exerçant d'autres métiers, il faut lui donner une idée de la répartition des pressions dans le terrain et de la charge des étançons.

Dans des publications antérieures, [1]) nous avons exposé que le terrain houiller disloqué, la roche désagrégée par l'expansion vers le vide créé par les travaux en cours, forme une masse de débris ou de blocs incohérents dans laquelle l'équilibre s'établit selon les lois de la répartition des tensions dans un terrain pulvérulent à frottement interne et que l'arsenal de la mécanique du sol est à la disposition des ingénieurs des mines.

Nos premières applications de cette hypothèse que la roche fissurée se comporte comme une masse de cailloux, au calcul de l'angle d'affaissement et à la répartition de la pression autour d'un puits ou d'une galérie sont acceptées à présent. [2])

Ce qui surprend l'ingénieur lorsqu'il constate qu'un soutènement relativement minime résiste à la pression énorme correspondant à la profondeur, s'explique par le fait que, grace à la courbure des surfaces dites de glissement, la pression active sur les supports est très restreinte.

Le meilleur exemple, emprunté à une solution trouvée dans la

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1) Proceedings, **44**, nos. 2 and 3, 120, 230 (1941).
Revue Universelle des Mines **9**, (1941).
Geologie en Mijnbouw, **10**, (October 1948).

2) Prof. HENRI LABASSE. Les pressions de terrains dans les mines de houilles. Revue Universelle des Mines Tome V, 1, (1949).
Prof. Ir TH. R. SELDENRATH, Ervaringen met een kolenploeg in een vlakke kolenlaag op de Oranje Nassaumijnen II, De Ingenieur, 2 (1949).

mécanique des sols, est représenté par la figure 3. Il fut traité dans nos publications antérieures.

Si l'on désigne par p_1 la pression sur le charbon normale et constante et par p_2 la pression sur le soutènement et les travaux abandonnés qui est aussi normale et constante on trouve

$$p_2 = \frac{1 - \sin \varrho}{1 + \sin \varrho} e^{-\pi \operatorname{tg} \varrho} p_1$$

où ϱ est l'angle de frottement interne, qui est égal à l'angle de tassement des débris.

Dans nos études antérieures, nous avions pris $\varrho = 45°$. Maintenant, pour des raisons que nous allons indiquer, nous prenons $\varrho = 50°$ ce qui fait une différence considérable comme indique cette table au dessous.

$\varphi =$	$40°$	$45°$	$50°$
$p_2/p_1 =$	$\frac{1}{64}$	$\frac{1}{135}$	$\frac{1}{320}$

La pression sur le soutènement n'est que 1/320ième de la pression sur le charbon.

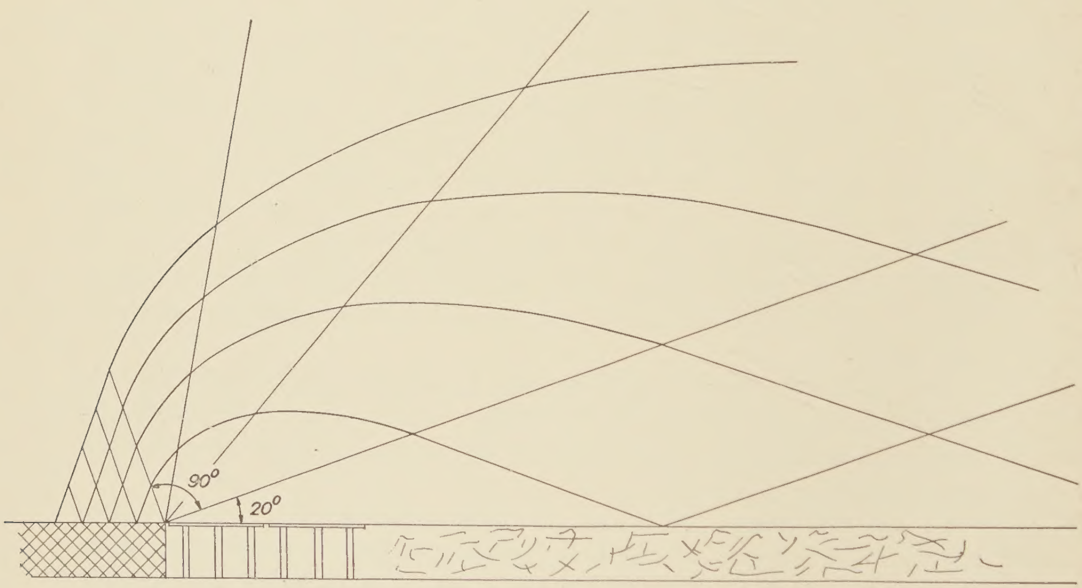

Fig. 3. Lignes, dites de glissement, pour pression constante sur charbon et sur soutènement.

Les lignes, dites de glissement, sont indiquées dans la figure 3 pour le cas où $\varrho = 50°$. Nous avions pris comme base du calcul que la pression p_1 règne conformément sur le charbon jusqu' au front. Quelques lecteurs avaient pris le résultat $p_1 : p_2 = \frac{1}{135}$ trop à la lettre et d'autres ont fait la remarque qu'en réalité les étançons près du front portent beaucoup moins et que la pression sur le soutènent n'est en pratique,

Fig. 1. Mine Orange-Nassau. Charrue à charbon (gauche) et transporteur à chaines.

Fig. 2. Mine Maurits. Installation de ratissage.

pas proportionelle à la profondeur comme il résultait de notre formule.

Maintenant que le problème de la distribution des pressions est considéré justement comme fondamental pour l'art minier, il est opportun de rechercher la solution pour une distribution plus conforme à la réalité, c'est-à-dire croissant avec la distance au front.

Quelle est en réalité la pression de la roche encaissante sur le charbon dans une travée? La distribution doit s'accorder avec le phénomène bien constaté, que la couche de charbon sort de quelques centimètres en glissant le long du toit et du mur quand on procède à l'abatage. Ce glissement peut être dû à l'expansion du charbon vers le vide, la lame de charbon restant entier ou la masse de débris écrasés soit extruse selon les lois de la masse pulvérulente. Dans le dernier cas la distribution des pressions peut être exprimée en formules, [3]) mais cette répartition de la pression du toit et du mur exige qu'une pression soit exercée contre le front du charbon et cette pression manque en réalité et le charbon de la veine à rabattre quoique fissuré ressemble plus à une masse intacte qu' à un amas de débris. En plus du frottement le charbon est retenu par la convergeance du toit et du mur. Ignorant la loi exacte nous prenons une augmentation linéaire de la pression et verrons que les difficultés mathématiques sont encore considérables, mais ainsi nous trouverons une explication pour "le prodige de la mine" l'amortissement rapide des pressions vers le chantier d'abatage.

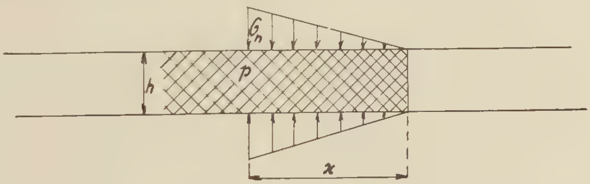

Fig. 4. Charbon glissant vers la taille.

Quel est le coefficient de frottement f pour le charbon en place? Nous ne le savons pas exactement et nous nous placerons dans le cas le plus défavorable. Le schiste contre lequel le charbon a glissé a l'air d'être lisse. Quelques essais que nous avons faits nous conduisent à prendre $f = 0,5$.

Dans les manuels on trouve des coefficients de frottement

grès sur grès	$f = 0,735$
pierre ou brique sur brique	$f = 0,53$ à $0,73$
brique ou galets sur bois	$f = 0,46$ à $0,6$
chêne sur chêne (bois debout)	$f = 0,43$
fer sur pierre calcaire	$f = 0,42$
chêne sur pierre calcaire	$f = 0,38$

[3]) Handbuch der Physik Band VI, Mechanik der Elastischen Körper, Kap. 6 V. Das Gleichgewicht lockerer Massen 31, p. 495 Isogonale Gleitflächenfelder. Lösungen von W. Hartmann. Fenner, cité par Labasse R. U. M. 9e série T, 6 no. 2, 42 (1950).

C'est un fait bien constaté que la roche encaissante est fissurée préalablement. Nous admettons que la roche en mouvement et disloqueé se comporte selon les lois de l'équilibre, acceptées dans la mécanique des sols. Si σ_1 et σ_2 sont les tensions principales, $\frac{\sigma_2}{\sigma_1} = \frac{1 - \sin \varrho}{1 + \sin \varrho} = i$. C'est l'équation caractéristique d'un milieu pulvérulent en équilibre; ϱ est l'angle de frottement interne.

Dans notre étude antérieure nous avions trouvé $\varrho = 45°$ pour les morceaux de roche parfaitement disloqués ou entassés pêle-mêle. Cet angle est égal à celui du talus des débris en tas.

Mais pour les morceaux encore enchevêtrés l'angle de frottement interne et le talus naturel est certainement plus élevé. En prenant $\varrho = 50°$ comme dans la figure 3 nous restons en dessous de la vérité et le résultat de nos calculs ne sera pas exageré.

Ceci nous permet de construire le cercle DE MOHR pour la roche en contact avec le charbon où $f = \operatorname{tg} \varphi = \frac{1}{2}$, ce que représente la Fig. 5.

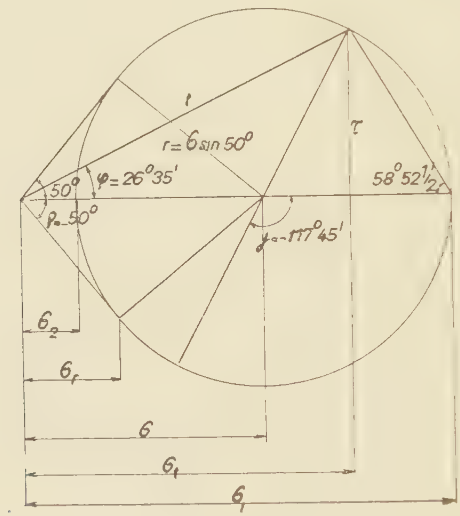

Fig. 5. Cercle de Mohr dans le schiste au contact entre schiste et charbon.

Nous rappelons quelques formules fondamentales de l'équilibre dans les massifs, pulvérulents où $\sigma = \frac{\sigma_1 + \sigma_2}{2}$ et $r = \sigma \sin \varrho$

$$\sigma_1 = \sigma (1 + \sin \varrho) \qquad \sigma_2 = \sigma (1 - \sin \varrho)$$

Dans son travail précité, Labasse fait remarquer justement que dans la mine les roches ne présentent jamais de phase plastique avant rupture. Ce sont des corps raides. Nous recommandons d'étudier, dans notre Traité de Plasticité pour l'Ingénieur, le chapitre 21 qui traite de la rupture des matières cassantes, montrant des fissures aussitôt que l'allongement spécifique atteint une valeur critique. Les ruptures sur-

viennent donc perpendiculairement aux tensions principales maximum, c'est à dire dans notre cas selon la pression maximum. Les fissures préalables observées, donnent la direction principale I et prouvent que notre choix: tg $\varphi = 0{,}5$ pour l'angle de frottement est assez juste.

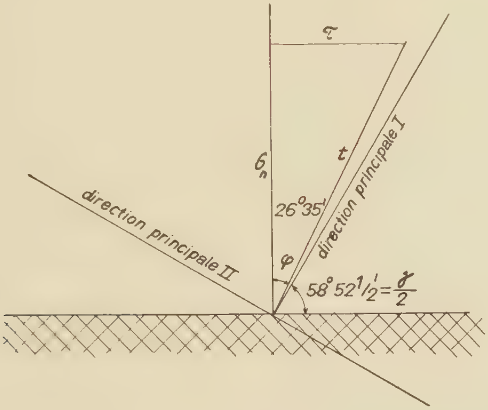

Fig. 6. Direction de la pression des épontes contre le charbon et directions principales.

2. *La distribution des tensions près du front d'abatage. Solution mathématique.*

Après avoir donné les bases physiques du problème, nous pouvons entamer la résolution mathématique qui, même avec ces hypothèses simples, s'avère assez malaisée. Les notations sont indiquées à la Fig. 7. Il est évident qu'il faut utiliser des coordonnées polaires r et θ et prendre l'extrémité O du front comme origine.

Mais d'abord, quelles sont les tensions exercées par le charbon sur la roche? Nous supposons que la pression principale maximum correspond à la pression de profondeur, donc avec un poids spécifique moyen de 2.5 σ_1 en kg/cm$^2 = 0{,}25 \times$ la profondeur en mètres, soit par exemple à une profondeur de 700 m une pression de $\sigma_1 = 175$ kg/cm^2. Quand on observe que toute la perturbation dans les tensions que nous considérons reste limitée à quelques mètres en dessus et en dessous du chantier et que l'augmentation et la diminution dans la pression de profondeur pour 4 mètres est de 1 kg/cm^2 ou seulement de l'ordre de $\frac{1}{2}$ %, il apparaît que l'on peut négliger en première approximation l'influence du poids des roches. Nous invitons le lecteur à indiquer la surface du sol pour les figures 3 et 7.

Selon le cercle DE MOHR (Fig. 5), la tension inclinée sur la surface de séparation entre charbon et schiste est $t = 0{,}87\,\sigma_1$ et $\tau = 0{,}38\,\sigma_1$.

Quand on veut tenir compte qu'auprès du chantier les pressions dans le terrain sont très réduites et que par conséquence à quelque distance la pression maximum peut devenir 1,7 fois la pression correspondant

à la profondeur [4]) on doit multiplier dans notre exemple $\sigma_1 = 175$ kg/cm^2 par le coefficient 1,7. Mais dans le Chapitre 3 nous allons indiquer des circonstances atténuantes.

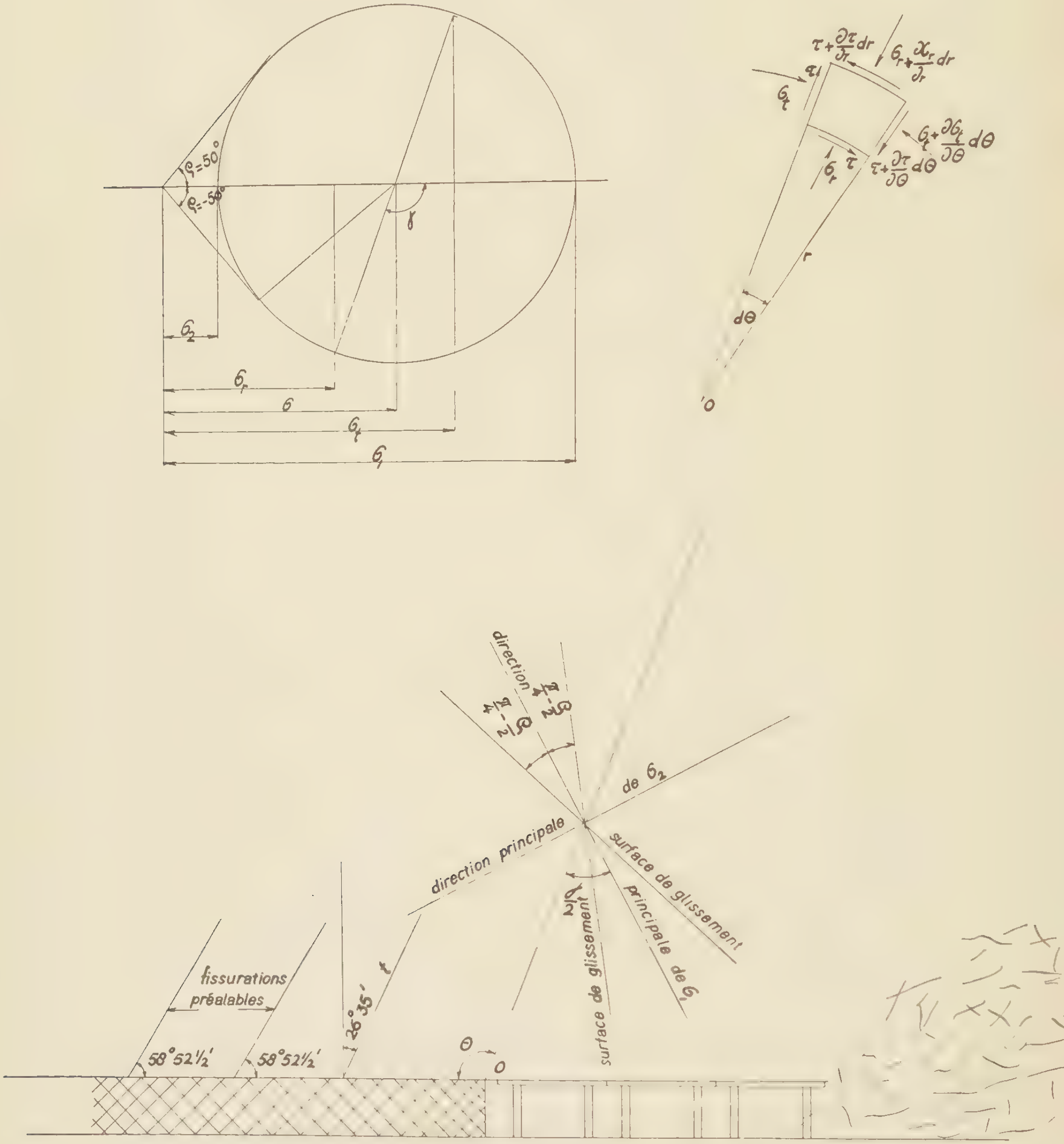

Fig. 7. Notations pour le calcul des tensions dans le terrain encaissant.

[4]) La pression du toit sur le charbon près du front dans les exploitations par tailles chassantes, troisième chapitre, Proceedings Ned. Akademie van Wetenschappen 44, 241 (1941).

Revue Universelle des Mines T XVII, no. 9, 8ième série (1941).

Entamons l'étude mathématique de la distribution des pressions dans le terrain.

Les équations d'équilibre pour un élément du massif rocheux en coordonnées polaires sont, comme on déduit de la Fig. 7, à droite, en haut:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_t}{r} + \frac{1}{r}\frac{\partial \tau}{\partial \theta} = 0 \qquad \text{et} \qquad \frac{1}{r}\frac{\partial \sigma_t}{\partial \theta} + \frac{\partial \tau}{\partial r} + \frac{2\tau}{r} = 0$$

On lit dans le cercle DE MOHR (Fig. 7 à gauche en haut)

$$\sigma_r = \sigma\,(1 + \sin\varrho\cos\gamma) \qquad \sigma_t = \sigma\,(1 - \sin\varrho\cos\gamma) \qquad \tau = \sigma\sin\varrho\sin\gamma$$

où l'on doit prendre pour ϱ la valeur négative.

L'angle γ est le double de l'angle que fait le rayon avec la direction principale σ_2. Nous avons admis que pour la ligne de contact entre roche et charbon les tensions croissent linéairement quand on s'éloigne de l'origine O. En considérant les deux équations différentielles partielles on voit que les tensions (tensions principales σ_1 et σ_2, radiales et tangentielles σ_r et σ_t, tension de cisaillement τ) croissent proportionnellement à r. La tension moyenne $\sigma = \frac{\sigma_1 + \sigma_2}{2}$ et γ sont fonction de θ. Donc

$$\sigma = r\,f(\theta) \text{ et } \gamma = g\,(\theta)$$

En substituant les expressions de σ_r, σ_t et τ dans les équations différentielles on obtient:

$$f\,(\theta)\,(1 + 3\sin\varrho\cos\gamma) + f'\,(\theta)\sin\varrho\sin\gamma + f\,(\theta)\sin\varrho\cos\gamma\,\frac{\partial\gamma}{\partial\theta} = 0$$

$$f\,(\theta)\,3\sin\varrho\sin\gamma + f'\,(\theta)\,(1 - \sin\varrho\cos\gamma) + f\,(\theta)\sin\varrho\sin\gamma\,\frac{\partial\gamma}{\partial\theta} = 0$$

Si l'on élimine $\partial\gamma/\partial\theta$ on obtient $f(\theta)\sin\gamma + f'(\theta)\,(\sin\varrho - \cos\gamma) = 0$.

$$\frac{f'\,(\theta)}{f\,(\theta)} = -\frac{\sin\gamma}{\sin\varrho - \cos\gamma} \quad \text{ou} \quad d\ln f\,(\theta) = -\frac{\sin\gamma}{\sin\varrho - \cos\gamma}\,d\theta.$$

Dans cette formule γ est fonction de θ. Nous y reviendrons. Si l'on élimine maintenant $f'(\theta)$ et qu'en même temps disparait $f(\theta)$, on trouve

$$1 + 2\sin\varrho\cos\gamma - 3\sin^2\varrho + (\sin\varrho\cos\gamma - \sin^2\varrho)\,\frac{\partial\gamma}{\partial\theta} = 0.$$

$$\frac{\partial\theta}{\sin\varrho} = \frac{\sin\varrho\,d\gamma}{(1 - 3\sin^2\varrho) + 2\sin\varrho\cos\gamma} - \frac{\cos\gamma\,d\gamma}{(1 - 3\sin^2\varrho) + 2\sin\varrho\cos\gamma}.$$

Afin de simplifier appelons les constantes $(1 - 3\sin^2\varrho) = a$ $2\sin\varrho = b$

$$\frac{2\,d\theta}{b} = \frac{b}{2}\,\frac{d\gamma}{a + b\cos\gamma} - \frac{\cos\gamma\,d\gamma}{a + b\cos\gamma}.$$

En intégrant

$$\frac{2}{b}(\theta+C)=\frac{b}{2}\int\frac{d\gamma}{a+b\cos\gamma}-\frac{\gamma}{b}+\frac{a}{b}\int\frac{d\gamma}{a+b\cos\gamma}$$

$$2(\theta+C)=-\gamma+\frac{\frac{b^2}{2}+a}{\sqrt{b^2-a^2}}\ln\frac{b+a\cos\gamma+\sqrt{b^2-a^2}\sin\gamma}{a+b\cos\gamma}.$$

Le but que nous poursuivons est d'ordre pratique: déterminer la pression sur le soutènement. Pour le schiste et le grès rompus et disloqués nous prenons cette fois pour l'angle de frottement interne comme dit dans l'introduction $\varrho=50°$.

En posant $\frac{a}{b}=\cos\beta$ on peut écrire l'équation

$$2(\theta+C)=-\gamma+\frac{\frac{b^2}{2}+a}{\sqrt{b^2-a^2}}\ln\frac{\cos\frac{1}{2}(\gamma-\beta)}{\cos\frac{1}{2}(\gamma+\beta)}.$$

Pour $\varrho=-50°$ $a=1-3\sin^2\varrho=-0{,}76043$ $b=2\sin\varrho=-1{,}53208$

$$\cos\beta=\frac{a}{b}=\frac{0{,}76043}{1{,}53208}\quad\beta=-60°\,14'\,32^1/_3{}''\quad\tfrac{1}{2}\beta=-30°\,7'\,16^1/_6{}''.$$

Prenons $\beta=-30°\,7\tfrac{1}{2}'$

$$2(\theta+C)=-\gamma+0{,}309\ln\frac{\cos\frac{1}{2}(\gamma-\beta)}{\cos\frac{1}{2}(\gamma+\beta)}$$

$$2(\theta+C)=-\gamma+0{,}712\log\frac{\cos\frac{1}{2}(\gamma-\beta)}{\cos\frac{1}{2}(\gamma+\beta)}.$$

Pour trouver la constante d'intégration C reportons nous à la figure 5 où nous voyons qu' à la surface de contact entre le charbon et la roche, pour $\theta=0$ $\gamma=-117°\,45'$ ce qui donne $2C=3{,}265$ donc l'équation entre γ et θ devient

$$2\theta=-3{,}265-\gamma+0{,}712\log\frac{\cos\frac{1}{2}(\gamma-\beta)}{\cos\frac{1}{2}(\gamma+\beta)}.$$

Ce qui donne

$\theta=$	$0°$	$6°\,30'$	$29°$	$43°$	limite
$\gamma=$	$-117°\,45'$	$-119°$	$-119°\,40'$	$-119°\,44'$	$-119°\,45'$

Nous donnons la représentation graphique de cette relation entre θ et γ dans Fig. 8.

Maintenant que la relation entre γ et θ est connue revenons à la formule

$$d\ln f(\theta)=-\frac{\sin\gamma}{\sin\varrho-\cos\gamma}\,d\theta$$

qui doit fournir $f(\theta)$ pour calculer la diminution des pressions dans le terrain quand θ croît.

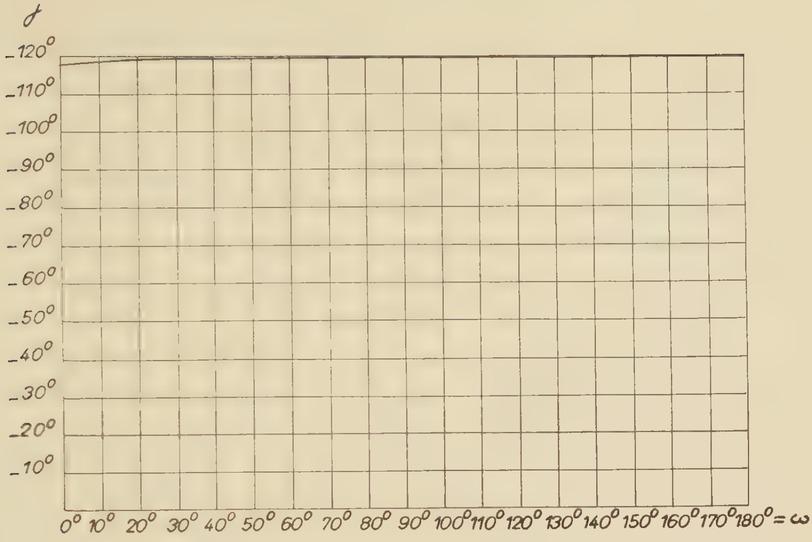

Fig. 8. Relation entre θ et γ

Avec la figure 8 il serait possible par un procédé semi-graphique d'intégrer cette équation différentielle. Mais pour un problème technique une telle précision n'est pas justifiée, on peut profiter du fait que γ est quasi constant, du moins pour $\operatorname{tg} \varphi = 0{,}5$ et $\varrho = 50°$, et prendre $\gamma = -120°$, ce qui donne

$$d \ln f(\theta) = -3{,}26 \; d\,\theta \quad \text{ou} \quad f(\theta) = C\, e^{-3{,}26\,\theta}$$

De $\theta = 0$ à $\theta = \pi/2$ les pressions diminuent à 1/167
,, $\theta = 0$ à $\theta = 2\,\pi/3$,, ,, ,, à 1/926
,, $\theta = 0$ à $\theta = \pi$,, ,, ,, à 1/28000

Dans la figure 9 nous avons dessiné pour analogie avec la figure 3 les lignes de glissement pour la distribution linéaire des pressions selon les rayons.

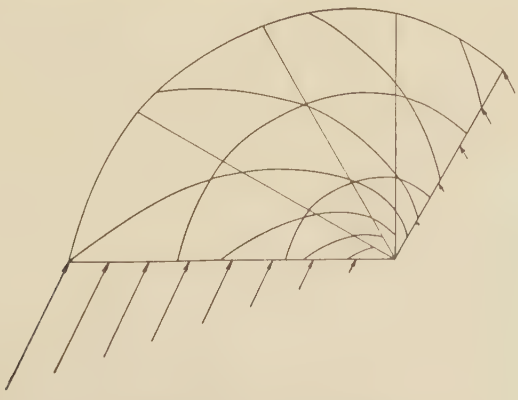

Fig. 9. Lignes dites de glissement pour distribution linéaire des pressions sur les rayons.

Le calcul de la diminution de la pression dans un massif à frottement interne pour une charge de la surface à accroissement linéaire peut être appliqué dans plusieurs problèmes de la mécanique des sols.

La figure 10 représente un cas où l'on veut calculer la pression sur les palplanches d'une cunette à côté d'un grand bâtiment en béton armé. Par suite du boisage, le sol se déplace vers la cunette et la pression sous le batiment diminue vers le bord de la fondation.

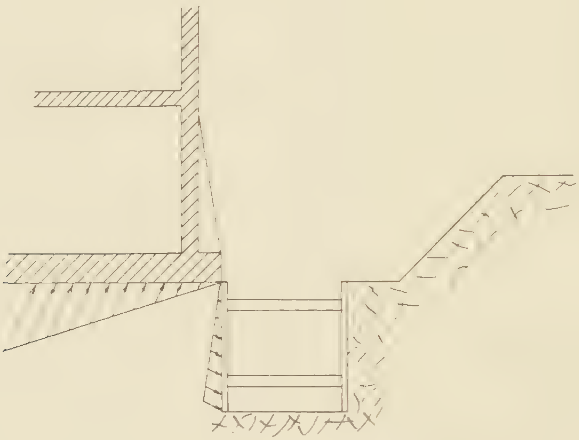

Fig. 10. Application de la solution mathématique à un problème de la mécanique des sols.

Nous avons dessiné dans la figure la direction inclinée de la pression du sol sur la fondation du bâtiment, ainsi que la pression sur les palplanches.

3. *Discussion du résultat.*

Nous avons trouvé que déjà dans le premier quadrant, de $\theta = 0°$ jusqu' à $\theta = 90°$ les contraintes dans la roche diminuent tellement qu'on peut négliger le restant. Par exemple à la profondeur de 700 mètres il ne reste de la pression correspondant à la profondeur de 175 kg/cm² qu' à peu près 1 kg/cm².

Dans le premier quadrant on pouvait laisser en dehors du calcul l'influence du poids des roches influencées, qui n'est qu'environ $\frac{1}{2}$ % de la pression due à la profondeur.

Mais dans le second quadrant où comme nous venons de remarquer les pressions dues à la profondeur sont presqu' amorties il n'est pas permis de négliger cette influence. Au contraire il n'y a que le poids des débris supportés qui compte pour calculer la pression sur le soutènement.

Cependant la situation est plus compliquée. Dans les chantiers

modernes, mécanisés, on emploie en général des étançons coulissants. Quand ceux-ci sont très rigides les déplacements dans la roche et la dislocation des débris enchevêtrés restent minimes. Alors l'angle de frottement interne est élevé, même supérieur à 50° et l'amortissement des pressions est encore plus considérable que nous l'avons calculé.

Si les étançons sont souples la disjonction des débris s'accentue, l'angle de frottement interne peut s'approcher de sa valeur limite qui est environ 45°. Dans ce cas il reste une pression de terrain non négligeable qui s'ajoute au poids de la masse disloquée, reposant sur le soutènement.

En visitant des chantiers mécanisés dans de différentes mines on s'étonne de la diversité de la résistance des soutènements. Il y a encore des ingénieurs qui pour produire le fondroyage certain du toit emploient comme supports des piles de tronçons de rail quasi incompressibles.

La pression sur le soutènement est elle calculable? Non! Elle dépend de la caractéristique de compressibilité des étançons. S'ils étaient incompressibles, il n'y aurait pas de déplacement ni d'expansion de la roche et les étançons auraient à résister à une pression énorme.

Maintenant considérons l'autre cas extrême, des étançons très souples qui peuvent à peine supporter le poids de 3 on 4 mètres de roche.

Il est curieux de remarquer que l'exploitation de la couche est possible et même que l'effondrement du toit dans la travée déhouillée est quelquefois retardé si le soutènement est compressible.

Dans la figure 11 nous indiquons de quel façon un dôme de pression peut se former si les étançons sont souples.

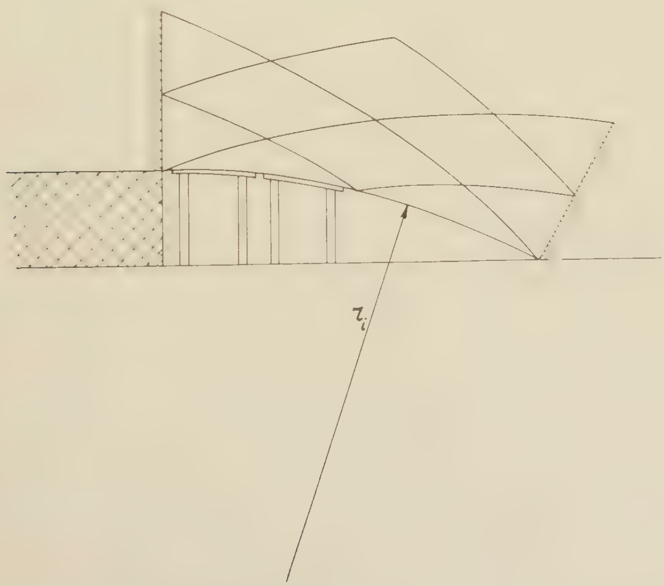

Fig. 11. Formation d'un dôme de pression à cause d'étancons souples.

La formule pour calculer la pression intérieure p_i dans une voûte cylindrique si la pression extérieure est p s'écrit:

$$p_i = (1 - \sin \varrho) \left(\frac{r_i}{r_u}\right)^{\frac{2 \sin \varrho}{1 - \sin \varrho}} p$$

où r_i et r_u sont les rayons interieur et extérieur. [5])

Mais on dira que souvent on observe, comme dessiné à la figure 11, qu'une partie de la voûte n'est pas supportée du tout.

Ceci s'explique par le fait que les débris dont se compose le toit sont bien enchevêtrés et que la partie du toit dont le fondroyage reste tardif est limitée dans le sens de la taille. On touche ici au problème de la distribution des pressions dans l'espace sphérique de la masse sans cohérence à frottement interne, facilement soluble, qui donne comme résultat: qu'un dôme se tient facilement sans s'écrouler.

On comprend que, quoique la formule soit exacte pour une ouverture cylindrique dans la masse sans cohérence avec une pression extérieure uniforme, l'influence de la pesanteur modifie legèrement le résultat.

Il est important de remarquer que, contrairement aux applications de la théorie de l'élasticité, mais conforme à la théorie de la plasticité, les solutions des problèmes de la distribution des tensions dans la mase à frottement interne ne doivent pas tenir compte de l'encastrement, de la limitation de la voûte ou du dôme.

Nous laissons aux ingénieurs des mines de grande expérience à juger si l'exploitation des travées à soutènement très souple avec formation de voûtes et effondrement différé et étendu du toit, n'offre pas le risque que les étançons se renversent.

Dans le chapitre 2, où nous donnions la solution mathématique pour l'augmentation linéaire des pressions nous avions calculé que dans le premier quadrant de la figure 7 les pressions diminuent jusqu' à 1/167ième de la pression initiale. Mais quelle est la pression des épontes sur le charbon et jusqu' à quelle distance accroît elle linéairement?

Reprenons la figure 4 qui représente la lame de charbon glissant vers la taille et la figure 5 donnant le cerde de Mohr pour les tensions dans le schiste et le charbon en contact. Nous acceptons que l'angle de frottement interne ϱ est le même pour charbon et schiste.

Appelons (Fig. 4): f le coefficient de frottement, h l'épaisseur de la couche, σ_n la pression normale maximum exercée par le toit et le mur

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[5]) La pression du toit sur le charbon pres du front dans les exploitations par tailies chassantes. Proceedings Ned. Akademie van Wetenschappen, **44**, 231 (1941).

Revue Universelle des Mines T. XVII, no. 9, (1941).

The Theory of Rock Pressure on Coal Mining. Geologie en Mijnbouw, **10**, 212 (October 1948).

(les épontes), p la poussée horizontale moyenne exercée sur le charbon, x la distance du front. Alors

$$p\,h = \tfrac{1}{2}\,\sigma_n\,f\,x \times 2 \quad x = \frac{h}{f}\,\frac{p}{\sigma_n}$$

Or $f\,\sigma_n = \tau$ donc $x = \frac{p}{\tau}\,h$

On lit de la figure 5 $\tau = 0{,}378\,\sigma_1$ et la poussée horizontale (tension radiale) $\sigma_r = p = 0{,}139\,\sigma_1$

$$\text{Alors} \quad x = \frac{0{,}139}{0{,}378}\,h = 0{,}37\,h.$$

Ce résultat est important et à retenir. Il dit que dans la couche de charbon considéré comme masse sans cohérence à frottement interne une augmentation linéaire de la pression de zéro *jusqu' à un montant quelconque* s'effectue sur la courte distance de $x = 0{,}37\,h$.

Maintenant, la distribution dans la couche de charbon écrasé est résolue parceque nous connaissons la solution du problème dès qu'il régne une certaine pression. Nous pouvons nous référer à plusieurs publications. [6]) La pression dès qu'elle cesse d'augmenter linéairement accroit selon une loi exponentielle beaucoup plus rapidement encore.

Or nous voici arrivés à une conclusion contraire à l'observation. Si le charbon est considéré comme de la masse pulvérulente pressée à la limite de glissement interne, obéissant à la loi $\frac{\sigma_1}{\sigma_2} = \frac{1 + \sin \varrho}{1 - \sin \varrho}$ on obtient une augmentation de pression beaucoup plus rapide que celle imaginable.

En outre le charbon quoique fissuré, sillonné de limets est loin de l'état pulvérulent, au contraire il est quasi intact et résistant.

Et si le charbon ne peut pas être traité comme de la masse écrasée il est encore moins permis d'appliquer les lois pour la poussée des terres, de calculer la pression active sur le soutènement selon la théorie de la mécanique des sols ce qui commence à être en vogue.

La conclusion qu'on doit tirer de tous les efforts pour calculer les contraintes des épontes sur la couche est que celles-ci sont presque zéro à la taille et croissent assez lentement quand on entre dans le charbon et n'influencent pas ou peu la charge sur le soutènement.

Nous ne voulons pas conclure cette étude sans parler brièvement de l'autre cas extrême, celui du toit intact, le banc de roche étant parfaitement supporté comme l'indique la figure 14.

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[6]) Handbuch der Physik. Band VI, Mechanik der Elastischen Körper A. Nádai Plastizität und Erddruck, 495 (1928), Lösungen von Hartmann.

Dissertation Dr R. Fenner Untersuchungen zur Erkenntnis des Gebirgsdrucks 22 (1938).

H. Labasse, Les pressions de terrains autour des galeries en couche. Revue Universelle des Mines Février 42 (1950).

Cette conception s'approche à l'ancienne idée des ingénieurs qui considéraient les bancs du toit comme des solives. Le chapitre 4 y sera consacré.

4. *La rupture du toit le long d'un soutènement regide.*

Nous traiterons donc le problème comme si la roche était une matière homogène et élastique et nous calculerons la distribution des tensions au bord du soutènement (figure 14). Nous disposons des formules pour les tensions dans l'hémi-espace chargé selon une ligne de P kilos par unité de longueur. [7])

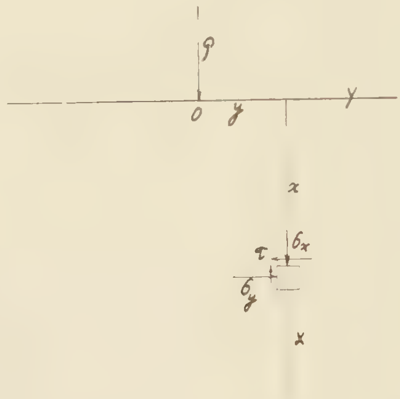

Fig. 12. Hémi-espace chargé le long d'une ligne.

$$\sigma_x = \frac{x^3}{(x^2+y^2)^2}\,\frac{2}{\pi}\,P \qquad \sigma_y = \frac{x\,y^2}{(x^2+y^2)^2}\,\frac{2}{\pi}\,P \qquad \tau = \frac{x^2\,y}{(x^2+y^2)^2}\,\frac{2}{\pi}\,P.$$

Par intégration on peut obtenir la distribution des pressions dans le terrain pour une charge quelconque, an moins quand on se tient aux problèmes à deux dimensions.

Nous considérons le terrain également chargé seulement à gauche de l'origine comme indiqué dans la figure 13.

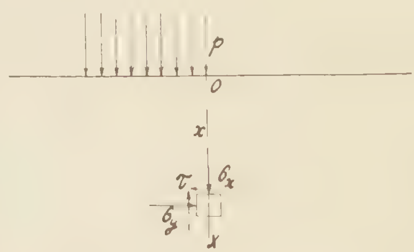

Fig. 13. Hémi-espace à demi chargé d'une charge égale p par unité de surface.

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[7]) Draagvermogen van Bouwgrond, De Ingenieur B 20, B 242 (22 Sept. 1928). Résistance du Terrain à la Charge des Constructions. Mémoires de la Société des Ingénieurs Civils de France, Bulletin de mars-avril (1928).

Le plus intéressant est la distribution des tensions le long de l'axe des X. Elle est extrèmement simple:

$$\sigma_x = \frac{p}{2} \qquad \sigma_y = \frac{p}{2} \qquad \tau = \frac{p}{\pi}.$$

Les tensions principales le long de l'axe des X sont

$$\left(\frac{1}{2} \pm \frac{1}{\pi}\right) p$$

et font des angles de 45° avec cet axe.

Quand on admet que dans la matière cassante les ruptures se produisent selon la direction de la plus grande pression, la rupture qui engendre l'effondrement du toit est indiquée en ligne pointillée dans la figure 14.

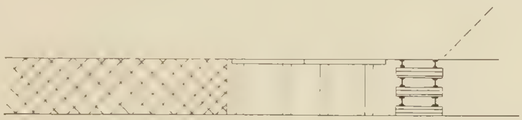

Fig. 14. Rupture qui se produit dans la roche solide bien soutenue sortant du bord du soutènement.

Si la charge n'est pas conforme comme indiquée dans la figure 13, mais diminue, par exemple linéairement de l'origine vers la gauche, les tensions sont maximum à l'origine et exprimeés par les mêmes formules.

Pour celui qui se heurte au fait que dans la figure 13, au point singulier O, $\tau = \frac{p}{\pi}$ au lieu de zéro comme il résulte de la formule générale; nous donnons le résultat de l'intégration pour un point $x\,y$ de la figure 15.

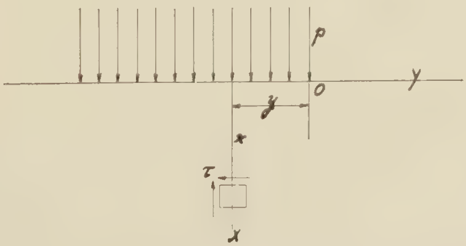

Fig. 15. La tension de cisaillement τ en un point x y dans l'hémi-espace également chargé pour la moitié.

$\tau = \frac{p}{\pi} \frac{y^2}{x^2 + y^2}$ Pour le point singulier O on obtient deux valeurs: $\tau = 0$ si l'on diminue y et $\tau = \frac{p}{\pi}$ si l'on réduit x jusqu' à zéro.

En fait le long de l'axe des Y $\tau = 0$ et le long de l'axe des X $\tau = \frac{p}{\pi}$.

Nous avons dit que le long de l'axe des X les ruptures se produisent sous des angles de 45° à partir de l'origine quand la pression du soutènement est un maximum à son bord.

Mais en parlant des fissures préalables, indiquées à la figure 7, nous avons dit que la roche n'est plus intacte. Sous les énormes forces de détente des fissures visibles et invisibles prennent naissance dans le toit. Dans notre exemple l'inclinaison de ces joints est 59° environ. Il est probable que les éboulements se produisent selon ces surfaces et pas sous l'angle de 45°.

Peut-être le lecteur est-il tenté de rejeter cette application de la théorie de l'élasticité sur la masse disloqueé; mais il faut bien se rendre compte que, pour la roche enchevêtrée, il est permis de calculer les tensions comme dans la masse solide aussi longtemps qu'il n'y a pas de glissement, c.à.d. tant que $\frac{\tau}{\sigma} < \operatorname{tg} \varrho$. C'est encore un précepte emprunté à la mécanique des sols.

Nous répétons qu'un toit fissuré mais enchevêtré se comporte en quelque sorte comme une matière élastique mais se rompt facilement le long d'un soutènement rigide.

Les essais photo-élastiques et les essais sur modèles en caoutchouc ont souvent de la valeur pour élucider la distribution des pressions dans les terrains rompus.

5. *La pression sur le soutènement dans une exploitation à foudroyage.*

Nous avons fait remarquer dans des publications antérieures [8]) qu' autour d'une exploitation il se forme une espèce de cylindre de détente autour duquel il n'y a pas de déplacements dans la roche et qu' à la limite de la partie soumise aux lois de la mécanique des sols et de la partie soumise aux lois de la théorie de l'élasticité, la pression augmente jusqu'à 1,7 fois la pression équivalant à la profondeur.

On ne saurait pas dire que pour la théorie de la diminuation linéaire qui est à la base de la présente étude, il faut partir de 1,7 fois la pression due à la profondeur, de cette pression même ou une pression réduite. Celà dépend de la masse extraite par l'exploitation et de la résistance du soutènement. Il est probable que la pression environnante est assez diminuée et puis tombe fortement selon une loi exponentielle dans la zône à lignes de glissement courbées (figure 9) jusqu'à être amortie presque complètement. La charge du soutènement est pour ainsi dire réduite au poids de la masse de débris qu'il supporte, du moins quand le soutènement n'est pas trop rigide. Sinon la charge dépend de la caractéristique des étançons coulissants. Le résultat de nos calculs est donc de peu d'utilité pratique pour le mineur.

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[8]) Proceedings Ned. Akademie van Wetenschappen, **44**, no. 3, 243 (1941). Revue Universelle des Mines T XVII, no. 9, (1941).

Pour lui donner tout de même une idée globale de la résistance minimum que doivent avoir les étançons, nous calculons le poids des débris à supporter. Nous pensons à la voûte ou au dôme qui se forme à tout éboulement.

Dans la figure 16 nous avons indiqué une limite parabolique pour la masse à supporter. Si la distribution est uniforme et si l'on appelle γ le poids spécifique, h la hauteur, b la largeur, le poids à supporter par mètre de longueur est $G = \frac{2}{3}\, b\, h\, \gamma$ et la charge à supporter par le soutènement par m² devient $r = \frac{2}{3}\, h\, \gamma$.

Mais la distribution de la charge sur le soutènement n'est pas uniforme. Elle est presque négligeable sur les étançons placés en dernier

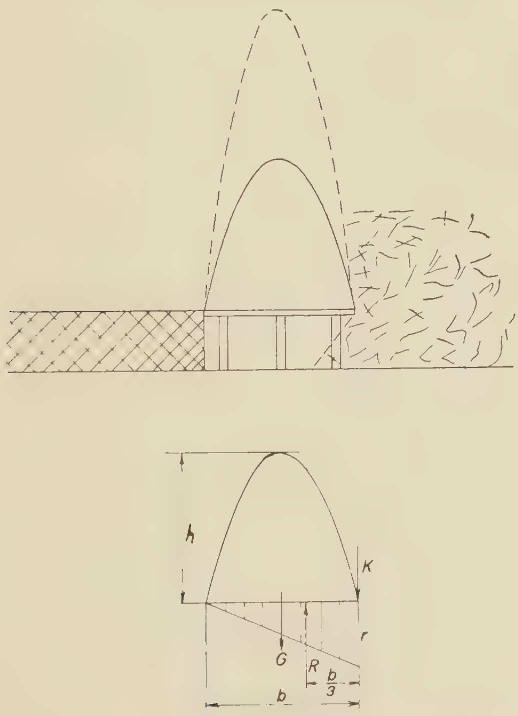

Fig. 16. Calcul sommaire de la pression sur le soutènement.

lieu et augmente linéairement vers les vieux travaux. Alors comme dessiné dans la figure 16 en bas, la résultante des forces exercées par le soutènement est située à $\frac{2}{3}$ de la largeur b et pour réaliser l'équilibre il faut compter sur une force de frottement K des débris écroulés ou sur une pression verticale de ces débris au coté droit de la parabole.

Alors $R = \frac{3}{2}\, G = b\, h\, \gamma \quad K = \dfrac{G}{2}$

La résistance maximum par m² du soutènement est $r = \dfrac{2\,R}{b} = 2\, h\, \gamma$ c'est trois fois plus que pour la distribution uniforme.

Pour avoir une certaine idée de la pression minimum au bord du soutènement on prendra h la hauteur de la parabole intermédiaire entre b et $2\,b$. Pour $b = 3\frac{1}{2}$ m et $\gamma = 2{,}5$ on obtient $17\frac{1}{2}$ tonnes par m^2 jusqu'à 35 tonnes par m^2 indépendamment de la profondeur de l'exploitation.

Mais pour assurer plus de rigidité, moins de déplacement, moins de dislocation dans le premier quadrant de la figure 9 et dans le toit près du front il est recommandable d'employer un soutènement plus résistant.

Pour la pratique, l'ingénieur de mines pourra se baser sur l'expérience décrite par le Professeur Ir TH. SELDENRATH. [9])

Nous remercions Ir. A. HELLEMANS, physicien et électricien, ingénieur du fond Mine Maurits, Prof. H. LABASSE, Université de Liége, Dr. P. A. COENEN, mathématicien, laboratoire mécanique des sols, Delft, pour leur appui pendant l'élaboration de cette étude.

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[9]) Ervaringen met een kolenploeg in een vlakke kolenlaag op de Oranje-Nassau Mijn II. Voordrachten Kon. Inst. van Ingenieurs, 1, no. 2, blz. 149—152 (Maart 1949).

MATHEMATICS

SUR LES FONCTIONS SYMETRIQUES [1])

PAR

J. G. VAN DER CORPUT

(Communicated at the meeting of April 29, 1950)

Le sujet de cette conférence est si simple et si souvent traité que je m'étonne qu'il est encore possible d'en dire quelque chose de nouveau. Cependant je me propose de l'essayer.

Le théorème fondamental de la théorie des fonctions symétriques dit que toute fonction entière rationnelle symétrique de m indéterminées $z_1, \dots, z_m$ peut être écrite d'une seule manière comme une fonction entière rationnelle de $a_1, \dots, a_m$, où $a_1, \dots, a_m$ désignent les fonctions symétriques élémentaires des m quantités $z_1, \dots, z_m$

$$\begin{aligned}
a_1 &= \sum z_1 = z_1 + z_2 + \dots + z_m, \\
a_2 &= \sum z_1 z_2 = z_1 z_2 + z_1 z_3 + \dots + z_{m-1} z_m, \\
a_3 &= \sum z_1 z_2 z_3, \\
&\dots\dots\dots\dots\dots \\
a_m &= z_1 z_2 \dots z_m.
\end{aligned}$$

Dans cette conférence je donnerai une démonstration de ce théorème, qui diffère des démonstrations usuelles et qui nous fournira non seulement le théorème fondamental, mais encore d'autres résultats qui vont la peine d'être étudiés.

D'après le théorème fondamental la somme simple

$$\sum z_1^4 z_2^2 = z_1^4 (z_2^2 + \dots + z_m^2) + \dots + z_m^4 (z_1^2 + \dots + z_{m-1}^2),$$

qui est formé par $m(m-1)$ termes et que je désignerai par $\{4, 2\}$, peut être écrite sous la forme

$$(1) \qquad \left\{ \begin{aligned} \{4, 2\} = a_1^2 a_2^2 - 2 a_1^3 a_3 - 2 a_2^3 + 4 a_1 a_2 a_3 + 2 a_1^2 a_4 - 3 a_3^2 \\ + 2 a_2 a_4 - 6 a_1 a_5 + 6 a_6. \end{aligned} \right.$$

Cette identité est aussi vraie pour $m < 6$, si nous posons, comme nous le ferons toujours dans cette conférence, $a_{m+1} = a_{m+2} = \dots = 0$.

Chacun des termes figurant dans le membre de droite de (1) a le poids 6, si nous donnons à a_1 le poids 1, à a_2 le poids 2, etc. On voit que dans le résultat deux termes de poids 6 manquent, à savoir a_1^6 et $a_1^4 a_2$. Ce fait ne m'intéresse pas pour le moment, quoiqu'il résulte immédiatement

[1]) Conférence faite à Strasbourg le 23 mars 1950 et à Bordeaux le 31 mars 1950.

d'un théorème général que je traiterai à la fin de cette conférence [2]).

Le théorème que je me pose est le suivant:

En supposant que $\sum z_1^4 z_2^2$ puisse être écrit comme une somme de la forme

$$\{4, 2\} = \sum c\, a_{l_1} a_{l_2} \cdots a_{l_t},$$

où le poids $l_1 + \ldots + l_t$ de chaque terme est égal à 6, y a-t-il une manière facile de calculer les coefficients c?

Si nous savons que les termes avec a_1^6 et $a_1^4 a_2$ ne figurent pas dans la somme, nous avons à calculer 9 coefficients, sinon même 11. Peut-être il y a des personnes parmi vous qui savent que le coefficient de $a_1^2 a_2^2$ est égal à 1, de sorte que la difficulté n'est pas la même pour différents mathématiciens, mais en tout cas il faut calculer au plus 11 et au moins 8 coefficients. Pour ce calcul nous avons plusieurs méthodes à notre disposition. Dans la première nous réduisons la somme considérée $\sum z_1^4 z_2^2$ à d'autres sommes dont nous connaissons déjà le développement. Dans une autre méthode, celle des coefficients indéterminés, nous considérons 11 ou 8 équations particulières dont nous connaissons les racines $z_1, \ldots, z_m$, donc aussi la valeur de $\sum z_1^4 z_2^2$. Ainsi on obtient pour les coefficients 11 ou 8 équations linéaires, qui nous permettent de calculer ces coefficients, en supposant que nous n'ayons pas été maladroits dans notre choix des équations particulières.

En théorie cela va bien, mais en pratique chaque méthode exige des calculs si longs que personne ne les exécute.

En préparant il y a dix ans mon cours d'algèbre à Groningue, mon désir de simplifier ces calculs m'a conduit, par un raisonnement [2]) qu'il n'est pas nécessaire de répéter ici, à introduire un certain opérateur différentiel

$$A_1 = [1] = \sum_{\mu=1}^{m} a_{\mu-1} \frac{\partial}{\partial a_\mu},$$

qui peut être appliqué à des polynomes en $a_1, a_2, \ldots, a_n$; je pose $a_0 = 1$. Cet opérateur diminue le poids d'une unité, car dans chaque terme on perd un facteur a_μ et on regagne un facteur $a_{\mu-1}$. Naturellement $A_1\{4, 2\}$ désigne le résultat qu'on obtient en appliquent l'opérateur A_1 au membre de droite de (1). Il est remarquable que le résultat est identiquement égal à zéro. Ainsi on obtient 6 ou 8 équations linéaires très simples entre les coefficients figurant dans le membre de droite de (1), de sorte que la considération de quelques équations particulières suffit pour trouver immédiatement les valeurs de tous ces coefficients.

L'opérateur A_1 ne détruit pas seulement $\sum z_1^4 z_2^2$, mais presque toutes les sommes simples. Plus précisément: L'opérateur A_1 annule chaque somme simple $\sum z_1^{k_1} z_2^{k_2} \ldots z_r^{k_r}$, excepté si au moins un des exposants

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[2]) Voir par ex. J. G. VAN DER CORPUT, Symmetrische functies, CHRISTIAAN HUYGENS 18, 251—277 (1940).

est égal à 1. Dans ce dermier cas on peut supposor, sans nuire à la généralité, $k_r = 1$ et alors

$$A_1 \sum z_1^{k_1} \dots z_{r-1}^{k_{r-1}} z_r = \sum z_1^{k_1} \dots z_{r-1}^{k_{r-1}};$$

dans le cas particulier $r = 1$ la dernière somme signifie 1.
En formule

$$A_1\{k_1, \dots, k_m\} = 0, \text{ si tous les exposants sont différents de } 1,$$
$$= \{k_1, \dots, k_{m-1}\}, \text{ si } k_m = 1.$$

L'opérateur A_1 est impitoyable: il exige toujours un exposant 1 comme victime et s'il ne peut pas trouver un tel exposant, il anéantit tout dans sa colère. A_1 est appelé l'opérateur supprimant 1. Par exemple

$$A_1 \sum z_1^5 z_2^3 z_3^2 = 0; \quad A_1 \sum z_1^5 z_2^3 z_3 z_4 = \sum z_1^5 z_2^3 z_3; \quad A_1 \sum z_1 = 1.$$

Il y a un an j'ai constaté qu'il y a encore d'autres opérateurs avec des propriétés analogues, par exemple l'opérateur

$$A_2 = [1, 1] = \tfrac{1}{2} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} a_{\mu-1}\, a_{\nu-1} \frac{\partial^2}{\partial a_\mu \partial a_\nu},$$

qui diminue le poids de 2. Cet opérateur annule la somme $\sum z_1^{k_1} \dots z_r^{k_r}$, excepté si au moins un des exposants est égal à 2. Dans ce dernier cas on peut supposer, sans nuire à la généralité, que $k_r = 2$ et alors on a

$$A_2 \sum z_1^{k_1} \dots z_{r-1}^{k_{r-1}} z_r^2 = \sum z_1^{k_1} \dots z_{r-1}^{k_{r-1}}.$$

En formule

$$A_2\{k_1, \dots, k_r\} = 0, \text{ si tous les exposants sont différents de } 2,$$
$$= \{k_1, \dots, k_{r-1}\}, \text{ si } k_r = 2.$$

Pour ce raison A_2 est appelé l'opérateur supprimant 2. Par exemple

$$A_2 \sum z_1^5 z_2^3 z_3 z_4 = 0; \qquad A_2 \sum z_1^5 z_2^3 z_3^2 z_4^2 z_5 = \sum z_1^5 z_2^3 z_3^2 z_4.$$

L'opérateur

$$A_3 = [1, 1, 1] = \frac{1}{3!} \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \sum_{\varrho=1}^{m} a_{\mu-1}\, a_{\nu-1}\, a_{\varrho-1} \frac{\partial^3}{\partial a_\mu \partial a_\nu \partial a_\varrho},$$

qui diminue le poids de 3, est l'opérateur supprimant 3, etc.
Il y a encore d'autres opérateurs, par exemple

$$[2] = \sum_{\mu=2}^{m} a_{\mu-2} \frac{\partial}{\partial a_\mu},$$

qui diminue le poids de 2. Dans le cas $m = 1$, [2] est l'opérateur nul, c'est-à-dire l'opérateur qui rend identiquement nulle toute fonction entière rationnelle de $a_1, \dots, a$.

Le résultat obtenu par application de l'opérateur [2] à la somme $\sum z_1^4 z_2^2$, peut être calculé de deux façons différentes. La première est

d'appliquer l'opérateur directement au membre de droite de (1). L'autre est d'exprimer d'abord l'opérateur [2] en fonction des opérateurs A_1 et A_2. On obtient après un peu de calcul

$$[2] = A_1^2 - 2A_2, \tag{2}$$

donc

$$[2] \sum z_1^4 z_2^2 = A_1^2 \sum z_1^4 z_2^2 - 2A_2 \sum z_1^4 z_2^2 = -2 \sum z_1^4.$$

La dernière méthode est beaucoup plus simple que la première. Mais j'attire l'attention sur la formule (2), qui m'a donné un petit choc, parce qu'elle avait exactement la même forme que

$$\sum z_1^2 = \{2\} = a_1^2 - 2a_2.$$

Il va sans dire que je me suis demandé s'il est possible de construire un anneau, formé par des opérateurs $[k_1, \ldots, k_r]$ qui soit isomorphe à l'anneau formé par les sommes simples

$$\{k_1, \ldots, k_r\} = \sum z_1^{k_1} z_2^{k_2} \ldots z_r^{k_r},$$

de telle façon que les opérateurs $A_1, A_2, \ldots,$ supprimant respectivement $1, 2, \ldots,$ correspondent aux fonctions symétriques élémentaires $a_1, a_2, \ldots$. C'est en effet possible. Je rappelle que le nombre des termes de la somme simple $\sum z_1^{k_1} \ldots z_r^{k_r}$ où les exposants $k_1, \ldots, k_r$ sont des entiers positifs, est égal à

$$\frac{m!}{(m-r)!\, j_1!\, j_2! \ldots j_u!}; \tag{3}$$

je suppose que le système formé par les exposants contient j_1 nombres égaux, j_2 nombres égaux différents des j_1 nombres déjà mentionnés, $\ldots$, finalement j_u nombres égaux différents des $j_1 + j_2 + \ldots + j_{u-1}$ nombres déjà mentionnés, de telle manière qu'on ait

$$j_1 + j_2 + \ldots + j_u = r.$$

En introduisant les opérateurs

$$[k_1, \ldots, k_r] = \frac{1}{j_1!\, j_2! \ldots j_u!} \sum_{\mu_1 = k_1}^{m} \ldots \sum_{\mu_r = k_r}^{m} a_{\mu_1 - k_1} \ldots a_{\mu_r - k_r} \frac{\partial^r}{\partial a_{\mu_1} \ldots \partial a_{\mu_r}}$$

(cet opérateur est l'opérateur nul, si au moins un des nombres $k_1, \ldots, k_r$ est supérieur à m) nous obtiendrons le théorème suivant d'isomorphie:

Etant donnée une identité entière rationnelle contenant une ou plusieurs sommes simples

$$\sum z_1^{k_1} \ldots z_r^{k_r} = \{k_1, \ldots, k_r\},$$

cette identité reste valable, si l'on y remplace chaque somme $\{k_1, \ldots, k_r\}$ *par l'opérateur correspondent* $[k_1, \ldots, k_r]$. *Par cette substitution les nombres* $a_1, a_2, \ldots,$ *qui sont aussi des sommes simples, sont remplacés par les opérateurs* $A_1, A_2, \ldots,$ *supprimant respectivement* $1, 2, \ldots$.

Par exemple l'identité

$$\{2\}\{2\}-\{4\}=(\Sigma z_1^2)^2-\Sigma z_1^4=2a_2^2-4a_1a_3+4a_4$$

donne pour les opérateurs la relation

$$[2][2]-[4]=2A_2^2-4A_1A_3+4A_4.$$

En particulier on trouve que la multiplication des opérateurs est commutative, puisque cela est aussi le cas pour les sommes simples.

Dans cette théorie nous rencontrons encore d'autres théorèmes, par exemple le théorème d'orthogonalité. L'opérateur $[k_1, \ldots, k_r]$ annule presque toujours le produit $a_{l_1} a_{l_2} \ldots a_{l_t}$, où

$$l_1+\ldots+l_t=k_1+\ldots+k_r. \tag{4}$$

La seule exception se présente si $r=t$ et si les systèmes $(k_1, \ldots, k_r)$ et $(l_1, \ldots, l_t)$ sont les mêmes, peut-être à l'ordre près; dans ce cas exceptionnel le résultat est égal à 1.

Ce théorème d'orthogonalité, que nous vérifierons tout à l'heure en quelques lignes par un simple calcul, est important à cause du fait suivant. La somme simple $\Sigma z_1^{k_1} \ldots z_r^{k_r}$ peut être écrite comme une somme $\sum ca_{l_1} \ldots a_{l_t}$, où les indices $l_1, \ldots, l_t$ satisfont à (4), donc

$$\{k_1,\ldots,k_r\}=\sum z_1^{k_1}\ldots z_r^{k_r}=\sum_l c\, a_{l_1}\ldots a_{l_t}. \tag{5}$$

D'après le théorème d'orthogonalité l'opérateur $[l_1, \ldots, l_t]$, appliqué au membre de droite de la dernière relation, donne le résultat c à cause du terme $ca_{l_1} \ldots a_{l_t}$, tandis que la contribution des autres termes est égale à zéro. Ainsi on obtient que dans le développement trouvé le coefficient c est égal à

$$c=[l_1,\ldots,l_t]\{k_1,\ldots,k_r\}. \tag{6}$$

De la même manière on obtient que la somme simple $\sum z_1^{l_1} \ldots z_t^{l_t}$ peut être écrite comme une somme

$$\{l_1,\ldots,l_t\}=\sum z_1^{l_1}\ldots z_t^{l_t}=\sum_k c^*\, a_{k_1}\ldots a_{k_r}, \tag{7}$$

étendue à tous les systèmes $(k_1, \ldots, k_r)$, formés par des entiers positifs avec (4); dans cette identité on a

$$c^*=[k_1, \ldots, k_r]\ \{l_1, \ldots, l_t\}.$$

C'est déjà Cayley qui a montré que les deux nombres c et c^* sont égaux. Donc le coefficient de $a_{l_1} a_{l_2} \ldots a_{l_t}$ dans le développement (5) est égal au coefficient de $a_{k_1} a_{k_2} \ldots a_{k_r}$ dans le développement (7). Avec notre notation cette loi de réciprocité devient une loi de commutativité

$$[l_1, \ldots, l_t]\{k_1, \ldots, k_r\}=[k_1, \ldots, k_r]\{l_1, \ldots, l_t\};$$

par conséquent dans un produit d'un opérateur et d'une somme simple du même degré l'ordre des facteurs peut être changé, si l'on remplace l'opérateur par la somme simple correspondante et en même temps la somme simple par l'opérateur correspondant.

On peut appliquer la théorie de ces opérateurs pour déduire des identités. Par exemple je traiterai les formules connues de NEWTON qui peuvent être obtenues au moyen de l'opérateur A_1. Puisque $z_1, \ldots, z_m$ satisfont à l'équation

$$z^m - a_1 z^{m-1} + \ldots \pm a_m = 0,$$

on trouve par addition

$$\sum z_1^m - a_1 \sum z_1^{m-1} + \ldots \pm m\, a_m = 0. \tag{8}$$

De la définition de l'opérateur A_1 il découle immédiatement que

$$A_1\, uv = u(A_1\, v) + v(A_1\, u).$$

En vertu de

$$A_1\, a_h = a_{h-1} \qquad (h = 1, \ldots, m)$$

on obtient donc

$$\begin{aligned} A_1 (a_h \textstyle\sum z_1^t) &= a_{h-1} \textstyle\sum z_1^t, \qquad \text{si } t \neq 1 \\ &= a_{h-1} \textstyle\sum z_1^t + a_h, \qquad \text{si } t = 1. \end{aligned}$$

Par conséquent $-A_1$ transforme (8) en

$$\sum z_1^{m-1} - a_1 \sum z_1^{m-2} + \ldots \pm a_{m-2} \sum z_1 \mp (m-1)\, a_{m-1} = 0.$$

En appliquant de nouveau le même opérateur nous trouvons

$$\sum z_1^{m-2} - a_1 \sum z_1^{m-3} + \ldots \pm (m-2)\, a_{m-2} = 0.$$

Continuant ainsi nous obtenons pour $h = 1, 2, \ldots, m-1$ les formules de NEWTON

$$\sum z_1^{m-h} - a_1 \sum z_1^{m-h-1} + \ldots \mp a_{m-h-1} \sum z_1 \pm (m-h)\, a_{m-h} = 0.$$

Retournons finalement au problème primitif que nous nous sommes proposés de traiter, à savoir le calcul des coefficients c figurant dans le développement

$$\sum z_1^{k_1} \ldots z_r^{k_r} = \sum_l c\, a_{l_1} \ldots a_{l_t},$$

où $k_1 \geqq k_2 \geqq \ldots \geqq k_r \geqq 1$ et où la somme Σ est étendue à tous les systèmes $l = (l_1, \ldots, l_t)$ formés par des entiers positifs tels que

$$l_1 + \ldots + l_t = k_1 + \ldots + k_r.$$

Comme nous l'avons déjà fait remarquer, dans beaucoup de cas plusieurs termes manquent dans ce développement. A la fin de cette communication nous démontrerons: Dans chaque terme, dont le coefficient c

est différent de zéro on a $\lambda_1 \leqq k_1$, où λ_1 désigne le nombre des entiers $\geqq 1$ figurant dans le système l, donc $\lambda_1 = t$. Chacun de ces termes satisfait en outre à l'inégalité $\lambda_1 + \lambda_2 \leqq k_1 + k_2$ (où λ_2 désigne le nombre des entiers $\geqq 2$ figurant dans le système l), aussi à l'inégalité $\lambda_1 + \lambda_2 + \lambda_3 \leqq k_1 + k_2 + k_3$ (où λ_3 désigne le nombre des entiers $\geqq 3$ figurant dans le système l), etc.

Par conséquent la loi de réciprocité nous apprend que $l_1 \geqq \varkappa_1$, si $l_1 \geqq l_2 \geqq \ldots \geqq l_t$ et si $\varkappa_1$ désigne le nombre des entiers $\geqq 1$ figurant dans le système $k = (k_1, \ldots, k_r)$, donc $\varkappa_1 = r$. En outre $l_1 + l_2 \geqq \varkappa_1 + \varkappa_2$, où $\varkappa_2$ désigne le nombre des entiers $\geqq 2$ figurant dans le système k, etc.

En particulier nous obtenons en vertu de $\lambda_1 = t$ que chaque produit $a_{l_1} a_{l_2} \ldots a_{l_t}$, figurant dans le développement, contient au plus k_1 facteurs. Il est très facile de trouver les coefficients des produits qui contiennent exactement k_1 facteurs, en supposant qu'on sache le développement de la somme simple $\sum z_1^{k_2} z_2^{k_3} \ldots z_{r-1}^{k_r}$. En effet, on a

$$\sum z_1^{k_2} z_2^{k_3} \ldots z_{r-1}^{k_r} = \sum_l{}' c\, a_{l_1-1} \ldots a_{l_t-1},$$

où $\sum\limits_l{}'$ est étendu aux systèmes $l = (l_1, \ldots, l_t)$ avec $t_1 = k_1$ et où les coefficients c sont les mêmes que dans le développement de la somme simple $\sum z_1^{k_1} \ldots z_r^{k_r}$. Car l'opérateur A_{k_1} appliqué à $\sum z_1^{k_1} \ldots z_r^{k_r}$ donne $\sum z_1^{k_2} \ldots z_{r-1}^{k_r}$ et, d'après sa définition, le même opérateur appliqué à $a_{l_1} \ldots a_{l_t}$ donne $a_{l_1-1} \ldots a_{l_t-1}$ ou 0, selon que $t =$ of $< k_1$.

Il s'ensuit que le développement de la somme $\sum z_1^{k_1} \ldots z_r^{k_r}$ contient un terme $a_{l_1} \ldots a_{l_r}$ avec $\lambda_1 = k_1$, $\lambda_2 = k_2, \ldots, \lambda_r = k_r$, et en outre que le coefficient de ce terme est égal à 1. En effet, si cette propriété est vraie pour la somme $\sum z_1^{k_2} z_2^{k_3} \ldots z_{r-1}^{k_r}$ (ce qui est le cas pour $r = 1$, car alors la somme est égale à 1), alors la propriété est aussi valable pour la somme $\sum z_1^{k_1} \ldots z_r^{k_r}$.

Considérons d'abord le cas particulier $\sum z_1^4 z_2^2$. Dans le développement de cette somme simple les termes a_1^6 et $a_1^4 a_2$ manquent, puisque dans le premier terme $\lambda_1 = 6 > 4$ et dans le deuxième $\lambda_1 = 5 > 4$. En outre nous savons que le coefficient de $a_1^2 a_2^2$ est égal à 1. Par conséquent la somme simple considérée a un développement de la forme

$$\sum z_1^4 z_2^2 = a_1^2 a_2^2 + c_2 a_1^3 a_3 + c_3 a_2^3 + c_4 a_1 a_2 a_3 + c_5 a_1^2 a_4 + c_6 a_3^2 + c_7 a_2 a_4 + c_8 a_1 a_5 + c_9 a_6.$$

Le fait que l'opérateur A_1 annule ce résultat nous donne les 6 relations suivantes

$$2 + c_2 = 0;\ 2 + 3c_3 + c_4 = 0;\ 3c_2 + c_4 + c_5 = 0;\ c_4 + 2c_6 + c_7 = 0;$$
$$2c_5 + c_7 + c_8 = 0;\ c_8 + c_9 = 0.$$

Les équations particulières $z^2 = 1$ et $z^4 - 2z^2 + 1 = 0$ donnent en outre $2 = -c_3$ et $4.3 = -8c_3 - 2c_7$, d'où suit immédiatement la formule (1).

Pour trouver les coefficients dans le développement de la somme simple

$$\sum z_1^3 z_2 z_3 z_4 = a_1^2 a_4 + c_2 a_2 a_4 + c_3 a_1 a_5 + c_4 a_6$$

il suffit d'appliquer l'opérateur A_1, si nous connaissons le développement de la somme simple

$$\sum z_1^3 z_2 z_3 = a_1^2 a_3 - 2 a_2 a_3 - a_1 a_4 + 5 a_5,$$

car l'application de l'opérateur A_1 fournit les relations

$$c_2 = -2; \quad 2 + c_2 + c_3 = -1; \quad c_3 + c_4 = 5.$$

Par conséquent

$$\sum z_1^3 z_2 z_3 z_4 = a_1^2 a_4 - 2 a_2 a_4 - a_1 a_5 + 6 a_6.$$

Si nous ne connaissons pas le développement de $\sum z_1^3 z_2 z_3$, nous pouvons appliquer l'opérateur

$$A_2 = [1]^2 - 2[2];$$

l'opérateur A_2 annule le membre de gauche de l'identité considérée et nous appliquons l'opérateur $[1]^2 - 2[2]$ au membre de droite de cette identité. De cette manière nous obtenons

$$4 + 2 c_2 = 0; \; 2 + 2 c_3 = 0, \text{ donc } c_2 = -6 \text{ et } c_3 = -1.$$

La considération de l'équation particulière $(z-1)^6 = 0$ donne

$$60 = 6^2 . 15 - 2 . 15^2 - 6^2 + c_4,$$

donc $c_4 = 6$, de sorte que nous retrouvons notre résultat.

Passons maintenant aux démonstrations.

Sur les sommes de longueur 1.

J'appelle r la longueur de la somme simple $\{k_1, \ldots, k_r\}$, de sorte que

$$\{1\} = \sum z_1; \; \{2\} = \sum z_1^2; \; \{3\} = \sum z_1^3; \; \ldots$$

sont les sommes de longueur 1. Ces sommes de longueur 1 sont indépendantes, c'est-à-dire si un entier positif p et un polynome $\varphi(u_1, \ldots, u_p)$, qui ne dépendent pas du nombre m des indéterminés possèdent pour chaque valeur de m la propriété

$$\varphi(\{1\}, \{2\}, \ldots, \{p\}) = 0,$$

alors le polynome $\varphi(u_1, \ldots, u_p)$ s'annule identiquement. En effet, si $v_1, \ldots, v_p$ désignent des entiers positifs quelconques, si nous posons $m = v_1 + \ldots + v_p$ et si nous choisissons v_1 des m indéterminées égales à 1, ensuite v_2 des indéterminées égales à 2, ..., finalement v_p des indéterminées égales à p, nous obtenons

$$\varphi(1 . v_1 + \ldots + p v_p, \; 1^2 v_1 + \ldots + p^2 v_p, \ldots, 1^p v_1 + \ldots p^p v_p) = 0.$$

Le polynome $\varphi(u_1, \ldots, u_p)$ satisfait à cette identité pour chaque choix des entiers positifs $v_1, \ldots, v_p$, donc pour chaque choix des nombres $v_1, \ldots, v_p$. Si $u_1, \ldots, u_p$ désignent des nombres quelconques on peut choisir les nombres $v_1, \ldots, v_p$ tels que la relation trouvée se transforme en $\varphi(u_1, \ldots, u_p) = 0$, d'où il suit que le polynome φ s'annule identiquement.

Démontrons maintenant que toute fonction entière rationnelle symétrique f des m indéterminées $z_1, \ldots, z_m$ peut être écrite comme un polynome en $\{1\}$, $\{2\}$, ... (d'après le résultat précédent ce polynome est défini univoquement par f). Il suffit de démontrer que toute somme simple $\sum z_1^{k_1} \ldots z_r^{k_r}$ peut être écrite sous cette forme. Cette propriété est évidente pour la somme $\sum z_1^{k_1}$ de longueur 1. Nous pouvons donc supposer que r soit $\geqq 2$ et que la propriété ait été déjà démontrée pour $r - 1$ au lieu de r. Comme (3) désigne le nombre des termes de la somme considérée $\sum z_1^{k_1} \ldots z_r^{k_r}$, le produit

$$(9) \qquad j_1\,!\, j_2\,! \ldots j_u\,! \sum z_1^{k_1} \ldots z_r^{k_r} = \sum^* z_1^{k_1} \ldots z_r^{k_r}$$

est formé par $\frac{m!}{(m-r)!}$ termes. On obtient la dernière somme en prenant toutes les combinations possibles de r éléments figurant dans le système formé par les m indeterminées $z_1, \ldots, z_m$. Par conséquent

$$(10) \qquad (\sum^* z_1^{k_r})\,(\sum^* z_1^{k_1} \ldots z_{r-1}^{k_{r-1}}) = \sum^* z_1^{k_1} \ldots z_r^{k_r} + \sum\nolimits_1 + \ldots \sum\nolimits_r,$$

où

$$\sum\nolimits_1 = \sum^* z_1^{k_1+k_r} z_2^{k_2} \ldots z_{r-1}^{k_{r-1}}, \ldots, \sum\nolimits_{r-1} = \sum^* z_1^{k_1} \ldots z_{r-2}^{k_{r-2}} z_{r-1}^{k_{r-1}+k_r}.$$

Le membre de gauche de (10) est un produit de deux facteurs de longueurs 1 et $r - 1$; les sommes $\sum_1, \ldots, \sum_{r-1}$ ont la longueur $r - 1$. D'après le raisonnement par récurrence ces expressions peuvent être écrites comme des polynomes en $\{1\}$, $\{2\}$, ... Cela est donc aussi le cas avec $\sum^* z_1^{k_1} \ldots z_r^{k_r}$, par conséquent aussi avec la somme simple $\sum z_1^{k_1} \ldots z_r^{k_r}$.

Je démontrerai encore un peu plus, à savoir que le polynome qui est égal à la somme simple $\sum z_1^{k_1} \ldots z_r^{k_r}$ de degré $d = k_1 + \ldots + k_r$ contient le terme $(-)^{r-1}\,\gamma\{d\}$, où $\gamma > 0$, tandis que les autres termes dépendent seulement de $\{1\}, \ldots, \{d-1\}$. Cette propriété est évidente pour $r = 1$ (avec $\gamma = 1$) et suit pour $r \geqq 2$ immédiatement de (10) par un raisonnement de récurrence.

Démonstration du théorème fondamental.

Nous venons de voir que

$$a_d = \sum z_1 z_2 \ldots z_d,$$

où $d \geqq 1$, peut être écrit sous la forme

$$(11) \qquad a_d = (-)^{d-1}\,\gamma\,\{d\} + \psi(\{1\}, \ldots, \{d-1\}), \quad \text{où} \quad \gamma > 0.$$

Chaque somme simple $\{d\}$ de longueur 1 peut être écrite comme un polynome en $a_1, a_2, \ldots, a_d$. Cela est évident pour $d = 1$ et suit pour $d \geqq 2$ immédiatement de (11) par un raisonnement de récurrence. Toute fonction entière rationnelle symétrique f des m indéterminées $z_1, \ldots, z_m$ peut être écrite comme un polynome en $\{1\}, \{2\}, \ldots$, donc aussi comme un polynome χ en $a_1, a_2, \ldots$. Ce polynome χ est défini univoquement par la fonction f. En effet, sinon on obtiendrait une relation de la forme $\omega(a_d) = 0$, où d est un entier positif et où $\omega(u)$ désigne un polynome qui n'est pas identiquement égal à zéro et dont les coefficients sont des fonctions entières rationnelles de $a_1, \ldots, a_{d-1}$, donc aussi de $\{1\}, \ldots, \{d-1\}$. Grâce à (11) on trouverait

$$\omega((-1)^d \gamma\{d\} + \psi) = 0,$$

où ψ est une fonction entière rationnelle de $\{1\}, \ldots, \{d-1\}$. En vertu de $\gamma \neq 0$ les sommes $\{1\}, \{2\}, \ldots, \{d\}$ de longueur 1 dépendraient mutuellement, cequi n'est pas le cas.

Démonstration du théorème d'isomorphie.

Comme nous l'avons vu, l'identité (10) nous permet de réduire chaque fonction entière rationnelle f d'une ou plusieurs sommes simples à un polynome φ contenant seulement des sommes de longueur 1; le polynome φ est défini univoquement par la fonction donnée f. Il suffit de démontrer que l'identité (10) reste valable si l'on y remplace chaque somme simple par l'opérateur correspondant, c'est-à-dire si l'on remplace chaque somme $\sum^* z_1^{k_1} \ldots z_r^{k_r}$ par l'opérateur

$$\sum_{\mu_1=k_1}^{m} \cdots \sum_{\mu_r=k_r}^{m} a_{\mu_1-k_1} \cdots a_{\mu_r-k_r} \frac{\partial^r}{\partial a_{\mu_1} \ldots \partial a_{\mu_r}}.$$

En effet, application répétée de ce résultat nous apprend que l'identité $f = \varphi$ reste valable, si l'on y remplace chaque somme simple par l'opérateur correspondant et dans le cas particulier où nous partons de l'identité $f = 0$, le polynome φ s'annule identiquement, de sorte que l'identité $f = 0$ reste valable, si l'on y remplace chaque somme simple par l'opérateur correspondant.

Par conséquent il suffit de démontrer que l'opérateur

$$\sum_{\mu=k_r}^{m} a_{\mu-k_r} \frac{\partial}{\partial a_\mu} \sum_{\mu_1=k_1}^{m} \cdots \sum_{\mu_{r-1}=k_{r-1}}^{m} a_{\mu_1-k_1} \cdots a_{\mu_{r-1}-k_{r-1}} \frac{\partial^{r-1}}{\partial a_{\mu_1} \ldots \partial a_{\mu_{r-1}}}$$

est égal à

$$\sum_{\mu_1=k_1}^{m} \cdots \sum_{\mu_r=k_r}^{m} a_{\mu_1-k_1} \cdots a_{\mu_r-k_r} \frac{\partial^r}{\partial a_{\mu_1} \ldots \partial a_{\mu_r}}$$

augmenté de $r-1$ opérateurs qui sont analogues entre eux et dont le premier est égal à

$$\sum_{\mu_1=k_1+k_r}^{m} \sum_{\mu_2=k_2}^{m} \cdots \sum_{\mu_{r-1}=k_{r-1}}^{m} a_{\mu_1-k_1-k_r} a_{\mu_2-k_2} \cdots a_{\mu_{r-1}-k_{r-1}} \frac{\partial^{r-1}}{\partial a_{\mu_1} \ldots \partial a_{\mu_{r-1}}}.$$

Cette propriété est évidente, puisque

$$\frac{\partial}{\partial a_\mu}\left\{a_{\mu_1-k_1}\dots a_{\mu_{r-1}-k_{r-1}}\frac{\partial^{r-1}}{\partial a_{\mu_1}\dots\partial a_{\mu_{r-1}}}\right\}$$

est égal à

$$a_{\mu_1-k_1}\dots a_{\mu_{r-1}-k_{r-1}}\frac{\partial^r}{\partial a_{\mu_1}\dots\partial a_{\mu_{r-1}}\partial a_\mu},$$

augmenté de $r-1$ termes qui sont analogues entre eux et dont le premier est égal à

$$\begin{cases} a_{\mu_2-k_2}\dots a_{\mu_{r-1}-k_{r-1}}\dfrac{\partial^{r-1}}{\partial a_{\mu_1}\dots\partial a_{\mu_{r-1}}} & \text{si } \mu=\mu_1-k_1,\\ 0 & \text{si } \mu\neq\mu_1-k_1. \end{cases}$$

Ainsi on obtient le théorème d'isomorphie.

Le théorème d'orthogonalité.

Il faut démontrer: Soit

$$k_1+\dots+k_r=l_1+\dots+l_t\,; \tag{12}$$

alors on a

(13) $[k_1,\dots,k_r]\,a_{l_1}a_{l_2}\dots a_{l_t}=1$, si les systèmes $(k_1,\dots,k_r)$ et $(l_1,\dots,l_t)$ sont les mêmes, peut-être à l'ordre près,
$=0$ dans tous les autres cas.

Le membre de gauche de (13) est égal à

$$\frac{1}{j_1!\,j_2!\dots j_u!}\sum_{\mu_1=k_1}^{m}\dots\sum_{\mu_r=k_r}^{m}\frac{\partial^r(a_{l_1}a_{l_2}\dots a_{l_t})}{\partial a_{\mu_1}\dots\partial a_{\mu_r}}. \tag{14}$$

Il suffit de considérer dans cette somme les termes dans lesquels le système $(\mu_1,\dots,\mu_r)$ est un système partiel de $(l_1,\dots,l_t)$, puisque les autres termes s'annulent. En vertu de (12) on déduit des inégalités

$$\mu_1\geqq k_1,\ \mu_2\geqq k_2,\ \dots,\ \mu_r\geqq k_r$$

que dans tous ces termes les systèmes $(\mu_1,\dots,\mu_r)$ et $(l_1,\dots,l_t)$ sont les mêmes, peut-être à l'ordre près. On obtient donc

$$\mu_1+\dots+\mu_r=l_1+\dots+l_t=k_1+\dots+k_r,$$

d'où il suit, à cause de (12), que

$$\mu_1=k_1;\ \mu_2=k_2;\ \dots;\ \mu_r=k_r.$$

Ainsi on trouve que tous les termes figurant dans la somme (14) s'annulent, si les systèmes $(k_1,\dots,k_r)$ et $(l_1,\dots,l_t)$ sont différents. Si ces deux systèmes sont les mêmes, peut-être à l'ordre près, le membre de gauche de (13) est égal à

$$\frac{1}{j_1!\dots j_u!}\;\frac{\partial^r(a_{k_1}\dots a_{k_r})}{\partial a_{k_1}\dots\partial a_{k_r}}.$$

Le dernier facteur peut être écrit comme un produit de u facteurs, dont le premier est égal à

$$\frac{\partial^{j_1}(a_k^{j_1})}{\partial a_k^{j_1}} = j_1\,!\,,$$

si k figure exactement j_1 fois dans le système $(k_1, \ldots, k_r)$. Le deuxième facteur est égal à $j_2!$, etc. Par conséquent le membre de droit de (13) est dans ce cas égal à

$$\frac{j_1!\ldots j_u!}{j_1!\ldots j_u!} = 1\,.$$

La loi de réciprocité.

Nous savons qu'une somme simple $\sum z_1^{k_1} \ldots z_r^{k_r}$ de degré d possède le développement

$$\sum z_1^{k_1} \ldots z_r^{k_r} = \sum_l c_{kl}\, a_{l_1} a_{l_2} \ldots a_{l_t}\,, \tag{15}$$

où $\sum\limits_l$ est étendu à tous les systèmes $l = (l_1, \ldots, l_t)$ formés par des entiers positifs tels que

$$l_1 + \ldots + l_t = d.$$

Il faut démontrer que les coefficients c_{kl} et c_{lk} sont identiques. En appliquant (15) à un système de n indéterminées $u_1, \ldots, u_n$ avec les fonctions symétriques élémentaires

$$b_1 = \sum u_1,\ b_2 = \Sigma\, u_1 u_2,\ b_3 = \Sigma\, u_1 u_2 u_3,\ \ldots,$$

on obtient pour la somme simple $\sum u_1^{l_1} \ldots u_t^{l_t}$ de degré d le développement

$$\sum u_1^{l_1} \ldots u_t^{l_t} = \sum_k c_{lk}\, b_{k_1} \ldots b_{k_r}\,.$$

où $\sum\limits_k$ est étendu à tous les systèmes $k = (k_1, \ldots, k_r)$ formés par des entiers positifs tels que

$$k_1 + \ldots + k_r = d.$$

Ainsi on obtient

$$\sum_k b_{k_1} \ldots b_{k_r} \left(\sum z_1^{k_1} \ldots z_r^{k_r}\right) = \sum_k \sum_l c_{kl}\, a_{l_1} \ldots a_{l_t}\, b_{k_1} \ldots b_{k_r} \tag{16}$$

et

$$\sum_l a_{l_1} \ldots a_{l_t} \left(\sum u_1^{l_1} \ldots u_t^{l_t}\right) = \sum_k \sum_l c_{lk}\, a_{l_1} \ldots a_{l_t}\, b_{k_1} \ldots b_{k_r}\,. \tag{17}$$

Il suffit de démontrer que ces deux expressions sont identiques; en effet, parce que les nombres $a_1, a_2, \ldots$ et pareillement les nombres $b_1, b_2, \ldots$ sont indépendants, cette propriété entraîne $c_{kl} = c_{lk}$.

La démonstration est maintenant facile. On peut écrire le produit

$$\prod_{\mu=1}^{m}\prod_{\nu=1}^{n}(1+\lambda z_\mu u_\nu)=\prod_{\mu=1}^{m}\Big((1+\lambda z_\mu u_1)\dots(1+\lambda z_\mu u_n)\Big)$$

$$=\prod_{\mu=1}^{m}(1+\lambda z_\mu b_1+\dots+\lambda^n z_\mu^n b_n)$$

comme un polynome en λ, dans lequel le coefficient de λ_j^d est exactement égal au membre de gauche de (16) et donc aussi égal au membre de gauche de (17), parce que le produit considéré reste le même si l'on change les systèmes $(z_1, \dots, z_m)$ et $(u_1, \dots, u_n)$. Ainsi la loi de réciprocité a été démontrée.

On peut formuler cette loi d'une manière un peu plus générale comme suit: Si s et s^* sont deux fonctions entières rationnelles homogènes symétriques du même degré d et si S et S^* désignent les opérateurs correspondants, on a $S^*s = Ss^*$. En effet on peut écrire s et s^* comme des fonctions linéaires de sommes simples,

$$s=\sum\gamma\,\{k_1, \dots, k_r\} \text{ et } s^*=\sum\gamma^*\,\{l_1, \dots, l_t\},$$

donc d'après le théorème d'isomorphie

$$S=\sum\gamma\,[k_1, \dots, k_r] \text{ et } S^*=\sum\gamma^*\,[l_1, \dots, l_t],$$

de sorte que la loi commutative nous apprend

$$S^*s=\sum\gamma\gamma^*\,[l_1, \dots, l_t]\,\{k_1, \dots, k_r\}=\sum\gamma\gamma^*\,[k_1, \dots, k_r]\,\{l_1, \dots, l_t\}=Ss^*.$$

Sur les opérateurs supprimant respectivement 1, 2, ...

Il faut démontrer

$$A_k\sum z_1^{k_1}\dots z_r^{k_r}=0, \text{ si tous les exposants sont différents de } k,$$

$$=\sum z_1^{k_1}\dots z_{r-1}^{k_{r-1}}, \text{ si } k_r=k.$$

Nous pouvons supposer que $k_1+\dots+k_r>1$ (sinon la propriété est évidente) et que la propriété ait été déjà démontrée si le degré $k_1+\dots+k_r$ est remplacé par un plus petit entier positif.

Le membre de gauche de la propriété à démontrer est une fonction entière rationnelle symétrique des m indéterminées de degré $d=k_1+\dots+k_r-k$ et peut donc être écrite comme une fonction linéaire de sommes simples de degré d. Ainsi on obtient

$$A_k\sum z_1^{k}\dots z_r^{k_r}=\sum c\,\{m_1, \dots, m_q\}, \tag{18}$$

où $m_1+\dots+m_q=d<k_1+\dots+k_r$.

Appliquons aux deux membres de la dernière relation l'opérateur $A_{m_1}A_{m_2}\dots A_{m_q}$. D'après notre hypothèse de récurrence cet opérateur

annule tous les termes du membre de droite excepté le terme $c\{m_1, \ldots, m_q\}$, qui donne la contribution c, donc

$$c = A_{m_1} \ldots A_{m_q} A_k \{k_1, \ldots k_r\},$$

d'où il suit d'après la loi commutative que nous venons de déduire

$$c = [k_1, \ldots k_r]\, a_{m_1} \ldots a_{m_q} a_k.$$

Le théorème d'orthogonalité nous apprend que ce résultat est presque toujours égal à zèro. La seule exception est que les systèmes $(k_1, \ldots, k_r)$ et $(m_1, \ldots, m_q, k)$ sont identiques, peut-être à l'ordre près et dans ce cas exceptionnel $c = 1$. Par conséquent, si aucun des exposants $k_1, \ldots, k_r$ n'est égal à k, tous les coefficients c s'annulent. Si par contre k figure dans le système des exposants, le membre de droite de (18) est formé par un seul terme; dans ce terme le coefficient c est égal à 1 et on obtient $\{m_1, \ldots, m_q)$ en supprimant dans $\{k_1, \ldots, k_r\}$ l'exposant k. Ainsi nous trouvons la propriété à démontrer.

Sur les termes qui manquent dans le développement d'une somme simple.

Soit f une fonction entière rationnelle symétrique des m indéterminés $z_1, \ldots, z_m$ et soit q un entier positif $\leqq m$. Si nous remplaçons $z_1, z_2, \ldots, z_q$ par un indéterminé ω, alors f devient un polynome en ω, dont les coefficients dépendent de $z_{q-1}, \ldots z_m$.

Supposons que le degré de ce polynome soit $\leqq p$, et démontrons dans ce cas que f possède pour chaque choix des indéterminés $z_1, \ldots, z_m$ le développement

$$f = \sum_l c\, a_{l_1} \ldots a_{l_t}, \tag{19}$$

où $\sum\limits_l$ est étendu à des systèmes $l = (l_1, \ldots, l_t)$, formés par des entiers positifs tels que

$$\lambda_1 + \lambda_2 + \ldots + \lambda_q \leqq p\,; \tag{20}$$

λ_h désigne le nombre des entiers $\geqq h$ figurant dans le système l.

Cette propriété est évidente pour $m = 1$, car alors $q = 1$ et f est un polynome en $z_1 = a_1$ d'un degré $\leqq p$; par conséquent dans chaque système l figurant dans le membre de droite de (19) chaque élément est également à 1, donc $l_1 = t \leqq p$.

Dans la démonstration nous supposons que m soit $\geqq 2$ et que la propriété soit valable pour $m - 1$ au lieu de m. Nous pouvons écrire f comme une fonction entière rationnelle de $a_1, a_2, \ldots$, donc $f = u + a_m v$, où u est la somme des termes, qui ne sont pas divisibles par a_m. Si nous remplaçons $z_1, z_2, \ldots, z_q$ par ω et en même temps z_m par 0, les fonctions f et u sont les mêmes polynomes en ω donc un polynome de degré $\leqq p$.

D'après le raisonnement par recurrence u est une fonction linéaire de produits $a_{l_1} \dots a_{l_t}$ avec (20).

v est une fonction rationnelle entière symétrique des indéterminés $z_1, \dots, z_m$, qui devient un polynome en ω de degré $p-q$, si l'on remplace les indéterminés $z_1, \dots, z_q$ par ω. D'après le raisonnement par recurrence v est une fonction linéaire de produits de la forme $a_{L_1} \dots a_{L_T}$ avec

$$\Lambda_1 + \Lambda_2 + \dots + \Lambda_q \leqq p - q;$$

Λ_h désigne le nombre des entiers $\geqq h$ figurant dans le système $L_1, \dots, L_T$. Le produit $a_m v$ est donc une fonction linéaire de produits $a_{l_1} \dots a_{l_t}$, où $\lambda_h = \Lambda_h + 1$, par conséquent

$$\lambda_1 + \dots + \lambda_q = \Lambda_1 + \dots + \Lambda_q + q \leqq p.$$

Ce résultat entraîne immédiatement que la somme simple

$$f = \sum z_1^{k_1} \dots z_r^{k_r},$$

où $k_1 \geqq k_2 \geqq \dots \geqq k_r \geqq 1$, est une fonction linéaire de produits de la forme $a_{l_1} \dots a_{l_t}$, tels que

$$\lambda_1 \leqq k_1; \lambda_1 + \lambda_2 \leqq k_1 + k_2; \dots; \lambda_1 + \dots + \lambda_{r-1} \leqq k_1 + \dots + k_{r-1}.$$

Il va sans dire, que

$$\lambda_1 + \lambda_2 + \dots \leqq k_1 + k_2 + \dots + k_r,$$

parce que le membre de gauche est égal à $l_1 + l_2 + \dots + l_t$.

CORRELATION PROBLEMS IN A ONE-DIMENSIONAL MODEL OF TURBULENCE. III. *)

BY

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20. *Further investigation of the correlation function* $\overline{v_1 v_2}$. — In Part II expressions have been constructed for the correlation functions $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$. These expressions have been applied to investigate some consequences of the fundamental equation (12), particular attention being given to values of η small in comparison with the mean length l of the segments λ_i. In the results an important part was played by the statistical parameter $f_0 = \lim f_1(\lambda)$ for $\lambda \to 0$, together with certain mean values referring to $\lambda \to 0$.

We now turn to some long distance relations. In the first place we will show that the expression (47) for $\overline{v_1 v_2}$ and its derivative (48) vanish for large values of η.

We write

$$\int_0^\eta d\lambda\, f_k(\lambda) = g_k \tag{69}$$

so that g_k measures the probability for Λ_k to be less than η [Λ_k is the sum of the lengths of k consecutive segments λ_i, from λ_{i+1} to λ_{i+k} inclusive; compare (37)]. It will be evident that the expression:

$$l^2 \sum_{k=1}^\infty \varphi_k = \int_0^\eta d\lambda \sum_1^\infty f_k(\lambda)\, \overline{\tau_i \tau_{i+k}}^{\,*} \tag{70}$$

combines all cases defined by the inequalities

$$\begin{aligned} 0 &< \lambda_{i+1} && < \eta \\ 0 &< \lambda_{i+1} + \lambda_{i+2} && < \eta \\ 0 &< \lambda_{i+1} + \lambda_{i+2} + \lambda_{i+3} && < \eta, \text{ etc.,} \end{aligned}$$

each case with its proper frequency of occurrence. Without omitting any one we can re-arrange these cases in such a way that we first consider the class for which

$$\lambda_{i+1} < \eta < \lambda_{i+1} + \lambda_{i+2}\,;$$

*) Continued from these Proceedings, p. 393—406. — In eq. (50), p. 401, the last factor should be η^2/l^2, the same as in eq. (49b).

next the class for which

$$\lambda_{i+1} + \lambda_{i+2} < \eta < \lambda_{i+1} + \lambda_{i+2} + \lambda_{i+3}$$

(which naturally entails $\lambda_{i+1} < \eta$); then the class for which

$$\lambda_{i+1} + \lambda_{i+2} + \lambda_{i+3} < \eta < \lambda_{i+1} + \lambda_{i+2} + \lambda_{i+3} + \lambda_{i+4}$$

(which entails $\lambda_{i+1} < \eta$; $\lambda_{i+1} + \lambda_{i+2} < \eta$), etc. The probabilities to be assigned to the classes thus distinguished (that is, the relative numbers of cases falling in any one of them) are given by

(A) $$g_1 - g_2 \quad ; \quad g_2 - g_3 \quad ; \quad g_3 - g_4 \; ; \; \cdots$$

It follows from (36) that

(B) $$\sum_1^{\infty} (g_k - g_{k+1}) = 1$$

if η is sufficiently large.

We take those terms of the sum (70) which refer to the particular class defined by

(C) $$\Lambda_k < \eta < \Lambda_{k+1}$$

and add together the quantities $\tau_i \tau_{i+1}$, $\tau_i \tau_{i+2}$, $\tau_i \tau_{i+3}, \ldots$ occurring in these terms. The contribution obtained in this way can be written:

(D) $$(g_k - g_{k+1}) \, \overline{\tau_i T_k'}^{**}$$

where

(71) $$T_k' = \tau_{i+1} + \tau_{i+2} + \ldots + \tau_{i+k} = T_k - \tfrac{1}{2}\tau_i + \tfrac{1}{2}\tau_{i+k} = \sigma_{i+k} - \sigma_i$$

[compare (44)], while the new type of restricted mean values refers to the cases satisfying the inequalities (C). When (D) is summed with respect to k from $k = 1$ to infinity, we obtain the expression (70).

We write:

(72) $$T_k' = \eta - \tfrac{1}{2}\tau_i + \zeta_i + \theta$$

(compare fig. 7). Since $T_k + \zeta_{i+k} - \zeta_i = \Lambda_k$, so that

$$T_k' = \Lambda_k - \tfrac{1}{2}\tau_i + \zeta_i + \tfrac{1}{2}\tau_{i+k} - \zeta_{i+k},$$

we find:

$$\theta = \Lambda_k - \eta + \tfrac{1}{2}\tau_{i+k} - \zeta_{i+k} = \Lambda_{k+1} - \eta - \tfrac{1}{2}\tau_{i+k+1} - \zeta_{i+k+1}.$$

Hence in consequence of (C) θ satisfies the inequalities:

$$-\tfrac{1}{2}\tau_{i+k+1} - \zeta_{i+k+1} < \theta < \tfrac{1}{2}\tau_{i+k} - \zeta_{i+k}.$$

When η is sufficiently large, g_k will practically be equal to 1 and $g_k - g_{k+1} = 0$, for all k which are small compared with η/l. Terms with small values of k consequently will play an insignificant part in the sum of the quantities (D). We can safely assume that in all significant terms θ will be completely independent of τ_i, so that

(72a) $$\overline{\tau_i \theta}^{**} = \overline{\tau_i} \cdot \overline{\theta}^{**} = 0$$

For later use we add the formula:

$$\overline{\tau_i\,\theta^2}^{**} = \overline{\tau_i}\cdot\overline{\theta^2}^{**} = \tfrac{1}{3}\,\overline{\{(\tfrac{1}{2}\,\tau_{i+k+1}+\zeta_{i+k+1})^3+(\tfrac{1}{2}\,\tau_{i+k}-\zeta_{i+k})^3\}}^{**},$$

which for large η we can assume to become equal to its ordinary mean value, so that

$$\overline{\tau_i\,\theta^2}^{**} = \overline{\tau_i\,\zeta_i^2} + \tfrac{1}{12}\,\overline{\tau_i^3} = l^3\,\{\tilde{\omega} + \tfrac{1}{12}\,(1+\omega^*)\} \tag{72b}$$

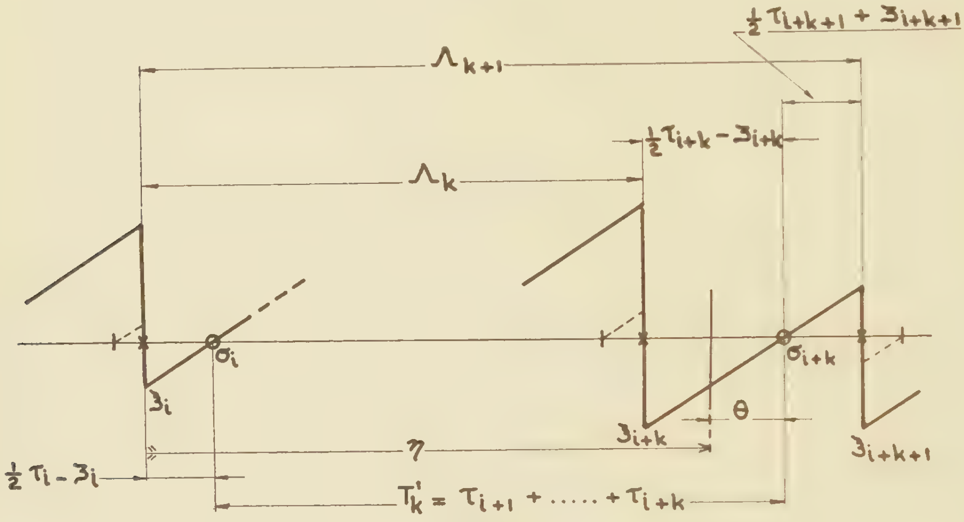

Fig. 7.

Since $\overline{\tau_i\,\zeta_i}^{**}$ will be zero in the same way as $\overline{\tau_i\,\zeta_i}$, the only terms remaining in the quantity (D) are:

$$(g_k - g_{k+1})\,\overline{\tau_i\,(\eta - \tfrac{1}{2}\,\tau_i)}^{**}.$$

Having regard to (B), we thus arrive at the result:

$$\sum_{k=1}^{\infty} \varphi_k \cong \frac{\overline{\tau_i\,(\eta-\frac{1}{2}\,\tau_i)}}{l^2} = \frac{\eta}{l} - \tfrac{1}{2}\,(1+\omega) \qquad (\text{for } \eta\to\infty) \tag{73}$$

In connection with (48) this already proves that $\partial(\overline{v_1 v_2})/\partial\eta$ vanishes for $\eta\to\infty$.

Differentiation of (73) with respect to η gives:

$$\sum_{k=1}^{\infty} f_k(\eta)\;\overline{\tau_i\,\tau_{i+k}}^{*} \cong l \qquad (\text{for } \eta\to\infty) \tag{74}$$

Since terms with small k do not contribute materially to this sum when η is large, we can safely replace $\overline{\tau_i\,\tau_{i+k}}^{*}$ by l^2, so that we find:

$$\sum_{k=1}^{\infty} f_k(\eta) \cong 1/l \qquad (\text{for } \eta\to\infty) \tag{74a}$$

which result is identical with eq. (39a).

By means of a similar re-arrangement as was used in the case of (70) we obtain the transformation:

$$l^3 \sum_{k=1}^{\infty} \Phi_k = \tfrac{1}{2} \sum (g_k - g_{k+1}) \overline{(\tau_i^2 T'_k + \tau_i T'^2_k)}^{**} .$$

Expressing T'_k through η and θ as before and making use of (72*a*) and (72*b*), we obtain, after some calculation:

$$\sum_{k=1}^{\infty} \Phi_k \cong \frac{\eta^2}{2\,l^2} + \tilde{\omega} - \tfrac{1}{12}\,(1+\omega^*) \qquad (\text{for } \eta \to \infty) \tag{75}$$

Finally we consider:

$$l^3 \sum_{k=1}^{\infty} (\Phi_k - \chi_k) = \int_0^{\eta} d\lambda \sum_1^{\infty} f_k(\lambda)\, \overline{(T_k - \lambda)\, \tau_i \tau_{i+k}}^{*} ,$$

where λ stands for Λ_k. We have already seen that $T_k - \Lambda_k = \zeta_i - \zeta_{i+k}$; hence, making use of the second invariant property:

$$l^3 \sum_{k=1}^{\infty} (\Phi_k - \chi_k) = 2 \int_0^{\eta} d\lambda \sum_1^{\infty} f_k(\lambda)\, \overline{\zeta_i \tau_i \tau_{i+k}}^{*} \tag{76}$$

By re-arrangement this is transformed into:

$$2 \sum (g_k - g_{k+1})\, \overline{\zeta_i \tau_i T'_k}^{**} ,$$

which, on working out, comes down to $2\,\overline{\tau_i \zeta_i^2}^{**}$, for which we can take $2\,\tilde{\omega}\, l^3$. Consequently:

$$\sum_{k=1}^{\infty} (\Phi_k - \chi_k) \cong 2\,\tilde{\omega} \qquad (\text{for } \eta \to \infty) \tag{77a}$$

and

$$\sum_{k=1}^{\infty} \chi_k \cong \frac{\eta^2}{2l^2} - \tilde{\omega} - \tfrac{1}{12}(1+\omega^*) \qquad (\text{for } \eta \to \infty) \tag{77b}$$

These results prove that both $\overline{v_1 v_2}$ and $\partial(\overline{v_1^2 v_2})/\partial\eta$ vanish for $\eta \to \infty$. I have not calculated $\sum X_k$, but it seems safe to assume that the expression for $\overline{v_1^2 v_2}$ likewise will vanish for $\eta \to \infty$. We shall make use of this assumption in the calculation of its FOURIER transform.

21. The following observations can be made in connection with eq. (73). We write:

$$\overline{\tau_i \tau_{i+k}} = l^2(1+\omega_k) \quad ; \quad \overline{\tau_i \tau_{i+k}}^{*} = l^2(1+\omega_k^*) \tag{78}$$

so that ω_k is a constant, whereas ω_k^* is a function of $\eta = \Lambda_k$, to which refers the restricted mean value [7]). It follows from (42) that

$$\int_0^{\infty} f_k(\eta)\, \omega_k^*\, d\eta = \omega_k .$$

[7]) The ω^* which was introduced in (31), has no connection with the ω_k^* defined here.

We can assume that ω_k and ω_k^* will vanish for large k. Hence for all relevant values of k we can write

$$\int_0^\eta f_k(\eta)\,\omega_k^*\,d\eta \cong \omega_k\,,$$

provided η has a large value. Equation (73) consequently gives:

$$\sum_{k=1}^{\infty} g_k(\eta) = \int_0^\eta \{\Sigma\, f_k(\eta)\}\,d\eta = \frac{\eta}{l} - \tfrac{1}{2} - \left(\tfrac{1}{2}\,\omega + \sum_1^{\sim} \omega_k\right) \tag{79}$$

for large values of η. This result confirms (40a) and gives an interpretation of the constant which had been left undetermined.

Both equation (72a) and equation (79) are connected in a certain way with the condition to be satisfied by the distribution of the λ_i and τ_i, in order that the mean value $\overline{\zeta_i^2}$ shall exist. Reference to this condition has been made in sections 8 and 12. One way of satisfying (79) is to assume that

$$\sum_1^{\infty} g_k(\eta) = \frac{\eta}{l} - \tfrac{1}{2} \tag{80a}$$

and

$$\tfrac{1}{2}\,\omega + \sum_1^{\sim} \omega_k = 0 \tag{80b}$$

The first equation is obtained when we suppose that for large values of k the dispersion of Λ_k about its mean value $k\,l$ is symmetrical and independent of k. The system of functions g_k then obtains the character indicated in fig. 8. We can assume that $g_k\,(h\,l) = 0$ when $k > 2h$. With $\eta = h\,l$ it is found that

$$\Sigma\, g_k(\eta) = \tfrac{1}{2} + \sum_1^{h-1} (g_k + g_{2h-k}) = \tfrac{1}{2} + (h-1) = \frac{\eta}{l} - \tfrac{1}{2}.$$

The relation can be expected to hold also for values of η not equal to an integer multiple of l.

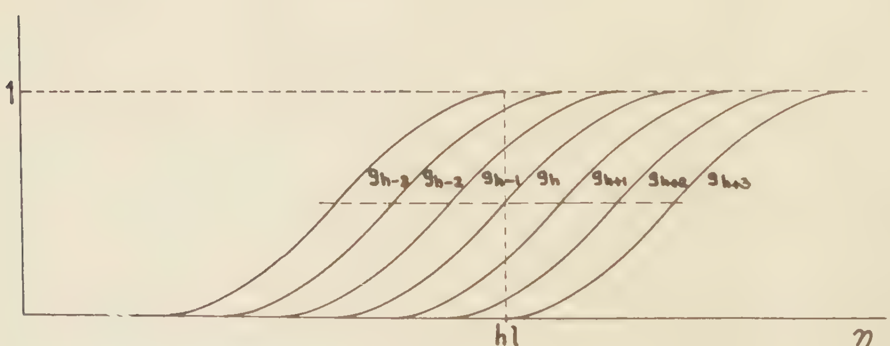

Fig. 8.

Equation (80b) expresses the condition that the dispersion of T'_h about its mean value $h\,l$ shall not increase with h. This follows when we calculate:

$$\overline{(T'_h - hl)^2} = h\,(\omega + 2\,\omega_1 + 2\,\omega_2 + 2\,\omega_3 + \ldots)\,l^2 - 2\,(\omega_1 + 2\,\omega_2 + 3\,\omega_3 + \ldots)\,l^2.$$

When (80b) is satisfied, the first sum vanishes, while the latter sum converges to a value independent of k.

As was remarked in 12, it may be that the substitution of (80a) and (80b) for (79) introduces an unnecessary restriction. What is actually required is that $\overline{(T'_h - \Lambda_h)^2}$ shall not increase with h.

Since the sums considered in eqs. (70) and (76) will occur repeatedly in the following sections, not only for large values of η, but also for other values, it is convenient to shorten notation by writing:

$$\sum_{k=1}^{\infty} f_k(\eta)\,\overline{\tau_i \tau_{i+k}}^{*} = l \cdot \Pi(z) \tag{81}$$

$$\sum_{k=1}^{\infty} f_k(\eta)\,\overline{\zeta_i \tau_i \tau_{i+k}}^{*} = l^2 \cdot \Omega(z) \tag{82}$$

where $z = \eta/l$. Equations (70) and (76) then become:

$$\sum_{1}^{\infty} \varphi_k = \int_0^z \Pi\, dz \tag{83}$$

$$\sum_{1}^{\infty} (\Phi_k - \chi_k) = 2 \int_0^z \Omega\, dz \tag{84}$$

It follows from (73) that

$$\int_0^z \Pi\, dz \simeq z - \tfrac{1}{2}(1 + \omega) \qquad (\text{for } z \to \infty) \tag{83a}$$

and from (77a) that

$$\int_0^\infty \Omega\, dz = \tilde{\omega} \tag{84a}$$

22. *Fourier transform of* $\overline{v_1 v_2}$. — According to section 3 the FOURIER transform $\Gamma(n)$ is defined by

$$\pi\,\Gamma(n) = 2 \int_0^\infty \overline{v_1 v_2} \cos n\eta\, d\eta \tag{17a}$$

It should be kept in mind that, in $\Gamma(n)$ and in all following equations, n is an arbitrary quantity of dimension (length)$^{-1}$ and not an integer. — We make use of formula (47) for $\overline{v_1 v_2}$. The circumstance that it is slightly in error for very small values of η can be left out of account so long as n

is not too large. Integration by parts, having regard to the vanishing of $\overline{v_1 v_2}$ and of its derivative for $\eta \to \infty$, gives

$$\pi\, \Gamma(n) = \frac{\beta^2 l(1+\omega)}{n^2} + \frac{2\beta^2 l}{n^2} \int\limits_0^\infty (\Pi - 1) \cos n l z \; dz \tag{85}$$

The correction which must be applied for very large values of n, can be found if we make use of the expression (49), which gives the correction to $\overline{v_1 v_2}$ connected with the rounding off of the jump at ζ_i. The FOURIER transform of (49) (multiplied by π) is [8]:

$$-\frac{\beta^2 \tau_i^2}{n^2} \left\{ 1 - \frac{(2\pi n v/\beta \tau_i)^2}{[\sinh (2\pi n v/\beta \tau_i)]^2} \right\}.$$

When the mean value is taken, the first term cancels the first term of (85), which is replaced by:

$$\overline{\frac{4\pi^2 v^2/l}{[\sinh (2\pi n v/\beta \tau_i)]^2}} \tag{85a}$$

So long as n is small compared with $\beta\, l/v$, formula (85) is sufficiently accurate. Since $\Pi - 1$ is a bounded quantity without discontinuities, the integral in (85) will be of order n^{-1}, making the whole second term of (85) of order n^{-3}. Hence for not too small n we may write:

$$\pi\, \Gamma(n) \cong \frac{l(1+\omega)}{n^2 t^2} \tag{85b}$$

This indicates a spectral function proportional to n^{-2}. The change of $\Gamma(n)$ with time in this range can be found with the aid of (57).

Since $\Gamma(n)$ measures the energy distribution, the amplitude of the spectral components will be proportional to n^{-1}; while for each separate component in this range, the dissipation in unit time will be *independent of n*. This result is analogous to that obtained in previous work, referring to an equation of similar type as eq. (1), but completed with a term representing the action of an outward agency and applied to a limited domain $0 \leqslant y \leqslant b$ [9]). It is a consequence of the nature of the jumps occurring in the function $v(y)$.

The range where (85b) is valid, ends when n approaches $\beta\, l/v = l/v\, t$. From then onward the spectral function decreases more rapidly than n^{-2}.

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[8]) Compare D. BIERENS DE HAAN, Nouv. Tables d'Intégrales Définies (Leiden 1867), Table 264, no. 2, from which the integral required can be obtained by differentiation with respect to p.

[9]) See: Mathematical examples illustrating relations in the theory of turbulent fluid motion, Verhand. Kon. Nederl. Akademie v. Wetenschappen (1e sect.), vol. XVII, no. 2, 18—33 (1939); or the paper in Advances in Applied Mechanics, 1, 175 (1948).

Let us finally consider very small values of n. In the integral defining $\Gamma(n)$ we develop the cosine function. This will bring us to the development already indicated in formula (18a), with

$$\pi \Gamma_{2m} = 2 \int_0^\infty \overline{v_1 v_2}\, \eta^{2m}\, d\eta$$

(m being an integer), provided we may assume that the integrals converge [10]). By means of partial integration this expression can be transformed into:

$$\pi \Gamma_{2m} = -\frac{2\beta^2 l^{2m+3}}{(2m+1)(2m+2)} \int_0^\infty (\Pi - 1)\, z^{2m+2}\, dz \tag{86}$$

23. *Fourier transform of* $\overline{v_1{}^2 v_2}$ *and application of equation* (19). — We have

$$\pi \Psi(n) = 2 \int_0^\infty \overline{v_1^2 v_2}\, \sin n\eta\, d\eta \tag{17b}$$

With the aid of (53) this can be reduced to:

$$\pi \Psi(n) = -\frac{2\beta}{n}\, \pi \Gamma(n) + \frac{4\beta^3 l^2}{n^2} \int_0^\infty \Omega \sin n l z\, dz \tag{87}$$

When this result is substituted into eq. (19), which is the FOURIER transform of the fundamental equation, the following relation can be obtained:

$$\frac{\partial}{\partial t}\left(\frac{\pi\Gamma}{\beta^2}\right) + 2\nu n^2 \frac{\pi\Gamma}{\beta^2} = \frac{4\beta l^2}{n} \int_0^\infty \Omega \sin n l z\, dz \tag{88}$$

The term with ν apparently can be discarded when $n \ll (\nu t)^{-\frac{1}{2}}$.

When n is small, we develop the sine function in order to arrive at the series (18b) with

$$\pi \Psi_{2m-1} = 2 \int_0^\infty \overline{v_1^2 v_2}\, \eta^{2m-1}\, d\eta,$$

again assuming that the integrals converge. By means of partial integration this can be transformed into:

$$\pi \Psi_{2m-1} = \frac{\beta}{m}\, \pi \Gamma_{2m} - \frac{2\beta^3 l^{2m+3}}{m(2m+1)} \int_0^\infty \Omega\, z^{2m+1}\, dz \tag{89}$$

Equation (20) then leads to:

$$\frac{d}{dt}\left(\frac{\pi\Gamma_{2m}}{\beta^2}\right) - 4\nu m(2m-1)\frac{\pi\Gamma_{2m-2}}{\beta^2} = \frac{4\beta l^{2m+3}}{2m+1} \int_0^\infty \Omega\, z^{2m+1}\, dz \tag{90}$$

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10) No difficulty will arise with the integrals appearing in (86), (89) and (90), if we assume that the functions $f_k(\Lambda_k)$ vanish exponentially for $\Lambda_k \to \infty$.

24. *Similarity considerations.* — An important tool in the investigation of turbulence has become the assumption of the preservation of statistical similarity during the development in time [11]). When applied to the system under consideration this assumption requires that, under the laws of motion stated in section 7, mean values of quantities of the same degree in λ_i, τ_i, ζ_i should keep a constant ratio. A consequence is that the dimensionless quantities ω, ω^*, $\tilde{\omega}$ introduced in (31) must remain constant during the development of the system. The same will apply to the ω_k defined in (78); and also to the ω_k^* and similar restricted mean values, provided the value of λ_i or Λ_k to which they refer is made to change proportionally with l.

We give attention in the first place to the expression (35) for the invariant J_0, which we assume to be different from zero. The mean value $\overline{\tau_i \tau_{i+k} \zeta_i \zeta_{i+k}}$ in a self-preserving system must be proportional to l^4. Since J_0 is independent of the time, we thus arrive at the result:

$$\beta^2 l^3 = l^3/t^2 = J_0/c$$

(c being a numerical quantity), from which:

$$l \propto t^{2/3}\,. \tag{91}$$

It follows immediately that

$$E \propto \overline{v^2} \propto \beta^2 l^2 \propto t^{-2/3} \tag{92a}$$

while:

$$\varepsilon \propto t^{-5/3} \tag{92b}$$

From eq. (9) we then obtain

$$\tfrac{2}{3}\, E/t = \tfrac{2}{3}\,\beta\, E = \varepsilon\,,$$

and reference to eqs. (32) and (33) leads to the relation:

$$\tilde{\omega} = \tfrac{1}{6}\,(1 + \omega^*) \tag{93}$$

from which, in connection with (34) [12]):

$$\lambda^2 = 3\nu t \tag{94}$$

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[11]) Compare papers by TH. DE KARMAN, G. K. BATCHELOR, F. N. FRENKIEL, C. C. LIN and others. We mention in particular: G. K. BATCHELOR, Energy decay and self-preserving correlation functions in isotropic turbulence, Quart. Appl. Mathem. **6**, 97, section 6 (1948).

[12]) It will be recognised that the quantity λ, defined in eqs. (7)—(9) as a kind of minimum length associated with the correlation function essentially depending on the rounding off of the jumps in $v(y)$, which quantity is used again in (34) and here in (94), has nothing to do with the λ applied in eqs. (36) etc. and in eqs. (69), (70) etc. as an integration variable representing the length of a segment λ_i between two jumps (or the sum Λ_k of the lengths of a set of consecutive segments). It is trusted that no difficulties will arise through this double use of λ.

Hence $\lambda \backsim t^{1/2}$, which means that l and λ are *not* proportional. Similarity consequently is not possible for all aspects of the system; the type of similarity considered here applies to relations involving dimensions large compared with $\sqrt{\nu t}$. Since $l \backsim t^{2/3}$, the difference between l and λ increases with t, so that the range to which this similarity applies is continually extended. It is of interest to note that the REYNOLDS' number Re_I increases proportionally with $t^{1/3}$, since $L \backsim l \backsim t^{2/3}$ and $v \backsim E^{1/2} \backsim t^{-1/3}$. — The failure of similarity in the smallest dimensions will be responsible for the difference between the limit found for n in connection with formulae (85*a*), (85*b*), *viz.* $\beta\, l/\nu \backsim t^{-1/3}$, and the limit found in connection with (88), *viz.* $(\nu\, t)^{-1/2}$.

Equation (65) giving dl/dt now leads to the important formula

$$\tag{95} \tfrac{1}{2} f_0\, \overline{(\tau_i + \tau_{i+1})}^{\,*} = f_0\, \overline{\tau_i}^{\,*} = \tfrac{2}{3}$$

When this is substituted into (68) we are brought back to (93).

Another interesting result is obtained from (66):

$$\tag{96} f_0\, \overline{(\tau_i + \tau_{i+1})\, \tau_i\, \tau_{i+1}}^{\,*} = \tfrac{2}{3}\,(1+\omega)\, l^2$$

Turning to the spectrum we observe that $\Gamma_0 = 2\, J_0/\pi$ is independent of the time. Hence we must expect that the same will apply to $\Gamma(n)$, provided we make n vary inversely proportionally to l, *i.e.* $n \backsim t^{-2/3}$. Indeed it will be seen from (85*b*) that this makes $\Gamma(n)$ constant. Apparently the head of the spectrum contracts towards the lower wave numbers. This is a consequence of the reduction of the number of segments λ_i and τ_i, considered in section 7.

25. The similarity hypothesis makes it possible to write $\overline{v_1 v_2}$ in the form:

$$\tag{a} \overline{v_1\, v_2} = \beta^2\, l^2\, \psi\,(\eta/l)\,,$$

where ψ does not contain the time in an explicit way. We then find

$$\tag{b} \frac{\partial}{\partial t}\, \overline{v_1\, v_2} = -\tfrac{2}{3}\,\beta^3\, l^2 \left(\psi + \frac{\eta}{l}\,\psi'\right).$$

When this result is substituted into the fundamental equation (12*a*) and use is made of (47) and (54), the following formula is obtained:

$$\sum_1^\infty \left(2\,\frac{\eta}{l}\,\varphi_k - 3\,\Phi_k - \chi_k\right) = -(1+\omega)\,\frac{\eta}{l} - 2\,\bar{\omega} + \tfrac{1}{3}\,(1+\omega^*)\,,$$

which is not confined to large values of η, but must be valid for all values of η (except those of order $\nu\, t/l$).

Substitution of $\eta = 0$ leads us back to (93), so that the equation is

also valid here. We can therefore omit the last two terms on the right hand side, which leaves us with:

$$\sum_1^\infty \left(2\,\frac{\eta}{l}\,\varphi_k - 3\,\Phi_k - \chi_k\right) = -(1+\omega)\frac{\eta}{l} \tag{97}$$

When we divide by η/l and go to the limit $\eta = 0$, we come back to (96). — With the aid of the results of section 20 it is easy to show that the equation is satisfied for $\eta \to \infty$.

When (97) is differentiated with respect to η, and attention is given to (81), (82) and (84), the following equation can be derived:

$$\Omega = \tfrac{1}{6}\,(1+\omega) - \tfrac{1}{3}\,z^2\,\frac{d}{dz}\left(\frac{1}{z}\int_0^z \Pi\,dz\right) \tag{98}$$

It can be surmised that this relation between the functions Ω and Π, obtained for self-preserving systems, will play an important part in the statistical relations characterising such systems.

For $z = 0$ the equation gives $\Omega = \frac{1}{6}(1+\omega)$. The same value is obtained from (82) if it is observed that all $f_k(\eta)$ go to zero for $\eta = 0$, with the exception of $f_1(\eta)$, which has the limiting value f_0. Since $\zeta_i - \zeta_{i+1} = \frac{1}{2}(\tau_i + \tau_{i+1})$ when $\lambda_{i+1} = 0$, the only term remaining in (82) can be readily reduced with the aid of (96).

It is further found that $d\,\Omega/d\,z = 0$ for $z = 0$.

For large values of z eq. (83*a*) gives $\int \Pi\,d\,z = z - \frac{1}{2}(1+\omega)$, which duly leads to $\Omega = 0$ for $z \to \infty$.

Equation (98) can also be solved for Π. Introducing the result into (83) we obtain:

$$\sum_1^\infty \varphi_k = C\,z - \tfrac{1}{2}\,(1+\omega) + 3\,z\int_z^\infty \frac{\Omega}{z^2}\,dz \tag{99}$$

This is an extension of (73), now valid for all values of η. The coefficient C (which makes its appearance as an integration constant) evidently must be equal to 1.

Formula (48) now gives:

$$\frac{\partial}{\partial\eta}\,\overline{v_1\,v_2} = -3\,\beta^2\,l\,z\int_z^\infty \frac{\Omega}{z^2}\,dz\,,$$

from which by integration:

$$\overline{v_1\,v_2} = \tfrac{3}{2}\,\beta^2\,l^2\left(\int_z^\infty \Omega\,dz - z^2\int_z^\infty \frac{\Omega}{z^2}\,dz\right) \tag{100}$$

Since we have seen that Ω has a finite value for $z=0$, the second term between the brackets in (100) vanishes for $z=0$ and we obtain:

(101)
$$\overline{v^2} = \tfrac{3}{2}\beta^2 l^2 \int_0^\infty \Omega\, dz$$

from which, in connection with (32) and (93):

(102)
$$\int_0^\infty \Omega\, dz = \tilde{\omega} = \tfrac{1}{6}(1+\omega^*)$$

We can also form:

$$\sum_1^\infty \chi_k = \int_0^z \Pi\, z\, dz = \tfrac{1}{2} z^2 - \tfrac{3}{2}\int_0^z \Omega\, dz + \tfrac{3}{2} z^2 \int_z^\infty \frac{\Omega}{z^2}\, dz$$

$$\sum_1^\infty \Phi_k = \int_0^z (\Pi\, z - 2\,\Omega)\, dz = \tfrac{1}{2} z^2 + \tfrac{1}{2}\int_0^z \Omega\, dz + \tfrac{3}{2} z^2 \int_z^\infty \frac{\Omega}{z^2}\, dz$$

$$\sum_1^\infty X_k = \int_0^z (\Pi\, z^2 + 2\,\Omega\, z)\, dz = \tfrac{1}{3} z^3 + z^3 \int_z^\infty \frac{\Omega}{z^2}\, dz$$

$$\frac{d}{d\eta}\,\overline{(v_1^2\, v_2)} = -\beta^3 l^2 \left(\int_z^\infty \Omega\, dz - 3 z^2 \int_z^\infty \frac{\Omega}{z^2}\, dz\right)$$

(103)
$$\overline{v_1^2\, v_2} = -\beta^3 l^3 \left(z \int_z^\infty \Omega\, dz - z^3 \int_z^\infty \frac{\Omega}{z^2}\, dz\right)$$

From the latter expression it follows that for self-preserving systems

(104)
$$\overline{v_1^2\, v_2} = -\tfrac{2}{3}\beta\,\eta\,\overline{v_1\, v_2}$$

It will be seen that this relation can be obtained immediately from eqs. (a) and (b) at the beginning of this section, since these give

$$\frac{\partial}{\partial t}\overline{v_1\, v_2} = -\tfrac{2}{3}\beta \frac{\partial}{\partial \eta}(\eta \cdot \overline{v_1\, v_2})$$

for self-preserving systems. Substitution into (12a) and integration immediately leads to (104).

Finally, after some transformations, we find:

(105)
$$\pi\, \Gamma(n) = \frac{6\beta^2 l^2}{n^3}\int_0^n dn\, n \int_0^\infty \Omega \sin n l z\, dz$$

By calculating the limit for $n \to 0$ we obtain

(106)
$$J_0 = \tfrac{1}{2}\pi\, \Gamma_0 = \beta^2 l^3 \int_0^\infty \Omega\, z\, dz$$

which can also be deduced directly from (100). If we may assume that $d\,\Omega/d\,z \leqslant 0$ for all z, we can calculate the limiting form for large n, which gives:

$$\pi\,\Gamma(n) = \frac{6\,\beta^2 l}{n^2}\,\Omega(0) = \frac{\beta^2 l\,(1+\omega)}{n^2}.$$

This is in accordance with (85*b*).

26. We add an observation concerning J_0. Introducing the "momenta"

$$\mu_i = \beta\,\tau_i\,\zeta_i \tag{107}$$

[(compare footnote 3) to section 7)], we can write:

$$J_0 = \frac{1}{l}\left(\tfrac{1}{2}\,\overline{\mu_i^2} + \sum_{k=1}^{\infty} \overline{\mu_i\,\mu_{i+k}}\right) \tag{108}$$

We have already seen that the μ_i do not change during the normal motion. When two vertical segments τ_i, τ_{i+1} combine to form a single segment, the corresponding momenta μ_i, μ_{i+1} simply add.

We consider the change in the course of time suffered by the various terms of the sum (108) in consequence of the combination of segments. For shortness we write:

$$a = \tfrac{1}{2}\,f_0\,(\tau_i + \tau_{i+1})/t \qquad (\text{for } \lambda_i \to 0).$$

From a consideration of terms disappearing and newly appearing in the sums, upon coalescence of segments, it is found that:

$$\frac{d}{dt}\left(\tfrac{1}{2}\,N\,\overline{\mu_i^2}\right) = N\,\overline{a\,\mu_i\,\mu_{i+1}}^{\,*}$$

$$\frac{d}{dt}\,N\,\overline{\mu_i\;\mu_{i+1}} = 2N\,\overline{a\,\mu_i\,\mu_{i+2}} - N\,\overline{a\,\mu_i\,\mu_{i+1}}^{\,*}$$

$$\frac{d}{dt}\,N\,\overline{\mu_i\;\mu_{i+k}} = (k+1)\,N\,\overline{a\,\mu_i\,\mu_{i+k+1}} - k\,N\,\overline{a\,\mu_i\,\mu_{i+k}}\,.$$

In the first two equations the asterisk indicates the restricted mean value for $\lambda_i \to 0$. In the other terms we have omitted the asterisk, since Λ_k $(k > 1)$ does not become zero; it is assumed that ordinary mean values could be used. Whether this may be correct or an approximation only, it will be seen that for every case of coalescence the terms appearing in one mean value, disappear from the next higher one. Hence it is found again that

$$d\,J_0/dt = 0.$$

However, if the similarity hypothesis holds, every separate mean value must satisfy the relation

$$\overline{\mu_i\,\mu_{i+k}} \backsim \beta^2\,l^4\,,$$

so that

$$\frac{d}{dt} N \overline{\mu_i \mu_{i+k}} \backsim \frac{d}{dt} \beta^2 l^3 = 0 .$$

This requires that $k\, a\, \overline{\mu_i \mu_{i+k}}$ shall be independent of k. We cannot suppose that this should be a constant differing from zero, for this would make the sum divergent. We are thus led to the conclusion that these quantities must be zero for $k \neq 0$. This will be the case if it can be assumed that all ζ_i are independent of each other, so that

$$\overline{\zeta_i \zeta_{i+k}} = 0 \quad (k \neq 0) \tag{109}$$

and likewise $\overline{\mu_i \mu_{i+k}} = 0$ for all k different from zero. The expression for J_0 then becomes:

$$J_0 = \frac{\beta^2}{2l} \overline{\tau_i^2 \zeta_i^2} = \frac{\overline{\mu_i^2}}{2l} \tag{110}$$

It is probable that in sections **24** — **26** we have obtained the principal relations that can be deduced from the hypothesis of similarity. The results of section **25** have shown that the relevant statistical quantities for self-preserving systems depend upon a single function, for which we may take either $\Pi(z)$ or $\Omega(z)$. The form of this function, however, remained unknown and it is not to be expected that the similarity hypothesis can help us much further in this respect. Substitution of the full expressions (47) and (54) for $\overline{v_1 v_2}$ and $\partial(\overline{v_1^2 v_2})/\partial\eta$ into the fundamental equation (12*a*) can give a lot of relations we have not made use of, but it has become evident that every new equation brings new statistical functions, so that it looks as if there will always be more unknowns than equations.

The only way to obtain further information will be an attack upon the intrinsic statistical problem. Although the complicated relations between the λ_i, τ_i, ζ_i make a solution of this problem beyond our power, it will be attempted to discuss certain aspects in the last part of this paper.

(To be continued).

CORRELATION PROBLEMS IN A ONE-DIMENSIONAL MODEL OF TURBULENCE. IV *)

BY

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27. In Parts I — III of this paper expressions have been constructed for $\overline{v^2}$, $\overline{v_1v_2}$ and $\overline{v_1^2v_2}$, referring to a particular type of solutions of eq. (1). These expressions contain several series of other mean values, depending on the lengths of the segments λ_i, the values of τ_i and ζ_i associated with them and combinations of such quantities. Substitution of the expressions for $\overline{v_1v_2}$ and $\overline{v_1^2v_2}$ into the fundamental equation (12) or (12*a*) for the propagation of correlation, gave a number of relations between mean values involving the λ_i, τ_i etc. However, even when this equation is supplemented by the hypothesis that the correlation function is of self-preserving type, no method was found which would enable us to calculate the values of these quantities. One gets the impression that the equation for the propagation of correlation does not adequately embody all the statistical properties of the system and that it is necessary to look for another statistical treatment, of a more basic character. Such a treatment also should show whether a tendency exists towards the development of self-preserving correlation functions.

To find a way which may lead to such a treatment is still the great problem of the theory of turbulence. The following pages do not pretend more than to sketch a few aspects of questions appearing here, in order to bring the investigation of our system provisionally to a close.

An ideal method for a statistical treatment would be to devise a description of the system of such type, that every possible state could be completely specified by a single datum, say by the position of a point in a multidimensional system of coordinates. Application of the laws of motion as stated in section 7 would give us the displacement of the representative point in course of time, and the simultaneous history of an assembly of systems could be pictured by a flow of such points. If now we could find a quasi-stationary probability function for the distribution of these points over the available space, which would be in conformity with the laws governing the flow, it would seem that mean

*) Continued from these Proceedings **53**, 718—731 (1950).

values could be calculated by averaging over the system of representative points. At the present no such description is available and it looks as if the number of details to be specified by means of the position of a representative point, must be so large that the application of this idea will be extremely complicated.

A different way of attack is to look for a class of elements which throughout the development of the system retain a certain individuality. For these elements a distribution function might be introduced and a consideration of the laws of motion might lead to the derivation of a BOLTZMANN equation for the distribution function. Although this idea likewise is beset with difficulties due to the peculiar nature of the relation between the λ_i and τ_i, which involves questions of order of arrangement to which reference has already been made in sections **11** and **12**, it appears nevertheless that the nearest approach to an assembly of "*weekly interacting elements*" is obtained by taking the segments λ_i as the fundamental entities. In order to be able to make use of the laws of motion stated in section 7, we must associate with every segment at least one further quantity, for which we choose $s_i = \frac{1}{2}(\tau_{i-1} + \tau_i)$.

So long as no interaction takes place between neighbouring segments, the laws of motion give:

$$(111) \qquad \frac{d\lambda_i}{dt} = \frac{\lambda_i - s_i}{t} \quad ; \quad \frac{ds_i}{dt} = 0$$

Interaction occurs when the length λ_i of a segment has decreased to zero. The segment then vanishes from the assembly, while its two neighbours suffer a sudden increase of their s-values: the segment on the left hand side has its value of s_{i-1} increased by $\frac{1}{2}\tau_i$ and the segment on the right hand side has its value of s_{i+1} increased by $\frac{1}{2}\tau_{i-1}$. When s_i is the only datum available, the way in which it divides into $\frac{1}{2}\tau_i$ and $\frac{1}{2}\tau_{i-1}$ becomes a point of uncertainty. It will immediately be seen that in this respect the segments are not independent: when the division should be known for a single segment, it can be found for all other segments by means of formulae of the type:

$$\tau_{i+1} = 2\, s_{i+1} - \tau_i \quad ; \quad \tau_{i+2} = 2\, s_{i+2} - \tau_{i+1}, \text{ etc.}$$

In order not to have our way blocked already at the start, we shall ignore this circumstance and consider the ratio τ_i/τ_{i-1}, for a segment which has decreased to zero, to be a matter of pure chance for that element.

Although the expressions for $\overline{v_1 v_2}$ and $\overline{v_1^\tau v_2}$ require the knowledge of a great many functions, the complexity of the problem forces us to restrict the attention to the distribution function $f_1(\lambda)d\lambda$ for the lengths λ_i of single segments, and to mean values like $\overline{\tau_i}^*$ or $\overline{s_i}^*$, referring to a given λ_i.

28. It may be useful, as a first attempt, to simplify still further and to consider only the values of the lengths λ_i. We shall assign an average

rate of increase of length to all segments with lengths between λ_i and $\lambda_i + d\lambda_i$, to be given by:

$$\frac{d\lambda_i}{dt} = \frac{\lambda_i - s_i^*}{t} \tag{111 a}$$

where s_i^* represents the restricted mean value

$$s_i^* = \tfrac{1}{2}\,\overline{(\tau_{i-1} + \tau_i)}^{\,*} = \overline{\tau_i}^{\,*} \tag{111 b}$$

for the given value of λ_i, as defined in section 12. If we denote by N the total number of segments in a constant great length (which we may take as unit length), the Boltzmann equation for Nf_1 obtains the form:

$$\frac{\partial}{\partial t}(Nf_1) = -\frac{\partial}{\partial \lambda}\left\{\frac{\lambda - s^*}{t}\,Nf_1\right\} \tag{112}$$

Since the interaction between segments affects only the value of s_i, but not the lengths of other segments, no interaction term is required in eq. (112). — Integration with respect to λ from 0 to ∞, having regard to (36) and to the fact that N is a function of t alone, gives:

$$-\frac{t}{N}\frac{dN}{dt} = f_0\, s^* \tag{113}$$

where f_0 and s^* refer to $\lambda_i = 0$. This equation, which is the same as (64), expresses the gradual decrease of the number of segments in unit length through the disappearance of segments for which λ_i has become zero. For shortness we write a for the right hand side of (113); in principle a can be a function of t. For self-preserving systems eq. (95) gives $a = \frac{2}{3}$.

Eliminating N from (112) we arrive at the following equation for f_1:

$$t\frac{\partial f_1}{\partial t} + \frac{\partial}{\partial \lambda}\{(\lambda - s^*)\, f_1\} - a f_1 = 0 \tag{114}$$

The solution of this equation is hampered by the circumstance that a guess must be made concerning the proper choice of s^* (which should satisfy the condition that the overall mean value of s must be equal to that of λ), while also the fact that a depends on special values of f and s^* brings complications. By way of example we give the following particular self-preserving solution, for which $a = {}^2$, while it has been assumed that s^* should be equal to $A\, t^{2/3}$ for all λ (with A independent of λ):

$$\left\{\begin{array}{ll} f_1 = \dfrac{2}{3\,A\,t^{2/3}}\left(1 - \dfrac{\lambda}{3\,A\,t^{2/3}}\right) & \text{for } \lambda < 3\,A\,t^{2/3} \\ f_1 = 0 & \text{for } \lambda > 3\,A\,t^{2/3} \end{array}\right. \tag{115}$$

This solution leads to:

$$\overline{\lambda_i} = \overline{s_i} = A\, t^{2/3}.\text{[13]}$$

[13]) A problem in a way complementary to the one considered here, is obtained when we start from the molecular analogue mentioned in footnote 3) to section 7. In this analogue we have to do with molecules of various masses τ_i, arbitrarily

29. We will now give attention to the circumstance that segments of a given length λ_i may have various values for s^1 and we introduce a distribution function $G(\lambda, s)\, d\lambda\, ds$, defined in such a way that the number of segments per unit length of the y-axis with values of λ_i and s_i between assigned limits, at a definite instant, is given by

$$N(t)\, G(\lambda, s)\, d\lambda\, ds \tag{116}$$

In general the function G will depend on t. For all t it must satisfy the condition:

$$\int_0^\infty d\lambda \int_0^\infty ds\, G(\lambda, s) = 1 \tag{116 a}$$

The Boltzmann equation for G will refer to a λ, s-plane and takes the form:

$$\frac{\partial}{\partial t}(NG) = -\frac{\partial}{\partial \lambda}\left\{\frac{\lambda - s}{t} NG\right\} + \triangle(NG) \tag{117}$$

The term $\triangle(NG)$ represents the effect of the interactions, which now must be taken into account. The interaction produces sudden additions to the values of s_i of certain segments, in consequence of which their representative points in the λ, s-plane jump upwards, but there is no creation, nor annihilation of representative points. It follows that we must have:

$$\int_0^\infty d\lambda \int_0^\infty ds\, \triangle(NG) = 0 \tag{118}$$

Integration of the Boltzmann equation over the whole λ, s-plane consequently gives:

$$-\frac{t}{N}\frac{dN}{dt} = \int_0^\infty ds\, s\, G(0, s) \tag{119}$$

which is equal to $f_0 s^* = a$, that is to the quantity already occurring in (113).

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distributed over a straight line of infinite length, and endowed with positive or negative velocities. With the data given in the footnote, the velocities $\beta\zeta_i$ are connected in a particular way with the coordinates ξ_i and the masses τ_i, but one could also take a more general point of view and treat the velocities as quite arbitrary.

There will be collisions and it is assumed that two colliding molecules will immediately stick together, with their masses and their momenta added. Then the following statistical problem presents itself: can one find a law governing the probable mass distribution of the molecules? Such a law cannot be of stationary character, since the number of molecules per unit length gradually decreases in consequence of their combining, while the average mass of a molecule correspondingly increases. But it might be that there would be an approach to a self-preserving function, *e.g.* for the dispersion of the masses about their average value.

In order to obtain an expression for $\triangle(NG)$ we consider a group of disappearing segments, for which s_i may have a value between s_1 and $s_1 + ds_1$. The number of these segments (for unit time and unit length of the y-axis) temporarily will be written:

$$\nu = NG(0, s_1)\,\frac{s_1}{t}$$

They will occasion upward jumps of the s-values of their immediate neighbours. If we write: $ln\,(\tau_i/\tau_{i-1}) = 2\varkappa$, we have:

$$\tfrac{1}{2}\,\tau_{i-1} = \frac{e^{-\varkappa}\,s_1}{e^{\varkappa} + e^{-\varkappa}} \;\; ; \;\; \tfrac{1}{2}\,\tau_i = \frac{e^{\varkappa}\,s_1}{e^{\varkappa} + e^{-\varkappa}}.$$

The value of $\varkappa$ is not known beforehand. We assume that there exists a probability function $p(\varkappa)\,d\varkappa$, defined in such a way that:

$$\int\limits_{-\infty}^{+\infty} p(\varkappa)\,d\varkappa = 1, \text{ with } p(\varkappa) = p(-\varkappa).$$

There will then be called forward $\nu\,p(\varkappa)\,d\varkappa$ jumps of amount $\frac{1}{2}\,\tau_{i-1}$ and an equal number of amount $\frac{1}{2}\,\tau_i$. If we neglect the possibility of a correlation between neighbouring segments, the representative points of the neighbouring segments can be situated in any element $d\lambda\,ds$ of the λ, s-plane. Since the probability to find such points in an element $d\lambda\,ds$ is given by $G(\lambda, s)\,d\lambda\,ds$, we must expect that out of this element there will disappear

$$(A) \qquad 2\,\nu\,p(\varkappa)\,d\varkappa \;\; G(\lambda, s)\,d\lambda\,ds$$

representative points, half of which make upward jumps of amount $\frac{1}{2}\,\tau_{i-1}$ and half of which make jumps of amount $\frac{1}{2}\,\tau_i$. It is possible, however, that there will be some preference for neighbours with a large positive value of $d\lambda_i/dt$ (that is, with a large positive value of $\lambda_i - s_i$), since such segments will have a great tendency to cause a decrease in length of their neighbours. It may therefore be advisable to multiply the expression (A) by a factor γ, increasing with $\lambda - s$. The introduction of such a factor must not bring about any change in the total number of points which jump upward; hence the following condition must be fulfilled:

$$(B) \qquad \int\limits_0^{\sim} d\lambda \int\limits_0^{\sim} ds\,\gamma\,G(\lambda, s) = 1$$

Having regard to the factor γ it is found that the total number of representative points disappearing from an element $d\lambda\,ds$ amounts to:

$$(C) \quad d\lambda\,ds\,\gamma\,G(\lambda, s)\,\frac{2\,N}{t}\int\limits_{-\infty}^{+\infty} d\varkappa\,p(\varkappa)\int\limits_0^{\infty} ds_1\,s_1\,G(0, s_1) = d\lambda\,ds\,\gamma\,G(\lambda, s)\,\frac{2\,Na}{t}$$

while the number of points newly appearing in an element $d\lambda\, ds$ in consequence of jumps from below is given by

$$(D) \qquad d\lambda\, ds \frac{2N}{t} \int_{-\infty}^{+\infty} d\varkappa\, p(\varkappa) \int_0^{s'} ds_1\, s_1\, G(0, s_1)\, \gamma\, G(\lambda, s - s'')$$

where

$$s' = \frac{(e^{\varkappa} + e^{-\varkappa})\, s}{e^{\varkappa}} \quad ; \quad s'' = \frac{e^{\varkappa}\, s_1}{e^{\varkappa} + e^{-\varkappa}}.$$

Combining (C) and (D) we obtain:

$$(120) \qquad \triangle(NG) = \frac{2N}{t}\left[\int_{-\infty}^{+\infty} d\varkappa\, p(\varkappa)\, S - a\gamma G(\lambda, s)\right]$$

with

$$(120\,a) \qquad S = \int_0^{s'} ds_1\, s_1\, G(0, s_1)\, \gamma\, G(\lambda, s - s'')$$

In the latter integral γ will be a function of $\lambda - (s - s'')$.

It can be proved that

$$(120\,b) \qquad \int_0^{\infty} ds\, \triangle(NG) = 0$$

This shows that integration of eq. (117) with respect to s from 0 to ∞ brings us back to eq. (112). At the same time we have obtained a verification of (118).

Elimination of N from eq. (117) gives:

$$(121) \qquad t\frac{\partial G}{\partial t} + \frac{\partial}{\partial \lambda}\{(\lambda - s)G\} - aG = 2\left\{\int_{-\infty}^{+\infty} d\varkappa\, p(\varkappa)\, S - a\gamma G\right\}$$

30. To simplify as much as possible we may look for a self-preserving solution of the type:

$$(122) \qquad \lambda = xt^a \quad , \quad s = zt^a \quad ; \quad G = \Phi(x, z)\, t^{-2a}$$

Further we replace γ by 1 and we assume that the only relevant value of $\varkappa$ is zero. We then obtain the following equation for the function Φ:

$$(123) \qquad \{(1-a)\, x - z\}\frac{\partial \Phi}{\partial x} - az\frac{\partial \Phi}{\partial z} + (1-a)\,\Phi = T$$

where

$$T = 2\int_0^{2z} dz_1\, z_1\, \Phi(0, z_1)\, \Phi(x, z - \tfrac{1}{2} z_1).$$

The characteristics of eq. (123) are determined by:

$$dx/dz = -\{(1-a)\, x - z\}/az,$$

from which:

$$(x - z)\, z^{(1-a)/a} = C.$$

A schematic picture has been given in fig. 9 for the case $\alpha = \frac{2}{3}$, so that the equation of the characteristics becomes $(x - z)\sqrt{z} = C$. The value $C = 0$ gives the diagonal $x = z$.

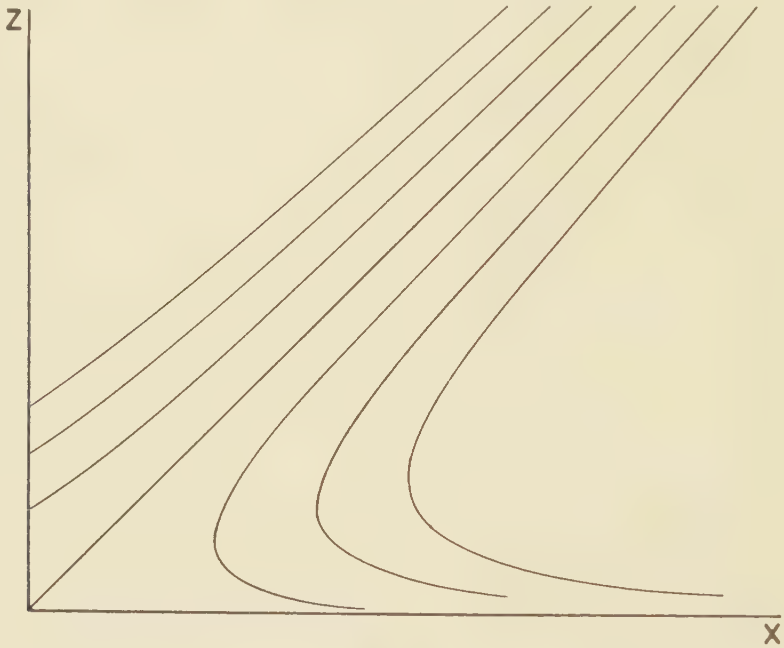

Fig. 9

Along a characteristic the following relation holds:

$$\frac{d\Phi}{\Phi} = \frac{1-\alpha}{\alpha}\frac{dz}{z} - \frac{T}{\alpha\Phi}\frac{dz}{z}.$$

For small z, the quotient T/Φ is of order z^2 (or may be smaller); this makes it possible to use as an approximation:

$$\Phi \simeq z^{\frac{1-\alpha}{\alpha}} H \qquad (\text{for } z \to 0),$$

where H has a constant value along a characteristic, which value in general must be supposed to be a function of C.

Since it can be proved that:

$$\int_0^\infty T\,dz = 2\,\alpha \int_0^\infty \Phi\,dz,$$

it follows that for large z we must have $T > 2\alpha\Phi$. We may conclude that:

$$\ln\Phi < \left(\frac{1-\alpha}{\alpha} - 2\right) z + \text{constant}$$

along those branches of the characteristics which go to infinity.

If we assume the function $H(C)$ to be of the type $\exp(-a\,C^2)$, where a is a constant, we shall get a concentration of representative points along the diagonal $x = z$ and also along the line $z = 0$. The distribution obtained in this way can have a convergent integral $\iint dx\,dz\,\Phi\,(x, z)$, so that the function Φ can be normalized. However, it may be that integrals of the type $\iint dx\,dz\,x^n\,\Phi\,(x,z)$ or $\iint dx\,dz\,z^n\,\Phi\,(x, z)$ with $n > 0$ will not be convergent. It cannot be made out at this moment whether the convergence can be improved by a better choice for the function $H(C)$, or by more suitable assumptions for γ and $p(\varkappa)$. It might also be that we are encountering here a real difficulty, which would rule out the existence of self-preserving solutions.

Résumé.

Dans une étude récente sur les théories de la turbulence par MM. Agostini et Bass [14]), les auteurs observent que l'impossibilité d'obtenir de solutions des équations de Navier-Stokes qui ressemblent à la turbulence et par suite de prendre des moyennes sur de telles solutions, nous oblige à nous contenter de prendre les moyennes sur les équations elles-mêmes. On obtient ainsi des équations qui sont vérifiées par les grandeurs statistiques, mais qui ne suffisent pas à les déterminer complètement. La déduction de ces équations constitue le point de départ des travaux modernes sur la turbulence.

Le but de l'article présent (auquel la communication sur la formation de couches tourbillonnaires dans un type simplifié de mouvement turbulent, Proc. **53**, 122 (1950), a servi d'introduction), était de substituer au système d'équations de Navier-Stokes une équation simplifiée, qui néanmoins contiendrait des termes essentiellement importants du point de vue dynamique. Cette équation devrait satisfaire à la condition qu'elle admettrait la construction d'une classe de solutions présentant un caractère propre à rendre efficace l'application de formules statistiques. Dans ce but le nombre de variables dépendantes a été réduit d'abord à deux (dans la communication qui sert d'introduction) et ensuite à une seule. En effectuant cette simplification, l'équation de continuité de l'hydrodynamique a été complètement abandonnée. Comme cette équation joue un rôle très important dans la théorie de la turbulence hydrodynamique, on doit s'attendre à ce que certaines propriétés cinématiques, caractéristiques de la turbulence vraie, ne se retrouvent pas dans les solutions de l'équation à étudier. Mais il reste d'autres propriétés, de nature plutôt dynamique et caractéristiques elles aussi, qui peuvent être vérifiées sur ces solutions. On espère qu'une telle étude pourra aider à discerner entre les traits fondamentaux du phénomène de la turbulence et des traits géométriques de nature plutôt accidentelle.

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[14]) L. Agostini et J. Bass, Les théories de la turbulence (Public. scient. et techn. du Ministère de l'Air, no. 237, 49, (Paris 1950)).

L'équation sur laquelle porte l'analyse présente la forme:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2} \tag{1}$$

Elle comporte un terme non-linéaire et un terme du second ordre, multiplié par une constante ν qui est traitée comme très petite. L'équation admet la dérivation d'une équation pour la dissipation d'une "énergie cinétique", ainsi que la formation d'une équation pour la propagation de corrélation, comme celle-ci a été donnée pour la première fois par DE KÁRMÁN et HOWARTH dans le cas de la turbulence hydrodynamique.

La présence du terme non-linéaire dans l'équation (1) (comme aussi des termes correspondants dans le système à deux variables considéré dans la communication précédente) donne lieu à une tendance à former des discontinuités dans la solution, qui pourtant se trouvent être "arrondies" par l'effet du terme $\nu\,(\partial^2 v/\partial y^2)$. On peut interpréter ce phénomène comme une tendance à faire apparaître des régions de dissipation concentrée. Pour bien mettre en évidence ce caractère et pour pouvoir en faire usage dans une analyse statistique, nous nous sommes bornés à un type de solutions approchées, dans lesquelles l'allure de la variable v est donnée par une courbe en forme de scie, présentant des segments inclinés vers la droite (tous parallèles), alternant avec des segments verticaux (ou plutôt presque verticaux, si l'on tient compte de l'effet du terme en ν). La répartition des longueurs λ_i des segments inclinés, celle des hauteurs τ_i/t des segments verticaux, ainsi que celle des hauteurs ζ_i/t des points de milieu des derniers, peuvent faire l'objet d'une statistique et permettent ainsi d'introduire un élément aléatoire dans les solutions.

Certaines relations fournies par l'équation (1) peuvent être utilisées pour déduire un système de "lois de mouvement", qui permettent de suivre le développement dans le temps d'une telle courbe au moyen de règles assez simples. Un phénomène particulier à ce développement est la possibilité de rapprochement de deux segments verticaux consécutifs, qui se confondent en un seul dès que la longueur du segment incliné entre eux est devenue égale à zéro. De cette façon le nombre moyen des segments λ_i par unité de longueur de l'axe des abscisses va en diminuant, et la "longueur d'onde moyenne" du système augmente avec le temps.

Le but principal de l'article a été de déduire des formules exprimant les corrélations $\overline{v_1 v_2}$ et $\overline{v_1^2 v_2}$ par des fonctions statistiques ayant trait à la répartition des λ_i, τ_i, ζ_i. Ici v_1 désigne la valeur de v à un point arbitraire y; v_2 la valeur de v au même instant au point $y + \eta$; on prend la moyenne par rapport à y pour une valeur de η constante, en supposant que de telles moyennes existent pour les solutions prises en vue. Les formules (47) et (53) obtenues dans la seconde partie de l'article se rapportent à une approximation dans laquelle l'influence du terme $\nu\,(\partial^2 v/\partial y^2)$ est négligée; la formule (50) pour $\overline{v_1 v_2}$ tient compte de cette influence.

Ces formules sont introduites dans l'équation fondamentale pour la propagation de corrélation (12) ou (12a). On peut s'attendre à ce que cette substitution puisse conduire à l'établissement de certaines relations entre les fonctions statistiques. Les plus simples ont été données par les équations (55), (57) ou bien par (66) — (68). Ces formules permettent une vérification de quelques relations valables pour la turbulence hydrodynamique. On peut reconnaître aussi dans ces relations l'influence de la diminution graduelle du nombre de segments par unité de longueur de l'axe des abscisses, conséquence déjà signalée du fait que de temps en temps deux segments verticaux peuvent se confondre en un seul.

En faisant usage de certaines transformations il a été possible de démontrer que $\overline{v_1 v_2}$ et $\partial(\overline{v_1^2 v_2})/\partial\eta$ tendent vers zéro quant η augmente indéfiniment (Partie III, section **20**).

Il était possible aussi d'obtenir une formule pour l'intégrale $J_0 = \int\limits_0^\infty \overline{v_1 v_2}\, d\eta$ et de démontrer que pour toute solution donnée cette quantité reste indépendante du temps. La question de savoir si cette intégrale peut prendre une valeur zéro, n'a pas été abordée; on a supposé que sa valeur serait positive. Sous ce rapport l'absence d'une équation de continuité peut introduire une différence notable entre le système étudié et le cas de la turbulence hydrodynamique.

Après avoir donné quelques formules pour les fonctions spectrales correspondant à $\overline{v_1 v_2}$ et $\overline{v_1^2 v_2}$, on a étudié les relations obtenues lorsqu'on introduit l'hypothèse que le développement dans le temps des fonctions de corrélation doit satisfaire à une loi de similitude (sections **24**, **25**). De cette façon on réussit à exprimer toute une série de grandeurs caractéristiques du système au moyen d'une seule fonction Ω, définie par l'équation (82). Mais on n'obtient pas une relation qui puisse servir à trouver une forme générale pour cette fonction.

Ici, de nouveau, on a l'impression que l'application de l'équation pour la propagation de corrélation ne traduit pas suffisamment toutes les propriétés statistiques qui déterminent le développement d'une solution. Sans doute il faut chercher une méthode statistique encore plus fondamentale. Quelques tentatives sont exposées dans la dernière partie de la communication; on a essayé de traiter les segments λ_i comme des éléments conservant une certaine individualité pendant le développement du système dans le temps et de former une équation de Boltzmann à laquelle devrait satisfaire la fonction de répartition correspondante. Les difficultés qui se présentent ici ne permettent pas encore de porter ces tentatives à un résultat définitif.

L'étude abordée dans cet article des problèmes statistiques se rattachant aux solutions de l'équation (1), n'est pas terminée. Il se pose par exemple des questions concernant la validité de la solution approchée, quand deux (ou même plusieurs) segments verticaux de la courbe pour v se trouvent très rapprochés, et aussi quand la solution arrive dans sa

dernière phase où l'amplitude de v est devenue très petite. D'autre part des questions ayant trait à la répartition des longueurs λ_i, τ_i, ζ_i sur des grandes distances, sont laissées ouvertes. Des détails de cette répartition peuvent influencer sur la grandeur de l'invariant J_0, ainsi que sur des autres grandeurs statistiques. Enfin on devra envisager encore le problème de la transmission de l'énergie d'une région du spectre à une autre. Il sera possible, peutêtre, de revenir sur ces problèmes dans une communication ultérieure.

Malgré ces lacunes nous espérons que les considérations presentées dans cet article démontront que l'étude d'un "modèle mathématique" pourra contribuer au développement de nos connaissances sur la turbulence.

CHEMISTRY

ELASTIC VISCOUS OLEATE SYSTEMS CONTAINING KCl. X [1])

a. *A new method for marking the oleate system with small gas bubbles.*
b. *The elastic behaviour as a function of the KCl concentration and the influence of benzene, naphtalene, hexane, heptane, glycerol and the undecylate ion hereupon.*
c. *Remarks on differences between oleate preparations from different sources.*

BY

H. G. BUNGENBERG DE JONG, W. A. LOEVEN *) AND W. W. H. WEIJZEN*), [2])

(Communicated at the meeting of April 29, 1950)

1. *Introduction.*

Besides giving a description of a simplified method to perform experiments with half filled spheres (section 2) and of a method to mark the oleate system with a small quantity of minute gas bubbles (section 3), in the present communication the investigation about the action of organic substances on the elastic behaviour of the oleate system is continued.

In parts VI — IX of this series we have already investigated the influence of a number of organic substances and have classed them into three types A, B, C; having type A when the curves representing G, λ, $1/\Lambda$ and n as a function of the KCl concentration are shifted by the added substance into the direction of smaller KCl concentrations, type C when these curves are shifted into the opposite direction and type B when no shift at all occurs. Only in two cases (n-hexylalcohol and ethanol, both belonging to type A) the conclusions were based on direct experiments, viz. on the cumbersome method of investigating the above mentioned influence at a whole series of KCl concentrations. The remaining substances were classed in connection with the results obtained from an indirect method (working at only one KCl concentration, namely the one corresponding to the minimum damping of the blank), which method on account of its simplicity is a very attractive one to be used at the continuation of the investigation about the relation between the structure of an organic substance and its action on the elastic oleate system.

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*) Aided by grants from the "Netherlands Organisation for Basic Research (Z.W.O.)".

[1]) Part I has appeared in these Proceedings **51**, 1197 (1948), Parts II—VI in these Proceedings **52**, 15, 99, 363, 377, 465 (1949), Parts VII—IX in these Proceedings **53**, 7, 109, 233 (1950).

[2]) Publication no. 4 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

This indirect method is based on the assumption that type C really exists. Up to now we have strong indications but no direct proofs, that this assumption is true. The aim of the present communication will therefore be to control, by using the direct method that some of the substances which, on account of the indirect method were classed under type C (heptane and the undecylate ion), really belong to type C. In the same way, that benzene really belongs to type A. To have some more examples which are investigated by the direct method, we added to this program naphtalene, hexane and glycerol, which were not yet investigated by the indirect method neither.

As our stock of Na oleinicum pur. pulv. from MERCK (used in parts I — IX) was completely exhausted and could not be replenished, the experiments were performed with Na-oleate, neutral powder, from BAKER.[3])

As this preparation will be used in the following parts of this series too, it may not be superfluous to discuss (section 7) the points of resemblance and difference between the two preparations, so far as these are important for our purposes. A comparison with oleate, obtained from chemically pure oleic acid will be included.

2. *A simplified method for performing experiments with half filled spheres.*

For the experiments in section 4 we followed in principle the technique of the method with the half filled spheres, described in details in part VI (section 1) of this series.

The original technique of the method with the half filled spheres requires a cumbersome preparation of the series of spheres to start with. One must first prepare a large volume of the KCl containing blank system and then provide each sphere with the required quantity of weight of the blank system so that it is exactly half filled according to quantity of volume. For that purpose from the radius of each sphere the required volume of the to be added blank system must be calculated, and from the density the weight of this volume.

In this original method G is calculated from the formula

$$T \qquad \frac{2 \times R}{4.493} \mid \frac{2}{G}$$

(compare part II of this series), so that for every individual sphere we have to use the radius of this sphere.

It can be shown, (see small print below) that in practice one may follow a much simpler method, provided the series of spheres to be used shows only a relatively small variation in capacity. This condition was fulfilled in our case as we had a series of Pyrex vessels with a mean

[3]) A generous gift of Na-oleate from The Rockefeller Foundation provided the means for the experiments described in this paper.

capacity of 110 ml, the capacity varying between 105 and 115 ml. The simplified method consists of providing each sphere with exactly 55 ml of the oleate system (for particulars see below) and to use for the calculation of the G from the period only one value of R, namely the value which corresponds to a sphere of 110 ml, being twice the volume of the oleate system introduced into each sphere.

It is true that the calculated G value is not the exact value, but the error which is introduced systematically, is relatively small, because this error consists of two factors, which act in opposite directions and abolish each other for the greater part. By using for instance a 113 ml vessel and filling it with 55 ml, the G value calculated from the observed period using a value for R which corresponds to a 110 ml vessel is only 0.5 % wrong.

The two above mentioned erroneous factors which are used in the calculation are:

1. the radius of the vessel, which ought to be that of a 110 ml sphere, but in reality is smaller or larger;
2. the degree of filling, which ought to be exactly 50 %, but in reality is larger or smaller because the actual radius is smaller or larger.

For an appreciation of the influence of these two errors on the calculated G value, we must know the influence of each, apart.

As to error sub 1) we read already from the above given formula that T is proportional to R. As to error sub 2) we have already given in part I of this series, provisional data about the influence of the degree of filling on the period. By means of more experiments we came to the result (already mentioned in part VI, section 1) that a change of the degree of filling from 50 % to 50.5 % (or to 49.5 %) gives an increase (resp. decrease) in the period of 0.25 %.

Now let us take as example a vessel with a capacity of 113 ml into which we introduce 55 ml oleate system. The radius of the vessel is $\sqrt[3]{113/110} = 1.009$ times the radius of a 110 ml vessel.

To calculate G we ought to use this radius, but at the same time we ought to perform the measurement of the period with 56.5 ml oleate in the vessel. If the latter was the case the period would be 1.009 times the one measured with an exactly half filled vessel of 110 ml.

In reality, however, the 113 ml vessel contains only 55 ml oleate system instead of 56.5 ml, that is to say, its degree of filling is not exactly 50 %, but only 48.7 %, in other words 1.3 % too low. Therefore the period measured with the 113 ml vessel will be 0.65 % lower than the one that would have been found if the vessel had been exactly half filled.

Therefore the period which we measure with the 113 ml vessel at a degree of filling of 48,7 % will be 0.9 % − 0.65 % = 0.25 % higher than the period measured with an exactly half filled 110 ml vessel.

Using the radius of the 110 ml vessel the actual period measured with the 113 ml vessel will (because G is proportional to $1/T^2$) therefore introduce systematically an error of only 0.5 % in the calculated value of G.

The great practical advantage of the above simplified method is not so much that the calculation of G from the experimentally determined periods becomes easier but above all that the preparation of a series of vessels, with which an experiment is performed becomes much more convenient.

As each vessel must be provided with a constant volume of the highly viscous elastic oleate system, the latter can be simply produced in each vessel apart by adding successively (by pipettes or burettes) adequate volumes of two low viscous solutions (the KCl solution and the stock oleate solution) and mixing thoroughly afterwards.

As to the other quantity, which we always measure, viz. n, this one is only slightly dependent on the radius of the vessel. Besides, in the neighbourhood of a degree of filling of 50 %, this n is very slightly dependent on small variations of the degree of filling. We found for instance a decrease in n of 0.09 % at a decrease of 1 % in the degree of filling. Therefore no error in n is to be feared for by using the above simplified method.

3. *Electrolytic H_2 mark.*

Up to now we used small air bubbles, shaken carefully into the oleate system, to be used in the measurements as indicators of the elastic movements. This method has two drawbacks: 1) the danger that a too large volume of bubbles is introduced, which effects the n- and T-values (see part I, section 3), 2) the visibility through the telescope of the kathetometer of very small amplitudes depends on the size of the air bubbles, if the oleate system is not perfectly clear. Formerly this second drawback was not felt, as we worked with oleate from MERCK, but it was strongly felt by working with the new oleate preparation from BAKER, the KCl systems of which were somewhat turbid at 15°.

So we looked for another method to mark the oleate system, which had to fulfil the conditions: 1) that we could introduce a small, limited number of bubbles, 2) that these bubbles are of uniform size. Both conditions were fulfilled by introducing electrolytically obtained H_2 bubbles into the oleate system.

The electrodes consist of a platinum wire as anode and a thin enamelled copper wire of which only the end is freed from enamel as cathode. By closing the current (4 Volt, from an accumulator) very small H_2 bubbles of uniform size are detached from the bare end of the enamelled copper thread [4]).

By special experiments it has been proved that bubbles introduced in this way have no influence upon the n- and T-values of the elastic oleate systems, even if we let them be formed during several minutes. In practice it is already sufficient to close the current during about 5 seconds.

Fig. 1 gives the apparatus we used. The two electrodes (Cu, Pt) are attached to the thin copper bars a and b, which can be raised or lowered through holes in the ebonite plates c and d, and can be fastened in the desired position by the screws e and f.

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[4]) One must take care to make the copper thread the cathode, otherwise green streaks (copper hydroxide or oleate) are formed around it when the current is closed.

These ebonite plates are attached to either ends of a brass tube g. This tube can be lowered or raised through a hole in the brass piece h (and can be fastened at the desired position by means of screw i).

For experiments in which a number of half filled or completely filled vessels of the same capacity are used, both tube g and the relative position of the electrodes can be adjusted once and for all, so that, placing the apparatus upon the edge of a vessel, the electrodes are automatically in the right position (fig. 1B). If the apparatus is not in use it is placed on the edge of a sufficiently wide glas tube near the thermostate.

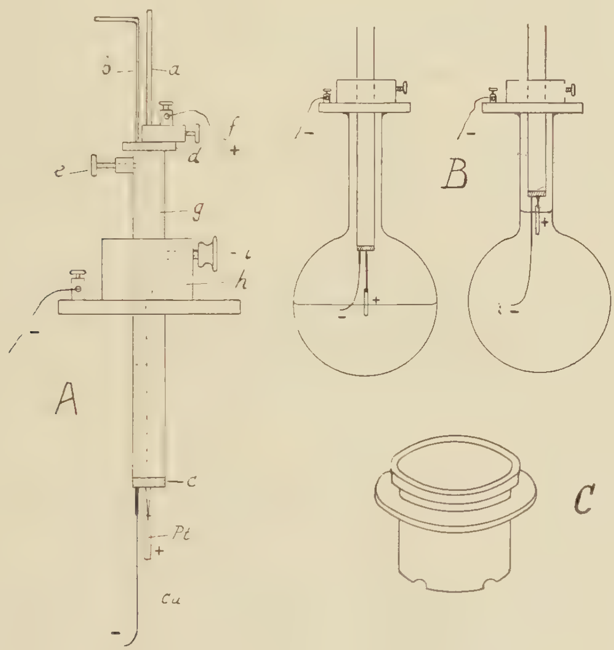

Fig. 1

It has appeared that in the special case of the half filled 110 ml vessels, which have been left overnight in the thermostate (to become free of air) and which are marked electrolytically, those vessels give a too large value of n and a slightly erroneous value of T. These errors do not occur if the vessels are vigorously wheeled round their axis beforehand, wheeling five times to and fro.

To this end the contrivance for exciting the rotational oscillation described in part VII was complemented with a short wide brass-cylinder (fig. 1C) which fits exactly over the inner ring of the ball bearing (d in fig. 1 of part VII). This cylinder is provided with two flanges, so that it may serve as a pulley for a string slung around the cylinder. With this string the vessel can thus be wheeled round whereby one wheels to and fro.

The values of n and T found after first wheeling round the half filled 110 ml vessel and thereafter marking the oleate system electrolytically are exactly the same as those found with the former method whereby the system was marked with wheeled in air bubbles. In state of rest (after standing overnight) the too high value of n develops again.

These phenomena are absent in completely filled 110 ml vessels. This induces one to attribute them to a (reversible) consolidation at the interface oleate system/air, which persists when the system is marked electrolytically, but is removed by mechanical agitation of this interface.

Still this explanation does not seem sufficient, as curiously enough the above complications were not (or only in a very slight degree) met with half filled 500 ml vessels. The question, whether this abnormal behaviour is only present with the new oleate preparation of BAKER (the solutions of which become turbid at 15°) and is absent with the oleate preparation of MERCK (the solutions of which remain clear at 15°), could not be answered, because we had no more left of the latter preparation.

4. *G and n as functions of the KCl concentration and the influence of benzene, naphtalene, hexane, heptane, glycerol and the undecylate ion thereupon.*

When not stated otherwise, the experiments in this section have been performed with 1.2 % oleate systems at 15°, using the simplified technique of the method with the half filled spheres (section 2).

The oleate systems are marked electrolytically and the rotational oscillation is excited with the contrivance described in part VII of this series.

This simplified technique is extremely convenient for the filling of the vessels with oleate systems in which the KCl concentration must be varied.

We used a series of "110 ml" vessels (varying from 109 — 115 ml) so that each vessel had to be provided with 55 ml of the oleate system. Starting from a stock (2.4 %) oleate solution containing 0.1 N KOH, the vessels were filled according to the receipt: x ml KCl 3.75 N + + $(27.5 - x)$ ml H_2O + 27.5 ml stock oleate solution.

Afterwards they were closed with the glass stoppers and vigorously shaken. After standing overnight in a thermostate of 15°, the vessels were wheeled round and next the oleate systems were marked electrolytically (section 3). The so obtained measurements give information on the elastic behaviour of the blank 1.2 % oleate system as a function of the KCl concentration. Further we proceeded quite according to the original method of half filled spheres, a description of which is given in part VI of this series. For the addition of known quantities of hexane, heptane or benzene the dripping pipette described in a previous communication [5]) was used. The dripping pipette could not be used in the case of glycerol, as the great quantities which had to be added in order

[5]) H. G. BUNGENBERG DE JONG and L. J. DE HEER, these Proceedings **52**, 783 (1949).

to obtain measurable effects, no longer allowed the total volume of the oleate system to remain practically constant. We followed therefore another way. We first measured a blank series (receipt see above) using the usual 2.4 % stock oleate solution (24 g Na oleate per liter, containing 0.1 N KOH). Thereafter we repeated the measurements with a number of analogous 2.4 % stock oleate solutions, in which we used: y ml glycerol + $(950 - y)$ ml H_2O instead of the 950 ml H_2O. Thus we could prepare stock oleate solutions with various concentrations of glycerol.

With the experiments with the two solid substances: naphtalene and undecylic acid we still had to follow another way. In the usual 2.4 % stock oleate solution we introduced a weighed amount of naphtalene which was dissolved by warming up carefully till 80° (melting point of naphtalene) under continual shaking. After cooling the naphtalene remained dissolved. Another portion of the 2.4 % stock oleate solution (without naphtalene) was also warmed up and cooled in precisely the same way.

Two more stock solutions, both containing naphtalene, were obtained by mixing the above stock solutions in various proportions. Each of the four stock solutions apart were used for the preparation of a series of KCl containing oleate systems following the usual receipt. The measure-

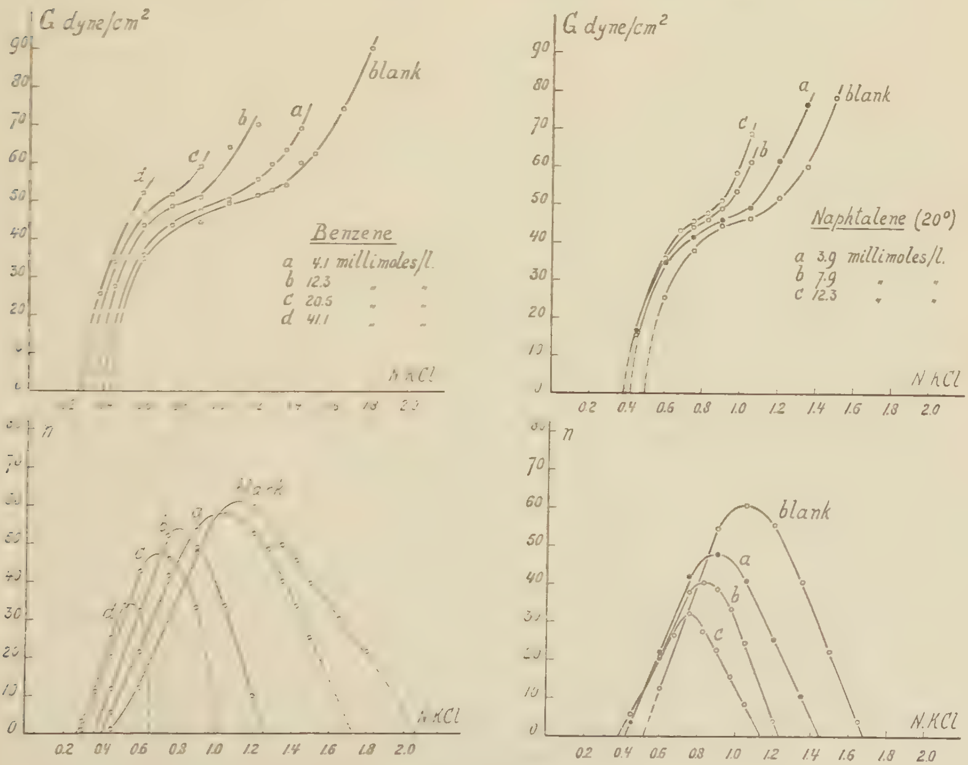

Fig. 2—3

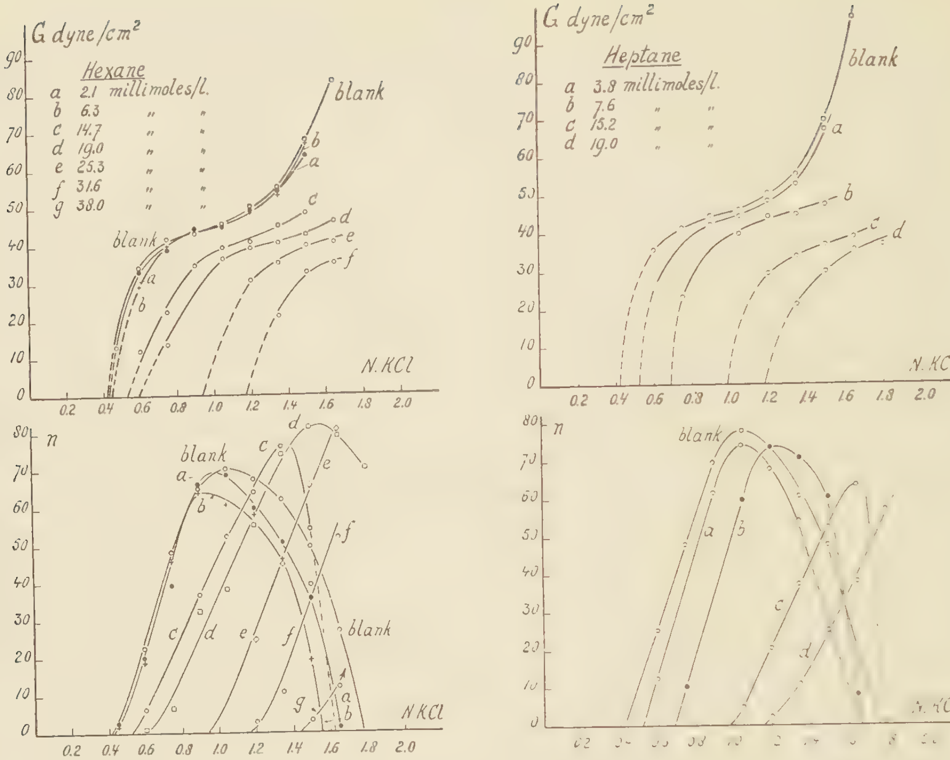

Fig. 4—5

ments had to be performed at 20°, because in consequence of the advanced season no sufficiently cold tap water for cooling was available any longer to keep the thermostate at 15°.

In the series with the highest naphtalene concentration a small fraction of the naphtalene had crystallized in the form of very thin leaflets which were embedded in the elastic system, so that the actual naphtalene concentration was somewhat smaller than 12.3 milli moles/l.

In the case of undecylic acid (which, as there is an exces of KOH in the stock oleate solution, forms undecylate) we followed the same procedure as stated above for naphtalene. The requisite temperature for dissolving the undecylic acid was only 50° here. The experiments were performed at the usual temperature of 15°.

We had to content ourselves with the measurement of the period T and of the maximum number of observable oscillations n, as none of us was able to perform the very difficult measurement of the decrement.

In the preceeding parts VI — IX we have given many examples from which the narrow correlation between n and $1/\Lambda$ is evident (in varying the KCl concentration or the concentration of an added organic substance, always providing that we keep the oleate concentration and the radius of the vessels constant). Therefore in the present and in following

parts of this series, we feel justified to use n as a relative measure for $1/\Lambda$ for those experiments which fulfil the above conditions.

We shall not give elaborate tables of the results, but we shall represent them graphically in the figures 2 -- 7.

In these figures G (the elastic shear modulus) and n, are given as functions of the KCl concentration for the blank series and for a few series with constant concentrations of the added organic substance.

For several reasons (see small print below) the results obtained with the six organic substances can not be compared quantitatively with one another. In the next section we therefore confine ourselves to the discussion of the characteristics occurring in each of the figures 2 — 7 apart.

The reasons hinted at in the above are the following:

1. the experimental conditions at the measurements are not always the same (the naphtalene series was performed at 20°, the other series at 15°),
2. in some series the stock oleate solutions have been warmed up during a short time (naphtalene 80°, undecylate 50°, see above) in the remaining series not,
3. the electrolytic marking was not yet used in the series with benzene. The series with hexane was the first which was marked in this way. Perhaps we had not yet enough experience how to wheel round the vessels before marking

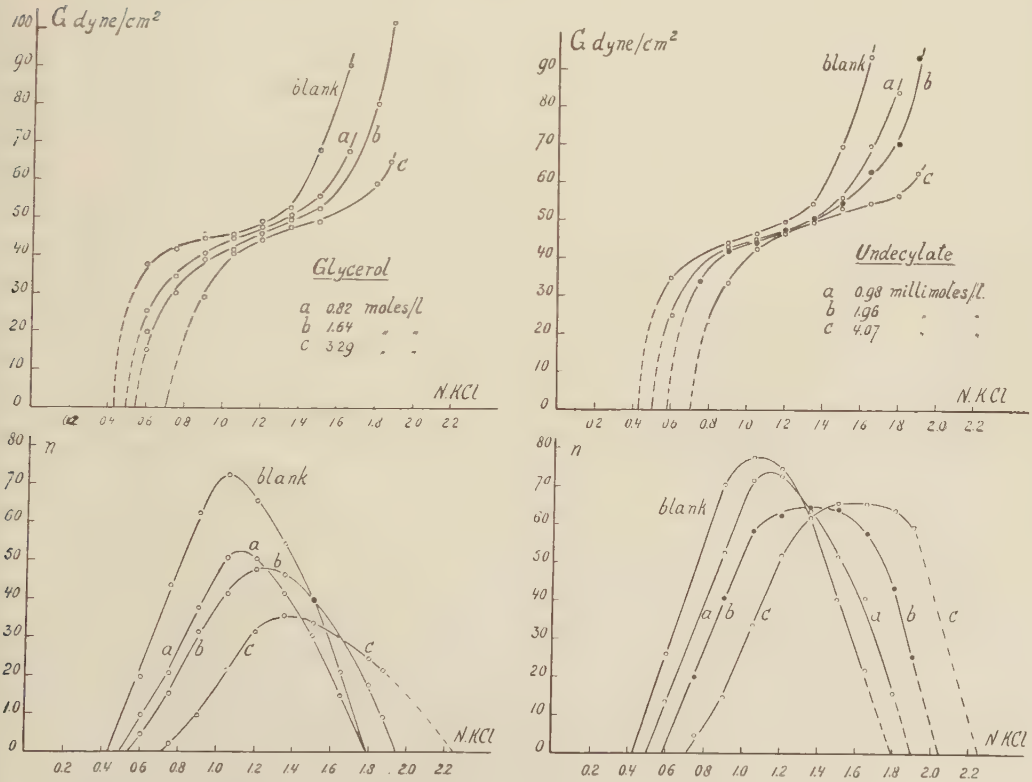

Fig. 6—7

them. The situation of the experimentally determined points was more irregular here than usual. The maximum of the n curve becomes higher here in its shifting to the right. Such an increase has never been found with other substances and must therefore be considered with suspicion,

4. the oleate used in the experiments is not strictly the same, though we always used oleate of BAKER bearing the same lot number. It has appeared that the contents of flasks with the same lot number may show quantitative differences. To make the experiments more comparable we soon used mixtures of the contents of five flasks. This was not yet done in the series with benzene, which series shows a higher value of G at the inflexion point of the blank G-curve than the other series (50 instead of $\pm$ 46). It also shows a higher value of that KCl concentration where the right footpoint of the n-curve is situated ($\pm$ 2.05 N instead of $\pm$ 1.78 N),
5. the six series of experiments are extended over a period of some six months. If we assume that the contents of all flasks bearing the same lot number were originally identical, than the quantitative differences, mentioned in 4 between the samples of oleate in different flasks, must be the result of changes of the oleate with time which may proceed with different rates in the individual flasks. Therefore one must fear that samples of oleate taken regularly from one and the same mixture at different times will no longer be strictly comparable.

5. *Discussion of the results.*

A. Referring to the Introduction for the aims of the experiments in section 4, the results confirm: 1) that type C (a shift of the G and n curves in the direction of higher KCl concentrations) really exists, 2) that conclusions drawn from the indirect method are confirmed by the direct method. Heptane (fig. 5) and the undecylate ion (fig. 7) indeed follow type C and benzene (fig. 2) indeed follows type A.

B. Glycerol (fig. 6) belongs to type C and very great concentrations are needed to obtain measurable effects. We had expected this from the results obtained with the n-primary alcohols (parts VI and VIII of this series). n-Hexanol is a very pronounced representant of type A and acts already at very small concentrations. From n-pentanol over n-butanol, n-propanol and ethanol to methanol there is a gradual transition. The latter stands already very near to type B and acts only at relatively great concentrations. Introduction of more hydroxyl groups in the molecule, leaving the number of carbon atoms as small as possible might therefore result in a substance of type C. As these three hydroxyl groups will make the uptake of the glycerol molecule in the oleate micelles more difficult, glycerol was expected to act only at large concentrations.

C. Naphtalene (fig. 3) and benzene (fig. 2) belong to type A, heptane (fig. 5) and hexane (fig. 4) however belong to type C (with a certain restriction see below under E). This suggests a marked difference in action between aromatic and aliphatic hydrocarbons (see already part VII of this series). Systematic work on the action of hydrocarbons will be published later in this series.

D. In part VI we have seen that the inflexion point on the G-curve (in practice determined by the reading of the value of G corresponding

with the maximum of the n-curve) is shifted by n-hexanol and by ethanol in a horizontal direction only, i.e. the G value practically retains its original value. The following survey gives the G values (in dyne/cm^2) corresponding to the maxima of the n curves for the experiments in section 4, viz. on every horizontal row first the G value of the blank, then the one of the first, second, third and fourth addition.

Substance added	blank	1e add.	2e add.	3e add.	4e add.
benzene	50	50.5	50	50.5	49
naphtalene	46	45.5	46	46	—
hexane	46	45	44	45.5	43.5
heptane	46	45	44.5	38.5 ?	—
glycerol	46	46	47	47.5	—
undecylate	47	47	51	54	—

Taking into consideration the experimental errors and the error made by estimating the position of the maxima of the n-curves, it may be concluded that here too, when the G curve is displaced by the added substance, the G value at the inflexion point of the G curve practically retains the same value for the first five substances of the survey.

With the undecylate ion, however, the G value distinctly increases. This may eventually be ascribed to its resemblance with the oleate ion, its uptake in the oleate micelles contributing to G at the inflexion point of the shifted G curve. An increase of G at the inflexion point indeed occurs if extra oleate is added to the oleate system, but in this case (as later experiments have shown) no marked shift of the inflexion point occurs towards lower or higher KCl concentrations.

E. Fig. 8 gives the KCl concentration as function of the concentration of the added substance for three characteristic points of the n curves in the figures 2 — 7 and in the analogous figures 2 and 4 in part VI of this series, viz. the KCl concentration corresponding to the left footpoint (l. fp.), to the maximum (max.) and to the right footpoint (r. fp.) of the n curves. If it is assumed that a given organic substance, when acting on the elastic oleate system can under all circumstances be classed under one of the types A, B and C, then the graphs for hexanol, benzene, naphtalene, glycerol and undecylate give no indications against this assumption. Hexanol, benzene and naphtalene belong to type A, the three curves for l. fp., max. and r. fp. showing a descent in each graph. Glycerol and undecylate belong to type C, these curves showing a rise.

The remaining graphs in fig. 8, those for ethanol, hexane and heptane bear a more complicated character. This indicates that a rigid division of organic substances into three classes (showing the types A, B and C) is based on too simple an assumption (see already part VI section 2). Ethanol behaves as a weak representant of type A at the KCl concentration of minimum damping of the blank (maximum of the n-curve). At the KCl concentration of the right footpoint of the n-curve, however,

ethanol behaves as a representant of type C. With increase of the KCl concentration ethanol therefore shows the following transition in type:

$$A \to B \to C \qquad \text{(ethanol).}$$

An analogous transition consequent on the increase of the KCl concentration is also present with the investigated alkanes, providing that we only consider the action of small additions. From the direction the curves denoted l. fp., max. and r. fp. take initially, there follows for increasing KCl concentration the transitions in type:

$$C \to B \to A \qquad \text{(heptane)}$$

$$B \to A \qquad \text{(hexane)}$$

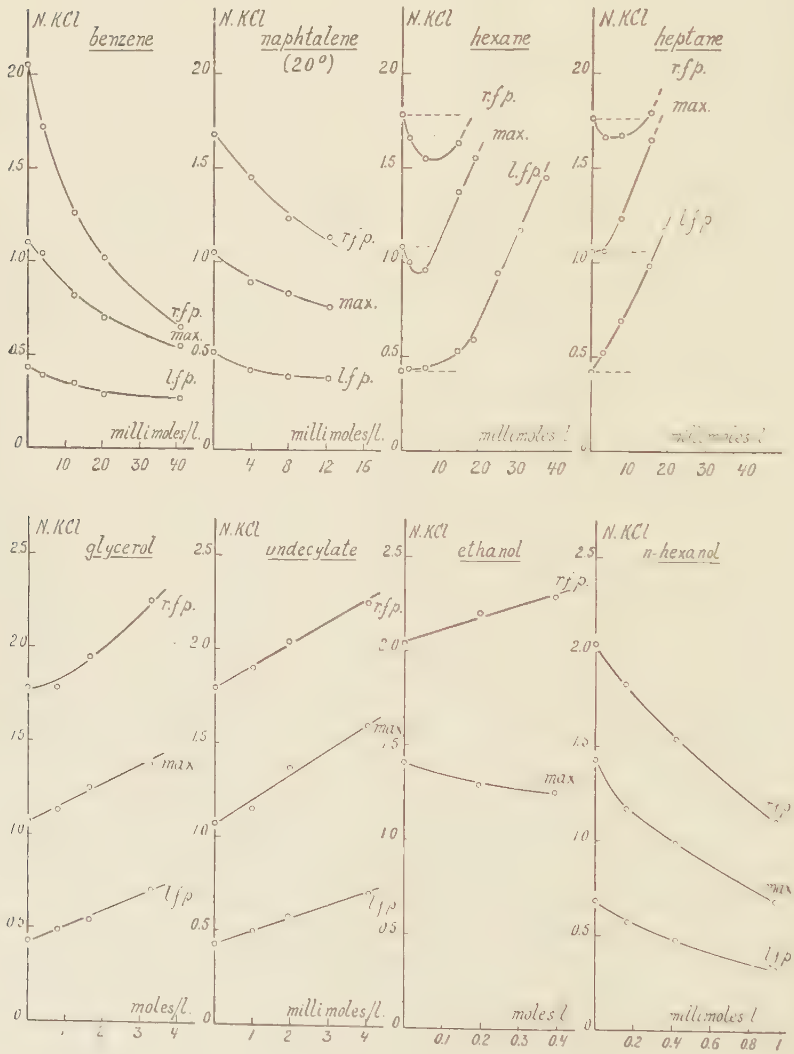

Fig. 8

Considering now the whole course of each of the curves denoted l. fp., max. and r. fp. we perceive that most of them show a transition in type consequent on the concentration of the added substance:

Substance added	Left footpoint	Maximum	Right footpoint
hexane	$B \to C$	$A \to B \to C$	$A \to B \to C$
heptane	C	$B \to C$	$A \to B \to C$

In section 6 we will return to the above transitions in type.

F. By plotting the results obtained with benzene or naphtalene (substances typically following type A) $G - n$ diagrams are obtained (only sketched schematically in fig. 9 A), which bear quite the same character as the $G - n$ diagram for n-hexanol (fig. 2 B in part IX of this series). If we do the same with the results obtained with undecylate (represented schematically in fig. 9 B) a diagram is obtained

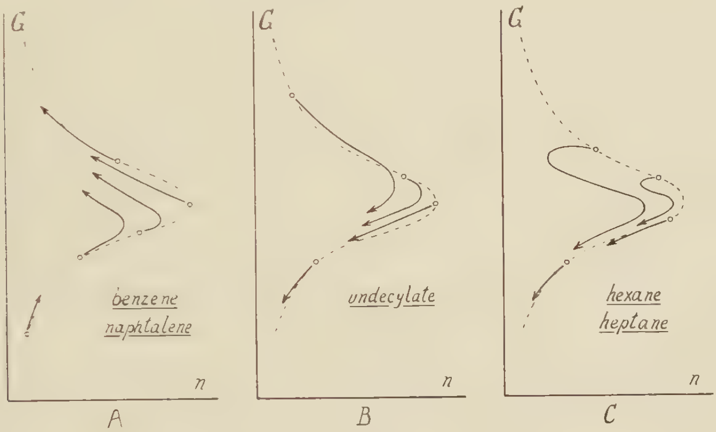

Fig. 9

which corresponds to what we might expect on account of the discussions in part IX (section 1, fig. 1) for a substance typically following type C. The results obtained with hexane and heptane give $G - n$ diagrams (see scheme fig. 9 C) in which at KCl concentrations higher than (or in the case of hexane corresponding to) the KCl concentration of minimum damping, the $G - n$ curves show a quite new type. The S shape of these curves, however, becomes understandable if we look at the transition in type, $A \to B \to C$, which occurs at higher KCl concentrations consequent on the increase in concentration of hexane or heptane. The S shape can be considered as a succession of initial courses of the $G - n$ curves represented in the diagrams 3, 6 and 9 of fig. 1 in part IX of this series.

6. *The action type (A, B or C) as a resultant of two components which act in opposite directions.*

The transitions of type ($A \to B \to C$ and $C \to B \to A$) met with in section 5 sub E call for a serious reflection on the question whether it

is correct to divide organic substances once and for all into three types A, B and C, according to their action on the elastic oleate system. It is evident that ethanol, hexane and heptane cannot be classed in either of these groups, their type of action depending on the circumstances (their own concentration and the concentration of the KCl on hand).

It seems appropriate to consider the type of action, shown by an organic substance, to be principally of compound nature, viz. as the resultant of two component actions exerted simultaneously by the added substance on essentially different places in the elastic structures, being present in the oleate system. Each of those component actions — if present alone — would shift the G and the λ (or $1/\Lambda$ or n) curve into opposite directions (into the directions of higher or lower KCl concentrations). If one of these component actions preponderates within the tract of given concentrations of the organic substance itself, (at a given oleate concentration) and of the KCl being present, there result a "typically type A" (e.g. n-hexanol) or a "typical type C" (e.g. undecylate).

If the component actions would happen to compensate one another exactly, in the considered tract of the above mentioned concentrations, there would result a "typically type B". One understands easily how unlikely it will be that the latter case will ever be realized, the two component actions being themselves functions of the concentrations of the added organic substance and of the KCl being present. As the points of attack in the elastic structure are essentially different these functions cannot be of exactly the same nature. A complete compensation of the two component actions may be possible of course, but at a given KCl concentration this will only be the case at one concentration of the added substance. A compensation within a whole range of KCl concentrations and of concentrations of the organic substance seems highly improbable.

This corresponds exactly to our experience. We have not yet met a substance which follows "typically type B". Instead of it we have met substances "standing near to type B", being "weak representants of type A or type C", which, however, alter their type of action in the opposite type when the concentrations of the substance itself or the KCl concentration are changed.

The graphs for hexane and heptane in fig. 8 may illustrate this. In the transitions of type $A \rightarrow$ type C, there is at a given KCl concentration only one concentration of the added substance, where one might speak of type B, viz. at the minima of the curves denoted max. (hexane) and r. fp. (hexane and heptane).

7. *Quantitative differences between oleates from different sources.*

Qualitatively the oleate preparations from Merck and from Baker show the same behaviour of the elastic systems prepared from them.

Already in part III of this series we observed that equally concentrated oleate systems prepared from these preparations showed different values for G and Λ (λ and n) at the same KCl concentration. If on comparing the blank curves, which represent

G and n as a function of the KCl concentration, in fig. 2 and 4 of part VI (MERCK's oleate), with those in the fig. 2—7 in the present communication (BAKER's oleate), once more quantitative differences appear to be present. MERCK's oleate showed a higher value for the KCl concentration corresponding to the minimum damping, a lower value of n at the maximum of the n-curve (i.e. a higher value of Λ) and a higher coacervation limit (the KCl concentration at the right foot point of the n-curve).

There also was a difference in the outward appearance of the 1.2 % oleate systems at 15°, those of MERCK's oleate being practically clear, those of BAKER's oleate being markedly turbid. An analysis kindly made by Prof. HAVINGA showed that BAKER's oleate contained an appreciable amount of palmitate (in the order of 20 %). However one must also take into account that the preparation of MERCK was rather old (± 7 years), though it had been stored in the original unopened paraffinized bottles. The BAKER preparation on the other hand was investigated within half a year after receiving it. Thus it was uncertain which factor was mainly responsible for the different positions of the G- and n-curves. Mr. C. DE BOCK has kindly prepared for us a sample of chemically pure oleic acid free from saturated fatty acids and from linoleic acid; mp 13°) with which we determined the position of the G and the n curve. The results showed that the KCl concentration of minimum damping, the value of n and the KCl concentration of the right footpoint of the n-curve do not differ much from the analogous values obtained with BAKER's preparation.

We have therefore the strong impression that the quantitative difference between BAKER's and MERCK's oleate preparations were mainly due to the age of the latter preparation. The relative position of the G and n curves suggest that as a result of the advanced deterioration of the oleate, substances have been produced which act according to type C (a shift of the G and n curves in the direction of higher KCl concentrations, accompanied by a lowering of the maximum of the n curve).

8. *Remarks on the use of half filled and completely filled spherical vessels of* 110 *and* 500 *ml capacity.*

Formerly we used as a rule, vessels of approximately 500 ml. The experiments in section 4 were performed with 110 ml vessels, which save 4/5 of the necessary chemicals. The aim of the above was to find out if still reliable experiments are possible with such small vessels. For the purposes of the present investigation the results were still good enough, but for future work on the relation between the structure of organic substances and their influence on the elastic properties of the oleate system, we will return to the use of 500 ml vessels if we want to use the method of the half filled spheres. In general we have the impression that the strive for an economic use of the chemicals goes at the cost of an increase in the experimental errors.

In order to investigate quantitative relationship completely filled vessels should be used instead of half filled ones (decidedly no "surface consolidation", see section 3) and here too we prefer 500 ml vessels above 110 ml vessels.

9. *Summary.*

1. Two new methods have been described: 1) the marking of the elastic oleate system with small H_2 bubbles obtained electrolytically, 2) a simplified way for the performing of experiments with the technique of the half filled vessels.

2. The influence of added benzene, naphtalene, hexane, heptane and undecylate was studied on the position of the curves which represent G (shear modulus) and n (maximum number of observable oscillations,

an approximate reciprocal measure for the logarithmic decrement) as functions of the KCl concentrations.

3. In the range of studied concentrations (both of the added substance and of KCl) benzene and naphtalene act typically according to type *A*, glycerol and undecylate act typically according to type *C*.

4. With hexane and heptane depending on each of the above mentioned concentrations, either type *A* or type *B* or type *C* may occur. A change in one of the concentrations keeping the other constant may lead to a transition in action type (e.g. $C \rightarrow B \rightarrow A$ at increase of the KCl concentration and $A \rightarrow B \rightarrow C$ at increase of the hexane concentration).

5. The hypothesis is drafted that each action type is the resultant of two components acting in opposite directions (type *A* and *C* if one preponderates over the other, type *B* when they compensate each other exactly).

6. The hypothesis sub 5 explains the occurrence of transitions in the type of action (mentioned in sub 4) and the unlikeliness of finding an organic substance which "follows typically type *B*", i.e. which within a whole range of concentrations of itself and of the KCl shows the action type *B*.

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CHEMISTRY

ELASTIC VISCOUS OLEATE SYSTEMS CONTAINING KCl. XI [1])

Influence of aliphatic and aromatic hydrocarbons on the elastic behaviour of the 1.2 % oleate system.

BY

H. G. BUNGENBERG DE JONG, W. A. LOEVEN* AND W. W. H. WEIJZEN* [2])

Communicated at the meeting May 20, 1950)

1. *Introduction and Methods.*

Preliminary investigations about the action of some hydrocarbons on the elastic oleate system, have already been performed in the parts VII and X of this series.

To get some insight into the relation between the structure of the hydrocarbon and its action on the elastic behaviour of the oleate system, we deal with 1) the homologous series of the alkanes, 2) isomeres of *n*-octane, 3) introduction of a double or a triple bond in *n*-octane, 4) benzene and alkylbenzenes, 5) *o*-, *m*- and *p*-xylenes, in the present investigation (performed at 15°) [3]).

We used the original method with the exactly half filled vessels of approximately 500 ml capacity (Pyrex vessels varying between 496 and 505 ml). This method is described in detail in part VI of this series. The hydrocarbons were added with the dripping pipette, described elsewhere [4]), the dropweight of each hydrocarbon being determined by weighing 10 drops. As the maximal quantity which was added, changed the degree of filling of the vessels only 0,13 %, no corrections for the increase in volume were necessary.

For the exciting of the rotational oscillation, we used the contrivance described in part VII. As none of us could perform the difficult measurements of the decrement, we had to content ourselves with the change

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1) Part I has appeared in these Proceedings **51**, 1197 (1948), Parts II—VI in these Proceedings **52**, 15, 99, 363, 377, 465 (1949), Parts VII—X in these Proceedings **53**, 7, 109, 233, 743 (1950).

2) Publication no. 6 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

3) We are much indebted to the Koninklijke/Shell Laboratorium, Amsterdam, which sent us the pure hydrocarbons used in our experiments.

4) H. G. BUNGENBERG DE JONG and L. J. DE HEER, these Proceedings **52**, 783 (1949).

in n, caused by the addition of hydrocarbons, as an approximate measure for the change in $1/\Lambda$ (part VII, section 3).

As an oleate preparation we used Na oleate, neutral powder, from BAKER [5]) (for remarks on this preparation see part X). All experiments were made with one and the same mixture out of the contents of 5 flasks.

The idea was, to measure the influence of the hydrocarbons at the KCl concentration of minimum damping, as at this concentration, maximum information is obtained (part VII, section 5).

For the above mixture this KCl concentration lies very near 1.05 N, but at the beginning of the experiments we did not know yet that the contents of individual flasks of Na oleate, bearing the same lot number, may show differences (part X, section 4). At a preliminary experiment on the position of this KCl concentration of minimum damping, on which occasion we used one flask, we found approximately 1.2 N.

Therefore several series of measurements have been performed at a KCl concentration of 1.2 N, that is at a concentration somewhat higher than that corresponding to the minimum damping of the blank oleate system. This detail is of importance for the discussion of the results. The series with the isomeres of octane, octene and octyn, however, were performed at 1.05 N KCl. The aromatic hydrocarbons were also investigated at 0.6 N KCl, that is at a KCl concentration markedly lower than that corresponding to the minimum damping of the blank oleate system.

The elastic systems used in the experiments always contained 1.2 % oleate and 0.05 N KOH. These are prepared from a stock oleate solution (24 g per liter) containing 0.1 N KOH.

For obtaining the elastic system, one volume of this stock solution is vigorously shaken, in a large flask, with one volume of a KCl solution. A quantity, sufficient to fill a series of 7 exactly half filled 500 ml vessels, is obtained according to this receipt:

1000 ml oleate stock solution + 1000 ml KCl x N (e.g. for 1.2 N KCl $x =$ 2.4; for 0.6 N KCl $x = 1.2$).

2. *The homologous series of the n-alkanes from n-pentane up to and including n-undecane.*

The alkanes mentioned in the title, were investigated in two consecutive series, the first (table I) comprising n-pentane, n-hexane, n-heptane and n-octane, the second (table II) n-octane, n-nonane, n-decane and n-undecane.

The results of both series have been united in fig. 1 [6]). This gives no difficulties for the plotting of the upper graph, representing G as a function of the concentration of the alkanes, because the mean G values of the

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[5]) A generous gift of Na oleate from The Rockefeller Foundation provided the means for the experiments described in this paper.

[6]) Some G and n values, provided with a question mark in the tables are obviously erroneous and have not been used in the plotting of fig. 1.

TABLE I. Influence of pentane, hexane, heptane and octane on the elastic behaviour of the 1.2 % oleate system at 1.2 N KCl and at 15°. Conc. = concentration of the hydrocarbon in millimoles/l; G = shear modulus in dyne/cm²; n = maximum number of observable oscillations through the telescope. Values of the blank G = 49.1 and n = 66.7

n-pentane			*n*-hexane			*n*-heptane			*n*-octane		
conc.	G	n	conc.	G	n	conc.	G	n	conc.	G	n
2.39	49.4	65.7	2.34	49.8	66.1	2.09	47.1	66.0	2.06	45.6?	69.7
4.79	50.5	65.7	4.68	50.6	65.0	4.17	45.8	70.3	2.89	45.5	75.4
9.58	50.4	62.1	9.36	45.7	67.1	8.34	41.0	66.0	4.12	44.4	71.7
14.4	48.2	59.8	14.0	40.1	71.1	12.5	34.3	29.5	6.60	40.4	62.2
19.2	45.9	61.7	18.7	37.3?	52.0	16.7	—	4.3	8.25	36.1	28.2?
23.9	44.2	79.5	23.4	29.1	21.4				9.48	33.7	26.1
			28.1	—	5.8						

TABLE II. Influence of octane, nonane, decane and undecane on the elastic behaviour of the 1.2 % oleate system at 1.2 N KCl and at 15°. Conc. = concentration of the hydrocarbon in millimoles/l; G = shear modulus in dyne/cm²; n = maximum number of observable oscillations through the telescope. Values of the blank G = 48.9 and n = 70.6

n-octane			*n*-nonane			*n*-decane			*n*-undecane		
conc.	G	n	conc.	G	n	conc.	G	n	conc.	G	n
2.89	45.7	80.3	1.95	46.8	79.7	1.47	47.7	77.4	1.04	47.6	75.9
5.77	41.7	71.3	3.50	44.0	75.7	2.21	45.0	75.9	1.74	45.1	78.1
7.42	39.0	55.1	4.67	42.3	69.8	2.94	43.9	71.6	2.43	43.6	70.0
9.07	34.7	32.1	5.84	40.0	56.5	4.78	40.4	58.2	3.47	41.7	59.7
10.3	30.5	13.6	7.00	38.0	42.5	5.89	38.2	41.9	4.86	41.2?	53.6?
			8.17	30.9	20.6	7.36	27.5	11.3	5.91	35.3	27.1

blanks are practically the same in the tables I and II. The two experiments with n-octane, give a further justification: the experimentally determined points are lying reasonably close to the one curve which is drawn through them.

The mean values for n, however, are differing somewhat for both series, and therefore we have used in the lower graph of fig. 1 the values 100 n/n_0, in which n_0 represents the n value of the blank. Now, the experimentally determined points for the two series with n-octane lie also close to one and same curve.

A first glance at fig. 1 reveals that the terms of the series of the n-alkanes exert their action at very different concentrations, an alkane being the more active when its carbon chain is longer. We will return to this in section 3.

Next, considering the shape of the G and n curves in fig. 1, we perceive

that the investigated alkanes fall apart in two groups: I *n*-undecane, *n*-decane, *n*-nonane, *n*-octane and *n*-heptane, of which the *G* curves at once go into a downward direction, II *n*-hexane and *n*-pentane, of which the *G* curves first proceed upwards to a maximum and thereafter proceed downwards.

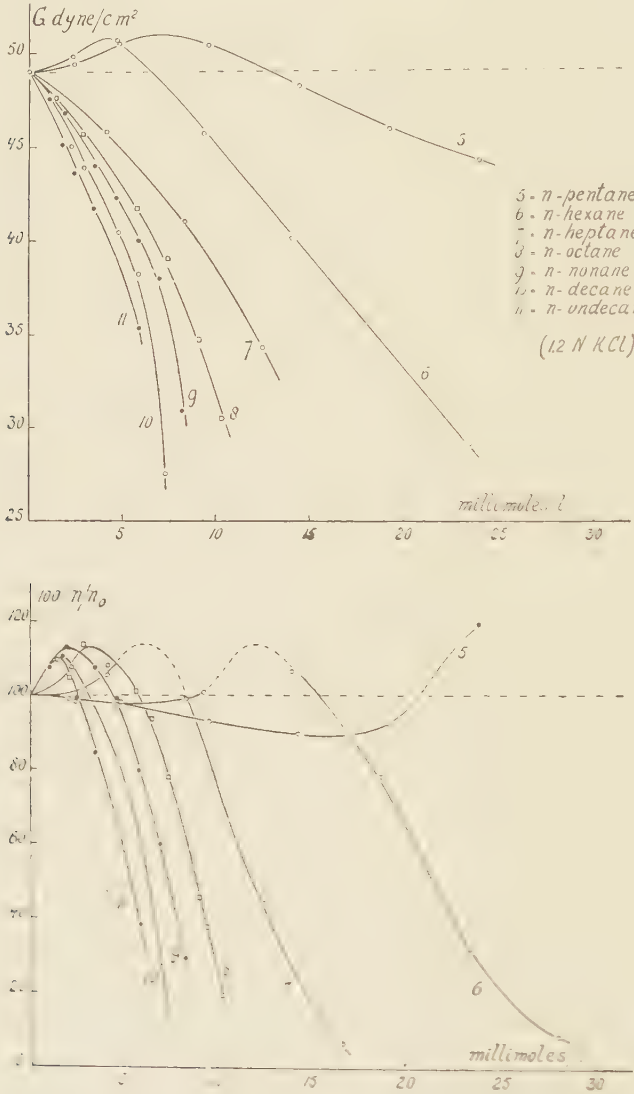

Fig. 1

Likewise, the shape of the *n* curves gives rise to a distinction of the same groups. For the hydrocarbons of group I these curves are proceeding upwards to a maximum and there after definitely downwards. For the hydrocarbons of group II they are first proceeding to a minimum, then to a maximum and thereafter definitely downwards (*n*-pentane

concentration has obviously not been taken high enough to realise the n curve in its whole).

In part VII of this series (in section 5) it was discussed how one can conclude from experiments on the influence of an added organic substance, performed at constant KCl concentration, to the type of action (A, B or C) which this substance would exhibit on the position of the G and $1/\Lambda$ (or λ or n) curves, in graphs representing G and $1/\Lambda$ (or λ or n) as functions of the KCl concentration (type A, a shift in the direction of smaller, type C, a shift in the direction of higher KCl concentrations and type B, no shift in either direction). A scheme was given, being repro-

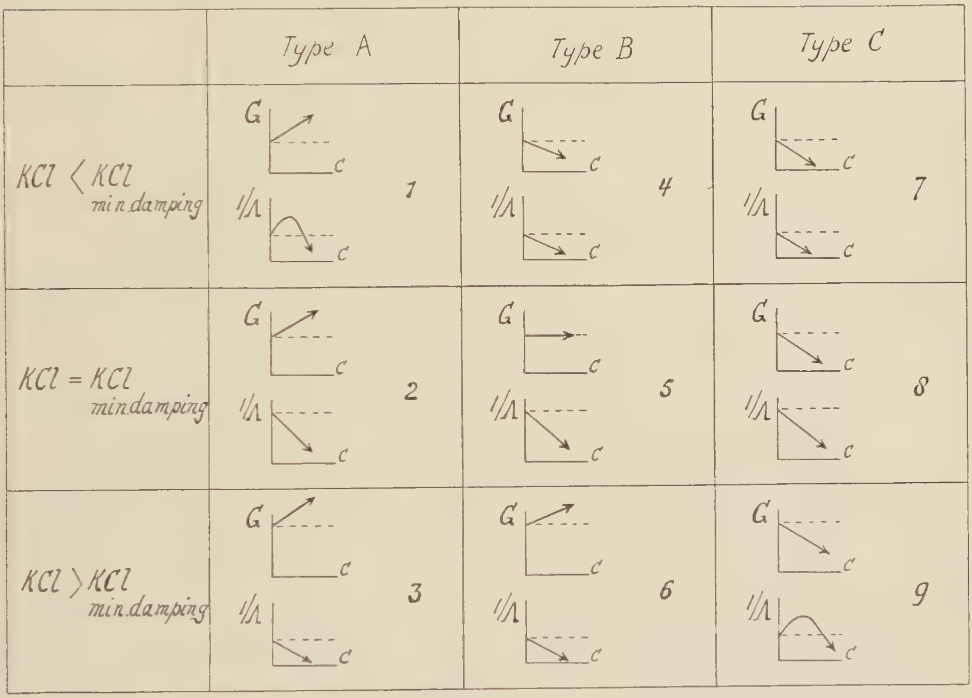

Fig. 2

duced here as fig. 2. From this one can read for KCl concentrations lower than, equal to or higher than that corresponding to the KCl concentration of minimum damping, the sign of the change of G and of $1/\Lambda$, λ or n if the added substance shows the type of action A, B or C.

Applying this to the present experiments with added hydrocarbons we must bear in mind that these have been performed at 1.2 N KCl, that is at a concentration somewhat higher than that corresponding to the KCl concentration of minimum damping of the blank. From the nine diagrams of fig. 2 we must therefore only consider the diagrams 3, 6 and 9.

The results of the experiments with octane, nonane, decane and undecane correspond to diagram 9, from which we conclude that the above alkanes $C_8 - C_{11}$ typically follow the action type C.

The results obtained with pentane and hexane give more complicated G and n curves, which do not correspond to one of the diagrams 3, 6 and 9 in fig. 2. They can, however, be considered as to be composed of the G and n curves from the diagrams 6 and 9, or from the diagrams 3, 6 and 9, linked together in the above mentioned succession.

This would mean, that hexane and pentane show a transition in the type of action at increasing hydrocarbon concentration, namely $B \rightarrow C$, or $A \rightarrow B \rightarrow C$. As to hexane we need not ponder much which of the two is more probable, because in part X we have found by the direct method, that both at 1.05 N and 1.7 N KCl, hexane shows the transition in type $A \rightarrow B \rightarrow C$, at increasing hexane concentration. Therefore, at 1.2 N KCl it must show the same transition.

As to pentane, we have no such direct indications in favour of one of the two possibilities, but here too we must conclude to $A \rightarrow B \rightarrow C$ on the ground of regularities, exhibited by fig. 8 of part X of this series. In this figure, which gives the shifts of three characteristic points of the n- C_{KCl} curve at the addition of a number of organic substances, we perceive that going from hexane to heptane, the initial shift, resulting from the action type A, in the transition $A \rightarrow B \rightarrow C$, diminishes (at 1.7 N KCl) or even disappears, leaving only the transition $B \rightarrow C$ (at 1.05 N KCl). We therefore may conclude with high probability, that the transition $A \rightarrow B \rightarrow C$ which exists at 1.2 N for hexane, will also hold for pentane, and further that the initial action resulting from the action type A in the transition $A \rightarrow B \rightarrow C$ will be more pronounced in pentane than in hexane. This latter expectation is confirmed by the experiment, the n curve for pentane in fig. 1 showing a deeper minimum than the n curve for hexane.

Now when we consider in fig. 1, the results obtained with heptane, we find that the course of the n curve (proceeding nearly horizontally before it mounts to the maximum) lies between the courses of the n-curves for the higher alkanes (proceeding at once upwards to the maximum) and the courses of the n curves for the hexane and pentane (proceeding at once downwards to the minimum).

Heptane may therefore be considered as a substance standing very near to one which at small concentrations shows type B and which at further increase of the concentration shows the transition in type $B \rightarrow C$ (in accordance with the findings in part X for heptane at the KCl concentration corresponding to the minimum damping of the blank).

We have therefore indicated in the survey below, which summarizes the above discussions, the type of action exhibited by heptane at the first additions, with "B".

We will end this section by remarking that this survey (valid for 1.2 N KCl) on the action of alkanes gives only two of the three variables that are involved. Next to the length of the carbon chain and

to the concentration of the alkane, the action type still depends on the KCl concentration (— or probably better — on the internal structure of the elastic system, this structure being a function of the KCl concentration). The experiments with hexane and heptane in part *X* of this series allow to conclude that the centre column (with at the top "*B*") of the survey is displaced to the right at increasing KCl concentration, and to the left at decreasing KCl concentration.

Concentration of the alkane	C_5	C_6	C_7	$\underbrace{C_8 \quad C_9 \quad C_{10} \quad C_{11}}$
small	*A*	*A*	"B"	*C*
	↓	↓	↓	↓
	A	**B**	*C*	*C*
	↓	↓	↓	↓
↓	**B**	*C*	*C*	*C*
increasingly	↓	↓	↓	↓
larger	*C*	*C*	*C*	*C*

3. *Discussion of the results obtained with the n-alkanes from the point of view that the total action is the resultant of two action components, a KCl-sparing one and a KCl-demanding one.*

In part *X* of this series the transitions in the type of action shown by hexane and heptane, led to the hypothesis that what we have called the type of action (*A* or *B* or *C*) is the resultant of two components acting in opposite directions (having type *A* and type *C* if one of the two preponderates over the other, type *B* when they compensate each other exactly).

Booij and collaborators investigated the influence of organic substances on the shift of the coacervation limit (which as a rule, lies at a slightly higher concentration than the right footpoint of the *n* curve). With the *n*-alkanes [7]) he obtained transitions in the sign of the KCl shift, which follow the same general rules as were found at the investigation about the influence of the alkanes on the elastic behaviour of the oleate system (at lower KCl concentrations) i.e. transitions as a result of the concentration of the alkane and, at constant concentration, as a result of the number of carbon atoms of the alkane. Booij too, considered the total KCl shift as the resultant of two component shifts acting in opposite directions, and drew up a hypothesis to explain the existence of component actions and their mechanism.

We have the strong conviction that similar considerations (the component actions being the result of the actions of the molecules taken up by the oleate micelles at essentially different places) will also be useful in the explanation of the influence of organic substances on the elastic behaviour of the oleate system, though probably, a superstructure of micelles is a complicating factor here.

[7]) H. L. Booij, J. C. Lycklama and C. J. Vogelsang, these Proceedings 53, 407—413 (1950).

It will be very convenient for the future work both of BOOIJ and collaborators working on the coacervation limit, and of us working on the elastic behaviour, to have expressions for the sign of the total action and that of the component actions. These expressions must be as neutral as possible and must therefore not contain elements borrowed from hypotheses about the mechanism of the actions on coacervates or on the elastic systems.

In concerted discussion with BOOIJ we propose the terms "*KCl-sparing*" and "*KCl-demanding*".

These terms will characterize the nature of the total action of an organic substance (and even may be used eliptically for the characterization of the substance at a given concentration of its own and of the KCl on hand) and the nature of the component actions of which the total action is the resultant.

Applying this to the action of the alkanes on the elastic behaviour at 1.2 N KCl, pentane and hexane exert a KCl-sparing action at low concentrations of these hydrocarbons and a KCl-demanding action at higher concentrations. The higher alkanes (octane up to and including undecane) exert a KCl-demanding action, independent of their concentration.

In applying the hypothesis of the action type (A, B, C) as being a resultant of oppositely directed component actions, the results obtained with the alkanes at 1.2 N KCl may be formulated as indicated in the survey below, in which the symbols S and D are used as to denote the KCl-sparing and KCl-demanding component actions.

Concentration of the hydrocarbon	C_5	C_6	C_7	C_8 C_9 C_{10} C_{11}
small	$D < S$	$D < S$	$\mathbf{D = S}$	$D > S$
↓	$D < S$	$\mathbf{D = S}$	$D > S$	$D > S$
increasingly	$\mathbf{D = S}$	$D > S$	$D > S$	$D > S$
larger	$D > S$	$D > S$	$D > S$	$D > S$

At decrease of the KCl concentration the oblique bar formed in the survey by the symbols $D = S$ is displaced to the left, and at increase of the KCl concentration it is displaced to the right.

We now come back to the remark made in section 2 that the alkanes exert their action at very different concentrations, an alkane being the more "active" when its carbon chain is longer (compare the succession of the G and n curves in fig. 1).

In part VI of this series we have met with an analogous rule for the homologous series of the n-primary alcohols $(C_1 - C_6)$ and the fatty acid anions $(C_7 - C_{11})$, but the resemblance with the succession of the alkanes of fig. 1 is only superficial. In the case of the alcohols and the

fatty acid anions — these being soluble in the soap free KCl solution — the series of "increasing activity" of the alcohols 6>5>4>3>2>1 or of the fatty acid anions 11>10>9>8 brings practically only to expression that the distribution of the added molecules (or anions) between oleate micelles and medium shows an important shift in favour of the micelles when the carbon chain is lenghthened (compare part VIII, section 3).

With the hydrocarbons we have quite another situation. As these are practically insoluble in the soap free KCl solution, a distribution equilibrium between hydrocarbon molecules being present free in the medium and molecules taken up by the oleate micelles, can be practically neglected. By approximation we may consider the added hydrocarbon (expressed in the tables and in fig. 1 as a "concentration") as being wholly taken up in the oleate micelles. But this means that the different location of the G and n curves in fig. 1 directly brings to expression specific differences in action between the successive terms of the homologous series of the n-alkanes.

The hypothesis of the total action being the resultant of two oppositely directed component actions may shed some light on the marked spreading of the curves in fig. 1 .One must assume (see already part X, section 6) that the hydrocarbon molecules are taken up simultaneously by the oleate micelle at two essentially different places, and that dependent on the occupied place, the hydrocarbon molecules exert a KCl-sparing or a KCl-demanding action. Calling these places within the oleate micelle S-places and D-places, the marked spreading of the curves in fig. 1 may for the greater part be caused by a change in the distribution between S- and D-places of the alkane molecules which are taken up, this being a result of the lenghthening of the carbon chain of the alkane.

In this line of thought a small length of the carbon chain is much in favour of occupying the S-places. The KCl-demanding component action therefore prevails at small concentrations of the alkane. At higher concentrations the S-places cannot take up much more alkane molecules and now the D-places become more and more occupied. This gives an explanation of the reverse in total action type ($A \rightarrow B \rightarrow C$) shown by pentane and hexane.

Alkanes with long hydrocarbon chains will on the contrary favour the D-places, that is at all concentrations the KCl-demanding component action is larger than the KCl-sparing component action; they only show the total action type C.

4. *Isomers of octane. Introduction of a double and of a triple bond in n-octane.*

In connection with the fact that these experiments (table III, fig. 3) have been performed at 1.05 N KCl, i.e. at the KCl concentration corresponding to the minimum damping of the blank, here the n curve of n-octane at once takes a course in a downward direction, whereas in

TABLE III. Influence of some octanes, of transoctene-4 and octyne-4 on the elastic behaviour of the 1.2 % oleate system at 1.05 N KCl and at 15°. Conc. = concentration of the hydrocarbon in millimoles/l; G = shear modulus in dyne/cm^2; n = maximum number of observable oscillations through the telescope.

n-octane			trans-octene-4			octyne-4		
conc.	G	n	conc.	G	n	conc.	G	n
blank	47.8	75.8	blank	48.0	76.3	blank	47.8	76.0
2.89	46.0	68.7	2.09	48.0	69.5	2.44	49.1	62.4
4.54	42.8	59.7	4.19	45.8	61.6	3.90	51.3	54.4
6.18	39.1	41.6	5.86	44.2	50.7	6.82	49.5	49.5
7.42	34.2	29.5	7.54	39.5	35.8	9.75	46.9	59.7
8.25	30.0	16.2	9.22	34.3	23.5			
9.48	25.9	10.5	11.3	24.1	10.6			
11.5	—	2.0						
2,5-dimethylhexane			3,4-dimethylhexane			3-methyl-3-ethylpentane		
conc.	G	n	conc.	G	n	conc.	G	n
blank	47.9	76.3	blank	48.2	76.1	blank	47.4?	76.2
1.88	45.9	68.3	2.07	46.2	71.5	1.65	47.4	74.2
3.75	42.7	59.7	4.54	43.1	63.7	4.96	44.1	64.5
5.63	39.9	43.6	6.20	41.3	46.6	7.45	40.8	47.6
7.50	34.8	23.6	8.26	36.5	28.4	9.93	35.1	31.5

Mean values of the blanks $G_0 = 47.9$ $n = 76.1$

fig. 1 (experiments performed at 1.2 N KCl) the n curve at first mounts to a maximum before it definitely takes its course in a downward direction (compare the diagrams 8 and 9 in fig. 2).

It appears from fig. 3*A* that the four octanes which are investigated all follow the action type *C*, but that their actions are not indentical. We will not comment upon this, but only mention that analogous differences in the influence on the displacement of the coacervation limit, do occur [8]).

A distinct influence results from the introduction of a double bond. In fig. 3*B*, the curves for trans-octene-4 show a course which indicates that at small concentrations type *B* is followed and at larger concentrations type *C* (a succession of the diagrams 5 and 8 in fig. 2).

The introduction of a triple bond in *n*-octane has a still larger effect. The course of the *G* and *n* curves for octyne-4 must be interpreted as a succession of the types $A \rightarrow B \rightarrow C$. In the scheme of fig. 2 we have the following succession of diagrams: $2 \rightarrow 3 \rightarrow 6 \rightarrow 9$.

The switching over from diagram $2 \rightarrow 3$ follows from the displacement of the *G* and *n* curves as a result of the type of action *A*. After

[8]) H. L. Booij, J. C. Lycklama and C. J. Vogelsang, these Proceedings 53, 407–413 (1950).

this displacement the KCl concentration of 1.05 N is a KCl concentration higher than that corresponding to the minimum of the displaced curve.

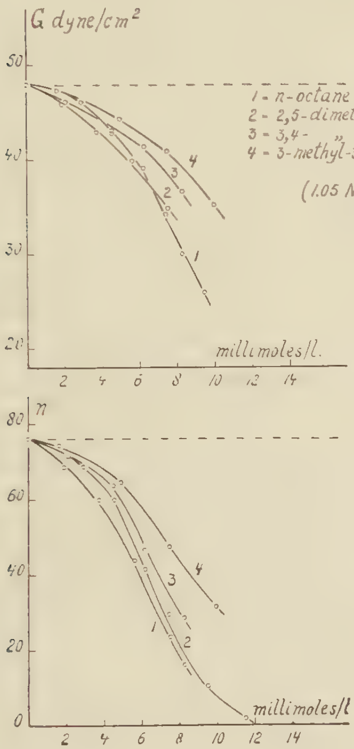

Fig. 3 *A*

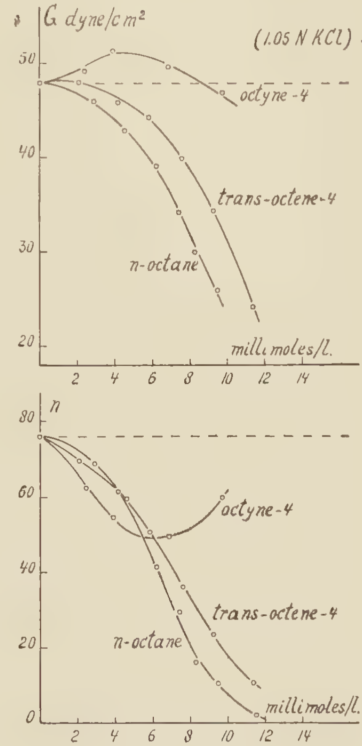

Fig. 3 *B*

From the point of view of the total action type being a resultant of component actions, we may conclude that the introduction of a double bond and still more that of a triple bond in an alkane, relatively enforces the KCl-sparing component action with regard to the KCl-demanding component action. An analogous conclusion was reached by Booij and collaborators [9]) in studying the shift of the coacervation limit.

5. *Benzene, toluene, ethylbenzene, n-propylbenzene and the three xylenes.*

The experiments have been performed at 1.2 N KCl (table IV, fig. 4) and at 0.6 N KCl (table V, fig. 5).

We will first discuss the results obtained with benzene, toluene, ethyl benzene and *n*-propylbenzene.

The results at 1.2 N KCl (fig. 4*A*) show that all four follow type *A* (compare diagram 3 in fig. 2). There is a distinct spreading of the curves. The concentration of the hydrocarbon which is needed to give a same

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[9]) H. L. Booij, J. C. Lycklama and C. J. Vogelsang, these Proceedings 53, 407—413 (1950).

TABLE IV. Influence of some aromatic hydrocarbons on the elastic behaviour of the 1.2 % oleate system at 1.20 N KCl and at 15°. Conc. = concentration of the hydrocarbon in millimoles/l; G = shear modulus in dyne/cm²; n = maximum of observable oscillations though the telescope.

benzene			toluene			ethylbenzene			n-propylbenzene		
conc.	G	n	conc.	G	n	conc.	G	n	conc.	G	n
blank	47.5	67.2	blank	48.0	67.3	blank	47.8	67.4	blank	47.9	67.2
2.35	51.0	53.6	1.95	51.5	50.0	1.17	51.3	60.3	1.04	51.7	53.3
3.92	54.3	48.3	3.25	57.9	33.6	2.34	56.2	40.4	2.08	56.7	34.2
6.27	61.2	33.1	3.90	61.3	28.3	2.92	59.7	33.8	2.60	59.7	24.2
7.84	63.7	23.5	4.55	63.2	23.0	3.50	60.5	24.7	3.13	62.6	19.7
9.4	65.5	13.7	5.20	64.4	16.2	4.10	61.0	16.7	3.65	—	11.2
11.0	—	5.8	6.5	—	6.8	5.26	—	7.0	4.17	—	5.7

o-xylene			m-xylene			p-xylene		
conc.	G	n	conc.	G	n	conc.	G	n
blank	47.6	67.4	blank	47.7	67.4	blank	47.5	67.3
1.82	54.1	48.3	1.74	54.3	48.3	1.74	54.5	48.1
3.04	62.9	25.9	2.90	58.6	30.6	2.90	57.1	31.2
3.64	67.0	19.7	3.48	61.2	21.4	3.48	60.2	21.6
4.25	—	10.0	4.06	—	13.2	4.06	—	14.3
4.86	—	5.5	4.64	—	6.3	4.64	—	6.8
5.46	—	coac.	5.22	—	1.0	5.22	—	3.0

Mean values of the blanks $G_0 = 47.7$ $n_0 = 67.3$

increase of G, or a same decrease in n, diminishes in the order: benzene > toluene > ethylbenzene > n-propylbenzene.

The results at 0.6 N KCl (fig. 5A) show that all four hydrocarbons begin to follow type A (compare diagram 1 of fig. 2). At increasing hydrocarbon concentration, however, a transition in the type of action occurs because G passes a maximum and further on decreases. It seems probable that we have a succession of the diagrams $1 \rightarrow 5 \rightarrow 8$ or $1 \rightarrow 5 \rightarrow 9$ of fig. 2 here, which means that at increasing hydrocarbon concentration the type of action changes from $A \rightarrow B \rightarrow C$. Just as at 1.2 N KCl, here too we perceive the sequence: benzene > toluene > ethylbenzene > propylbenzene at small concentrations. The relative positions of the G and n curves of benzene and toluene do not allow to conclude for which of the two hydrocarbons the transition $A \rightarrow B \rightarrow C$ occurs earlier. However, it is clear that this transition occurs at smaller and smaller concentrations in the sequence: toluene > ethylbenzene > n-propylbenzene.

We could now proceed to discuss the results from the point of view that the total action is the resultant of a KCl-sparing and a KCl-demanding component action. But this would only be an analogous repetition of the discussions in section 4 for the n-alkanes. We therefore give at once, in the survey below, the results to which this discussion leads.

KCl concentration	hydrocarbon concentration	benzene	toluene	ethyl-benzene	n-propyl-benzene
0.6 N	small	$D < S$	$D < S$	$D < S$	$D < S$
	↓	$D < S$	$D < S$	$D < S$	**D = S**
		$D < S$	$D < S$	**D = S**	$D > S$
	increasingly	**D = S**	**D = S**	$D > S$	$D > S$
	larger	$D > S$	$D > S$	$D > S$	$D > S$
1.2 N	at all concentrations investigated . . .	$D < S$	$D < S$	$D < S$	$D < S$

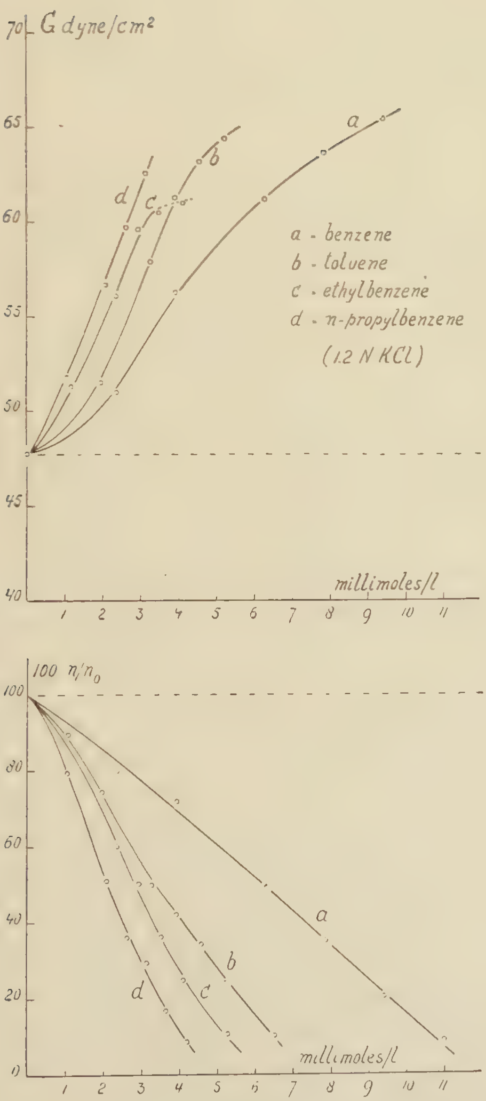

Fig. 4 *A*

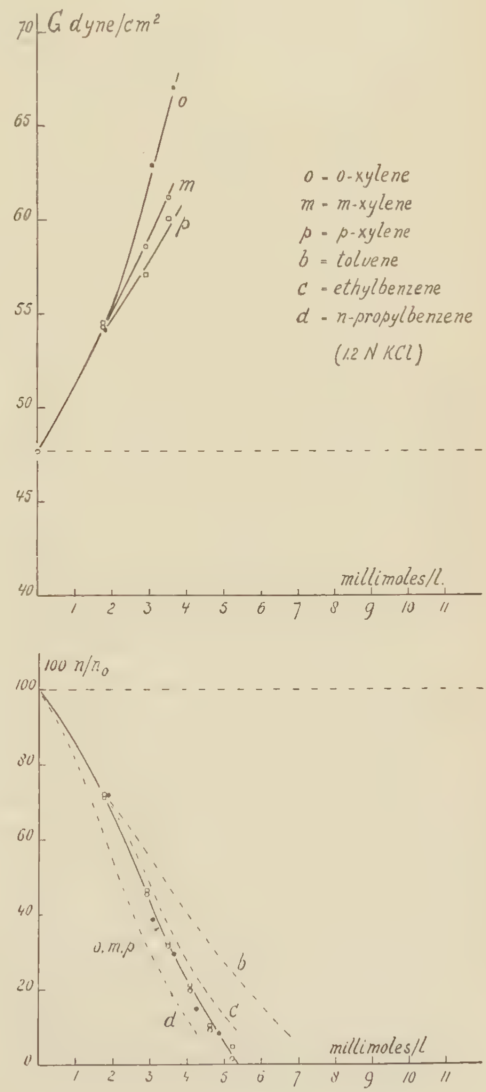

Fig. 4 *B*

TABLE V. Influence of some aromatic hydrocarbons on the elastic behaviour of the 1.2 % oleate system at 0.6 N KCl and at 15°. Conc. = concentration of the hydrocarbon in millimoles/l; G = shear modulus in dyne/cm^2; n = maximum number of observable oscillations though the telecsope.

benzene			toluene			ethylbenzene			n-propylbenzene		
conc.	G	n	conc.	G	n	conc.	G	n	conc.	G	n
blank	31.3	22.1	blank	31.5	21.9	blank	31.3	22.1	blank	31.2	21.8
1.57	33.3	25.6	0.65	32.7	26.0	0.59	32.5	25.4	0.52	32.3	25.3
3.92	34.8	31.6	1.95	33.9	29.4	1.75	34.4	30.9	1.56	33.6	29.1
10.2	38.6	46.2	6.5	38.9	45.2	5.26	36.9	36.3	4.17	33.9	28.1
13.3	40.9	53.8	8.4	40.2	47.6	6.4	36.5	37.5	4.7	32.7	25.8
17.2	40.9?	53.8	11.0	40.3	50.6	7.6	36.2	38.3	6.3	31.7	21.6
23.5	45.6	50.0	16.2	41.1	49.2	11.7	34.6	30.7	8.9	26.7	15.3
35.3	30.3	7.9	26.0	38.8	47.1	15.8	32.0	26.1	11.5	19.3	8.3
39.2	—	5.0	39.0	11.9	6.9	21.6	18.2	10.6	15.6	—	1.0

o-xylene			m-xylene			p-xylene		
conc.	G	n	conc.	G	n	conc.	G	n
blank	31.3	22.0	blank	31.1	21.8	blank	30.9	21.8
0.61	31.9	25.8	0.58	32.6	25.6	0.58	32.8	26.2
1.82	34.2	30.3	1.74	34.2	29.8	1.74	34.2	29.6
5.46	38.8	47.3	5.22	38.5	43.8	5.22	37.8	43.2
6.7	38.5	47.6	6.4	37.7	44.3	6.4	37.8	43.7
7.9	38.1	43.8	7.5	37.4	41.5	7.5	37.7	40.3
12.1	38.7	42.2	11.6	37.8	39.7	11.6	38.5	40.2
18.2	33.9	34.2	17.4	32.2	30.6	17.4	34.2	32.4
27.3	—	6.5	26.1	13.5	6.4	26.1	19.3	9.9

Mean values of the blanks $G_0 = 31.2$ $n_0 = 21.9$

By comparing this survey with the analogous one in section 3 for the homologous series of the alkanes we perceive that here too the same three variables work in exactly the same direction as with the n-alkanes, viz. D is relatively favoured to S by:

1. increase of the hydrocarbon concentration,
2. increase of the length of the carbon chain, [10])
3. lowering of the KCl concentration.

Though on first acquaintance with the action of hydrocarbons on the elastic behaviour in part VII, the results seemed to indicate an essential difference between aliphatic and aromatic hydrocarbons (at the KCl concentration of minimum damping of the blank: heptane follows type C, benzene and toluene follow type A), this first impression was erroneous. From the results of the present investigation we cannot but conclude that there is only a gradual difference between aliphatic and aromatic hydrocarbons.

[10]) See also H. L. BOOIJ, C. J. VOGELSANG and J. C. LYCKLAMA, these Proceedings 53, 882 (1950).

The results with the three xylenes (fig. 4*B* and fig. 5*B*) show that an alteration in the relative position of the two methyl groups does not bring about a large difference in action.

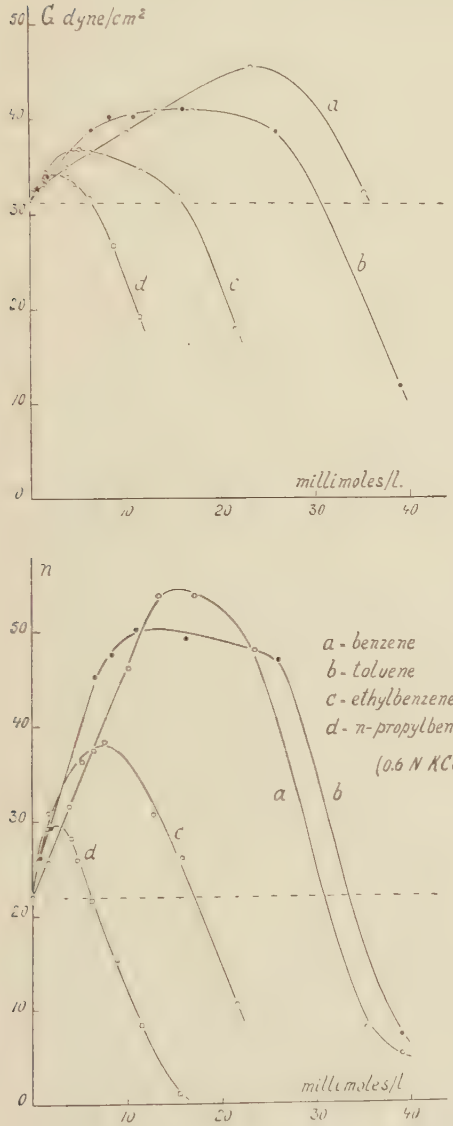

Fig. 5*A*

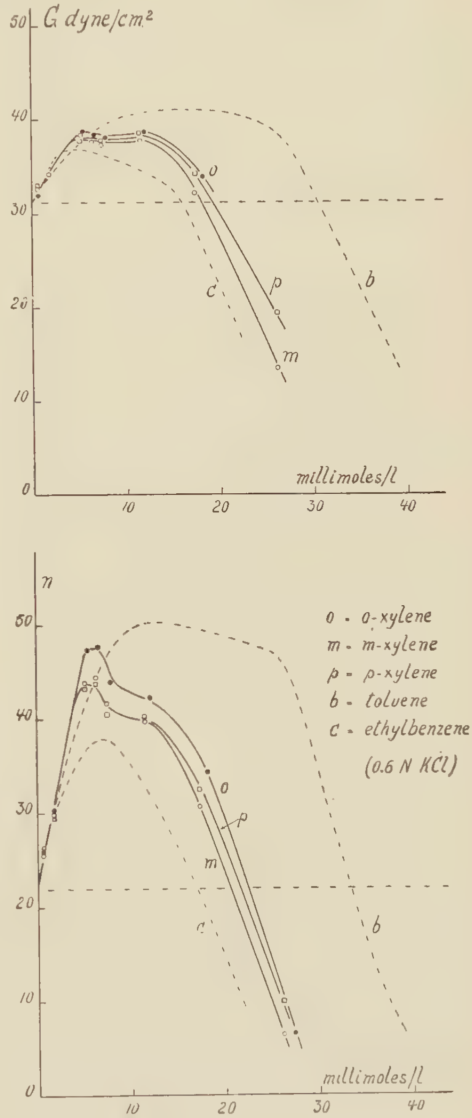

Fig. 5*B*

Summary.

1. An investigation has been made about the influence on the elastic behaviour of the 1.2 % oleate system by a number of hydrocarbons (*n*-pentane up to and including *n*-undecane, four isomers of octane, trans octene-4, octyne-4, benzene, toluene, ethylbenzene, *n*-propylbenzene and the three xylenes).

2. The type of action shown by *n*-alkanes and *n*-alkylbenzenes depends on:
   1. the length of the carbon chain (or: of the side chain),
   2. the concentration of the hydrocarbon,
   3. the KCl concentration.

3. The results, discussed from the point of view that the total action type is the resultant of a KCl-sparing and a KCl-demanding component action, allow to conclude that there is no essential but only a gradual difference between the action of aliphatic and aromatic hydrocarbons.

4. The KCl-demanding component action is relatively enforced with regard to the KCl-sparing component action by:
   1. increase of the hydrocarbon concentration,
   2. increase of the length of the carbon (side) chain,
   3. lowering of the KCl concentration.

5. Introduction of a double bond, and still more that of a triple bond in *n*-octane relatively enforces the KCl-sparing component action with regard to the KCl-demanding one.

6. The four isomers of octane which are investigated show actions which differ quantitatively. Though differences between the three xylenes are also present, they are here but relatively small.

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MAXILLARY TEETH IN SPECIMENS OF HYPEROODON ROSTRATUS (MÜLLER) AND MESOPLODON GRAYI VON HAAST STRANDED ON THE DUTCH COASTS

BY

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(Communicated at the meeting of May 20, 1950)

In various species of Ziphioid Whales there occur, besides the larger teeth found at the extremity or near the posterior end of the symphysis of the mandible, very small teeth, in the upper jaw as well as in the lower.

In *Hyperoodon rostratus* (Müller) these teeth have been recorded by several authors. LACÉPÈDE (1804) remarked that in this species there were small teeth, not only on the sides of the upper jaw, but also on the palate. It seems, however, that LACÉPÈDE erroneously regarded the rough papillae on the palate as teeth (BEDDARD, 1900, p. 222). Later ESCHRICHT (1845) found three small teeth projecting through the gum of the upper jaw and two similar teeth in the lower jaw of a specimen of *Hyperoodon*. Further research led to the discovery of a larger number of these teeth embedded in the gums of both jaws of this specimen. There were twelve to thirteen on the right side of the upper jaw, probably the same number on the left side; in the right half of the lower jaw there were eleven rudimentary teeth, for the left half the number was not noted before they were lost. The eleven small teeth in the right half of the lower jaw were figured (copied in text-figure 1*b* of the present paper). The rudimentary teeth described by ESCHRICHT were slightly curved, their length was 3.3 to 5.9 mm, their thickness up to 1.6 mm. In the specimen of *Hyperoodon* dissected by VROLIK (1848) there were six rudimentary teeth in the foremost part of the left lower jaw, completely hidden in the gum. VROLIK's figure of these teeth has been copied in text-figure 1*c* of the present paper. The teeth are smaller than those described by ESCHRICHT, their length varying from 1.37 to 2.75 mm. Various other authors have drawn attention to these small teeth in *Hyperoodon*.

Repeatedly similar small teeth have been observed in the upper as well as in the lower jaw of *Ziphius cavirostris* Cuvier. As a rule these were of very small size, but a fairly large rudimentary tooth was described and figured by TRUE (1910). This tooth had a length of 16 mm and a thickness of 2 mm. Generally in *Ziphius cavirostris* these teeth are extremely small; large rows of such minute teeth have been recorded by FRASER (1936). In one specimen there were fourteen vestigial teeth

on the right and twelve on the left side of the lower jaw, the largest tooth of the series measuring 8.5 mm. In a second specimen there were one vestigial tooth and seven follicles without calcified remains in the left lower jaw; the right lower jaw was not available. In the left upper jaw of the same specimen there were eight vestigial teeth and six or seven additional follicles, and in the right upper seven teeth and the same number of additional follicles. In a third specimen, of which one ramus of the lower jaw was available for dissection, there were twenty-eight vestigial teeth, the longest of which was 3.9 mm. Various other authors have mentioned the occurrence of similar teeth in *Ziphius cavirostris*, as a rule in much smaller numbers. On the other hand FRASER remarks that BURMEISTER in the Buenos Aires specimen of *Ziphius* found twenty-five teeth in each upper and thirty to thirty-two in each lower jaw.

VAN BENEDEN (1888) and BEDDARD (1900) state that rudimentary teeth of the same size and structure as those of other Ziphioid Whales not unfrequently occur in *Mesoplodon bidens* (Sowerby). Obviously this statement is based on the small functionless teeth described by GERVAIS (on page 402 in VAN BENEDEN and GERVAIS, 1880, figures 6 and 8 on plate XXVI). At a very short distance from the larger tooth in the lower jaw there were on one side four and on the other two teeth not longer than 5 mm; originally there may have been more of these teeth. In a specimen of *Mesoplodon bidens* stranded in the Shetland Islands HARMER (1927, p. 56) found six vestigial teeth concealed in the gum of the right ramus of the lower jaw, the row commencing 34 mm behind the large tooth; in the left ramus of the lower jaw one vestigial tooth was found.

In the rarer species of the genus *Mesoplodon* up to the present time no vestigial teeth similar to those dealt with above were found. On the other hand in *Mesoplodon grayi* von Haast, a species fairly common in New Zealand waters, the occurrence of a row of small teeth on each side of the upper jaw seems to be a constant character of the species. The peculiarities of these teeth are discussed below.

In the present paper the maxillary teeth of two specimens of Ziphioid Whales in the collection of the Leiden Museum are described in some detail.

Hyperoodon rostratus (Müller)

The specimen (Leiden Museum, reg. no. 7218) stranded on the Noorderleeg on the Frisian coast on October 19, 1946. It proved to be a male of a total length of 7.5 m. Mr. H. CORNET, technician of the Leiden Museum, who was in charge of the activities for roughly cleaning the skeleton on the spot, took care to preserve strips of the gum of each side of the upper jaw, each containing two or more small teeth, slightly projecting above the surface. In its lower jaw the specimen has one tooth of fairly large size in the left half, and in the right half two teeth of slightly smaller size, the one at some distance behind the other.

The preserved part of the gum of the right upper jaw showed ten distinct shallow pits in a slightly curved line, in the five hindmost of which the tip of a small tooth was visible (Pl. I fig. 1). The pits shown in Pl. I fig. 1 correspond with the numbers 5 to 14 in text-figure 1*a*. Dissection of the gum showed that before the first visible pit there were four small teeth completely embedded in the tissues, surrounded by a fibrous tooth-sac. The pits numbered 5, 6 and 7 (the uppermost in Pl. I fig. 1) proved to be empty follicles; undoubtedly here the teeth recently had fallen out. The pits numbered 8 and 9 each contained a small tooth. As remarked above the teeth in the pits numbered 10 to 14 were already visible before dissection. The last of the row (no. 15 in text-figure 1*a*, not visible in Pl. I fig. 1) again was a small tooth completely hidden in the gum. The accurate position of the teeth in the gum is given in text-figure 1*a*. the distance (in mm) of each of these teeth or tooth-pits from the tip of the lower jaw is:

1:	428	6:	500	11:	570
2:	446	7:	517	12:	585
3:	461	8:	531	13:	592
4:	475	9:	545	14:	599
5:	492	10:	560	15:	606

On the left side of the jaw there were two visible teeth, distinctly projecting through the gum, and no pits indicating the presence of other teeth. Dissection showed that behind these two teeth there were two more, entirely hidden in the gum. The first of these four teeth was situated as far from the tip of the jaw as the tenth tooth of the row on the right side. The accurate position of the teeth in the gum is given in text-figure 1*d*, the distance (in mm) of each of these teeth from the tip of the lower jaw is:

1:	560	3:	593
2:	578	4:	602

All the maxillary teeth of specimen no. 7218 are figured, 5 times enlarged, on Pl. II of the present paper. As far as the larger teeth are concerned the lingual surface is represented, for some of the smaller teeth the exact position in the photographs is not certain. As a rule the teeth are slightly curved, the concave side being turned posteriorly and ventrally, the convex side anteriorly and dorsally. The topmost parts of the teeth that protruded through the surface of the gum are comparatively smooth (the first two teeth of the upper row and the first four of the lower row of the plate). The smallest teeth on almost the whole of their surface are covered with an irregular layer of cement (middle row), the same is found on the roots of the larger teeth. In many of the larger teeth the roots are conspicuously enlarged by masses of cement (first tooth of the upper row, and four in the lower row of the

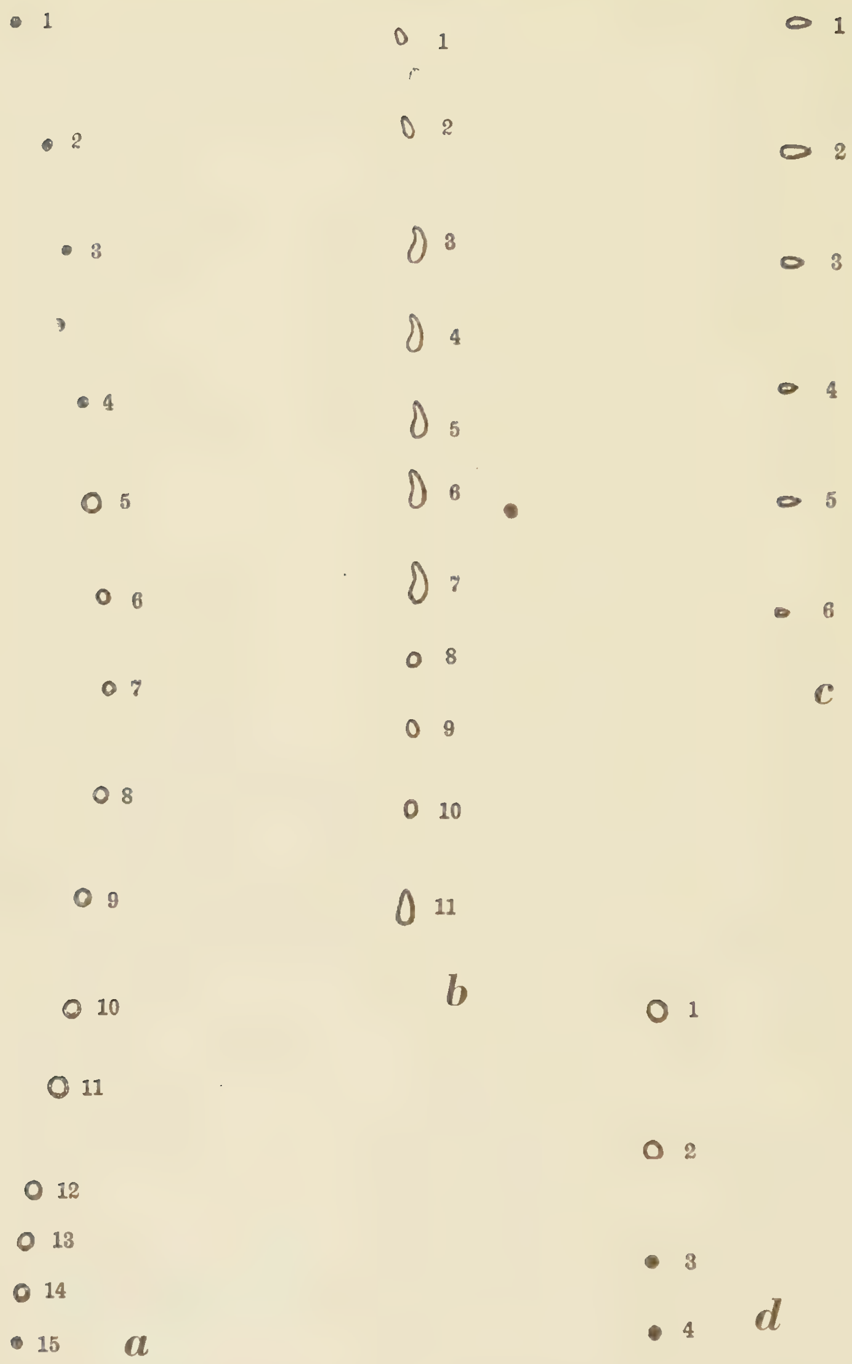


Fig. 1. *Hyperoodon rostratus* (Müller). *a*, Leiden Museum reg. no. 7218, right maxillary toothrow; *b*, right mandibular toothrow (except the large anterior tooth) after Eschricht (1845, woodcut on page 338); *c*, left mandibular teeth (except the large anterior tooth) after Vrolik (1848, Pl. VIII fig. 14); *d*, Leiden Museum reg. no. 7218, left maxillary toothrow. Natural size.

plate). The roots of the teeth, even in the smallest, are completely closed. In the larger teeth that do not show the long projections of cement the roots are closed by a bulbous mass of tooth tissue (distinctly visible in two teeth of the upper row and in two of the lower row of the plate). The length of the teeth without the projecting masses of cement varies from 3 to 6 mm, the length including the irregular masses of cement may amount to 11 mm. The thickness of the teeth is from 1 to 2 mm.

With the exception of the masses of cement the larger mandibular teeth of the present specimen correspond in size and in shape with those described by ESCHRICHT (1845, cf. also text-figure 1*b*); the smaller teeth are similar in shape and in size to those described by VROLIK (1848, cf. also text-figure 1*c*). Undoubtedly they are entirely functionless; those of which the tips protruded above the surface were so loosely attached that they easily could be moved about in their alveoli, at their extreme lower ends only they were firmly attached to the surrounding tissue.

Mesoplodon grayi von Haast

The specimen (Leiden Museum, reg. no. 1638) stranded on December 11 or 12, 1927, near Kijkduin, Loosduinen (North Sea coast of the province South Holland). It has been mentioned by VAN OORT (1928) as an accession for the Leiden Museum, and later by VAN DEINSE (1931), who comments upon the fact that two rows of about twenty small maxillary teeth have been preserved on the skull. In the two cited publications the (female) specimen is named *Mesoplodon bidens* (Sowerby), its length is recorded as 4.6 m.

Just as in the three skulls on which the description of the species *Mesoplodon grayi* was based (VON HAAST, 1876*a*) our specimen shows, on both sides of the upper jaw, commencing in the region of the conspicuous tooth in the lower jaw, a row of small teeth (text-figures 2 and 3). On each side there are twenty-two of these teeth, the greater part of which for a distance of 2 or 3 mm protrude above the surface. The toothrows have a length of about 10 cm, the teeth themselves are placed at fairly equal distances from each other. In our specimen the part of the gum containing the teeth probably has shrunk by the process of drying, so that in the living animal the length of the toothrows may have been somewhat larger. The teeth at the beginning and at the end of each toothrow are slightly smaller than those in the middle region. The largest teeth have a length of about 12 mm, one third of which protrudes from the gum; their thickness is about 1 to 1.5 mm. The teeth are slightly curved, the concave side being turned inwards. Owing to the shrinking of the surrounding tissues the embedded parts of the teeth are faintly visible. The tips of some of the teeth have broken off, but this may have occurred during or after the mounting of the skeleton.

Eight of these small teeth (nos. 11 to 14 and nos. 7 to 10 of the left toothrow) are shown, 5 times enlarged, on Pl. I figs. 2 and 3. Their shape

Fig. 2. *Mesoplodon grayi* von Haast, Leiden Museum reg. no. 1638, part of rostrum and lower jaw, left side. Natural size.

Fig. 3. *Mesoplodon grayi* von Haast, Leiden Museum reg. no. 1638, part of rostrum and lower jaw, right side. Natural size.

is very similar to that of the teeth of the Common Dolphin (*Delphinus delphis* Linnaeus). Many of these teeth have split lengthwise, a common occurrence in various Cetaceans when preserved as museum specimens.

In contradistinction to the minute teeth in *Hyperoodon* these teeth in *Mesoplodon grayi* do not give the impression of being rudimentary. It is true that they are not placed in distinct alveoli, but this is not exceptional in Toothed Whales. Even in the Sperm Whale (*Physeter macrocephalus* Linnaeus) the teeth of the lower jaw are so loosely set in the dental groove that it is difficult to regard their place of attachment as a real socket; the fastening of the teeth to the jaw is largely due to the hard and extremely fibrous gum. In the specimen of *Mesoplodon grayi* dealt with here the row of teeth on each side of the upper jaw was implanted in the lateral basirostral groove.

The minute teeth in the upper jaw of *Mesoplodon grayi* do not appear as being in a rudimentary state. They are, as remarked above, very similar to those of the Common Dolphin, and, as in the latter species, in all probability they have a definite function for capturing or holding the prey. This opinion is in full agreement with that of von Haast (1876*a*).

Moreover, von Haast (1876*a*) regarded the row of minute teeth as the distinctive specific character of the species. His three original skulls of *Mesoplodon grayi* showed nineteen, seventeen, and seventeen of these teeth respectively. In four specimens stranded at a later date there were from seventeen to nineteen teeth on both sides of the roof of the mouth (von Haast, 1876*b*). In the specimen from South Australia described by Hale (1932) on one side of the upper jaw there were fifteen small teeth (on the left side the soft parts had decayed when the specimen was collected). Flower (1879) examined a specimen of *Mesoplodon grayi* with eighteen maxillary teeth; van Beneden and Gervais described a specimen with eighteen maxillary teeth on the left side and seventeen on the right (Gervais, in van Beneden and Gervais, 1880, p. 518, Pl. LXII fig. 2*a*). The Leiden Museum specimen, with its rows of twenty-two maxillary teeth on each side of the jaw, therefore, has a slightly larger number than previously recorded.

In von Haast's drawing (1876*a*, figure on page 10) and in Hale's figure (1932, fig. 9) the minute teeth of *Mesoplodon grayi* appear even more strongly developed than in the Leiden Museum specimen, and in all these cases they give the impression of being functional. In von Haast's specimens these teeth had a length of 0.20 to 0.40 inch (5 to 10 mm), the largest tooth at its base was $\frac{1}{8}$ of an inch (3 mm) thick. In Hale's figure the small teeth appear to protrude from the gum for up to 8 mm, but this may be due to partial decay of the object. Here the teeth again have a thickness of 1 to 2 mm. It is interesting to note that in von Haast's figure *b* (1876*a*, page 10), showing four of the upper teeth with the whole of their roots exposed, the basal parts of the roots have an even contour, indicating that the teeth were still in full growth.

This again is an argument in favour of the opinion that the minute teeth in *Mesoplodon grayi* are functional.

The occurrence of well developed toothrows on each side of the rostrum formed an indication for the possible identity of the Leiden Museum specimen as *Mesoplodon grayi*. The skull was compared with that of three specimens undoubtedly belonging to *Mesoplodon bidens*, in various respects it proved to be distinctly different from these. A comparison of the skull with the figures of FLOWER (1879, pls. LXXI and LXXII), of VAN BENEDEN and GERVAIS (1880, pl. LXII), of FORBES (1893, pls. XII and XIII), of HALE (1932, figs. 2—4), and of BRAZENOR (1933, pl. VI) showed a close agreement in every respect. As in the specimens described and figured by the authors cited above the skull of the Leiden Museum specimen shows the narrow base of the rostrum and the conspicuous lateral basirostral groove which form the striking characters of the skull of *Mesoplodon grayi*. The peculiarities of the Leiden Museum specimen of *M. grayi* and its differences from *M. bidens* will be dealt with in more detail in a later paper.

Mesoplodon grayi seems to be of fairly common occurrence in New Zealand waters (VON HAAST, 1876*a*, *b*). OGILBY (1892) notes as habitat for the species New South Wales and New Zealand. WAITE (1922) records the species for South Australia, based on a lower jaw only; HALE (1932) obtained a complete specimen stranded on the South Australian coast. BRAZENOR (1933) figures a skull without lower jaw found on the beach in Victoria. OLIVER (1922) gives as range of distribution New Zealand and Patagonia. Finally LYDEKKER (1911) records the species for the South coast of South Africa. Up till now, therefore, *Mesoplodon grayi* was known to occur in the Southern hemisphere only; the specimen stranded on the Dutch coast in 1927 forms the first record of the occurrence of the species in the Northern hemisphere.

Immediately after it had been found on the beach the carcase of our specimen of *Mesoplodon grayi* was transported to the Leiden Museum in completely undamaged state. Here Mr. M. A. KOEKKOEK executed an excellent picture of the specimen based on accurate measurements of all the various parts. This oil painting is reproduced on Plate IV of the present paper; it appears to be the first complete figure of the species. The figures on Plate IV are so exact that accurate measurements may be taken from these.

Concerning the colour of *Mesoplodon grayi* VON HAAST (1876*b*, p. 458) states: "The colour of the back is black, getting a little lighter near the tail, where it assumes a dark slate tint; the lower side is reddish brown, near the tail assuming on both sides a more blackish hue". In our specimen (Plate IV) the colour of the back is black to dark slate grey, on the sides towards the ventral surface the colour gradually becomes lighter, the sides being of a brownish grey. The ventral surface is of a light grey with a brownish tinge, with the exception of a broad darker

median band gradually becoming mottled anteriorly and vanishing in the region of the flippers. The flippers and the tail-flukes on both sides are very dark grey to black, the edges of the flippers have a lighter border. The lower jaw and the throat are of a very light grey, in some parts even whitish. On the edges of the upper jaw, around the navel, the genital aperture and the anus there are whitish lines. On the sides of the body there are one white streak and several smaller white patches; possibly these are to be regarded as scars.

The figures distinctly show the two throat grooves which anteriorly nearly meet in the basal part of the lower jaw. In dorsal and ventral view the rostrum appears very narrow.

The existing figures of *Mesoplodon bidens* as a rule show the animal in lateral view, figures of specimens in dorsal or in ventral view seem to be extremely rare. Fortunately the Rijksmuseum van Natuurlijke Historie possesses a photograph of a female specimen of *Mesoplodon bidens* (reg. no. 2114), total length 4 m, stranded near Hoedekenskerke, province Zeeland, September 14, 1932, showing the not too badly damaged carcase in dorsal view (Pl. III, upper figure). The triangular outline of the head is markedly different from that of *Mesoplodon grayi* of Pl. IV; in the latter the sides of the triangle are decidedly more concave than in the specimen of Pl. III. These differences closely correspond with those found in the skulls of the two species, the basal parts of the maxillary in *M. bidens* expanding far more laterally than in *M. grayi*.

The lower figure of Pl. III shows the ventral side of the head of *Mesoplodon bidens* no. 2114, here especially the throat grooves are to be seen, which are of a similar configuration as those of *M. grayi* (Pl. IV, lower figure).

Harmer (1918) discovered a fairly reliable character for distinguishing the three Ziphioid Whales not uncommonly stranding on the British coasts, by reducing the distance of the tip of the beak to the blow-hole to a percentage of the total length. These percentages are 14.0—22.0 for *Hyperoodon rostratus*, 10.4—12.6 for *Ziphius cavirostris*, and 9.7—15.2 for *Mesoplodon bidens*. In our specimen of *Mesoplodon grayi* this percentage is 13.4 so that in this respect it corresponds with *M. bidens*; in the specimen of Pl. III the percentage is 12.3.

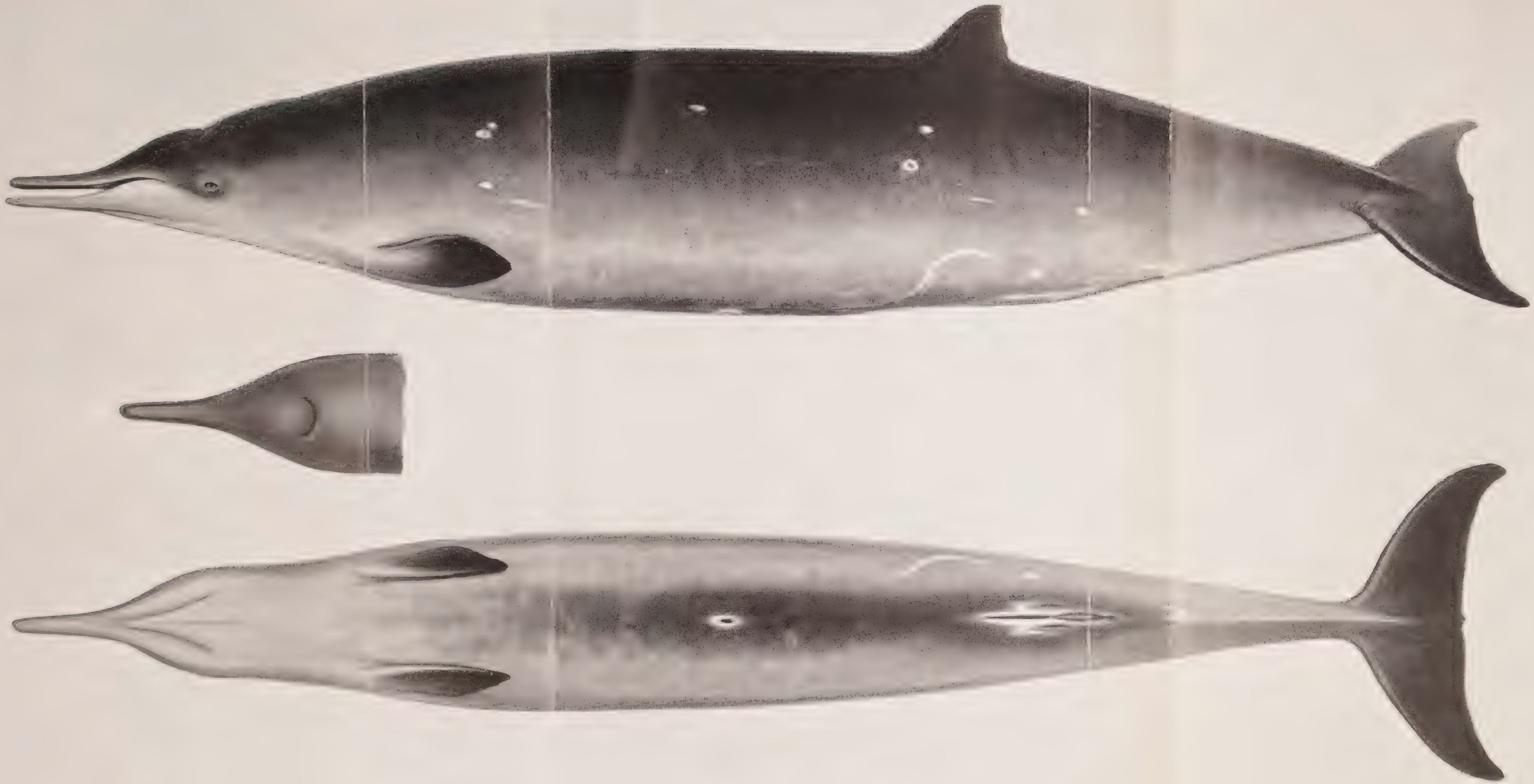
As far as concerns the colour of *Mesoplodon bidens* there seems to be a great deal of variation, some specimens being described as entirely black, others as having white ventral parts. The following data in regard to the colour are given by Harmer (1927, pp. 55, 56): "The coloration of Sowerby's Whale is not constant, but is mainly black, although some specimens have a considerable amount of white on the ventral surface (Report no. 4, p. 12). A pregnant female stranded on December 18, 1892, and described by Mr. Southwell and myself, was almost entirely of a uniform black colour, though it was stated that when quite fresh a bluish

Fig. 1. *Hyperoodon rostratus* (Müller), Leiden Museum reg. no. 7218, part of the gum showing right maxillary toothrow. Natural size.

Fig. 2. *Mesoplodon grayi* von Haast, Leiden Museum reg. no. 1638, eleventh to fourteenth maxillary teeth of left side. × 5.

Fig. 3. Same specimen, seventh to tenth maxillary teeth of left side. × 5.

Hyperoodon rostratus (Müller), Leiden Museum reg. no. 7218.
Upper row, first to fourth maxillary teeth of left side.
Middle row, first to fourth, and eighth and ninth maxillary teeth of right side.
Lower row, tenth to fifteenth maxillary teeth of right side.
All figures × 5.



Mesoplodon grayi von Haast. Leiden Museum reg. no. 1638, lateral and ventral view, and dorsal view of head. One fourteenth natural size.

tinge was visible. The black was interrupted by numerous splashes and smears, of irregular shape and distribution, and of a lighter colour. The colour was not appreciably lighter on the belly than above. Certain parts were grey, namely the anterior edges of the tail-flukes, part of the lower jaw (which was partly white) and the upper jaw, which had white edges. 1914, 43, a female, was black above, leaden-grey below; and 1916, 21, which was probably a female, from the evidence of its concealed teeth, was described as black with a white lower surface".

In connexion with von Haast's remarks cited above it is interesting that the lateral parts of the specimen of *Mesoplodon grayi* stranded on the Dutch coast had a brownish grey colour. In other respects there are differences, but these are of minor importance when compared to the very strong variation in colour observed in *Mesoplodon bidens.*

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FORMALISTISCHE BETRACHTUNGEN ÜBER INTUITIONISTISCHE UND VERWANDTE LOGISCHE SYSTEME III

VON

J. RIDDER

(Communicated by Prof. L. E. J. BROUWER at the meeting of April 29, 1950)

„Logistische" Kalküle im Sinne von GENTZEN für A_1, A_2, A_1^* und A_2^*.

§ 17. In diesem Par. beschränken wir uns auf die Betrachtung der Aussagenlogik A_1 (§ 1).

Definition. *Eine Sequenz* $\mathfrak{A}_1 . \mathfrak{A}_2 \ldots \mathfrak{A}_\mu \rightarrow \mathfrak{B}$ (μ eine ganze Zahl $\geqq 1$) ist eine andere Schreibweise des Satzes $(\ldots(((\mathfrak{A}_1 . \mathfrak{A}_2) . \mathfrak{A}_3) . \mathfrak{A}_4) \ldots \ldots \mathfrak{A}_\mu) \subset \mathfrak{B}$; insbes. ist $\nu \rightarrow \mathfrak{B}$ eine andere Schreibweise für das Theorem $\nu \subset \mathfrak{B}$. [12])

Aus dieser Definition geht hervor, dasz hier, in Abweichung von GENTZEN, loc. cit. 8), Antezedens $\mathfrak{A}_1 . \mathfrak{A}_2 \ldots \mathfrak{A}_\mu$ und Sukzedens $\mathfrak{B}$ beide immer nicht leer sind.

Aus den Sätzen 1, 2, 4—7, Einsetzungsregel E, Axiom 2^α und die Schemata 1^β, 2^β lassen sich die folgenden GENTZENschen Schluszschemata ableiten:

Schema A_1. Verdünnung im Antezedens: $\dfrac{\mathfrak{A} \rightarrow \mathfrak{B}}{\mathfrak{D} \cdot \mathfrak{A} \rightarrow \mathfrak{B}}$;

Schema A_2. Zusammenziehung im Antezedens: $\dfrac{\mathfrak{D} \cdot \mathfrak{D} \cdot \mathfrak{A} \rightarrow \mathfrak{B}}{\mathfrak{D} \cdot \mathfrak{A} \rightarrow \mathfrak{B}}$;
($\mathfrak{A}$ darf fehlen)

Schema A_3. Vertauschung im Antezedens: $\dfrac{\mathfrak{A} \cdot \mathfrak{D} \cdot \mathfrak{E} \cdot \mathfrak{C} \rightarrow \mathfrak{B}}{\mathfrak{A} \cdot \mathfrak{E} \cdot \mathfrak{D} \cdot \mathfrak{C} \rightarrow \mathfrak{B}}$;
($\mathfrak{A}$ oder $\mathfrak{C}$ oder beide dürfen fehlen)

Schema A_4. Schnitt: $\dfrac{\mathfrak{A} \rightarrow \mathfrak{B} \qquad \mathfrak{B} \cdot \mathfrak{C} \rightarrow \mathfrak{D}}{\mathfrak{A} \cdot \mathfrak{C} \rightarrow \mathfrak{D}}$.
($\mathfrak{C}$ darf fehlen)

Eine unmittelbare Folgerung von Axiom 1^α und Einsetzungsregel E ist das

Schema P. $\mathfrak{D} \rightarrow \mathfrak{D}$ (Ersetzung von $\mathfrak{D}$ durch eine willkürlich gewählte Formel liefert einen Satz).

Das Schema 2^β liefert das

Schema UES. $\dfrac{\mathfrak{D} \rightarrow \mathfrak{A} \qquad \mathfrak{D} \rightarrow \mathfrak{B}}{\mathfrak{D} \rightarrow (\mathfrak{A} \cdot \mathfrak{B})}$ („Und"-Einführung im Sukzedens).

Axiom 2^a, Regel E und die Schemata 1^β, 2^β liefern das

Schema *UEA*. $\dfrac{\mathfrak{A}\cdot\mathfrak{D}\to\mathfrak{C}}{(\mathfrak{A}\cdot\mathfrak{B})\cdot\mathfrak{D}\to\mathfrak{C}}\quad\dfrac{\mathfrak{B}\cdot\mathfrak{D}\to\mathfrak{C}}{(\mathfrak{A}\cdot\mathfrak{B})\cdot\mathfrak{D}\to\mathfrak{C}}$ („Und"-Einführung im Antezedens).

($\mathfrak{D}$ darf fehlen)

Satz 6, Regel E, Schema 1^β und Ersetzungsschema E_0 liefern:

Schema *FES*. $\dfrac{\mathfrak{A}\cdot\mathfrak{B}\to\mathfrak{C}}{\mathfrak{B}\to(\mathfrak{A}\subset\mathfrak{C})}$ („Folge"-Einführung im Sukzedens).

Schema *FEA*. $\dfrac{\mathfrak{D}\to\mathfrak{A}\quad\mathfrak{B}\cdot\mathfrak{E}\to\mathfrak{C}}{(\mathfrak{A}\subset\mathfrak{B})\cdot\mathfrak{D}\cdot\mathfrak{E}\to\mathfrak{C}}$ („Folge"-Einführung im Antezedens).

($\mathfrak{E}$ darf fehlen)

Beweis. $\mathfrak{D}\subset\mathfrak{A}$ und $(\mathfrak{B}.\mathfrak{E})\subset\mathfrak{C}$; (Ax. 1^a, Regeln E, E_0, Satz 6, Sch. 1^β) $[\mathfrak{A}.(\mathfrak{A}\subset\mathfrak{B})]\subset\mathfrak{B}$; ($2^a$, E, 1^β, 2^β) $[\mathfrak{D}.(\mathfrak{A}\subset\mathfrak{B})]\subset\mathfrak{B}$; ($2^a$, 1^β, E, 2^β) $\{[\mathfrak{D}.(\mathfrak{A}\subset\mathfrak{B})].\mathfrak{E}\}\subset\mathfrak{C}$; $\mathfrak{D}.(\mathfrak{A}\subset\mathfrak{B}).\mathfrak{E}\to\mathfrak{C}$; ($A_3$) $(\mathfrak{A}\subset\mathfrak{B}).\mathfrak{D}.\mathfrak{E}\to\mathfrak{C}$.

Schema Q. $\mathfrak{A}\to\nu$ (Ersetzung von $\mathfrak{A}$ durch eine willkürlich gewählte Formel liefert einen Satz).

Dies ist eine Folge von Axiom 3° und Einsetzungsregel E.

In der affirmativen Aussagenlogik A_1 läszt sich somit ein logistischer oder L-Kalkül LG_1 einbetten, der: 1° *Kalkülformeln gemäsz Regel E* (*erster Teil*) (§ 1) *hat;* 2° *die Einsetzungsschemata P und Q besitzt;* 3° *daneben die Schluszschemata A_1, A_2, A_3, A_4, UES, UEA, FES, FEA, während:* 4° *der zweite Teil der Einsetzungsregel die Form erhält: „läszt sich auf Grund der eben genannten Schemata schreiben $\mathfrak{A}\to\mathfrak{B}$, mit $\mathfrak{A}$ und $\mathfrak{B}$ Kalkülformeln, und sind $\mathfrak{D}$ und $\mathfrak{E}$ mittels Regel E* (*erster Teil*) *aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende gleichgestaltete Buchstaben in beiden nicht oder in beiden an allen Stellen durch gleichgestaltete Kalkülformeln ersetzt sind, so läszt sich auch schreiben* $\mathfrak{D}\to\mathfrak{E}$"; Theoreme dieses Kalküls sind die „affirmativen" Sequenzen $\nu\to\mathfrak{A}$.

Mit A_1, A_3 und *FES* leitet man leicht in dem Kalkül LG_1 folgendes Schema ab:

$$\frac{\mathfrak{A}\to\mathfrak{B}}{\nu\to(\mathfrak{A}\subset\mathfrak{B})};$$

mit *UES* folgt in LG_1 das Schema:

$$\frac{\nu\to\mathfrak{A}\quad\nu\to\mathfrak{B}}{\nu\to(\mathfrak{A}\cdot\mathfrak{B})}.$$

§ 17^{bis}. *In der affirmativen Aussagenlogik A_2* (§ 3) *läszt sich ein L-Kalkül LG_2 einbetten, welcher aus dem Kalkül LG_1 durch Hinzufügung nachfolgender Schemata OEA und OES entsteht* (Einsetzungsregel entsprechend erweitert). [54])

[54]) Es läszt sich unschwer beweisen, dasz der Umfang von LG_2 sich nicht ändert, wenn man die Schemata *UEA* und *OEA* durch die engeren Schemata *UEA** und *OEA** (§§ 18, 18*bis*) ersetzt.

Schema *OEA*. $\dfrac{\mathfrak{A}\cdot\mathfrak{C}\to\mathfrak{D}\quad\mathfrak{B}\cdot\mathfrak{C}\to\mathfrak{D}}{(\mathfrak{A}+\mathfrak{B})\cdot\mathfrak{C}\to\mathfrak{D}}$ („Oder"-Einführung im Antezedens);
($\mathfrak{C}$ darf fehlen)

Schema *OES*. $\dfrac{\mathfrak{C}\to\mathfrak{A}}{\mathfrak{C}\to(\mathfrak{A}+\mathfrak{B})}\quad\dfrac{\mathfrak{C}\to\mathfrak{B}}{\mathfrak{C}\to(\mathfrak{A}+\mathfrak{B})}$ („Oder"-Einführung im Sukzedens).

§ 18. Dual gegenüber § 17 steht:

In der verneinenden Aussagenlogik A_1^* *läszt sich,* mittels der Definition „*Eine Sequenz* $\mathfrak{B}\to\mathfrak{A}_1+\mathfrak{A}_2+\ldots+\mathfrak{A}_\mu$ (μ eine ganze Zahl $\geqq 1$) ist eine andere Schreibweise für den Satz $\mathfrak{B}\subset(\ldots((\mathfrak{A}_1+\mathfrak{A}_2)+\mathfrak{A}_3)+ +\ldots+\mathfrak{A}_\mu)$ [also $\mathfrak{B}\to\lambda$ für das (verneinende) Theorem $\mathfrak{B}\subset\lambda$]", *ein logistischer oder L-Kalkül* LG_1^* *einbetten, der:* 1° *Kalkülformeln gemäsz Regel* E^* (*erster Teil*) (§ 2) *hat;* 2° *die Einsetzungsschemata:*

$P\cdot\mathfrak{D}\to\mathfrak{D}$; *und:*

$R\cdot\lambda\to\mathfrak{A}$

besitzt; 3° *daneben die folgenden Schluszschemata benutzt:*

A_1^*. Verdünnung im Sukzedens: $\dfrac{\mathfrak{B}\to\mathfrak{A}}{\mathfrak{B}\to\mathfrak{A}+\mathfrak{D}}$;

A_2^*. Zusammenziehung im Sukzedens: $\dfrac{\mathfrak{B}\to\mathfrak{A}+\mathfrak{D}+\mathfrak{D}}{\mathfrak{B}\to\mathfrak{A}+\mathfrak{D}}$;
($\mathfrak{A}$ darf fehlen)

A_3^*. Vertauschung im Sukzedens: $\dfrac{\mathfrak{B}\to\mathfrak{C}+\mathfrak{E}+\mathfrak{D}+\mathfrak{A}}{\mathfrak{B}\to\mathfrak{C}+\mathfrak{D}+\mathfrak{E}+\mathfrak{A}}$;
($\mathfrak{A}$ oder $\mathfrak{C}$ oder beide dürfen fehlen)

A_4^*. Schnitt: $\dfrac{\mathfrak{B}\to\mathfrak{A}\quad\mathfrak{D}\to\mathfrak{C}+\mathfrak{B}}{\mathfrak{D}\to\mathfrak{C}+\mathfrak{A}}$; ($\mathfrak{C}$ darf fehlen)

OEA^*. $\dfrac{\mathfrak{A}\to\mathfrak{D}\quad\mathfrak{B}\to\mathfrak{D}}{(\mathfrak{B}+\mathfrak{A})\to\mathfrak{D}}$;

OES^*. $\dfrac{\mathfrak{C}\to\mathfrak{D}+\mathfrak{A}}{\mathfrak{C}\to\mathfrak{D}+(\mathfrak{B}+\mathfrak{A})}\quad\dfrac{\mathfrak{C}\to\mathfrak{D}+\mathfrak{B}}{\mathfrak{C}\to\mathfrak{D}+(\mathfrak{B}+\mathfrak{A})}$; ($\mathfrak{D}$ darf fehlen)

FEA^{**}. $\dfrac{\mathfrak{C}\to\mathfrak{B}+\mathfrak{A}}{(\mathfrak{C}\subset\mathfrak{A})\to\mathfrak{B}}$;

FES^{**}. $\dfrac{\mathfrak{A}\to\mathfrak{D}\quad\mathfrak{C}\to\mathfrak{E}+\mathfrak{B}}{\mathfrak{C}\to\mathfrak{E}+\mathfrak{D}+(\mathfrak{B}\subset\mathfrak{A})}$ ($\mathfrak{E}$ darf fehlen),

während: 4° *der zweite Teil der Einsetzungsregel die Form erhält:* „*läszt sich auf Grund der eben genannten Schemata schreiben* $\mathfrak{A}\to\mathfrak{B}$, *mit* $\mathfrak{A}$ *und* $\mathfrak{B}$ *Kalkülformeln, und gehen* $\mathfrak{D}$ *und* $\mathfrak{E}$ *mittels Regel* E^* (*erster Teil*) *in*

bekannter Weise aus $\mathfrak{A}$ *bzw.* $\mathfrak{B}$ *hervor, so läszt sich auch schreiben* $\mathfrak{D} \to \mathfrak{E}$"; Theoreme von LG_1^* sind die „negierenden" Sequenzen $\mathfrak{A} \to \lambda$.

In LG_1^* lassen sich folgende Schemata ableiten:

$$\frac{\mathfrak{B} \to \mathfrak{A}}{(\mathfrak{B} \subset \mathfrak{A}) \to \lambda} \qquad \text{und} \qquad \frac{\mathfrak{A} \to \lambda \quad \mathfrak{B} \to \lambda}{(\mathfrak{B} + \mathfrak{A}) \to \lambda}.$$

§ 18^{bis}. *In der verneinenden Aussagenlogik* A_2^* (§ 4) *läszt sich ein L-Kalkül* LG_2^* *einbetten, welcher aus dem Kalkül* LG_1^* *durch Hinzufügung nachfolgender Schemata* UES^* *und* UEA^* *entsteht* (Einsetzungsregel entsprechend erweitert).

Schema UES^*. $\dfrac{\mathfrak{D} \to \mathfrak{C} + \mathfrak{A} \quad \mathfrak{D} \to \mathfrak{C} + \mathfrak{B}}{\mathfrak{D} \to \mathfrak{C} + (\mathfrak{B} \cdot \mathfrak{A})}$ ($\mathfrak{C}$ darf fehlen);

Schema UEA^*. $\dfrac{\mathfrak{A} \to \mathfrak{C}}{(\mathfrak{A} \cdot \mathfrak{B}) \to \mathfrak{C}} \qquad \dfrac{\mathfrak{B} \to \mathfrak{C}}{(\mathfrak{A} \cdot \mathfrak{B}) \to \mathfrak{C}}$.

Einbettung von „natürlichen" Kalkülen in LG_1, LG_2, LG_1^* und LG_2^*.

§ 19. In diesem Par. beschränken wir uns auf den Kalkül LG_1 (§ 17).

Schema UE. $\dfrac{\begin{matrix}[\mathfrak{D}] & [\mathfrak{E}] \\ \mathfrak{A} & \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D} \cdot \mathfrak{E}] \\ \mathfrak{A} \cdot \mathfrak{B}\end{matrix}}$ („Und"-Einführung)

sei eine andere Schreibweise von: $\dfrac{\mathfrak{D} \to \mathfrak{A} \quad \mathfrak{E} \to \mathfrak{B}}{\mathfrak{D} \cdot \mathfrak{E} \to (\mathfrak{A} \cdot \mathfrak{B})}$.

UE läszt sich in LG_1 mit A_1, A_3 und UES ableiten.

Schema UB. $\dfrac{\begin{matrix}[\mathfrak{D}] \\ \mathfrak{A} \cdot \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D}] \\ \mathfrak{A}\end{matrix}} \qquad \dfrac{\begin{matrix}[\mathfrak{D}] \\ \mathfrak{A} \cdot \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D}] \\ \mathfrak{B}\end{matrix}}$ („Und"-Beseitigung)

sei eine andere Schreibweise von: $\dfrac{\mathfrak{D} \to (\mathfrak{A} \cdot \mathfrak{B})}{\mathfrak{D} \to \mathfrak{A}} \qquad \dfrac{\mathfrak{D} \to (\mathfrak{A} \cdot \mathfrak{B})}{\mathfrak{D} \to \mathfrak{B}}$.

Beweis mit den Schemata P, A_1, A_3, A_4 und der Einsetzungsregel.

Schema FE. $\dfrac{\begin{matrix}[\mathfrak{D} \cdot \mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D}] \\ \mathfrak{A} \subset \mathfrak{B}\end{matrix}}$ („Folge"-Einführung)

sei eine andere Schreibweise von: $\dfrac{\mathfrak{D} \cdot \mathfrak{A} \to \mathfrak{B}}{\mathfrak{D} \to (\mathfrak{A} \subset \mathfrak{B})}$.

Beweis mit FES und A_3.

Schema FB. $\dfrac{\begin{matrix}[\mathfrak{D}] & [\mathfrak{E}] \\ \mathfrak{A} & \mathfrak{A}\subset\mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D}\cdot\mathfrak{E}] \\ \mathfrak{B}\end{matrix}}$ („Folge"-Beseitigung)

sei eine andere Schreibweise von: $\dfrac{\mathfrak{D}\to\mathfrak{A} \quad \mathfrak{E}\to(\mathfrak{A}\subset\mathfrak{B})}{\mathfrak{D}\cdot\mathfrak{E}\to\mathfrak{B}}$.

Beweis mit den Schemata P, FEA, A_4, A_3 und der Einsetzungsregel. A_1, A_2 und A_3 lassen sich schreiben:

Schema B_1. $\dfrac{\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D}\cdot\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}$ Schema B_2. $\dfrac{\begin{matrix}[\mathfrak{D}\cdot\mathfrak{D}\cdot\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D}\cdot\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}$

($\mathfrak{A}$ darf fehlen)

Schema B_3. $\dfrac{\begin{matrix}[\mathfrak{A}\cdot\mathfrak{D}\cdot\mathfrak{E}\cdot\mathfrak{C}] \\ \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{A}\cdot\mathfrak{E}\cdot\mathfrak{D}\cdot\mathfrak{C}] \\ \mathfrak{B}\end{matrix}}$ ($\mathfrak{A}$ oder $\mathfrak{C}$ oder beide dürfen fehlen)

A_4 liefert das Schema: $\dfrac{\mathfrak{D}\to\mathfrak{A} \quad \mathfrak{A}\to\mathfrak{B}}{\mathfrak{D}\to\mathfrak{B}}$, welches sich schreiben läszt:

Schema Sn. $\dfrac{\begin{matrix}[\mathfrak{D}] & [\mathfrak{A}] \\ \mathfrak{A} & \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D}] \\ \mathfrak{B}\end{matrix}}$.

Schlieszlich lassen sich die Schemata P und Q schreiben:

P_0. $\begin{matrix}[\mathfrak{D}] \\ \mathfrak{D}\end{matrix}$ und Q_0. $\begin{matrix}[\mathfrak{A}] \\ \nu\end{matrix}$.

In dem logistischen Kalkül LG_1 läszt sich somit ein natürlicher oder N-Kalkül NG_1 einbetten, der: 1° Kalkülformeln gemäsz Regel E (erster Teil) (§ 1) hat; 2° die Einsetzungsschemata P_0 und Q_0 besitzt; 3° daneben die Schluszschemata B_1, B_2, B_3, Sn, UE, UB, FE und FB, während: 4° der zweite Teil der Einsetzungsregel denselben Wortlaut wie für LG_1 hat, nur mit $\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}$ und $\begin{matrix}[\mathfrak{D}] \\ \mathfrak{E}\end{matrix}$ statt $\mathfrak{A}\to\mathfrak{B}$ bzw. $\mathfrak{D}\to\mathfrak{E}$; ein am Ende einer Schluszreihe auftretendes Schema $\begin{matrix}[\mathfrak{P}] \\ \mathfrak{Q}\end{matrix}$ liefert die Sequenz (den Satz) $\mathfrak{P}\to\mathfrak{Q}$:

Theoreme sind die „affirmativen" Sequenzen $\nu\to\mathfrak{A}$.

Mit B_1 und FE leitet man in NG_1 ab: $\dfrac{\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\begin{matrix}[\nu] \\ \mathfrak{A}\subset\mathfrak{B}\end{matrix}}$;

mit UE und B_2 folgt das Schema: $\dfrac{\begin{matrix}[\nu] & [\nu] \\ \mathfrak{A} & \mathfrak{B}\end{matrix}}{\begin{matrix}[\nu] \\ \mathfrak{A}\cdot\mathfrak{B}\end{matrix}}$.

§ 19^{bis}. *In dem logistischen Kalkül LG_2 (§ 17^{bis}) läszt sich ein N-Kalkül NG_2 einbetten, welcher aus dem Kalkül NG_1 durch Hinzufügung nachfolgender Schemata OE und OB entsteht* (Einsetzungsregel entsprechend erweitert).

Schema *OE*. $\dfrac{\begin{matrix}[\mathfrak{D}]\\ \mathfrak{A}\end{matrix}}{\begin{matrix}[\mathfrak{D}]\\ \mathfrak{A}+\mathfrak{B}\end{matrix}} \qquad \dfrac{\begin{matrix}[\mathfrak{D}]\\ \mathfrak{B}\end{matrix}}{\begin{matrix}[\mathfrak{D}]\\ \mathfrak{A}+\mathfrak{B}\end{matrix}}$ („Oder"-Einführung).

Schema *OB*. $\dfrac{\begin{matrix}[\mathfrak{D}]\\ \mathfrak{A}+\mathfrak{B}\end{matrix} \quad \begin{matrix}[\mathfrak{E}\cdot\mathfrak{A}]\\ \mathfrak{C}\end{matrix} \quad \begin{matrix}[\mathfrak{F}\cdot\mathfrak{B}]\\ \mathfrak{C}\end{matrix}}{\begin{matrix}[\mathfrak{D}\cdot\mathfrak{E}\cdot\mathfrak{F}]\\ \mathfrak{C}\end{matrix}}$ („Oder"-Beseitigung).

($\mathfrak{E}$ oder $\mathfrak{F}$ oder beide dürfen fehlen)

§ 20. Dual gegenüber § 19 steht:

In Kalkül LG_1^* von § 18 läszt sich, mittels der Definition "Ein Schema $\begin{matrix}[\mathfrak{A}]\\ \mathfrak{B}\end{matrix}$ ist eine andere Schreibweise der Sequenz $\mathfrak{A}\rightarrow\mathfrak{B}$ [also $\begin{matrix}[\mathfrak{A}]\\ \lambda\end{matrix}$ für die „verneinende" Sequenz $\mathfrak{A}\rightarrow\lambda$]", *ein natürlicher oder N-Kalkül NG_1^* einbetten, der:* 1° *Kalkülformeln gemäsz Regel E* (erster Teil)* (§ 2) *hat;* 2° *die Einsetzungsschemata:*

P_0. $\begin{matrix}[\mathfrak{D}]\\ \mathfrak{D}\end{matrix}$; *und:* R_0. $\begin{matrix}[\lambda]\\ \mathfrak{A}\end{matrix}$

besitzt; 3° *daneben die folgenden Schluszschemata benutzt:*

B_1^*. $\dfrac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}\end{matrix}}{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}+\mathfrak{D}\end{matrix}}$; B_2^*. $\dfrac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}+\mathfrak{D}+\mathfrak{D}\end{matrix}}{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}+\mathfrak{D}\end{matrix}}$ ($\mathfrak{A}$ darf fehlen); B_3^*. $\dfrac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{C}+\mathfrak{E}+\mathfrak{D}+\mathfrak{A}\end{matrix}}{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{C}+\mathfrak{D}+\mathfrak{E}+\mathfrak{A}\end{matrix}}$ ($\mathfrak{A}$ oder $\mathfrak{C}$ oder beide dürfen fehlen);

Sn (siehe § 19); .

OE^*. $\dfrac{\begin{matrix}[\mathfrak{A}]\\ \mathfrak{D}\end{matrix} \quad \begin{matrix}[\mathfrak{B}]\\ \mathfrak{E}\end{matrix}}{\begin{matrix}[\mathfrak{B}+\mathfrak{A}]\\ \mathfrak{E}+\mathfrak{D}\end{matrix}}$; OB^*. $\dfrac{\begin{matrix}[\mathfrak{B}+\mathfrak{A}]\\ \mathfrak{D}\end{matrix}}{\begin{matrix}[\mathfrak{A}]\\ \mathfrak{D}\end{matrix}} \qquad \dfrac{\begin{matrix}[\mathfrak{B}+\mathfrak{A}]\\ \mathfrak{D}\end{matrix}}{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{D}\end{matrix}}$;

FE^*. $\dfrac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}+\mathfrak{D}\end{matrix}}{\begin{matrix}[\mathfrak{B}\subset\mathfrak{A}]\\ \mathfrak{D}\end{matrix}}$; FB^*. $\dfrac{\begin{matrix}[\mathfrak{A}]\\ \mathfrak{D}\end{matrix} \quad \begin{matrix}[\mathfrak{B}\subset\mathfrak{A}]\\ \mathfrak{E}\end{matrix}}{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{E}+\mathfrak{D}\end{matrix}}$,

während: 4° *der zweite Teil der Einsetzungsregel denselben Wortlaut wie für LG_1^* hat, nur mit $\begin{matrix}[\mathfrak{A}]\\ \mathfrak{B}\end{matrix}$ und $\begin{matrix}[\mathfrak{D}]\\ \mathfrak{E}\end{matrix}$ statt $\mathfrak{A}\rightarrow\mathfrak{B}$ bzw. $\mathfrak{D}\rightarrow\mathfrak{E}$;* Theoreme von NG_1^* sind die „negierenden" Sequenzen $\mathfrak{A}\rightarrow\lambda$.

In NG_1^* hat man die Schemata:

$$\frac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}\end{matrix}}{\begin{matrix}[\mathfrak{B}\subset\mathfrak{A}]\\ \lambda\end{matrix}} \qquad \text{und} \qquad \frac{\begin{matrix}[\mathfrak{A}]\\ \lambda\end{matrix}\quad\begin{matrix}[\mathfrak{B}]\\ \lambda\end{matrix}}{\begin{matrix}[\mathfrak{B}+\mathfrak{A}]\\ \lambda\end{matrix}}$$

§ 20^{bis}. *In LG_2^* (§ 18^{bis}) läszt sich ein N-Kalkül NG_2^* einbetten, welcher aus dem Kalkül NG_1^* durch Hinzufügung nachfolgender Schemata UE^* und UB^* entsteht* (Einsetzungsregel entsprechend erweitert).

$$UE^*.\quad \frac{\begin{matrix}[\mathfrak{A}]\\ \mathfrak{D}\end{matrix}}{\begin{matrix}[\mathfrak{B}\cdot\mathfrak{A}]\\ \mathfrak{D}\end{matrix}} \qquad \frac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{D}\end{matrix}}{\begin{matrix}[\mathfrak{B}\cdot\mathfrak{A}]\\ \mathfrak{D}\end{matrix}}.$$

$$UB^*.\quad \frac{\begin{matrix}[\mathfrak{B}\cdot\mathfrak{A}]\\ \mathfrak{D}\end{matrix}\quad\begin{matrix}[\mathfrak{C}]\\ \mathfrak{A}+\mathfrak{E}\end{matrix}\quad\begin{matrix}[\mathfrak{C}]\\ \mathfrak{B}+\mathfrak{F}\end{matrix}}{\begin{matrix}[\mathfrak{C}]\\ \mathfrak{F}+\mathfrak{E}+\mathfrak{D}\end{matrix}} \quad (\mathfrak{E} \text{ oder } \mathfrak{F} \text{ oder beide dürfen fehlen}).$$

Aequivalenz der Kalküle A_j, LG_j, NG_j und A_j^*, LG_j^*, NG_j^* $(j = 1,\ 2)$.

§ 21. Durch Umsetzung des am Ende einer Schluszreihe auftretenden Schemas $\begin{matrix}[\mathfrak{P}]\\ \mathfrak{Q}\end{matrix}$ in die Form $\mathfrak{P} \to \mathfrak{Q}$ läszt sich in dem Kalkül NG_1 folgendes ableiten:

P_0 liefert: $X \subset X$ (d.i. Axiom 1^a von Kalkül A_1).

P_0 und UB liefern: $(X.Y) \subset X$ und $(X.Y) \subset Y$ (d.i. Ax. 2^a von A_1).

Q_0 liefert: $X \subset \nu$ (d.i. Ax. 3°).

Sn liefert: $\dfrac{\mathfrak{A}\subset\mathfrak{B}\quad \mathfrak{B}\subset\mathfrak{C}}{\mathfrak{A}\subset\mathfrak{C}}$ (d.i. Schema 1^β).

UE und B_2 liefern: $\dfrac{\mathfrak{C}\subset\mathfrak{A}\quad \mathfrak{C}\subset\mathfrak{B}}{\mathfrak{C}\subset[\mathfrak{A}\cdot\mathfrak{B}]}$ (d.i. Schema 2^β).

FE liefert: $\dfrac{[\mathfrak{A}\cdot\mathfrak{B}]\subset\mathfrak{C}}{\mathfrak{A}\subset[\mathfrak{B}\subset\mathfrak{C}]}$ (d.i. der erste Teil von Schema E_0).

$$\begin{matrix}(P_0)\\ (FB)\\ \\ (B_3)\end{matrix}\quad \frac{\dfrac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{B}\end{matrix}\quad\begin{matrix}[\mathfrak{A}]\\ \mathfrak{B}\subset\mathfrak{C}\end{matrix}}{\begin{matrix}[\mathfrak{B}\cdot\mathfrak{A}]\\ \mathfrak{C}\end{matrix}}}{\begin{matrix}[\mathfrak{A}\cdot\mathfrak{B}]\\ \mathfrak{C}\end{matrix}} \quad \text{oder} \quad \frac{\mathfrak{A}\subset[\mathfrak{B}\subset\mathfrak{C}]}{(\mathfrak{A}\cdot\mathfrak{B})\subset\mathfrak{C}} \quad (\text{d.i. der zweite Teil von Sch. } E_0).$$

Da schlieszlich die Einsetzungsregel E (zweiter Teil) von § 1 auf in NG_1 abgeleitete Sätze $\mathfrak{A} \subset \mathfrak{B}$ anwendbar ist, zeigt sich, dasz die Logik A_1 im Kalkül NG_1 eingebettet ist. Nach § 17 ist LG_1 in A_1, nach § 19 NG_1 in LG_1 eingebettet. Dies liefert:

Die Kalküle A_1, LG_1 und NG_1 sind völlig aequivalent.

Man sieht leicht, *dasz auch die Kalküle* A_2, LG_2 *und* NG_2 *aequivalent sind* (§§ 17^{bis}, 19^{bis}, 3).

Die dualen Resultate sind:

Die verneinende Aussagenlogik A_1^*, *der Kalkül* LG_1^* *und der Kalkül* NG_1^* *sind aequivalent; ebenfalls aequivalent sind die Kalküle* A_2^*, LG_2^* *und* NG_2^*.

Mit M und M^* aequivalente logistische und natürliche Kalküle.

§ 22. Die im Kalkül M neu hinzugekommenen Axiome 6^α und 6^β (§ 5) ermöglichen die Ableitung folgender Schluszschemata für die in § 17 eingeführten Sequenzen:

$$NES.\quad \frac{\mathfrak{A}\cdot\mathfrak{B}\rightarrow\lambda}{\mathfrak{B}\rightarrow\mathfrak{A}'}\,;\qquad\qquad NEA.\quad \frac{\mathfrak{B}\rightarrow\mathfrak{A}}{\mathfrak{A}'\cdot\mathfrak{B}\rightarrow\lambda}\,.$$

Ableitung von NES: $(\mathfrak{A}.\mathfrak{B})\subset\lambda$; (E_0) $\mathfrak{B}\subset(\mathfrak{A}\subset\lambda)$; $(6^\alpha, E)$ $(\mathfrak{A}\subset\lambda)\subset$ $\subset\mathfrak{A}'$; (1^β) $\mathfrak{B}\subset\mathfrak{A}'$.

Ableitung von NEA: $\mathfrak{B}\subset\mathfrak{A}$; $(\mathfrak{A}'.\mathfrak{B})\subset(\mathfrak{A}'.\mathfrak{A})$; $(6^\beta, E)$ $(\mathfrak{A}'.\mathfrak{A})\subset\lambda$; (1^β) $(\mathfrak{A}'.\mathfrak{B})\subset\lambda$.

LM sei ein mit NES und NEA erweiterter Kalkül LG_2 (Einsetzungsregel entsprechend erweitert). Dann ist *LM in M eingebettet.*

§ 22^{bis}. NES und NEA gestatten in LM für die in § 19 eingeführten Schemata $\begin{matrix}[\mathfrak{P}]\\ \mathfrak{Q}\end{matrix}$ die Ableitung folgender Schluszschemata:

$$NE.\quad \frac{\begin{matrix}[\mathfrak{A}\cdot\mathfrak{B}]\\ \lambda\end{matrix}}{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}'\end{matrix}}\,;\qquad NB.\quad \frac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}\end{matrix}\quad\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}'\end{matrix}}{\begin{matrix}[\mathfrak{B}]\\ \lambda\end{matrix}}\left(\text{, oder }\frac{\begin{matrix}[\mathfrak{B}]\\ \mathfrak{A}\cdot\mathfrak{A}'\end{matrix}}{\begin{matrix}[\mathfrak{B}]\\ \lambda\end{matrix}}\right).$$

Ableitung von NE: $\mathfrak{A}.\mathfrak{B}\rightarrow\lambda$; (NES) $\mathfrak{B}\rightarrow\mathfrak{A}'$.

Ableitung von NB: $\mathfrak{B}\rightarrow\mathfrak{A}$; (NEA) $\mathfrak{A}'.\mathfrak{B}\rightarrow\lambda$; $\mathfrak{B}\rightarrow\mathfrak{A}'$ und (Sch. P) $\mathfrak{B}\rightarrow\mathfrak{B}$; (UES) $\mathfrak{B}\rightarrow(\mathfrak{A}'.\mathfrak{B})$; (A_4) $\mathfrak{B}\rightarrow\lambda$.

NM sei ein mit NE und NB erweiterter Kalkül NG_2 (Einsetzungsregel entsprechend erweitert). Dann ist *NM in LM eingebettet.*

§ 22^{ter}. In NM sind 6^α und 6^β ableitbar.

$$\text{Ableitung von } 6^\alpha:\quad \begin{matrix}(P_0)\\ (FB)\\ \\ (NE)\end{matrix}\quad \cfrac{\cfrac{\begin{matrix}[X] & [X\subset\lambda]\\ X & X\subset\lambda\end{matrix}}{\begin{matrix}[X\cdot(X\subset\lambda)]\\ \lambda\end{matrix}}}{\begin{matrix}[X\subset\lambda]\\ X'\end{matrix}}\qquad \text{d.h. } (X\subset\lambda)\subset X'.$$

$$\text{Ableitung von } 6^\beta:\quad \begin{matrix}(P_0, B_1, B_3)\\ (NB)\end{matrix}\quad \frac{\begin{matrix}[X\cdot X'] & [X\cdot X']\\ X & X'\end{matrix}}{\begin{matrix}[X\cdot X']\\ \lambda\end{matrix}}\qquad \text{d.h. } (X\cdot X')\subset\lambda.$$

Dies zeigt, dasz M in NM eingebettet ist. Somit:
Die Kalküle M, LM und NM sind aequivalent.

§ 23. Dual gegenüber § 22 steht:
Ist LM^* ein mit den Schemata

$$NEA^{**}.\ \frac{\nu \rightarrow \mathfrak{B} + \mathfrak{A}}{\mathfrak{A}' \rightarrow \mathfrak{B}} \quad ; \quad NES^{**}.\ \frac{\mathfrak{A} \rightarrow \mathfrak{B}}{\nu \rightarrow \mathfrak{B} + \mathfrak{A}'}$$

erweiterter Kalkül LG_2^* (Einsetzungsregel entsprechend erweitert), so ist *LM^* in M^* eingebettet.*

§ 23bis. Ist NM^* ein mit den Schemata

$$NE^*.\ \frac{\begin{matrix}[\nu] \\ \mathfrak{B} + \mathfrak{A}\end{matrix}}{\begin{matrix}[\mathfrak{A}'] \\ \mathfrak{B}\end{matrix}} \quad ; \quad NB^*.\ \frac{\begin{matrix}[\mathfrak{A}] & [\mathfrak{A}'] \\ \mathfrak{B} & \mathfrak{B}\end{matrix}}{\begin{matrix}[\nu] \\ \mathfrak{B}\end{matrix}}$$

erweiterter Kalkül NG_2^* (Einsetzungsregel entsprechend erweitert), so ist *NM^* in LM^* eingebettet.*

§ 23ter. *Die Kalküle M^*, LM^* und NM^* sind aequivalent.*

Logistische und natürliche Kalküle aequivalent mit I, K, I^* oder K^*.

§ 24. Definitionen. LI sei ein Kalkül LM, welcher das Schema R (§ 18) erfüllt; LK sei ein Kalkül LI, welcher das Schema

$S.\quad \nu \rightarrow (\mathfrak{A} + \mathfrak{A}')$

erfüllt.

Definitionen. NI sei ein Kalkül NM, welcher das Schema R_0 (§ 20) erfüllt; NK sei ein Kalkül NI, welcher das Schema

$$S_0.\quad \begin{matrix}[\nu] \\ \mathfrak{A} + \mathfrak{A}'\end{matrix}$$

erfüllt.

Die Kalküle I, LI und NI sind aequivalent; ebenso die Kalküle K, LK und NK. (Beweise evident).

§ 25. Definitionen. LI^* sei ein Kalkül LM^*, welcher das Schema Q (§ 17) erfüllt; LK^* sei ein Kalkül LI^*, welcher das Schema

$T.\quad (\mathfrak{A} \cdot \mathfrak{A}') \rightarrow \lambda$

erfüllt.

Definitionen. NI^* sei ein Kalkül NM^*, welcher das Schema Q_0 (§ 19) erfüllt; NK^* sei ein Kalkül NI^*, welcher das Schema

$$T_0.\quad \begin{matrix}[\mathfrak{A} \cdot \mathfrak{A}'] \\ \lambda\end{matrix}$$

erfüllt.

Die Kalküle I^*, LI^* und NI^* sind aequivalent; ebenso die Kalküle K^*, LK^* und NK^*.

Weitere Bemerkungen über die logistischen Kalküle.

§ 26. a_1. Mit Hilfe des Axiomensystems von Kalkül A_2 lassen sich für den mit A_2 aequivalenten logistischen Kalkül LG_2 nicht nur die in §§ 17 und 17^{bis} gegebenen Schemata ableiten, sondern auszerdem die Schemata A_1^*, A_2^*, A_3^*, A_4^* [55]), UES^* (verallgemeinert UES), OES^* (verallgemeinert OES) (§§ 18 und 18^{bis}), und das Schema:

$$FEA^\circ.\ \frac{\mathfrak{D} \to (\mathfrak{A} + \mathfrak{F}) \qquad \mathfrak{B} \cdot \mathfrak{E} \to \mathfrak{C}}{(\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E} \to (\mathfrak{C} + \mathfrak{F})}\ ^{56)} \text{ (verallgemeinert } FEA);$$

die Schemata UEA und OEA haben schon ihre allgemeinere Form (vergl. UEA^* und OEA^*).

Die allgemeinere Form

$$FES^\circ.\ \frac{\mathfrak{A} \cdot \mathfrak{B} \to (\mathfrak{C} + \mathfrak{D})}{\mathfrak{B} \to [(\mathfrak{A} \subset \mathfrak{C}) + \mathfrak{D}]}$$

von FES gilt dagegen im Kalkül LG_2 nicht.

Denn sogar in der in § 7 betrachteten Heytingschen Gruppe für den intuitionistischen Aussagenkalkül gilt FES° nicht; $\mathfrak{A} = 2$, $\mathfrak{B} = 0$, $\mathfrak{C} = 1$, $\mathfrak{D} = 2$ liefern für $(\mathfrak{A} \cdot \mathfrak{B}) \subset (\mathfrak{C} + \mathfrak{D})$ den Wert 0, dagegen für $\mathfrak{B} \subset [(\mathfrak{A} \subset \mathfrak{C}) + \mathfrak{D}]$ den Wert 2.

b_1. Für den mit Kalkül A_2^* aequivalenten logistischen Kalkül LG_2^* lassen sich in A_2^* nicht nur die in §§ 18 und 18^{bis} gegebenen Schemata ableiten, sondern auszerdem die Schemata A_1, A_2, A_3, A_4 [55]), OEA (verallgemeinert OEA^*), UEA (verallgemeinert UEA^*) (§§ 17 und 17^{bis}), und das Schema:

$$FES^{\circ\circ}.\ \frac{(\mathfrak{F} \cdot \mathfrak{A}) \to \mathfrak{D} \qquad \mathfrak{C} \to \mathfrak{E} + \mathfrak{B}}{(\mathfrak{F} \cdot \mathfrak{C}) \to \mathfrak{E} + \mathfrak{D} + (\mathfrak{B} \subset \mathfrak{A})} \text{ (verallgemeinert } FES^{**});$$

die Schemata OES^* und UES^* haben schon ihre allgemeinere Form (vergl. OES und UES).

Die allgemeinere Form

$$FEA^{\circ\circ}.\ \frac{(\mathfrak{D} \cdot \mathfrak{C}) \to \mathfrak{B} + \mathfrak{A}}{[\mathfrak{D} \cdot (\mathfrak{C} \subset \mathfrak{A})] \to \mathfrak{B}}$$

von FEA^{**} gilt dagegen im Kalkül LG_2^* nicht.

§ 27. a_2. *Für den mit M aequivalenten Kalkül LM* (§ 22) *ist es nicht möglich das Schluszschema NES zu erweitern zu:*

$$NES^\circ.\ \frac{\mathfrak{A} \cdot \mathfrak{B} \to \mathfrak{C}}{\mathfrak{B} \to (\mathfrak{A}' + \mathfrak{C})}.\ ^{56bis)}$$

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[55]) Auch das A_4 und A_4^* umfassende „Schnitt"-Schema von Gentzen:

$$\frac{\mathfrak{A} \to (\mathfrak{B} + \mathfrak{H}) \qquad \mathfrak{H} \cdot \mathfrak{C} \to \mathfrak{D}}{\mathfrak{A} \cdot \mathfrak{C} \to (\mathfrak{B} + \mathfrak{D})}.$$

[56]) Ableitung: $\{(\mathfrak{A} + \mathfrak{F}) \cdot [(\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E}]\} \subset$ (distr.) $[\{\mathfrak{A} \cdot (\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E}\} + {} + \{\mathfrak{F} \cdot (\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E}\}]$; $(E_0, 1^a)$ $[\mathfrak{A} \cdot (\mathfrak{A} \subset \mathfrak{B})] \subset \mathfrak{B}$, also $[\mathfrak{A} \cdot (\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E}] \subset {} \subset (\mathfrak{B} \cdot \mathfrak{D} \cdot \mathfrak{E}) \subset (\mathfrak{B} \cdot \mathfrak{E}) \subset \mathfrak{C} \subset (\mathfrak{C} + \mathfrak{F})$; $[\mathfrak{F} \cdot (\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E}] \subset \mathfrak{F} \subset (\mathfrak{C} + \mathfrak{F})$; somit $\{(\mathfrak{A} + \mathfrak{F}) \cdot [(\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E}]\} \subset (\mathfrak{C} + \mathfrak{F})$, und umsomehr, wegen $\mathfrak{D} \subset (\mathfrak{A} + \mathfrak{F})$, $\{\mathfrak{D} \cdot [(\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E}]\} \subset (\mathfrak{C} + \mathfrak{F})$ oder $[(\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D} \cdot \mathfrak{E}] \subset (\mathfrak{C} + \mathfrak{F})$.

[56bis]) Es wird deutlich sein wie das Schema gelesen werden musz, falls $\mathfrak{C}$ mit λ oder ν zusammenfällt, um wirklich eine „Erweiterung" zu sein.

und ebenso ist es nicht möglich das Schema NEA zu erweitern zu

$$NEA^\circ. \frac{\mathfrak{B} \to (\mathfrak{A} + \mathfrak{C})}{\mathfrak{A}' \cdot \mathfrak{B} \to \mathfrak{C}}.$$ [56bis]

Denn: 1° die unter a_1 betrachtete Gruppe liefert mit $\mathfrak{A} = 2$, $\mathfrak{B} = 0$, $\mathfrak{C} = 2$ den Wert 0 für $(\mathfrak{A} \cdot \mathfrak{B}) \subset \mathfrak{C}$, und den Wert 2 für $\mathfrak{B} \subset (\mathfrak{A}' + \mathfrak{C})$;

2° die in § 5 am ersten betrachtete Gruppe, mit $\lambda = \nu = 0$, liefert mit $\mathfrak{A} = 0$, $\mathfrak{B} = 0$, $\mathfrak{C} = 1$ den Wert 0 für $\mathfrak{B} \subset (\mathfrak{A} + \mathfrak{C})$, und den Wert 1 für $(\mathfrak{A}' \cdot \mathfrak{B}) \subset \mathfrak{C}$.

Schlieszlich zeigt das Beispiel unter a_1, *dasz FES ebenfalls nicht in seiner allgemeineren Form FES° in LM gilt.*

b_2. *Für den mit M^* aequivalenten Kalkül LM^** (§ 23) *ist es nicht möglich das Schluszschema NEA^{**} zu erweitern zu*

$$NEA^{\circ\circ}. \frac{\mathfrak{C} \to \mathfrak{B} + \mathfrak{A}}{(\mathfrak{C} \cdot \mathfrak{A}') \to \mathfrak{B}},$$ [56bis]

und ebenso ist es nicht möglich NES^{**} zu erweitern zu

$$NES^{\circ\circ}. \frac{(\mathfrak{C} \cdot \mathfrak{A}) \to \mathfrak{B}}{\mathfrak{C} \to \mathfrak{B} + \mathfrak{A}'};$$ [56bis] [57]

auch FEA^{**} gilt nicht in seiner allgemeineren Form $FEA^{\circ\circ}$.

§ 28. a_3. *Für den mit I aequivalenten Kalkül LI* (§ 24) *ist es nicht möglich NES zu erweitern zu NES°*; das folgt aus dem schon unter a_2, 1° betrachteten Beispiel. *Auch das unter a_1 angegebene allgemeinere Schema FES° gilt nicht,* wie aus dem dort betrachteten Beispiel folgt. *Dagegen läszt sich das Schema NEA erweitern zu NEA°* (siehe a_2).

Ableitung von NEA°: $(\mathfrak{A}' \cdot \mathfrak{B}) \subset \mathfrak{B} \subset (\mathfrak{A} + \mathfrak{C})$, also $(\mathfrak{A}' \cdot \mathfrak{B}) \subset \subset [\mathfrak{A}' \cdot (\mathfrak{A} + \mathfrak{C})]$; (distr.) $(\mathfrak{A}' \cdot \mathfrak{B}) \subset [(\mathfrak{A}' \cdot \mathfrak{A}) + (\mathfrak{A}' \cdot \mathfrak{C})] \subset (6^\beta)\ [\lambda + \mathfrak{C}] \subset \subset (5^\circ)\ \mathfrak{C}$, oder $\mathfrak{A}' \cdot \mathfrak{B} \to \mathfrak{C}$.

b_3. *Für den mit I^* aequivalenten Kalkül LI^** (§ 25) *ist es nicht möglich NEA^{**} zu erweitern zu $NEA^{\circ\circ}$* [57]*), oder FEA^{**} zu erweitern zu $FEA^{\circ\circ}$; dagegen läszt sich NES^{**} erweitern zu $NES^{\circ\circ}$.* [57]

§ 29. a_4. *In dem mit K aequivalenten Kalkül LK* (§ 24) *gelten alle Schemata in erweiterer Form;* somit gelten FES° und NES°.

Ableitung von NES°: $\mathfrak{B} \subset (3^\circ)\ (\mathfrak{B} \cdot \nu) \subset (7^\circ)\ [\mathfrak{B} \cdot (\mathfrak{A}' + \mathfrak{A})] \subset (\mathfrak{A}' + + \mathfrak{B} \cdot \mathfrak{A}) \subset (\mathfrak{A}' + \mathfrak{C})$.

Bemerkung. In § 24 wurde ein Kalkül *LK* charakterisiert als ein Kalkül *LI*, welcher das Schema *S* erfüllt. Dies läszt sich ändern in: *ein Kalkül LK ist ein Kalkül LI, welcher NES° erfüllt.* Denn NES° liefert:

$$\frac{\mathfrak{A} \cdot \nu \to \mathfrak{A}}{\nu \to (\mathfrak{A}' + \mathfrak{A})} \text{ oder } \nu \to (\mathfrak{A} + \mathfrak{A}').$$

Aus Fuszn. 54 geht hervor, dasz *LI* sich aufbauen läszt auf Einsetzungsregel, Einsetzungsschemata *P*, *Q*, *R*, die Schluszschemata A_1, A_2,

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[57]) Wir bemerken, *dasz NEA° und $NEA^{\circ\circ}$ aequivalent sind; ebenso NES° und $NES^{\circ\circ}$.*

A_3, A_4 und die engeren Schluszschemata UES, UEA^*, OES, OEA^*, FES, FEA, NES, NEA; ändert man NES in NES°, so entsteht ein System, das LK charakterisiert.

Ableitung von FES° (für LK): $[(\mathfrak{C}\cdot\mathfrak{A})+\lambda]\subset(5^\circ)\,\mathfrak{C}$, also (6^β) $[(\mathfrak{C}\cdot\mathfrak{A})+(\mathfrak{A}'\cdot\mathfrak{A})]\subset\mathfrak{C}$, oder $[(\mathfrak{C}+\mathfrak{A}')\cdot\mathfrak{A}]\subset\mathfrak{C}$ oder (E_0) $(\mathfrak{C}+\mathfrak{A}')\subset$ $\subset[\mathfrak{A}\subset\mathfrak{C}]$. Dadurch: $\mathfrak{B}\subset(3^\circ)$ $(\mathfrak{B}\cdot\nu)\subset(7^\circ)$ $[\mathfrak{B}\cdot(\mathfrak{A}+\mathfrak{A}')]\subset[(\mathfrak{B}\cdot\mathfrak{A})+$ $+(\mathfrak{B}\cdot\mathfrak{A}')]\subset[(\mathfrak{C}+\mathfrak{D})+\mathfrak{A}']\subset[(\mathfrak{A}\subset\mathfrak{C})+\mathfrak{D}]$.

b_4. *In dem mit K^* aequivalenten Kalkül LK^** (§ 25) *gelten alle Schemata in erweiterter Form;* somit gelten $FEA^{\circ\circ}$ und $NEA^{\circ\circ}\equiv NEA^\circ$.

Bemerkung. *Ein Kalkül LK^* läszt sich somit auch charakterisieren als ein Kalkül LI^*, welcher NEA° erfüllt.*

Logistische und natürliche Kalküle für die engeren Prädikatenlogiken.

§ 30. Definition a. LP_1 sei ein Kalkül LG_1 (§ 17), welcher die Schemata AEA und AES erfüllt. [58])

$$AEA.\ \frac{F(a)\cdot\mathfrak{A}\rightarrow\mathfrak{B}}{(x)\,F(x)\cdot\mathfrak{A}\rightarrow\mathfrak{B}}\ (\mathfrak{A}\text{ darf fehlen}).$$

$$AES.\ \frac{\mathfrak{A}\rightarrow F(a)\ (\text{in }\mathfrak{A}\text{ kommt }a\text{ nicht vor})}{\mathfrak{A}\rightarrow(x)\ F(x)}.$$

Die Kalküle P_1 (§ 15) *und LP_1 sind aequivalent.* [59])

Definition a^*. LP_1^* sei ein Kalkül LG_1^* (§ 18), welcher die Schemata EES und EEA erfüllt. [58])

$$EES.\ \frac{\mathfrak{B}\rightarrow\mathfrak{A}+F(a)}{\mathfrak{B}\rightarrow\mathfrak{A}+(Ex)\ F(x)}\ (\mathfrak{A}\text{ darf fehlen}).$$

$$EEA.\ \frac{F(a)\rightarrow\mathfrak{A}\ \ (\text{in }\mathfrak{A}\text{ komt }a\text{ nicht vor})}{(Ex)\ F(x)\rightarrow\mathfrak{A}}.$$

Die Kalküle P_1^* (§ 15^{bis}) *und LP_1^* sind aequivalent.* [60])

Definition b. LP_2 sei ein Kalkül LG_2 (§ 17^{bis}), welcher die Schemata AEA, AES, EES und EEA erfüllt. [58])

Die Kalküle P_2 (§ 15) *und LP_2 sind aequivalent.*

Definition b^*. LP_2^* sei ein Kalkül LG_2^* (§ 18^{bis}), welcher die vier obigen Schemata erfüllt. [58])

Die Kalküle P_2^* (§ 15^{bis}) *und LP_2^* sind aequivalent.*

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[58]) Auszerdem soll die Einsetzungsregel für jeden Kalkül in sinngemäszer Weise erweitert, und eine Umbenennungsregel für gebundene Variable hinzugefügt werden; wie dies jedesmal zu geschehen hat, geht aus der Lektüre von Hilbert-Ackermann, loc. cit. 53), S. 54, 56 u. 57 leicht hervor.

[59]) Hier und im folgenden kann AEA überall durch AEA^*. $\dfrac{F(a)\rightarrow\mathfrak{B}}{(x)\ F(x)\rightarrow\mathfrak{B}}$ ersetzt werden.

[60]) Hier und im folgenden kann EES überall durch EES^*. $\dfrac{\mathfrak{B}\rightarrow F(a)}{\mathfrak{B}\rightarrow(Ex)\ F(x)}$ ersetzt werden.

Definitionen c, c^*. LMP, LIP, LKP [LMP^*, LIP^*, LKP^*] entstehen aus LM bzw. LI bzw. LK [aus LM^* bzw. LI^* bzw. LK^*] durch Hinzufügung von AES, AEA, EES und EEA.[58])

So entstehen *folgende Paare von aequivalenten Kalkülen: MP und LMP; IP und LIP; KP und LKP; MP^* und LMP^*; IP^* und LIP^*; KP^* und LKP^*.*

§ 31. Definition a_0. NP_1 sei ein Kalkül NG_1 (§ 19), welcher die Schemata AE und AB erfüllt.[58])

$$AB.\ \frac{\begin{matrix}[\mathfrak{A}]\\ (x)\ F(x)\end{matrix}}{\begin{matrix}[\mathfrak{A}]\\ F(a)\end{matrix}} \qquad AE.\ \frac{\begin{matrix}[\mathfrak{A}]\\ F(a)\end{matrix}}{\begin{matrix}[\mathfrak{A}]\\ (x)\ F(x)\end{matrix}} \quad (\mathfrak{A} \text{ enthält } a \text{ nicht}).$$

Die Kalküle LP_1 (§ 30) und NP_1 sind aequivalent.

Definition b_0. NP_2 sei ein Kalkül NG_2 (§ 19[bis]), welcher die Schemata AE, AB, EE und EB erfüllt.[58])

$$EE.\ \frac{\begin{matrix}[\mathfrak{A}]\\ F(a)\end{matrix}}{\begin{matrix}[\mathfrak{A}]\\ (Ex)\ F(x)\end{matrix}} \qquad EB.\ \frac{\begin{matrix}[\mathfrak{A}]\\ (Ex)\ F(x)\end{matrix} \quad \begin{matrix}[F(a)]\\ \mathfrak{C}\end{matrix}}{\begin{matrix}[\mathfrak{A}]\\ \mathfrak{C}\end{matrix}} \quad (\mathfrak{C} \text{ enthält } a \text{ nicht}).$$

Die Kalküle LP_2 (§ 30) und NP_2 sind aequivalent.

Definition c_0. NMP, NIP und NKP entstehen aus NM bzw. NI bzw. NK durch Hinzufügung von AE, AB, EE und EB.[58])

Die folgenden Paare sind *Paare von aequivalenten Kalkülen: LMP und NMP; LIP und NIP; LKP und NKP.*[61])

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[61]) Die Einführung der zu den Kalkülen von § 31 dualen fordert die Anwendung von vier Schemata, bzw. dual zu AE, AB, EE und EB.

MATHEMATICS

A SYMMETRIC FORM OF GÖDEL'S THEOREM *)

BY

S. C. KLEENE

(Communicated by Prof. L. E. J. BROUWER at the meeting of April 29, 1950)

It has been remarked, particularly in articles of MOSTOWSKI [1]), that recursively enumerable sets behave surprisingly similarly to analytic sets and general recursive sets to Borel sets. It is a theorem of LUSIN that two disjoint analytic sets can always be separated by a Borel set, i.e. this Borel set contains one of the given analytic sets and is disjoint from the other [2]). We shall construct two disjoint recursively enumerable sets C_0 and C_1 which cannot be separated by a general recursive set. This example shows that there is no exact parallelism between the two theories [3]).

We actually establish the following property of the sets C_0 and C_1, which is stronger constructively: Given any two disjoint recursively enumerable sets D_0 and D_1 such that $C_0 \subset D_0$ and $C_1 \subset D_1$, there can always be found a number f such that $\overline{f \,\varepsilon\, D_0 + D_1}$.

Let T_1 be the primitive recursive predicate so designated in a previous paper by the author [4]), and let $(x)_i$ be the number of times x contains the $i+1$-st prime number as factor (0, if $x = 0$) [5]). Let predicates W_0

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*) Presented to the American Mathematical Society, October 29, 1949. The first paragraph of this note is taken essentially from a letter of MOSTOWSKI to the author, dated 6 June 1949. Cf. the concluding paragraph.

1) ANDRZEJ MOSTOWSKI, *On definable sets of positive integers*, Fundamenta Mathematicae, 34, 81—112 (1946), and *On a set of integers not definable by means of one-quantifier predicates*, Annales de la société polonaise de mathématique, 21, 114—119 (1948).

2) See p. 52 of N. LUSIN, *Sur les ensembles analytiques*, Fundamenta mathematicae, 10, 1—95 (1927); or CASIMIR KURATOWSKI, *Topologie I*, Monografie Matematyczne, Warsaw-Lwów 249 (1933).

3) The example does not go against the parallelism between the theory of recursive predicates and quantifiers and the corresponding theory formulated by MOSTOWSKI 1946 [1]) in terms similar to the theory of projective sets. In § 6 of MOSTOWSKI's paper it is shown that these theories are equivalent, unless we admit as the basic system S for his theory one which does not satisfy two recursivity conditions (R_1) and (R_2). All ordinary (constructive) formal systems for arithmetic satisfy these conditions.

4) S. C. KLEENE, *Recursive predicates and quantifiers*, Transactions of the American Mathematical Society, 53, 41—73 (1943).

5) This $(x)_i$ is a primitive recursive function of x and i; in the notation of S. C. KLEENE, *General recursive functions of natural numbers*, Mathematische Annalen, 112, 727—742 (1936), no. 6, p. 732, $(x)_i = i + 1 \; Gl \; x$.

and W_1 be defined thus,

$$W_0(x, y) \equiv T_1((x)_1, x, y) \;\&\; (z)\{z < y \rightarrow \overline{T}_1((x)_0, x, z)\},$$
$$W_1(x, y) \equiv T_1((x)_0, x, y) \;\&\; (z)\{z < y \rightarrow \overline{T}_1((x)_1, x, z)\},$$

and the sets C_0 and C_1 as follows,

$$C_0 = \hat{x}\,(Ey)\,W_0(x, y), \qquad\qquad C_1 = \hat{x}\,(Ey)\,W_1(x, y).$$

The predicates W_0 and W_1 are primitive recursive [6]); hence the sets C_0 and C_1 are recursively enumerable [7]). From $W_0(x, y_0)$ and $W_1(x, y_1)$ we can infer both $y_0 > y_1$ and $y_1 > y_0$; hence

(1) $$\overline{(Ey)\,W_0(x, y) \;\&\; (Ey)\,W_1(x, y)},$$

i.e. C_0 and C_1 are disjoint.

Consider any two disjoint recursively enumerable sets D_0 and D_1 such that $C_0 \subset D_0$ and $C_1 \subset D_1$. We can write $D_0 = \hat{x}\,(Ey)\,R_0(x, y)$ and $D_1 = \hat{x}\,(Ey)\,R_1(x, y)$ with R_0 and R_1 recursive.

Now we show that there is a number f such that $\overline{f \,\varepsilon\, D_0 + D_1}$.

By the enumeration theorem for predicates of the form $(Ey)\,R(x, y)$ with R recursive [8]), there are numbers f_0 and f_1 such that, if we put $f = 2^{f_0} \cdot 3^{f_1}$, then

(2) $$(Ey)\,R_0(x, y) \equiv (Ey)\,T_1(f_0, x, y) \equiv (Ey)\,T_1((f)_0, x, y),$$

(3) $$(Ey)\,R_1(x, y) \equiv (Ey)\,T_1(f_1, x, y) \equiv (Ey)\,T_1((f)_1, x, y).$$

Assume: (a) $f \,\varepsilon\, D_0$, i.e. $(Ey)\,R_0(f, y)$. Then by (2): (b) $(Ey)\,T_1((f)_0, f, y)$. Also by (a) and the disjointness of D_0 and D_1: (c) $\overline{f \,\varepsilon\, D_1}$, i.e. $(\overline{Ey})\,R_1(f, y)$. Thence by (3), $(\overline{Ey})\,T_1((f)_1, f, y)$; whence: (d) $(y)\,\overline{T}_1((f)_1, f, y)$. By (b) and (d), $(Ey)\,[T_1((f)_0, f, y) \;\&\; (z)\{z < y \rightarrow \overline{T}_1((f)_1, f, z)\}]$, i.e. $(Ey)\,W_1(f, y)$, i.e. $f \,\varepsilon\, C_1$. Since $C_1 \subset D_1$, therefore $f \,\varepsilon\, D_1$, contradicting (c). By reductio ad absurdum, therefore (a) is false; i.e.

(4) $$\overline{f \,\varepsilon\, D_0}.$$

By a similar argument, or thence by the symmetry of the conditions on C_0 and D_0 to those on C_1 and D_1,

(5) $$\overline{f \,\varepsilon\, D_1}.$$

Thus there is no separation of all natural numbers into two disjoint recursively enumerable sets D_0 and D_1 such that $C_0 \subset D_0$ and $C_1 \subset D_1$.

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[6]) See e.g. KURT GÖDEL, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik, 38, 173–198 (1931), Theorems II and IV.

[7]) KLEENE 1936 [5]) Theorem III. Members of the sets C_0 and C_1 are easily found; e.g. if we take $x = x \;\&\; y = y$ as the $R(x, y)$ in KLEENE 1943 [4]) Theorem I, then $(Ey)\,W_0(2^g \cdot 3^f, y)$ and $(Ey)\,W_1(2^f \cdot 3^g, y)$.

[8]) KLEENE 1943 [4]) Theorem I.

This of course implies, and by the theorem for recursive predicates and quantifiers [9]) analogous to SOUSLIN's theorem for analytic and Borel sets [10]) is actually equivalent to, the statement that C_0 and C_1 cannot be separated by any general recursive set.

The root of this example is ROSSER's method [11]) of weakening the hypothesis of ω-consistency to simple consistency for GÖDEL's proof of the existence of an undecidable proposition in a formal system containing arithmetic [12]). The author mentioned previously that ROSSER's form of GÖDEL's theorem (as well as the original form) can be brought under a general theorem on recursive predicates and quantifiers [13]). The present result is obtained by rearranging the argument to make it symmetrical between the proposition and its negation. A discussion of it from this standpoint is included in another manuscript by the author. Upon seeing that manuscript, MOSTOWSKI pointed out the contrast to a theorem holding for analytic and Borel sets.

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[9]) KLEENE 1943 [4]) Theorem V, or p. 290 of EMIL L. POST, *Recursively enumerable sets of positive integers and their decision problems*, Bulletin of the American Mathematical Society, **50**, 284—316 (1944), or MOSTOWSKI 1946 [1]) 5.51. The present application is valid intuitionistically.

[10]) M. SOUSLIN, *Sur une définition des ensembles mesurables B sans nombres transfinis*, Comptes Rendus hebdomadaires des séances de l'Academie des Sciences, Paris, **164**, 88—91 (1917), Theorem III; KURATOWSKI 1933 [2]) p. 251 Corollary 1.

[11]) BARKLEY ROSSER, *Extensions of some theorems of* GÖDEL *and* CHURCH, The Journal of Symbolic Logic, **1**, 87—91 (1936), Theorem II.

[12]) GÖDEL 1931 [6]) Theorem VI.

[13]) KLEENE 1943 [4]) p. 64.

MATHEMATICS

ON THE NUMBER OF UNCANCELLED ELEMENTS IN THE SIEVE OF ERATOSTHENES

BY

N. G. DE BRUIJN

(Communicated by Prof. W. VAN DER WOUDE at the meeting of April 29, 1950)

1. *Introduction.*

Let, for $x > 0$, $y \geqslant 2$, $\Phi(x, y)$ denote the number of positive integers $\leqslant x$ which have no prime factors $< y$. Information on $\Phi(x, y)$ for large values of x and y can be obtained from several points of view.

A. First, for x very large with respect to y (roughly $\log x > C\, y/\log^2 y$), the following elementary formula gives a satisfactory estimate. Putting $\prod_{p<y} p = Q$, we have LEGENDRE's formula

$$\Phi(x, y) = \sum_{d|Q} \mu(d) \left[\frac{x}{d}\right]$$

and hence, for $y \geqslant 2$,

$$\left|\Phi(x, y) - x \prod_{p<y} \left(1 - \frac{1}{p}\right)\right| \leqslant \sum_{d|Q} \left(\frac{x}{d} - \left[\frac{x}{d}\right]\right) \leqslant \sum_{d|Q} 1 = 2^{\pi(y)} < 2^{y}. \tag{1.1}$$

B. On the other hand, if x is relatively small, the information comes from the prime number theorem. If $y \leqslant x \leqslant y^2$, the uncancelled elements in the sieve are exactly the primes in the interval $y \leqslant p \leqslant x$. Starting from here, A. BUCHSTAB [1]) derived estimates for $\Phi(x, y)$ in the regions $y^2 \leqslant x \leqslant y^3$, $y^3 \leqslant x \leqslant y^4, \ldots, y^n \leqslant x \leqslant y^{n+1}, \ldots$. He gave, however, no estimates holding uniformly in n. In the present paper these will be achieved, owing to several improvements on BUCHSTAB's method.

BUCHSTAB's result was the following one. Let, for $u \geqslant 1$, the function $\omega(u)$ be defined by

$$\begin{cases} \omega(u) = u^{-1}, & (1 \leqslant u \leqslant 2) \\ \dfrac{d}{du}\{u\, \omega(u)\} = \omega(u-1), & (u \geqslant 2) \end{cases} \tag{1.2}$$

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[1]) Rec. Math [Mat. Sbornik], (2), 44, 1239–1246 (1937). BUCHSTAB's work was partly duplicated by S. SELBERG (Norsk. Mat. Tidsskr. 26, 79–84 (1944)). SELBERG also proved (cf. (1.16) below) that $x^{-1}\,\Phi(x, y) \log y$ is uniformly bounded for $x \geqslant y \geqslant 2$ (Norske Vid. Selsk. Forh., Trondhjem, 19 (2), 3–6 (1946)).

The present paper can be read independently from these publications.

where for $u = 2$ the right-hand derivative has to be taken. Then for $u > 1$, u fixed, BUCHSTAB proved:

$$\lim_{y\to\infty} \Phi(y^u, y)\, y^{-u} \log y = \omega(u). \tag{1. 3}$$

It is not very difficult to derive from (1. 2) that $\lim\limits_{u\to\infty} \omega(u)$ exists (see (1. 10) below). Furthermore, since we have [2])

$$\prod_{p<y} \left(1 - \frac{1}{p}\right) \backsim e^{-\gamma} \log y, \tag{1. 4}$$

where γ is EULER's constant, it can be expected that

$$\lim_{u\to\infty} \omega(u) = e^{-\gamma}. \tag{1. 5}$$

Namely, if we put

$$\Phi(x, y) = x \prod_{p<y} \left(1 - \frac{1}{p}\right) \cdot \psi(x, y), \tag{1. 6}$$

we obtain from (1. 1) and (1. 3), respectively

$$\lim_{u\to\infty} \psi(y^u, y) = 1, \quad \lim_{u\to\infty} \lim_{y\to\infty} \psi(y^u, y) = e^{\gamma} \lim_{u\to\infty} \omega(u).$$

Formula (1. 5) can be established indeed. It will follow from the closer investigation of $\psi(x, y)$ to be carried out in the sequal. A direct proof can also be given (section 4).

In section 2 we shall prove [3])

$$\left|\psi(y^u, y) - e^{\gamma} \log y \int_1^u y^{t-u}\, \omega(t)\, dt\right| < CR(y) \qquad (u \geqslant 1,\ y \geqslant 2). \tag{1. 7}$$

Here $\omega(t)$ is BUCHSTAB's function, defined by (1. 2). $R(y)$ is a positive function satisfying $R(y) \downarrow 0$ for $y \to \infty$, $R(y) > y^{-1}$ and [4])

$$\begin{cases} |\pi(y) - \operatorname{li} y| < y\, R(y)/\log y & (y \geqslant 2) \\ \int_y^\infty |\pi(t) - \operatorname{li} t| \cdot t^{-2}\, dt < R(y) & (y \geqslant 2). \end{cases} \tag{1. 8}$$

Suitable functions $R(y)$ are known from the theory of primes, for instance $R(y) = C \exp(-C \log^{\varkappa} x)$, with $\varkappa = \frac{1}{2}$ [5]) or $\varkappa = \frac{4}{7} - \varepsilon$ [6]), and $\varkappa = 1$ if the RIEMANN hypothesis is correct.

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[2]) See A. E. INGHAM, The Distribution of Prime Numbers, London, p. 22 (1932).

[3]) The C's are absolute constants, not necessarily the same at each occurrence.

[4]) As usual, $\pi(y)$ denotes the number of primes $\leqslant y$; li y denotes the logartihmic integral.

[5]) See INGHAM loc. cit. p. 65.

[6]) N. TCHUDAKOFF, C. R. Acad. Sci. URSS, N. s. 21, 421—422 (1938).

From (1. 7) we can deduce that

$$(1.9) \qquad |\psi(y^u, y) - 1| < C\,\Gamma^{-1}(u) + C\,R(y) \qquad (u \geqslant 1,\ y \geqslant 2).$$

This follows from the behaviour of the function $\omega(t)$. In a previous paper [7]) we proved that $\lim_{t\to\infty} \omega(t)$ exists, and, if we denote it by A, that

$$(1.10) \qquad \omega(u) = A + O\,\{\Gamma^{-1}(u+1)\}.$$

This can also be proved independently in a few lines. If we write (1. 2) in the form $u\,\omega'(u) = -\,\omega(u) + \omega(u-1)$, we infer that

$$\omega'(u) \leqslant u^{-1} \operatorname*{Max}_{u-1\leqslant t\leqslant u} |\omega'(t)| \qquad (u \geqslant 2).$$

It follows that $\omega'(u)$ is bounded for $u \geqq 2$. Denoting the upper bound of $|\omega'(t)|$ for $u \leqslant t < \infty$ by $M(u)$, we find $M(u) \leqslant u^{-1} M(u-1)$ $(u \geqslant 3)$, and so $M(u) \leqslant C\,\Gamma^{-1}(u+1)$. Now (1. 10) easily follows.

The error-term in (1. 10) is certainly not the best possible. The right order is probably something like $\exp(-\,u \log u - u \log\log u)$.

It easily follows from (1. 10) that

$$|\log y \int_1^u y^{t-u}\,\omega(t)\,dt - A| < C\,\Gamma^{-1}(u) + C\,y^{-1} \qquad (u \geqslant 1,\ y \geqslant 2).$$

Now (1. 7) leads to

$$(1.11) \qquad |\psi(y^u, y) - e^\gamma A| < C\,\Gamma^{-1}(u) + C\,R(y) \qquad (u \geqslant 1,\ y \geqslant 2).$$

Take a fixed value of y, and make $u \to \infty$. Comparing the result with (1. 1) we find $|e^\gamma A - 1| < C\,R(y)$ $(y \geqslant 2)$, and hence $A = e^{-\gamma}$. This proves (1. 5) and (1. 9).

C. There is a third approach to the problem of $\Phi(x, y)$. Put

$$\zeta(s) \prod_{p \leq y} (1 - p^{-s}) = \sum_{n=1}^{\infty} c_n\, n^{-s}, \qquad (\operatorname{Re} s > 1),$$

then

$$(1.12) \qquad \Phi(x, y) = \sum_{n \leqslant x} c_n,$$

and this sum can be evaluated by contour integration. This will be exposed in section 3. There is nothing new in the method which is quite familiar from the theory of DIRICHLET series. The result has some interest, since, for $u = (\log x)/\log y$ not too large, it does not fall far from (1. 9). We shall prove in section 3, namely, that

$$(1.13) \qquad \left\{ \begin{array}{l} |\psi(y^u, y) - 1| < C \log^3 y \cdot e^{-u \log u - u \log\log u + Cu} \\ \qquad\qquad (1 \leqslant u \leqslant 4 y^{\frac{1}{4}}/\log y; \quad y \geqslant 2). \end{array} \right.$$

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[7]) N. G. DE BRUIJN, On some linear functional equations, Example 1. To be published in Publicationes Mathematicae, Debrecen.

On the other hand we shall show that

$$(1.\,14)\qquad |\psi(y^u, y) - 1| < C\,e^{-\frac{1}{3}u\log y} = C\,x^{-\frac{1}{3}} \qquad (u > 4y^{\frac{1}{3}}/\log y;\ y \geqslant 2).$$

For $u > \varepsilon^{-1}\, y \log^{-2} y$, however, (1. 1) gives a better result than this one, namely

$$(1.\,15)\qquad |\psi(x, y) - 1| < C\,x^{-1+\varepsilon} \qquad (u > \varepsilon^{-1}\, y \log^{-2} y;\ y \geqslant 2).$$

In section 3 we have to make the restrictions $y \geqslant e^2$, $u > 2e$, but it is easily seen from (1. 9) and (1. 15) that in (1. 13) and (1. 14) these restrictions may be removed.

It is easily inferred from the results of methods **B** and **C** that there is a positive constant a, such that

$$(1.\,16)\qquad |\psi(y^u, y) - 1| < C\,e^{-au} \qquad (y \geqslant 2,\ u \geqslant 1).$$

Namely, in (1. 9) we can take $R(y) = C exp\,(-C \log^{\frac{1}{2}} y)$. Hence (1. 16) holds, with $a = 1$, for $y \geqslant 2$, $1 \leqslant u \leqslant C \log^{\frac{1}{2}} y$. On the other hand, if $u > C \log^{\frac{1}{2}} y$, we have $\log^3 y < C\,u^6 < Ce^u$. Consequently, by (1. 13) and (1. 14), we have

$$|\psi(y^u, y) - 1| < C\,\mathrm{Max}\,(e^{-u\log u - u\log\log u + Cu},\ e^{-\frac{1}{3}u\log y}).$$

This proves (1. 16), with $a = \frac{1}{3}\log 2$.

Buchstab considered, in the paper quoted before, the more general problem of the uncancelled numbers which belong to a given arithmetical progression. Suppose $k \geqq 1$, $(l, k) = 1$, and let $\Phi_l(k; x, y)$ denote the number of positive integers $\leqslant x$, which are $\equiv 1 \pmod{k}$, and which contain no prime factors $< y$. For $\varphi(k)\,\Phi_l(k; x, y)$, where $\varphi(k)$ is Euler's indicator, he obtained the same result as for the case $k = 1$ described above. The present results can also be generalised that way. In section 2 this can be carried out with very little alterations. Following Buchstab, we can simply deal simultaneously with all $\Phi_l(k; x, y)$, for k, x, y fixed. In section 3 we have to use Dirichlet's L-series instead of the Riemann zeta function. In both methods real difficulties only arise when estimations holding uniformly in k are required.

2. *Proof of* (1. 7).

Suppose $x \geqslant y \geqslant 2$. Clearly, for $h \geqslant 1$

$$(2.\,1)\qquad \Phi(x, y) = \sum_{y \leqslant p < y^h} \Phi\left(\frac{x}{p}, p\right) + \Phi(x, y^h),$$

and hence, by (1. 6)

$$(2.\,2)\qquad \psi(x, y) = \sum_{y \leqslant p < y^h} \psi\left(\frac{x}{p}, p\right) \cdot \frac{1}{p} \prod_{y \leqslant q < p} \left(1 - \frac{1}{q}\right) + \psi(x, y^h) \prod_{y \leqslant p < y^h} \left(1 - \frac{1}{p}\right).$$

Here p and q run through the primes. Put

$$\prod_{p<y}\left(1-\frac{1}{p}\right)=P(y),\quad \sum_{y\leqslant p<y^\sigma}\frac{1}{p}\prod_{y\leqslant q<p}\left(1-\frac{1}{q}\right)=W(\sigma)=1-\frac{P(y^\sigma)}{P(y)}.$$

$W(\sigma)$ depends on y also.

Using a Stieltjes integral, we can write instead of (2. 2), for $u \geqslant 1$,

$$\psi(y^u, y)=\int_1^h \psi(y^{u-\sigma}, y^\sigma)\, dW(\sigma)+\psi(y^u, y^h)\{1-W(h)\}. \tag{2. 3}$$

We first estimate $W(\sigma)$. We have for $\sigma \geqslant 1$, if $\pi^*(y)$ denotes the number of primes $< y$ (not $\leqslant y$):

$$\log\{P(y^\sigma)/P(y)\}=\sum_{y\leqslant p<y^\sigma}\log\left(1-\frac{1}{p}\right)=\int_1^\sigma \log(1-y^{-\mu})\, d\pi^*(y^\mu)=$$

$$=\int_1^\sigma \log(1-y^{-\mu})\, d\{\pi^*(y^\mu)-\mathrm{li}\, y^\mu\}+\int_1^\sigma \log(1-y^{-\mu})\frac{y^\mu \log y}{\log y^\mu}\, d\mu=$$

$$=\log(1-y^{-\mu})\{\pi^*(y^\mu)-\mathrm{li}\, y^\mu\}\Big|_1^\sigma-\int_1^\sigma \frac{\log y}{y^\mu-1}\{\pi^*(y^\mu)-\mathrm{li}\, y^\mu\}\, d\mu-\log\sigma+O\left(\frac{1}{y}\right).$$

By (1. 8) we now find

$$|\sigma P(y^\sigma)/P(y)-1|<C\,R(y) \qquad (\sigma\geqslant 1\ ,\ y\geqslant 2) \tag{2. 4}$$

Hence we have

$$|W(\sigma)-1+\sigma^{-1}|<C\,R(y). \qquad (\sigma\geqslant 1\ ,\ y\geqslant 2) \tag{2. 5}$$

An approximate solution of (2. 3) is $\theta(y^u, y)$, where

$$\theta(y^u, y)=e^\gamma \log y\cdot\int_1^u y^{t-u}\,\omega(t)\, dt, \tag{2. 6}$$

and $\omega(t)$ is given by (1. 2). In order to show this, we first evaluate

$$\Omega_1(u; y; h)=\theta(y^u, y)-\int_1^h \theta(y^{u-\sigma}, y^\sigma)\,\sigma^{-2}\, d\sigma-h^{-1}\,\theta(y^u, y^h)$$

for $1\leqslant h\leqslant \frac{1}{2}u$. We have $\Omega(u; y; 1)=0$, and

$$\frac{\partial}{\partial h}\Omega_1(u; y; h)=-h^{-2}\,\theta(y^{u-h}, y^h)+h^{-2}\,\theta(y^u, y^h)-h^{-1}\frac{\partial}{\partial h}\theta(y^u, y^h).$$

The right-hand-side can be evaluated; using (2. 6) and (1. 2) we find it to be $-e^\gamma h^{-1} y^{h-u}\log y$. Therefore

$$|\Omega_1(u; y; h)|=e^\gamma \log y\int_1^h t^{-1}\, y^{t-u}\, dt\leqslant e^\gamma\, y^{h-u}. \tag{2. 7}$$

Next we put

$$\Omega_3(u; y; h)=\theta(y^u, y)-\int_1^h \theta(y^{u-\sigma}, y^\sigma)\, dW(\sigma)-\theta(y^u, y^h)\{1-W(h)\}=$$
$$=\Omega_1(u; y; h)+\Omega_2(u; y; h)$$

where

$$\Omega_2(u;y;h) = -\int_1^h \theta(y^{u-\sigma}, y^\sigma)\, d\left\{W(\sigma) - 1 + \frac{1}{\sigma}\right\} - \theta(y^u, y^h)\left\{1 - \frac{1}{h} - W(h)\right\}.$$

By partial integration, using (2. 5) and using $W(1) = 0$, we obtain

$$(2.\ 8)\quad |\Omega_2(u;y;h)| \leqslant CR(y)\left\{|\theta(y^u, y^h) - \theta(y^{u-h}, y^h)| + \int_1^h \left|\frac{d}{d\sigma}\theta(y^{u-\sigma}, y^\sigma)\right| d\sigma\right\}.$$

In the sequel we shall need the following inequality

$$(2.\ 9)\quad \left|\Omega_3\left(u;y;\frac{u}{k}\right)\right| \leqslant C k^{-2} R(y), \qquad (k = 2, 3, 4, \ldots; k \leqslant u < k+1; y \geqslant 2)$$

C not depending on k. As to the contribution of Ω_1, this follows from (2. 7), since $y_k^{-1} = O\{R(y)\}$. For an estimate of Ω_2 we have to use (2. 8), (2. 6) and (1. 10); we omit the verification, which is straight forward.

The difference $\psi(y^u, y) - \theta(y^u, y) = \eta(y^u, y)$ satisfies ($h \geqslant 1$):

$$(2.\ 10)\quad \eta(y^u, y) = \int_1^h \eta(y^{u-\sigma}, y^\sigma)\, dW(\sigma) + \eta(y^u, y^h)\{1 - W(h)\} - \Omega_3(u;y;h).$$

For $1 \leqslant u \leqslant 2$ the function ψ is known; we obtain

$$\eta(y^u, y) = \frac{\pi(y^u) - \pi^*(y)}{y^u P(u)} - e^\gamma \log y \int_1^u y^{t-u} \frac{dt}{t} \qquad (1 \leqslant u \leqslant 2)$$

We have $\lim\limits_{y\to\infty} P(y) \log y = e^{-\gamma}$, and hence, by (2. 4),

$$P(y) = e^{-\gamma}(1 + O\{R(y)\})/\log y.$$

From (1. 8) it now readily follows that

$$|\eta(y^u, y)| < C\, R(y) \qquad (y \geqslant 2,\ 1 \leqslant u \leqslant 2)$$

Now put, for $k = 1, 2, 3, \ldots, y \geqslant 2$

$$s_k(y) = \sup_{\substack{k \leqslant u < k+1 \\ t \leqslant y}} |\eta(t^u, t)|,$$

whence $s_1(y) < C\, R(y)$ ($y \geqslant 2$). We apply (2. 10) for $k = 2, 3, 4, \ldots$, $k \leqslant u < k+1$, with $h = u/k$; using (2. 9) we obtain

$$s_k(y) < s_{k-1}(y) + C k^{-2} R(y) \qquad (k = 2, 3, \ldots;\ y \geqslant 2).$$

It follows that, for $k = 1, 2, 3, \ldots;\ y \geqslant 2$,

$$s_k(y) < \{C + C \sum_{n=2}^k n^{-2}\}.\ R(y) < C\, R(y).$$

Consequently

$$|\eta(y^u, y)| < C\, R(y), \qquad (y \geqslant 2; u \geqslant 1)$$

which proves (1. 7).

3. *Proof of* (1. 13) *and* (1. 14).

Throughout this section we suppose $x \geqslant y \geqslant e^2$, $u = (\log x)/\log y$, whence $u \geqslant 1$. We introduce the positive numbers a, T and λ, satisfying

$$1 < a < 2, \quad T > 2, \quad 1 \leqq \lambda \leqq \tfrac{1}{2} \log y,$$

and we put

$$b = 1 - \lambda/\log y \qquad (\text{whence } \tfrac{1}{2} \leqq b < 1).$$

For a, T and λ we shall choose suitable values later on. For simplicity we assume x to be half an odd integer. In the final results this restriction is easily eliminated.

The constants implied in our O-symbols are absolute constants.

It is easily verified by contour integration that

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} = E(x, n) + O\left\{\left(\frac{x}{n}\right)^a \int_0^\infty e^{-\xi\,|\log x/n|} \frac{d\xi}{(\xi^2+T^2)^{\frac{1}{2}}}\right\} =$$

$$= E(x, n) + O\left\{\left(\frac{x}{n}\right)^a \operatorname{Max}\left(\frac{1}{T\,|\log x/n|}, \log \frac{1}{T\,|\log x/n|}\right)\right\}$$

uniformly for $n = 1, 2, 3, \ldots$. Here $E(x, n) = 1$ or 0 according to $n < x$ and $n > x$, respectively.

It follows that (cf. (1. 12)).

$$(3.\ 1) \qquad \begin{cases} \Phi(x, y) = \dfrac{1}{2\pi i} \displaystyle\int_{a-iT}^{a+iT} x^s \zeta(s) \prod_{p<y} \left(1 - \frac{1}{p^s}\right) \frac{ds}{s} + \\ + O\left\{\displaystyle\sum_{n=1}^{\infty} \frac{x^a}{n^a} \operatorname{Min}\left(\frac{1}{T\,|\log x/n|}, \log \frac{1}{T\,|\log x/n|}\right)\right\}. \end{cases}$$

The best possible extimate for the O-term in (3. 1) is

$$(3.\ 2) \qquad \frac{x}{T} O\left\{\log \frac{1}{a-1} + \log \operatorname{Min}(x, T) + \operatorname{li}\{\operatorname{Max}(3, x^{a-1})\}\right\}.$$

We omit the verification, which can be carried out by splitting the sum into three parts, corresponding to $n < \frac{1}{2} x$, $\frac{1}{2} x \leqslant n < 2x$, $n \geqslant 2x$, respectively.

The integral in (3. 1) can be evaluated by means of the residue theorem; we obtain

$$x \prod_{p<y} (1 - p^{-1}) + J_1 + J_2 + J_3,$$

where

$$J_1 = \frac{1}{2\pi i} \int_{b-iT}^{b+iT}, \quad J_2 = \frac{1}{2\pi i} \int_{b+iT}^{a+iT}, \quad J_3 = -\frac{1}{2\pi i} \int_{b-iT}^{a-iT}.$$

The most important contribution is J_1. We have

$$\int_{b-iT}^{b+iT} \left| \zeta(s) \frac{ds}{s} \right| = O(\log^{3/2} T) + O\left(\log \frac{1}{1-b}\right). \tag{3. 3}$$

This is easily deduced from the fact that, for $T_1 > 2$, $\frac{1}{2} < b < 1$,

$$\int_{T_1}^{2T_1} |\zeta(b \pm it)|^2 \, dt = O(T_1 \log T_1) \; ^{8})$$

whereas the second O-term in (3. 3) arises from the pole of $\zeta(s)$ at $s = 1$.

Furthermore, for $s = b + it$ we simply use

$$\left| \prod_{p<y} (1 - p^{-s}) \right| \leqslant \prod_{p<y} (1 + p^{-b}),$$

and, as $\pi(\xi) < 2 \operatorname{li} \xi + C$,

$$\log \prod_{p<y} (1 + p^{-b}) \leqslant O(1) + \int_2^y \xi^{-b} \, d\pi(\xi) \leqslant O(1) + 2 \int_2^y \xi^{-b} \, d \operatorname{li} \xi =$$

$$= O(1) + 2 \int_{(1-b)\log 2}^{\lambda} e^{\eta} \eta^{-1} \, d\eta = O(1) + 2 \operatorname{li} e^{\lambda} + 2 \log\log y - 2 \log \lambda.$$

Summarizing, we have

$$J_1 = O\left\{ x e^{-\lambda u} \cdot 2 \operatorname{li} e^{\lambda} \cdot \frac{(\log y)^2}{\lambda^2} \left(\log^{3/2} T + \log \frac{\log y}{\lambda} \right) \right\}. \tag{3. 4}$$

For J_2 we use

$$|\zeta(\sigma + it)| < C t^{\frac{1}{2}}, \qquad (|t| \geqslant 2, \; \sigma \geqslant \tfrac{1}{2})$$

which leads to

$$J_2 = O\left\{ x^a \exp(2 \operatorname{li} e^{\lambda}) \cdot \left(\frac{\log y}{\lambda} \right)^2 \cdot T^{-\frac{1}{2}} \right\}. \tag{3. 5}$$

The same holds for J_3, of course.

The difference $\Phi(x, y) - x \prod_{p<y} (1 - p^{-1})$ is less than the sum of (3. 2), (3. 4) and (3. 5). Simplifying our result by specialization

$$a = 1 + (\log x)^{-1}, \qquad T = e^{2\lambda u},$$

we can deduce

$$\Phi(x, y) - x \prod_{p<y} (1 - p^{-1}) = O\{ x \log^2 y \cdot (\lambda u)^{3/2} \cdot \exp(-\lambda u + 2 \operatorname{li} e^{\lambda}) \}.$$

By (1. 6) and (1. 4) we now obtain

$$\psi(x, y) - 1 = O\{ \log^3 y \cdot (\lambda u)^{3/2} \cdot \exp(-\lambda u + 2 \operatorname{li} e^{\lambda}) \}. \tag{3. 6}$$

We are still free to give λ any value in the interval $1 \leqslant \lambda \leqslant \frac{1}{2} \log y$.

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[8]) TITCHMARSH, The Zeta Function of RIEMANN. London, p. 31 (1931).

Now assume that $u > 2e$. Then the minimum of $-\lambda u + 2 \operatorname{li} e^{\lambda}$ for $\lambda \geqslant 1$ is attained for $\lambda = \lambda_0$, defined by

$$\lambda_0 u = 2 e^{\lambda_0}, \quad \lambda_0 > 1. \tag{3. 7}$$

There we have (put $2\xi^{-1} e^{\xi} = \eta$)

$$\operatorname{li} e^{\lambda_0} = O(1) + \int_1^{\lambda_0} e^{\xi} \xi^{-1} d\xi = O(1) + \tfrac{1}{2} \int_{2e}^{u} (1-\xi^{-1})^{-1} d\eta = O(u).$$

Furthermore

$$\lambda_0 = \log(\tfrac{1}{2} u\lambda_0) > \log \tfrac{1}{2} u + \log\log \tfrac{1}{2} u > \log u + \log\log u + O(1),$$

whence it follows

$$\exp(-\lambda_0 u + 2 \operatorname{li} e^{\lambda_0}) < \exp\{-u \log u - u \log\log u + O(u)\}. \tag{3. 8}$$

This result can be applied to (3. 6) whenever the solution of (3. 7) is less than $\frac{1}{2} \log y$, that is for

$$2e < u \leqslant 4 y^{\frac{1}{2}}/\log y.$$

In that region we obtain, since $\lambda_0 u = O(e^u)$,

$$\psi(x, y) - 1 = O(\log^3 y \cdot e^{-u \log u - u \log\log u + Cu}).$$

If, however, $u > 4 y^{\frac{1}{2}}/\log y$, we take $\lambda = \frac{1}{2} \log y$, and we infer from (3. 6)

$$\psi(x, y) - 1 = O(e^{-1/_3 u \log y}) = O(x^{-1/_3}).$$

4. *Direct proof of* (1. 4).

We have established by combination of the results of methods **A** and **B** (cf. section 1), that

$$\lim_{u \to \infty} \omega(u) = e^{-\gamma}. \tag{4. 1}$$

A purely analytical proof can also be given. The function

$$h(u) = \int_0^{\infty} \exp\{-ux - x + \int_0^x (e^{-t} - 1) t^{-1} dt\} dx \tag{4. 2}$$

is analytical for $u > -1$; it satisfies the equation

$$u h'(u-1) + h(u) = 0 \qquad (u > 0). \tag{4. 3}$$

The expression

$$\int_{a-1}^{a} \omega(u) h(u) du + a \omega(a) h(a-1) = (\omega, h)$$

does not depend on a for $a \geqslant 2$. This is easily verified by differentiation, using (1. 2) and (4. 3). We now evaluate (ω, h) in two ways. First let $a \to \infty$. Then we have $\omega(a) \to A$, by (1. 10), and $h(a) \sim a^{-1}$ by (4. 2). Hence $(\omega, h) = A$. Secondly, take $a = 2$. We find

$$A = \int_1^2 u^{-1} h(u) du + h(1).$$

In virtue of (4. 3) we obtain

$$A = -\int_1^2 h'(u-1)\,du + h(1) = h(0) = \lim_{u\downarrow 0} u\,h'(u-1) =$$

$$= -\lim_{u\downarrow 0} u \int_0^\infty \exp\left\{-ux + \int_0^x (e^{-t}-1)\,t^{-1}\,dt + \log x\right\} dx.$$

Here we have

$$\lim_{x\to\infty}\left\{\int_0^x \frac{e^{-t}-1}{t}\,dt + \log x\right\} = -\gamma,$$

and (4. 1) easily follows.

The method used here consists of the construction of an "adjoint" equation (4. 3), such that there is an invariant inner product (ω, h) for any pair of solutions ω, h of the original equation and of the adjoint one, respectively. This method was decisive in the author's researches on the equation $F'(x) = e^{\alpha x+\beta}\,F(x-1)$, which are unpublished as yet.

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ON SOME VOLTERRA INTEGRAL EQUATIONS OF WHICH ALL SOLUTIONS ARE CONVERGENT

BY

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1. *Introduction.*

We shall be concerned with equations of the type

$$f(x) = \int_0^1 K(x,t)\, f(x-t)\, dt. \tag{1. 1}$$

The problem is to impose conditions on K which guarantee that every solution $f(x)$ will be convergent, i.e. $\lim\limits_{x \to \infty} f(x)$ exists. Throughout the paper we suppose that the kernel is absolutely integrable with respect to x and t for $0 \leqslant x \leqslant a$, $0 \leqslant t \leqslant 1$ and any $a > 0$. Furthermore we shall assume that, for $x \geqslant 1$,

$$K(x,t) \geqslant 0 \ (0 \leqslant t \leqslant 1); \ K(x,t) = 0 \ (t > 1 \text{ and } t < 0); \ \int_0^1 K(x,t)\, dt = 1. \tag{1. 2}$$

A function $f(x)$ will be called a *solution*, if it is measurable and bounded on any finite interval $0 \leqslant x \leqslant a$, and if it satisfies (1. 1) for $x \geqslant 1$. Since we do not assume $K(x,t)$ to be uniformly bounded, a special difficulty arises, which is demonstrated by the following example. Take

$$\begin{cases} K(x,t) = 1 \text{ if } x \text{ is an integer} \\ K(x,t) = (x - [x])^{-1} \text{ for } 0 \leqslant t \leqslant x - [x] \text{ if } x \text{ is not an integer} \\ K(x,t) = 0 \text{ otherwise.} \end{cases} \tag{1. 3}$$

Now solutions $f(x)$ can be constructed from arbitrary sequences $c_1, c_2, c_3, \ldots$ of real numbers by taking $f(x) = c_n$ for $n - 1 < x \leqslant n$. In this example $f(x)$ is not uniquely determined by its values on the interval $0 \leqslant x \leqslant 1$. Nevertheless the kernel (1. 3) has a stabilizing effect: every *continuous* solution is constant for $x \geqslant 1$.

In order to avoid the difficulty shown by this example, we shall consider *regular* solutions only. A solution $f(x)$ is called regular, if, for all values of m, M and $a \geqslant 1$ the following statement holds true: *If* $m \leqslant f(x) \leqslant M$ *almost everywhere on* $a - 1 \leqslant x \leqslant a$, *then we have* $m \leqslant f(x) \leqslant M$ *for* $x > a$. It follows that a regular solution is uniquely determined by its values on $0 \leqslant x \leqslant 1$.

The problem as to which kernels have the property that any solution is regular, will not be considered in this paper. Neither shall we deal

with the question whether any function $f(x)$ given on $0 \leqslant x \leqslant 1$ can be continued to a solution [1]).

In practice the two following theorems are often sufficient.

Theorem 1. *Every continuous solution is regular.*

Proof. Let $f(x)$ be continuous, and assume $f(x) \leqslant M$ for $a-1 \leqslant x \leqslant a$. Assume that $x_1 > a$ and $f(x_1) > M$. Let x_2 be the least possible number such that $a < x_2 \leqslant x_1$, $f(x_2) = f(x_1)$. Now we have $f(x) < f(x_2)$ for $x_2 - 1 \leqslant x < x_2$. Apply (1. 1) with $x = x_2$, and a contradiction follows.

Theorem 2. *If, for any finite $A > 0$, the kernel $K(x,t)$ is bounded over $1 \leqslant x \leqslant A$, $0 \leqslant t \leqslant 1$, then every solution is regular.*

Proof. We can find $\varepsilon > 0$ (ε depending on A), such that $\int_0^\varepsilon K(x,t)\,dt < \frac{1}{2}$ for $1 \leqslant x \leqslant A$. Now assume $a > 1$, $A > a$, and $f(x) \leqslant M$ for $x < a$. By the definition of a solution, $f(x)$ is bounded for $a \leqslant x \leqslant a + \varepsilon$; denote its upper bound by M_1. Assume $M_1 > M$. By (1. 1) we find, for $a \leqslant x \leqslant \leqslant a + \varepsilon$, that $f(x) \leqslant M + \frac{1}{2}(M_1 - M)$, which means that the upper bound for $a \leqslant x \leqslant a + \varepsilon$ would be less than M_1. This being contradictory, we infer $M_1 \leqslant M$. The theorem now follows by induction.

A kernel $K(x,t)$ will be called *stabilizing*, if every regular solution is convergent. In sections 2 and 3 we shall give sufficient conditions for a kernel to be stabilizing.

In section 4 we give two examples; of which the first one is the case where $K(x,t)$ does not depend on x.

We here refer to the possibility of extending theorem 3 (section 2) to the case of an equation involving a Stieltjes integral, viz.

$$f(x) = \int_0^1 f(x-t)\,dW_x(t).$$

Here, for any $x \geqslant 1$, $W_x(t)$ has to be increasing, and $W_x(0) = 0$, $W_x(1) = 1$, and we content ourselves with the consideration of the behaviour of continuous solutions $f(x)$. Only formal changes in the statement and proof of Theorem 3 are necessary. Again, every continuous solution is regular, provided that for no value of x we have

$$W_x(t) = 0 \qquad (0 \leqslant t < 1), \quad W_x(1) = 1.$$

2. *Sufficient condition for a kernel to be stabilizing.*

In order to show how far we can certainly *not* go, we give two examples of kernels with a regular but non-convergent solution.

A. If we take $f(x) = \frac{x}{x+1} \sin 2\pi x$, we can construct, in very many ways, kernels admitting $f(x)$ as a solution. But if x aproaches an integer,

[1]) If K is uniformly bounded, these problems are relatively simple. See Volterra, Leçons sur les equations intégrales, (Paris 1913).

we cannot avoid concentrating the power of $K(x, t)$, considered as a function of t, near the points $t = 0$ and $t = 1$.

Indeed, the conditions of both theorem 3 and theorem 4 will exclude too heavy concentrations of power at a finite number of points, for too many values of x.

B. In the following example $K(x, t)$ is uniformly bounded, but not stabilizing. Let N be a natural number. Put $f(x) = 0$ if $0 \leqslant Nx - [Nx] < \frac{1}{2}$, $f(x) = 1$ if $\frac{1}{2} \leqslant Nx - [Nx] < 1$. Take $K(x, t) = 2$ if $x \geqslant 1$, $0 \leqslant t \leqslant 1$ and $f(x) = f(x - t)$; $K(x, t) = 0$ otherwise. Clearly K satisfies (1. 2), and $f(x)$ is a regular solution which does not converge.

This example shows that, in the following theorem, the constant γ may not be taken $\geqslant \frac{1}{2}$.

Theorem 3. *Let γ be a positive constant $< \frac{1}{4}$, and let, for $x \geqslant 1$, $\Phi(x)$ be a continuous function satisfying*

$$\Phi(x) \geqslant 0 \quad (x \geqslant 1), \quad \sum_{n=1}^{\infty} \eta_n = \infty \tag{2. 1}$$

where

$$\eta_n = \min_{n \leqslant x \leqslant n+2} \Phi(x).$$

Now a sufficient condition for $K(x, t)$ (satisfying (1. 2)) to be stabilizing, is that

$$\int_E K(x, t)\, dt \geqslant \Phi(x) \tag{2. 2}$$

for any $x \geqslant 1$, and for any measurable subset E of the interval $0 \leqslant t \leqslant 1$ whose measure is $\geqslant \gamma$.

Proof. Let $f(x)$ be a regular solution. For $x \geqslant 1$, we denote by $M(x)$ and $m(x)$, respectively, the effective maximum and minimum of $f(u)$ for $x - 1 \leqslant u \leqslant x$. It immediately follows from the definition of regularity, that $M(x)$ is non-increasing, and $m(x)$ non-decreasing.

The difference

$$\triangle(x) = M(x) - m(x)$$

is also non-increasing. Putting

$$\int_{x-1}^{x} f(u)\, du = \mathfrak{M}(x)$$

we have, for $x > 1$ almost everywhere

$$\left| \frac{d}{dx} \mathfrak{M}(x) \right| \leqslant \triangle(x) \tag{2. 3}$$

Lemma 1. *If n is a positive integer, we can find a number y in the interval $n + 1 \leqslant y \leqslant n + 2$ such that either*

$$\mathfrak{M}(x) \leqslant M(n) - \tfrac{1}{4} \triangle(n) \qquad \text{for all } x \text{ in } y - 1 \leqslant x \leqslant y, \tag{2. 4}$$

or

$$\mathfrak{M}(x) \geqslant m(n) + \tfrac{1}{4} \triangle(n) \qquad \text{for all } x \text{ in } y - 1 \leqslant x \leqslant y. \tag{2. 5}$$

Proof. Assume (2. 4) to be false for all y in $n+1 \leqslant y \leqslant n+2$. Then it is false for $y = n + \frac{3}{2}$, and we can find a number x_0 $(n + \frac{1}{2} \leqslant \leqslant x_0 \leqslant n + \frac{3}{2})$, such that $\mathfrak{M}(x_0) > M(n) - \frac{1}{4}\Delta(n)$.

Now by (2. 3) we have, for $x_0 - \frac{1}{2} \leqslant x \leqslant x_0 + \frac{1}{2}$,

$$\mathfrak{M}(x) > M(n) - \tfrac{1}{4}\Delta(n) - \tfrac{1}{2}\Delta(n) = m(n) + \tfrac{1}{4}\Delta(n).$$

Hence we can take $y = x_0 + \frac{1}{2}$ in (2. 5), which proves the lemma.

Lemma 2. *Let, for a fixed value of* x, $\lambda_1 \geqslant 0$ *be such that, for any subset* E *of* $0 \leqslant t \leqslant 1$ *of measure* $\mu(E) = \frac{M(x) - \mathfrak{M}(x)}{M(x) - m(x)}$ *we have* $\int_E K(x,t)\,dt \geqslant \lambda_1$. *Let* $\lambda_2 \geqslant 0$ *be such that for any subset* E' *of measure* $\mu(E') = \frac{\mathfrak{M}(x) - m(x)}{M(x) - m(x)}$ *we have* $\int_{E'} K(x,t)\,dt \geqslant \lambda_2$. *Then we have*

$$f(x) \leqslant M(x) - \lambda_1 \{M(x) - m(x)\} \tag{2. 6^a}$$

$$f(x) \geqslant m(x) + \lambda_2 \{M(x) - m(x)\}. \tag{2. 6^b}$$

Proof. We only show the truth of (2. 6^a); the other part is analogous. Choose E such that it has the prescribed measure, and such that there is a number p with $K(x,t) \leqslant p$ for $t \in E$, $K(x,t) > p$ for t not on E. Let $\bar{E}$ be the complement of E. Now we have, by (1. 1),

$$f(x) = \int_E K(x,t)\, f(x-t)\,dt + \int_{\bar{E}} K(x,t)\, f(x-t)\,dt.$$

Furthermore

$$\int_{\bar{E}} K(x,t) \{M(x) - f(x-t)\}\,dt \geqslant p \int_{\bar{E}} \{M(x) - f(x-t)\}\,dt,$$

$$\int_E K(x,t) \{m(x) - f(x-t)\}\,dt \geqslant p \int_E \{m(x) - f(x-t)\}\,dt.$$

The sum of the right-hand-sides is zero, due to the choice of $\mu(E)$. Hence we obtain

$$f(x) \leqslant m(x) \int_E K\,dt + M(x) \int_{\bar{E}} K\,dt = M(x) - \{M(x) - m(x)\} \int_E K\,dt,$$

and the assertion follows.

We now conclude the proof of theorem 3. In accordance with lemma 1, first assume that there exists an y $(n+1 \leqslant y \leqslant n+2)$ such that (2. 4) holds. Now apply (2. 6^a) with $\lambda_1 = \eta_n$ for all x in $y - 1 \leqslant x \leqslant y$. We obtain that either for all x in $y - 1 \leqslant x \leqslant y$

$$f(x) \leqslant M(x) - \eta_n \{M(x) - m(x)\} \tag{2. 7}$$

or for some x_1 $(y - 1 \leqslant x_1 \leqslant y)$

$$\frac{M(x_1) - \mathfrak{M}(x_1)}{M(x_1) - m(x_1)} \tag{2. 8}$$

From (2. 7) we infer

$$M(y) \leqslant (1 - \eta_n)\, M(y-1) + \eta_n\, m(y),$$

and so

$$M(n+2) \leqslant (1-\eta_n)\, M(n) + \eta_n\, m(n+2).$$

Adding the non-negative number $(1-\eta_n)\,\{m(n+2)-m(n)\}$ to the right-hand-side, we obtain

$$\triangle(n+2) \leqslant (1-\eta_n)\,\triangle(n). \tag{2.9}$$

Now assume (2. 8) to be true. According to our previous assumption, (2. 4) holds for $x = x_1$; combining this with (2. 8) we obtain

$$M(x_1) - \gamma\,\{M(x_1) - m(x_1)\} \leqslant M(n) - \tfrac{1}{4}\,\triangle(n)$$

and hence, by $n \leqslant x_1 \leqslant n+2$,

$$(1-\gamma)\, M(n+2) + \gamma\, m(x_1) - m(n) \leqslant \tfrac{3}{4}\,\triangle(n). \tag{2.10}$$

And, since $m(n) \leqslant m(x_1) \leqslant m(n+2)$, we have

$$\gamma\, m(x_1) - m(n) = \gamma\,\{m(x_1) - m(n)\} - (1-\gamma)\, m(n) \geqslant -(1-\gamma)\, m(n+2).$$

It now follows from (2. 10) that

$$\triangle(n+2) \leqslant \frac{3}{4(1-\gamma)}\,\triangle(n). \tag{2.11}$$

Summarizing, the assumption that (2. 4) is true for some y ($n+1 \leqslant \leqslant y \leqslant n+2$) leads either to (2. 9) or to (2. 11). The same thing can be proved by assuming (2. 5) to be true for some y ($n+1 \leqslant y \leqslant n+2$). The proof is analogous; we have to use (2. 6^b) instead of (2. 6^a).

Hence, for $n = 1, 2, 3, \ldots$

$$\frac{\triangle(n+2)}{\triangle(n)} \leqslant \operatorname{Max}\left\{\frac{3}{4(1-\gamma)},\ 1-\eta_n\right\}. \tag{2.12}$$

By (2. 1) we have

$$\prod_{k=1}^{\infty} \operatorname{Max}\left\{\frac{3}{4(1-\gamma)},\ 1-\eta_k\right\} = 0.$$

Hence $\triangle_n \to 0$, and so $M(x)$ and $m(x)$ tend to the same limit, as $x \to \infty$. Since $f(x)$ is a regular solution, we have $m(x) \leqslant f(u) \leqslant M(x)$ for $u \geqslant x$ (without exception for a nul-set.) Consequently $\lim\limits_{x\to\infty} f(x)$ exists, and the theorem is proved.

3. *Results obtained by studying iterated kernels.*

If $K(x, t)$ is very small on sets of measure $\geqslant \frac{1}{4}$, theorem 3 does not apply. Nevertheless we often can prove $K(x, t)$ to be stabilizing by use of iterated kernels.

These iterated kernels are not normalised by the condition

$$\int_0^1 K(x,t)\, dt = 1, \qquad K(x,t) = 0\ (t > 1).$$

Therefore we use the letters Q, R, S, T, U for such kernels; the letter K is reserved for normalised kernels.

Putting $Q^{(1)}(x,t) = K(x,t)$ $(0 \leqslant t \leqslant 1)$, we define $Q^{(\nu)}(x,t)$ by

$$Q^{(\nu)}(x,t) = 0 \text{ for } t > \nu \qquad (\nu = 1, 2, 3, \ldots) \text{ and for } t < 0,$$

$$Q^{(\nu+1)}(x,t) = \int_0^\infty Q^{(\nu)}(x,s)\, Q^{(1)}(x-s, t-s)\, ds. \tag{3. 1}$$

Then $Q^{(\nu)}(x,t)$ exists almost everywhere, and

$$Q^{(\nu)}(x,t) \geqslant 0 \quad , \quad \int_0^\nu Q^{(\nu)}(x,t)\, dt = 1.$$

Let $f(x)$ be a regular solution of (1. 1), then we have

$$f(x) = \int_0^\nu Q^{(\nu)}(x,t)\, f(x-t)\, dt \qquad (\nu = 1, 2, 3, \ldots ;\ x \geqslant \nu). \tag{3. 2}$$

Now, if m is a natural number, we take

$$K_m(x,t) = \sum_{\nu=1}^{m} Q^{(\nu)}(mx, mt) \quad , \quad f(mx) = g(x) \tag{3. 3}$$

and we obtain

$$g(x) = \int_0^1 K_m(x,t)\, g(x-t)\, dt \qquad (x \geqslant 1). \tag{3. 4}$$

It is easily seen that $K_m(x,t)$ satisfies (1. 2) and that $g(x)$ is a regular solution of (3. 4). Hence $K(x,t)$ is stabilizing whenever $K_m(x,t)$ is stabilizing.

There are cases where theorem 3 applies to K_m but not to K.

We have also the possibility of incomplete iteration, which is demonstrated in the proof of theorem 4.

Theorem 4. *Let $k(t)$ be a non-negative function, absolutely integrable over $0 \leqslant t \leqslant 1$, and such that $\int_0^1 k(t)\, dt > 0$. Then, if $K(x,t) \geqslant k(t)$ for all $x \geqslant 0$, $0 \leqslant t \leqslant 1$, then $K(x,t)$ is stabilizing.*

Proof. We first assume that $k(t)$ is continuous; the general case can be derived from this one. Then there are numbers a, b, C $(0 \leqslant a < b < 1,\ C > 0)$, such that

$$K(x,t) \geqslant C \qquad (a \leqslant t \leqslant b \ ,\ x \geqslant 1).$$

Let the kernels R, S be defined by

$$R(x,t) = C \qquad (a \leqslant t \leqslant b \ ,\ x \geqslant 1),$$

$$R(x,t) = 0 \qquad (t < a \text{ and } t > b \ ;\ x \geqslant 1).$$

and $S(x,t) = K(x,t) - R(x,t)$. Both R and S are non-negative. We shall write

$$\int_0^\infty R(x,t)\, f(x-t)\, dt = Rf,$$

etc. Now we have, by incomplete iteration,

$$\begin{aligned} f &= Rf + Sf = R(Rf + Sf) + Sf = \\ &= R^3 f + R^2 S f + RSf + Sf = \\ &= R^m f + R^{m-1} S f + R^{m-2} S f + \ldots + Sf = \\ &= R^m f + T_m f. \end{aligned}$$

Here the kernel R^m does not depend on x; it is a continuous function of t for $ma \leqslant t \leqslant mb$ and it is positive for $ma < t < mb$. The kernel $T_m(x,t)$ vanishes for $t > B$ where $B = (m-1)\,b + 1$.

If m is large enough, $m > m_0$ say, the union of the intervals (a, b), $(2a, 2b), \ldots, (ma, mb)$ is a set of measure $> \frac{7}{8} B$. Hence for $m > m_0$, the kernel

$$U = m^{-1}(R + T_1 + R^2 + T_2 + \ldots + R^m + T_m)$$

has the following properties:

$$U(x,t) \geqslant 0 \quad (x \geqslant B \ ,\ t \geqslant 0) \quad , \quad U(x,t) = 0 \quad (x \geqslant B \ ,\ t > B)$$

$$\int_0^B U(x,t)\,dt = 1$$

$$f(x) = \int_0^B U(x,t)\, f(x-t)\,dt \qquad (x \geqslant B)$$

and finally, there is a positive constant c such that

$$\int_E U(x,t)\,dt > c$$

for any subset $E \subset (0, B)$ of measure $\mu(E) \geqslant \frac{1}{5} B$, and for all $x \geqslant B$. Now writing

$$K^*(x,t) = B^{-1}\, U(Bx, Bt) \quad , \quad f(x) = g(Bx),$$

we obtain $g = K^* g$, and theorem 3 can be applied. K^* is stabilizing, and hence K has the same property.

If $k(t)$ is not continuous, we may take it to be bounded (otherwise deal with Max $\{1, k(t)\}$ instead of $k(t)$).

If $Q^{(2)}$ is the first iteration of K, we have

$$Q^{(2)}(x,t) \geqslant \int_0^1 k(s)\, k(t-s)\,ds = k_1(t) \qquad (0 \leqslant t \leqslant 2 \ ,\ x \geqslant 1)$$

The function $k_1(t)$ is continuous. We can now apply the result we just proved; this asserts that the kernel $Q^{(2)}(2x, 2t) = K^{**}(x, t)$ is stabilizing. It follows that K has the same property, and the theorem is proved.

It is not difficult to prove, under the assumptions of theorem 4, that there is a positive constant A, depending on $k(t)$ only, such that for any regular solution we have

$$f(x) - \lim_{t \to \infty} f(t) = O(e^{-Ax}) \qquad (x \geqslant 1). \tag{3.5}$$

In order to get a simple result we did not, in the above theorem, exhaust the full strength of theorem 3. Namely, we only applied it for $\eta_n =$ constant. Furthermore, we also neglected the possibility of allowing m to tend slowly to infinity, as $x \to \infty$. But it is, however, very difficult to embody the results of these possibilities in a small number of theorems.

4. *Examples.*

a. If $K(x,t)$ does not depend on x, theorem 4 can be applied immediately. Furthermore every solution $f(x)$ is continuous for $x \geqslant 1$, and, if $m \leqslant f(x) \leqslant M$ on $0 \leqslant x \leqslant 1$ almost everywhere, we have

$$m \leqslant \lim_{x \downarrow 1} f(x) \leqslant M.$$

It follows that $f(x)$ is regular (cf. the proof of theorem 1).

We state the result obtained after a simple transformation in the following form:

Theorem 5. *Let $k(t)$ be absolutely integrable and non-negative for $0 \leqslant t \leqslant 1$, and assume that λ is the real root of the equation*

$$\int_0^1 k(t)\, e^{-\lambda t}\, dt = 1. \tag{4. 1}$$

Let $f(x)$ be measurable and bounded over every finite range $0 \leqslant x \leqslant a$, and assume that

$$f(x) = \int_0^1 k(t)\, f(x-t)\, dt \qquad (x \geqslant 1).$$

Then $f(x)$ is of the form

$$f(x) = C\, e^{\lambda x} + O\,(e^{(\lambda - A) x}),$$

where A is a positive number depending on k only (cf. (3. 5)).

b. The following functional equation arises in connection with a certain prime number problem [2]). We shall devote a separate paper to it, but here we shall derive the things we can deduce from the present results. The equation is

$$x\, F(x) = \int_0^1 F(x-t)\, dt \qquad (x > 1). \tag{4. 2}$$

We can construct a solution $F_0(x)$ of (4. 2) which is continuous, positive and non-increasing for $x \geqslant 0$. Such a function can be constructed, for instance, by taking

$$F_0(x) = 1 \quad (0 \leqslant x \leqslant 1) \quad ; \quad x\, F_0'(x) = -F_0(x-1) \quad (x \geqslant 1).$$

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[2]) S. D. Chowla and T. Vijayaraghavan, J. Indian Math. Soc. (N.S.) **11**, 31–37 (1947).

V. Ramaswami, Duke Math. J. **16**, 99–109 (1949).

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Let $F(x)$ be an arbitrary solution of (4. 2) which is bounded and measurable for $0 \leqslant x \leqslant a$ (any a). It is easily seen to be continuous for $x \geqslant 1$. Hence $f(x) = F(x)/F_0(x)$ is a regular solution of

$$f(x) = \int_0^1 \frac{F_0(x-t)}{x\,F_0(x)}\, f(x-t)\, dt. \tag{4. 3}$$

It is easily seen that the kernel

$$K(x,t) = \frac{F_0(x-t)}{x\,F_0(x)} \qquad (0 \leqslant t \leqslant 1) \tag{4. 4}$$

satisfies (1. 2). For $0 \leqslant t \leqslant 1$ we have, since F_0 is positive and decreasing, $K(x,t) \geqslant x^{-1}$. Therefore, we apply theorem 3, with $\eta_n = \gamma/(n+2)$. It follows that $f(x)$ tends to a limit as $x \to \infty$. Moreover from (2. 11) we infer

$$f(x) - \lim_{t\to\infty} f(t) = O(x^{-\frac{1}{2}}),$$

although the O-term is by far not the best possible. So any solution of (4. 2) is of the form

$$F(x) = \{C + O(x^{-\frac{1}{2}})\}\, F_0(x).$$

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MATHEMATICS

NEW RESULTS IN THE THEORY OF C-UNIFORM DISTRIBUTION

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§ 1. *Introduction.*

In a recent paper [1] B. MEULENBELD developed the theory of the C-uniform distribution (mod 1) of the values of a function of n variables. For the definition of this kind of distribution we refer to the paper mentioned above. In this note the author also formulated a useful test to establish the behaviour of a system of m functions of n variables with regard to the C-uniform distribution (mod 1). We repeat here the C-test in the special form ($m = 1$) in which it will be used in the present paper.

C-test.

Let n be a positive integer and let F be a sequence of n-dimensional intervals:

$$(1) \qquad Q: \quad 0 \leqslant t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where T_n and the measure of Q tend to infinity if Q runs through F.

Let $f(t) = f(t_1, \ldots, t_n)$ be a function, defined for all $(t) = (t_1, \ldots, t_n)$ of all Q.

Then it is necessary and sufficient for the C-uniform distribution (mod 1) of the function $f(t)$ in the intervals (1), that, for every integer $h \neq 0$, $f(t)$ satisfies the relation:

$$\lim \frac{1}{T_1 T_2 \ldots T_n} \int_0^{T_1} \int_0^{T_2} \ldots \int_0^{T_n} e^{2\pi i h f(t_1, t_2, \ldots, t_n)}\, dt_1\, dt_2 \ldots dt_n = 0,$$

if Q runs through F.

In the present paper we shall prove the following Theorems.

Theorem I.

Let F be a sequence of n-dimensional intervals

$$Q: \quad 0 \leqslant t_k < T_k \qquad (k = 1, 2, \ldots, n),$$

where $T_k (k = 1, 2, \ldots, n) \to \infty$, if Q runs through F.

Let $f(t) = f(t_1, \ldots, t_n)$ be a function defined for all (t) of all Q.

Let $f(t)$ have first partial derivatives with respect to each t_k, with the property:

$$(2) \qquad \left| t_k \frac{\partial f(t_1, \ldots, t_n)}{\partial t_k} \right| < M_k \text{ for } t_k \geqq \bar{t}_k \geqq 0 \ (\bar{t}_k \text{ fixed}),$$

uniformly in $(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)$, *where the* M_k *are fixed positive numbers* $(k = 1, \ldots, n)$.

Then $f(t)$ *is not C-uniformly distributed* (*mod* 1) *in the intervals* Q *of* F.

For $n = 1$ we get the following

Theorem II.

Let F *be a sequence of intervals*

$$Q: \quad 0 \leqslant t < T, \text{ with } T \to \infty.$$

Let $f(t)$ *be a differentiable function, with the property*:

$$|t\, f'(t)| < K \text{ for } t \geqslant t_0 \geqslant 0,$$

where t_0 *and* K *are fixed numbers.*

Then $f(t)$ *is not C-uniformly distributed* (*mod* 1) *in the intervals* Q *of* F.

As an immediate consequence of Theorem II we have:

If $f(t)$ $(t \geqslant 0)$ *is C-uniformly distributed* (*mod* 1), *then* $t\, f'(t)$ *cannot be bounded.*

This Theorem II is a considerable improvement of a theorem we proved in [2] (Theorem II).

The additional restrictive condition we made on $t\, f'(t)$ in the note just mentioned can be omitted.

In order to prove Theorem I we again apply the C-test, but the argumentation is quite different from that used in our previous papers. In the present note we make use of the following

Lemma.

If $\varphi(u)$ $(u \geqslant 0)$ *is a function with first and second derivative, if*

$$\lim_{u \to \infty} \frac{\varphi(u)}{u} = a \text{ (constant)},$$

and

$$u\, \varphi''(u) \textit{ is bounded for } u \geqslant u_0 \textit{ (fixed)} \geqslant 0,$$

then

$$\lim_{u \to \infty} \varphi'(u) = a.$$

For the proof of this lemma we refer to [3].

Theorem III.

Let F *be a sequence of n-dimensional intervals*

$$Q: \quad 0 \leqslant t_k < T_k \qquad (k = 1, \ldots, n),$$

where T_n *and the measure of* Q *tend to infinity if* Q *runs through* F.

Let $f(t) = f(t_1, \ldots, t_n)$ *be a measurable function defined for all* (t) *of all* Q.

Let $f(t)$ *have a partial derivative with respect to* t_n *with the property*:

$$\lim_{t_n \to \infty} t_n^p \frac{\partial f(t_1, \ldots, t_n)}{\partial t_n} = c \neq 0, \tag{3}$$

uniformly in $(t_1, \ldots, t_{n-1})$, *where* c *and* p *are fixed numbers, and* $0 \leqslant p < 1$. *Then* $f(t)$ *is C-uniformly distributed* (*mod* 1) *in the intervals* Q *of* F.

For $n = 1$ we get

Theorem IV.

Let F *be a sequence of intervals*

$$Q: \quad 0 \leqslant t < T, \quad \text{with} \quad T \to \infty.$$

Let $f(t)$ $(t \geqslant 0)$ *be a differentiable function with the property*

$$\lim_{t \to \infty} t^p f'(t) = c \neq 0, \quad 0 \leqslant p < 1.$$

Then $f(t)$ *is C-uniformly distributed* (*mod* 1) *in the intervals* Q *of* F.

This Theorem is a generalisation of Theorem III of [2], where we assumed $p = 0$. N. H. Kuiper reported us by letter that he also possesses a proof of Theorem IV.

In § 2 we prove Theorem I, in § 3 Theorem III, while in § 4 we give some examples.

We remark that Theorem IV does not hold if we take $p = 1$.

In this case $f(t)$ is not C-uniformly distributed (mod 1) as follows from Theorem II.

§ 2. *Proof of Theorem I.*

We shall show that the expression

$$I = \frac{1}{T_1 \ldots T_n} \int_0^{T_1} \cdots \int_0^{T_n} e^{2\pi i h f(t_1, \ldots, t_n)}\, dt_1 \ldots dt_n$$

does not tend to zero, if Q runs through F.

Let us suppose that

$$I^* = \frac{1}{T_1 \ldots T_n} \int_0^{T_1} \cdots \int_0^{T_n} \cos 2\pi h f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n$$

tends to zero if Q runs through F. Then we should also have:

$$\lim_{T_n \to \infty} \cdots \lim_{T_1 \to \infty} \frac{1}{T_1 \ldots T_n} \int_0^{T_1} \cdots \int_0^{T_n} \cos 2\pi h f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n =$$

$$\lim_{T_n \to \infty} \cdots \lim_{T_2 \to \infty} \left[\lim_{T_1 \to \infty} \frac{1}{T_1} \int_0^{T_1} \left\{ \frac{1}{T_2 \ldots T_n} \int_0^{T_2} \cdots \int_0^{T_n} \cos 2\pi h f(t_1, \ldots, t_n)\, dt_2 \ldots dt_n \right\} dt_1 \right] =$$

$$\lim_{T_n \to \infty} \cdots \lim_{T_2 \to \infty} \left[\lim_{T_1 \to \infty} \frac{1}{T_2 \ldots T_n} \int_0^{T_2} \cdots \int_0^{T_n} \cos 2\pi h f(T_1, t_2, \ldots, t_n)\, dt_2 \ldots dt_n \right] =$$

(as follows from the lemma and from (2) with $k = 1$)

$$\lim_{T_1\to\infty} \lim_{T_n\to\infty} \dots \lim_{T_2\to\infty} \frac{1}{T_2}\int\limits_0^{T_2} \left\{\frac{1}{T_3\dots T_n}\int\limits_0^{T_3}\dots\int\limits_0^{T_n} \cos 2\pi h f(T_1, t_2, \dots, t_n)\, dt_3\dots dt_n\right\} dt_2 =$$

$$\lim_{T_1\to\infty} \lim_{T_n\to\infty} \dots \lim_{T_2\to\infty} \frac{1}{T_3\dots T_n}\int\limits_0^{T_3}\dots\int\limits_0^{T_n} \cos 2\pi h f(T_1, T_2, t_3, \dots, t_n)\, dt_3\dots dt_n = 0$$

(as follows again from the lemma and from (2) with $k = 2$).

Repeating this argument we should finally have:

$$\lim_{T_1\to\infty} \dots \lim_{T_n\to\infty} \cos 2\,\pi\, h\, f\,(T_1, \dots, T_n) = 0. \tag{4}$$

If we furthermore suppose that

$$I^{**} = \frac{1}{T_1\dots T_n}\int\limits_0^{T_1}\dots\int\limits_0^{T_n} \sin 2\,\pi\, h\, f\,(t_1, \dots, t_n)\, dt_1 \dots dt_n$$

tends to zero if Q runs through F, then we should find in a similar way:

$$\lim_{T_1\to\infty} \dots \lim_{T_n\to\infty} \sin 2\,\pi\, h\, f\,(T_1, \dots, T_n) = 0. \tag{5}$$

Both relations (4) and (5) however cannot be satisfied simultaneously. So our assumption is false, and we see that I does not tend to zero if Q runs through F.

§ 3. *Proof of Theorem III.*

Put $P = T_1 T_2 \dots T_n$. Without loss of generality we assume $c > 0$. From (3) it follows, that, for an arbitrary small $\varepsilon > 0$ and $t_n > T_n^* = T_n^*(\varepsilon)$, we have

$$\left| \frac{1}{t_n^p \dfrac{\partial f(t_1, \dots, t_n)}{\partial t_n}} - \frac{1}{c} \right| < \varepsilon, \tag{6}$$

uniformly in $(t_1, \dots, t_{n-1})$.

For $T_n > T_n^*$ the expression

$$I = \frac{1}{P}\int\limits_0^{T_1}\dots\int\limits_0^{T_n} e^{2\pi i h f(t_1,\dots,t_n)}\, dt_1 \dots dt_n$$

can be written as

$$I = \frac{1}{P}\int\limits_0^{T_1}\dots\int\limits_0^{T_n^*} + \frac{1}{P}\int\limits_0^{T_1}\dots\int\limits_{T_n^*}^{T_n}. \tag{7}$$

It is easily seen that the first term on the right of (7) tends to zero if Q

runs through F. The second term on the right of (7) equals

$$(8)\quad \begin{cases} \dfrac{1}{P}\displaystyle\int_0^{T_1} \cdots \int_0^{T_{n-1}} \left[\int_{f(T_n^*)}^{f(T_n)} \frac{e^{2\pi i h u}\, du}{\dfrac{\partial f(t_1,\ldots,t_n)}{\partial t_n}}\right] dt_1 \ldots dt_n = \\[2ex] \dfrac{1}{P}\displaystyle\int_0^{T_1} \cdots \int_0^{T_{n-1}} \left[\int_{f(T_n^*)}^{f(T_n)} t_n^p\, e^{2\pi i h u} \left(\frac{1}{t_n^p \dfrac{\partial f(t_1,\ldots,t_n)}{\partial t_n}} - \frac{1}{c}\right) du\right] dt_1 \ldots dt_{n-1} + \\[2ex] \dfrac{1}{cP}\displaystyle\int_0^{T_1} \cdots \int_0^{T_{n-1}} \left[\int_{f(T_n^*)}^{f(T_n)} \{F(u)\}^p\, e^{2\pi i h u}\, du\right] dt_1 \ldots dt_{n-1}, \end{cases}$$

where we put, for the sake of brevity,

$$f(T_n) = f(t_1, \ldots, t_{n-1}, T_n)$$
$$f(T_n^*) = f(t_1, \ldots, t_{n-1}, T_n^*),$$

and where $t_n = F(u, t_1, \ldots, t_{n-1})$ is the inverse function of $u = f(t_1, \ldots, t_n)$. This inverse function exists on account of our assumption $c > 0$.

The first term on the right of (8) is in absolute value less than

$$(9)\qquad \varepsilon \left| \frac{f(T_n) - f(T_n^*)}{T_n} \right| T_n^p < \varepsilon\, \frac{|f(T_n)| + |f(T_n^*)|}{T_n^{1-p}}.$$

From

$$\lim_{T_n \to \infty} \frac{|f(T_n)|}{T_n^{1-p}} = \lim_{T_n \to \infty} \frac{T_n^p\, |f'(T_n)|}{1-p} = \frac{c}{1-p},$$

(9), and the assumption $0 \leqslant p < 1$, it follows, that the first term on the right of (8) tends to zero if Q runs through F.

Furthermore we have:

$$\int_{f(T_n^*)}^{f(T_n)} \{F(u)\}^p\, e^{2\pi i h u}\, du = \left[\frac{\{F(u)\}^p\, e^{2\pi i h u}}{2\pi i h}\right]_{u=f(T_n^*)}^{u=f(T_n)} -$$

$$-\frac{p}{2\pi i h}\int_{f(T_n^*)}^{f(T_n)} e^{2\pi i h u}\, \{F(u)\}^{p-1}\, F'(u)\, du = \frac{T_n^p\, e^{2\pi i h f(T_n)}}{2\pi i h} - \frac{T_n^{*p}\, e^{2\pi i h f(T_n^*)}}{2\pi i h} -$$

$$-\frac{p}{2\pi i h}\int_{T_n^*}^{T_n} e^{2\pi i h f(t)}\, t_n^{p-1}\, dt_n = K_1 + K_2 + K_3, \text{ say.}$$

Now we have, replacing the form [ ] in the second term on the right

of (8) by $K_1 + K_2 + K_3$, successively

$$\left| \frac{1}{cP} \int_0^{T_1} \dots \int_0^{T_{n-1}} K_1 \, dt_1 \dots dt_{n-1} \right| < \frac{1}{2\pi |h| c T_n^{1-p}},$$

$$\left| \frac{1}{cP} \int_0^{T_1} \dots \int_0^{T_{n-1}} K_2 \, dt_1 \dots dt_{n-1} \right| < \frac{T_n^{*p}}{2\pi |h| c T_n},$$

$$\left| \frac{1}{cP} \int_0^{T_1} \dots \int_0^{T_{n-1}} K_3 \, dt_1 \dots dt_{n-1} \right| < \frac{T_n^p - T_n^{*p}}{2\pi |h| c T_n};$$

and from these inequalities it follows that also the second term on the right of (8) tends to zero if Q runs through F.

This completes the proof.

§ 4. *Examples.*

a) The functions

$$f(t) = \lg (1 + t_1 + \dots + t_n),$$

$$f(t) = \sum_{k=1}^{n} \lg (1 + t_k),$$

$$f(t) = \lg (a_0 + \sum_{k=1}^{n} a_k t_k^{\beta_k}) \text{ with } a_0 > 0 \text{ and } a_k, \beta_k > 0 \quad (k = 1, \dots, n),$$

are not C-uniformly distributed (mod 1) in the intervals

(10) $$0 \leqslant t_k < T_k, T_k \to \infty \qquad (k = 1, \dots, n),$$

as follows from Theorem I.

b) The function

$$f(t) = \sum_{k=1}^{n} t_k^{1-p} + \sum_{k=1}^{n} \frac{\sin t_k}{t_k} + \psi (t_1, \dots, t_{n-1}),$$

where ψ is an arbitrary real measurable function, and where $0 < p < 1$, satisfies

$$\lim_{t_n \to \infty} t_n^p \frac{\partial f(t_1, \dots, t_n)}{\partial t_n} = 1 - p > 0, \text{ uniformly in } (t_1, \dots, t_{n-1}),$$

so that $f(t)$ is C-uniformly distributed (mod 1) in the intervals (10), as follows from Theorem III.

Bandung, 1 April 1950. *University of Indonesia.*

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MATHEMATICS

ON PARTIALLY ORDERED GROUPS

BY

LADISLAS FUCHS

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1. The present paper contains a discussion of order-homomorphisms, including certain further strongly related concepts and theorems on partially ordered groups, some of which are of interest in themselves too.

After collecting certain known definitions and facts needed throughout the paper, we define order-homomorphisms as group-homomorphisms preserving order. The kernels and inducers of order-homomorphisms are proved to be the convex invariants subgroups. A new and fundamental notion is that of tautomorphism which is a group-isomorphism preserving order in *one* direction. The necessity of introducing this concept may be seen in view of Theorem 2. But there is a basic difficulty connected with the notion of tautomorphism, since it is not symmetric and therefore the theorems corresponding to the isomorphism-theorems are not enough deep for proving the JORDAN-HÖLDER theorem for partially ordered groups.

For the proofs we remark that the theorems on order-homomorphisms include two essentially different propositions: one concerning group-homomorphism and one concerning order-preserving. Since the pure group-theoretic parts of the theorems are familiar facts [1]), we may and shall omit them and shall consistently confine ourselves to discussing the order-preserving.

A new concept analogous to the KANTOROVITCH-RIESZ-BIRKHOFF l-ideals [2]) is introduced, it is defined in terms of a generalized "absolute" introduced in my paper [5]. It is shown that these ideals are convex invariant subgroups with the MOORE-SMITH property. We finally give an interesting theorem on ideals in abelian, normally ordered groups.

2. We begin by recalling a few definitions and facts on which the sequel depends.

A group G is said to be *partially ordered* (p.o.) if an order relation $\geq$ is defined for some pairs of elements in G satisfying [3]) (i) *reflexiveness*:

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[1]) Cf. e.g. ZASSENHAUS [9]. Numbers in brackets refer to the bibliography given at the end of the paper.

[2]) KANTOROVITCH [7], RIESZ [8], BIRKHOFF [1].

[3]) See e.g. EVERETT and ULAM [4].

$x \geqq x$ for every x in G, (ii) *antisymmetry*: $x \geqq y$ and $y \geqq x$ imply $x = y$, (iii) *transitivity*: $x \geqq y$ and $y \geqq z$ imply $x \geqq z$, (iv) *the* MOORE-SMITH *property*: for every pair x, y there exists a z with $z \geqq x$, $z \geqq y$, finally, (v) *homogeneity*: $x \geqq y$ implies $u + x + v \geqq u + y + v$ for every u, v in G. (The group-operation is written as addition, but we do not require commutativity.) We shall say x and y *incomparable*, $x \| y$, if neither $x \geqq y$, nor $y \geqq x$.

The elements t satisfying $t \geqq 0$ (where 0 is the group identity) are called *positive*. Homogeneity implies that $x \geqq y$ if and only if $x - y$ (or $-y + x$) is positive; therefore the definition of a partially ordered group may also be given in terms of "positiveness" in G, for example, the MS-property is equivalent to requiring that each element be the difference of two suitable positive elements (CLIFFORD [3]).

We call G *simply* or *linearly ordered* if $x \| y$ is impossible. If a p.o. group G is at the same time a lattice under the same order relation $\geqq$, we say G is a *lattice-ordered group*. By an *archimedean-ordered group* we mean a p.o. group such that $nx < y$, $n = 0, \pm 1, \pm 2, \ldots$ implies $x = 0$.

The partial order defined in G is said to be *normal* (FUCHS [5], [6]) if $nx \geqq 0$ for some positive integer n implies $x \geqq 0$. It is immediate that linear order is normal and normality implies that in the group every element except 0 has an infinite order.

If two partial orders P and R are defined on the same group and if $x \geqq y$ in P always implies $x \geqq y$ in R, it is natural to say that R is *an extension of* P. It is convenient to consider the same group G with different partial orders P and R as distinct p.o. groups, $G(P)$ and $G(R)$. If R is an extension of P, we shall say $G(R)$ an *order-extension* (briefly o-extension) of $G(P)$.

3. Let G and H be two p.o. groups. A single-valued mapping η of G onto H is called an *order-homomorphism* (o-homomorphism) if it is a group-homomorphism which preserves order, i.e., $\eta(x + y) = \eta(x) + \eta(y)$, and $x \geqq y$ implies $\eta(x) \geqq \eta(y)$. A group-isomorphism Θ meeting the requirement "$x \geqq y$ implies $\Theta(x) \geqq \Theta(y)$ and vice-versa" will be called an *o-isomorphism*. The definitions of o-automorphisms as well as of o-endomorphisms are now obvious.

The existence of order-preserving transformations is a consequence of the simple fact that the inner automorphisms preserve order in both directions. In fact, from homogeneity we conclude that $x \geqq y$ if and only if $-a + x + a \geqq -a + y + a$.

It may happen that G and H are isomorphic groups in the pure group-theoretical sense, and G is o-homomorphic to H, but they fail to be o-isomorphic. In this case we say *G tautomorphic to H*. It is evident that each group is tautomorphic to any of its o-extensions. Moreover, it may readily be verified that G is tautomorphic to H if, and only if, G has an o-extension o-isomorphic to H.

Since a linear order has no proper o-extension, it follows that if G is tautomorphic to H and G is linearly ordered, then G is necessarily o-isomorphic to H.

By a *convex subgroup of* G is meant a subgroup C containing with any x, y ($x \geqq y$) all elements between x and y. If a is any o-automorphism of G, then the convexity of C implies that of $a(C)$, and conversely. Hence all conjugate groups of a convex subgroup are convex.

4. Recalling the definition of the *kernel* of a mapping as the set of elements sent into 0, we state the fundamental theorem [4]) on o-homomorphisms:

Theorem 1. *The kernel of an o-homomorphism is a convex invariant subgroup, and every convex invariant subgroup C of G induces an o-homomorphic mapping of G upon the factor-group G/C.*

The convexity of the kernel follows from the definition of o-homomorphisms, according to which together with x, y all elements between x and y are mapped upon 0. (Hence it is clear that an o-homomorphism is a tautomorphism if, and only if, the kernel consists of 0 alone).

For the proof of the converse we define a natural partial order in the factor group G/C by the specification that for two cosets we put $C+x \geqq C+y$ if and only if for some representatives $u \in C+x$, $v \in C+y$ a relation $u \geqq v$ holds. To justify this definition of order, one has to verify the fulfilment of all conditions for partial orders listed in 2. These conditions are partly automatically satisfied, partly may be checked readily, so that there is no need of a detailed demonstration [5]). However, for the proof as well as for later discussions it is useful to remark that the definition of the natural order in G/C may be expressed in the apparently more stricter but clearly equivalent form too: $C+x \geqq C+y$ if and only if to every u in $C+x$ there is a v in $C+y$ such that $u \geqq v$. Whenever we are speaking of G/C as a p.o. group we always mean G/C with the natural order induced by the order of G.

Now the definition of the partial order in G/C at once establishes that the mapping $x \to C+x$ of G onto G/C is an o-homomorphism, indeed.

The natural order in G/C is the worst possible one in the sense of

Theorem 2. *H is an o-homomorphic image of G if, and only if, the factor-group G/C is tautomorphic to H, where C is the kernel of the mapping $G \to H$.*

Since by Theorem 1 G is o-homomorphic to G/C, G is necessarily

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[4]) The first part of this theorem is a special case of a general theorem on order-preserving mappings of partially ordered sets.

[5]) For example, (ii) may be verified as follows. $C+x \geqq C+y$ and $C+y \geqq C+x$ imply $x^* \geqq y \geqq x^{**}$ for some x^*, x^{**} in $C+x$. Hence by homogeneity we conclude $x^* - x^{**} \geqq y - x^{**} \geqq 0$ with $x^* - x^{**} \in C$, so that by convexity we are directly led to $y - x^{**} \in C$, i.e., $C+y = C+x^{**} = C+x$.

o-homomorphic to H, whenever G/C is tautomorphic to H. Conversely, if G is o-homomorphic to H under the mapping $x \to x^*$ ($x \in G$, $x^* \in H$) with the kernel C, then consider the mapping $C + x \to x^*$ of G/C upon H. This is a group-isomorphism preserving order. $C + x \geqq C + y$ implies $u \geqq v$ for some $u \in C + x$, $v \in C + y$, hence we get $u^* \geqq v^*$, which proves the statement, because $u^* = x^*$ and $v^* = y^*$.

5. In this section we are interested in studying the tautomorphism theorems corresponding to the well-known isomorphism theorems in group theory. As was already emphasized in the introduction, we must discuss the propositions only from the point of view of order-preserving.

Theorem 3. (*First tautomorphism theorem.*) *Assume that* $G \to H$ *is an o-homomorphism and* D *is an invariant convex subgroup of* H. *Then the subgroup* C *of* G *consisting of all elements mapped upon* D *is an invariant convex subgroup; moreover,* G/C *is tautomorphic to* H/D.

From Theorem 1 it follows that G is o-homomorphic to H/D; the kernel of this transformation is evidently the totality of elements sent into D, that is C. This fact establishes the invariance and convexity of C and by Theorem 2 one is immediately led to the tautomorphism of G/C to H/D.

Theorem 4. (*Second tautomorphism theorem.*) *If* U, C *are subgroups of the p.o. group* G, *and* C *is invariant convex in* G, *then* $U \cap C$ *is an invariant convex subgroup of* U *and* $U/U \cap C$ *is tautomorphic to* $U + C/C$.

U is obviously o-homomorphic to the factor group $U + C/C$ under the correspondence $x \to C + x$, x in U. Since the elements mapped onto 0 are those of $U \cap C$, Theorems 1 and 2 prove all assertions of our theorem.

At this stage it is natural to try to prove the analogue of Schreier's theorem expressing the fact that any two normal series of a group have equivalent refinements. But in the present case there are insurmountable difficulties in the proof, caused by the asymmetric character of tautomorphism and by the fact that $C + D$ need not be invariant convex if so are C and D [6]). So that our result would be extremely weak, practically expressing nothing at all, therefore we shall omit it.

6. It is an elementary fact that any finite group without proper subgroups is a cyclic group of prime order, hence is isomorphic to the additive group of integers modulo a prime and is commutative. We may

[6]) This may be illustrated by the example of the additive group of all real two-dimensional vectors; we put $(a, b) \geqq (c, d)$ if and only if $\geqq$ componentwise. Let a, b, c, d be positive numbers such that $a/c > b/d$ and choose a rational number p/q with $a/c > p/q > b/d$. The elements of the type $(-ka, kb)$ as well as those of the type $(-kc, kd)$ with integer k constitute discrete convex subgroups whose union-group is clearly not convex, since $(-qa, qb) < (-pc, pd)$ contradicts discreteness.

state something similar on certain p.o. groups S without proper convex subgroups:

Theorem 5. *Let S be a group with a normal partial order having no proper convex subgroups. Then S is isomorphic to a subgroup of the additive group of real numbers ordered by magnitude, hence is commutative.*

At first, S is *linearly ordered*. For, assuming the contrary we can find an element $x \| 0$ and normality implies $nx \| 0$ for $n = \pm 1, \pm 2, \ldots$. It is readily seen that the cyclic subgroup generated by x is convex, hence is identical to S. This case is impossible, since S now fails to possess the MS-property.

Further, S is *archimedean*. Indeed, $nx < y$, $n = 0, \pm 1, \pm 2, \ldots$ implies that y does not belong to the least convex subgroup containing x, consequently, $x = 0$.

By a theorem due to H. Cartan [7]) our theorem is completely proved.

7. In lattice-ordered groups the homomorphisms (with respect both to group- and lattice-operations) can be described by the so-called *l-ideals* defined as invariant subgroups containing with any a also all x such that $|x| \leqq |a|$ [8]). Evidently, this definition has no meaning in p.o. groups that are not lattice-ordered. But if we appropriately define the concept of "absolute" $\| x \|$, viz. as the set of all upper bounds for x and $-x$ [9]), we may define the corresponding concept: an invariant subgroup I of G will said to be an *ideal*, if 1) it contains with any a also all x with [10]) $\| x \| \supset \| a \|$, and 2) it satisfies the MS-property [11]).

Ideals are expected to be closely connected with convex subgroups. Actually:

Theorem 6. *A set I is an ideal if, and only if, it is an invariant convex subgroup with the MS-property.*

Before entering into the proof we observe that for any subgroup with the MS-property convexity is equivalent to the property of containing with a also all elements between a and $-a$. For, to any x, y (say $x \geqq y$) in the subgroup C we can find by the MS-property an a in C such that $a \geqq x$ and $a \geqq -y$, and if C contains each element between a and $-a$, then it a fortiori contains each element between x and y, being $a \geqq x \geqq y \geqq -a$.

Now let x lie between a and $-a$, $a \geqq x \geqq -a$. Then $\|x\|$ contains a, by definition, and hence contains the set of elements $\geqq a$, which is now

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7) Cartan [2]: "*A linearly ordered archimedean group is isomorphic to a subgroup of the additive group of all real numbers*".

8) By the *absolute* $|x|$ is meant the join of x and $-x$, $x \cup -x$. Cf. e.g. Birkhoff [1].

9) For its definition and its main properties see my paper [5].

10) $\supset$ is the sign of inclusion.

11) Given $x, y \in I$, some $z \in I$ satisfies $z \geqq x$ and $z \geqq y$.

plainly equal to $\| a \|$; that is, $\| x \| \supset \| a \|$ and therefore all ideals are convex.

Conversely, let a belong to a convex invariant subgroup C with the MS-property. Assume $\| x \| \supset \| a \|$ and take a c in C such that $c \geqq a$, $c \geqq -a$. Then $\| a \|$ and hence $\| x \|$ contains c, consequently, x lies between c and $-c$. By convexity we conclude that C is actually an ideal.

8. Let G be a *commutative* group with a normal partial order P and I an ideal of $G(P)$. If we extend P to a *linear* order L of G and at the same time adjoin to I all x satisfying a relation $a \geqq x \geqq -a$ in L $(a \in I)$, then I becomes an ideal $I(L)$ of $G(L)$. We shall call $I(L)$ briefly a *linear extension of* $I(P)$.

Theorem 7. *Any ideal I in an abelian group with a normal partial order P is the intersection of its linear extensions $I(L)$.*

The proof of this theorem is based on the following lemmas.

Lemma 1. *If P is a normal partial order on a commutative group G and x, y are any two elements incomparable in P, then there exists a normal extension R_{xy} of P such that $x \geqq y$ in R_{xy}.*

Indeed, putting $a \geqq b$ in R_{xy} if and only if $p(a-b) \geqq q(x-y)$ in P for some non-negative integers p, q, not both zero, we get a normal partial order satisfying the conditions [12]).

Lemma 2. *If x does not belong to the ideal $I(P)$, then P has an extension R_x such that either $a \geqq x$ in R_x for all $a \in I$, or dually, $x \geqq a$ in R_x for all $a \in I$.*

x not in I and the convexity of I imply the impossibility of one of the relations: $x \geqq a$ in P for some $a \in I$, and $b \geqq x$ in P for some $b \in I$. Assuming the second relation impossible, either $x \geqq a$ in P is true for all $a \in I$ (when the proof is finished), or there is a $c \in I$ with $x \| c$. In the latter case, using Lemma 1, let us extend P to P_c, such that $x \geqq c$ in P_c. In the new order P_c a relation like $b \geqq x$ for some $b \in I$ is again impossible, since by the definition of P_c this would mean $p(b-x) \geqq q(x-c)$ in P, that is, $pb + qc \geqq (p+q)x$ in P for some non-negative integers p, q, not both zero. Choosing a $d \in I$ such that $d \geqq b$, $d \geqq c$ in P, we conclude that $(p+q)d \geqq (p+q)x$ in P, whence by normality $d \geqq x$ in P, against the assumptions on x. By a transfinite continuation of this process of extending the order, Zorn's maximal principle [10] establishes Lemma 2.

Resuming the proof of Theorem 7, it is clear that we need only to prove that the intersection of the $I(L)$ contains no element not in I. Given x not in I, using the maximal principle again, Lemmas 1 and 2 guarantee the existence of a linear extension L_0 of P such that either $a \geqq x$ in L_0 for all $a \in I$ or dually. In either case, a is not in $I(L_0)$, q.e.d.

[12]) For the proof we refer to Fuchs [6].

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MATHEMATICS

PROJECTIVE GEOMETRIZATION OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS, III: PROJECTIVE NORMAL SPACES

BY

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Synopsis. This paper is a continuation of two previous papers with the same title which will be referred to as *PI* and *PII* (Kon. Ned. Akad. v. Wet., Proceedings Vol. **53**, Nos 3 and 4, 1950). In the last section of *PII* we saw that the points $\mathbf{n}$ are not appropriate to define "projective normal spaces". In this paper we use the results of *PI* and *PII* to find a set of ("privileged") points which (together with $\mathbf{x}$) may be used for the definition of projective normal spaces. These spaces are defined in the last section of this paper.

§ 1. *Introductory notions.*

Lemma (1, 1). *The normal points* (*cf.* § 4 *in PI*) *satisfy the condition*

$$\mathbf{n}_{a_r \dots a_1} = \mathbf{n}_{(a_r \dots a_1)}, \qquad r = 2, \dots, N. \tag{1, 1}$$

Proof. The symbols $\left\{ P_{a'_r \dots a'_s}^{\ a_s} P_{a'_{s-1} \dots a'_1}^{a_{s-1} \ \ a_1} \right\}$ and $P_{a'_r \dots a'_s \dots a'_1}^{\ a_s \dots a_1}$ which appear in (1, 7) *PI* are obviously symmetric in their subscripts. If we replace in these symbols the derivatives $P_{a'_q \dots a'_u}^{\ a_u}$ by $\Gamma_{a_u \dots a_q}^{\ b_q}$ (which are symmetric in their subscripts) and the $P_{a'}^{b}$ by δ_a^b we obtain the symbols $\left\{ \Gamma_{a_r \dots a_s}^{\ b_s} \delta_{a_{s-1} \dots a_1}^{b_{s-1} \dots b_1} \right\}$ and $\gamma_{a_r \dots a_s \dots a_1}^{\ b_s \dots b_1}$, symmetric in their subscripts. Consequently the normal points defined by (4, 2) *PI* satisfy (1, 1).

Another lemma to be used later on deals with the tensor $Q_a^b = y^c \Gamma_{ac}^b$ (cf. Theorem (3, 4) in *PI*).

Lemma (1, 2). *If* $N > 2$ *then*

$$y^c K_{(cba)}^{\ d} = y^c K_{cba}^{\ d} - \tfrac{2}{3} D_{(b} Q_{a)}^d \tag{1, 2a}$$

and consequently if

$$Q_a^d = Q\, \delta_a^d, \qquad Q = \text{const.} \tag{1, 3}$$

then

$$y^c K_{(cba)}^{\ d} = y^c K_{cba}^{\ d} \tag{1, 4a}$$

$$y^c D_{(c} \mathbf{n}_{ba)} = y^c D_c \mathbf{n}_{ba} \tag{1, 4b}$$

Proof. We have

$$(1,5)\qquad -y^c\,\partial_c\,\Gamma^d_{ab}=\Gamma^d_{ab},\quad -y^c\,\partial_a\,\Gamma^d_{bc}=\Gamma^a_{ba}-\partial_a Q^d_b.$$

On the other hand if $N>2$, then $K_{cba}^{\;\;\;d}$ may be thought of as defined by (2, 3) *PII* and this equation together with (1, 5) leads at once to (1, 2*a*), from which (1, 4*a*) follows by virtue of (1, 3). Using the equation (2, 2*b*) in *PII*, we see that

$$(1,6)\qquad y^c\,D_c\mathbf{n}_{ba}=y^c\,\mathbf{n}_{cba}+y^c\,K_{cba}^{\;\;\;d}\,\mathbf{x}_d$$

$$(1,7)\qquad y^c\,D_{(c}\mathbf{n}_{ba)}=y^c\,\mathbf{n}_{(cba)}+y^c\,K_{(cba)}^{\;\;\;\;\;d}\,\mathbf{x}_d.$$

The equation (1, 4*b*) follows from (1, 6), (1, 7), (1, 1) and (1, 4*a*).

In the following definition we use the projective tensors K's (cf. § 1 in *PII*):

Definition (1, 1). *A $\mathfrak{P}_m$ will be referred to as symmetric if the following conditions hold:*

I) *The tensor Q^b_a satisfies the relation*

$$(1,8a)\qquad Q^b_a=Q\,\delta^b_a$$

where $Q\neq -1$ is a constant.

II) *Among the tensors K's there is at least one, say $K_{a_{u+1}\dots a_1}^{\qquad\quad b}$, such that*

$$(1,8b)\qquad K\equiv K_{a_{u+1}\dots a_1}^{\qquad\quad a_{u+1}}\,y^{a_u}\dots y^{a_1}\neq 0.$$

III) *If $N>3$ then* [1])

$$(1,9)\qquad \begin{cases} a)\;\; y^{a_r}\,D_{a_r}\,\mathbf{n}_{a_{r-1}\dots a_1}=y^{a_r}\,D_{(a_r}\,\mathbf{n}_{a_{r-1}\dots a_1)}\;{}^{2)}\qquad (r=4,\dots,N,\ \mathbf{x}_a=\mathbf{n}_a)\\ b)\;\; y^{a_p}\,D_{a_p}\,D_{(a_{p-1}}\dots D_{a_{s+1}}\,K_{a_s\dots a_1)}^{\qquad\quad c}=\\ \qquad y^{a_p}\,D_{(a_p}\dots D_{a_{s+1}}\,K_{a_s\dots a_1)}^{\qquad\quad c}\qquad (s=3,4,\dots,N-1,\ p=s+1,\dots,N).\end{cases}$$

Throughout this paper we will deal with symmetric cases only without stating it explicitly.

One of the consequences of (1, 8*a*) and (1, 9*a*) is stated in the following lemma where we put

$$(1,10)\qquad c_r\equiv N+1-(r-1)\,(Q+1),\qquad r=2,\dots,N.$$

Lemma (1, 3). *If $N\geqq 2$, then*

$$(1,11a)\qquad y^b\,\mathbf{n}_{ba}=y^b\,\mathbf{n}_{(ba)}=c_2\,\mathbf{x}_a.$$

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[1]) The condition (1, 9*a*) is satisfied for $r=2$ and if $N>2$ also for $r=3$ (cf. the equation (1, 4*b*)).

[2]) We impose this condition in order to simplify the final results. The device used later on (cf. the equations (1, 12)) may easily be generalized for the case where (1, 9*a*) does not hold.

If $N \geqq 3$ *then*

(1, 11*b*) $$y^c\,\mathbf{n}_{cba} = y^c\,\mathbf{n}_{(cba)} = c_3\,\mathbf{n}_{ba} - y^c\,K_{(cba)}^{\;\;\;\;\;d}\,\mathbf{x}_d.$$

Moreover if $N \geqq 4$ *then*

(1, 11*c*) $$\begin{cases} y^{a_r}\mathbf{n}_{a_r\ldots a_1} = y^{a_r}\mathbf{n}_{(a_r\ldots a_1)} = c_r\,\mathbf{n}_{a_{r-1}\ldots a_1} - \\ -\sum_{2}^{r-2}{}_s\, y^{a_r}\left\{\left\{K_{(a_r\ldots a_s}^{\;\;\;\;\;\;\;\;b_s}\,\delta_{a_{s-1}\ldots a_1)}^{b_{s-1}\;\;\;b_1}\right\}\right\}\mathbf{n}_{b_s\ldots b_1} - y^{a_r}\,K_{(a_r\ldots a_1)}^{\;\;\;\;\;\;\;\;b}\,\mathbf{x}_b, \quad r=4,\ldots,N. \end{cases}$$

Proof: The first equations (1, 11) follow at once from (1, 1) and

$$\mathbf{n}_{ba} = \mathbf{x}_{ba} - \Gamma_{ba}^{\;\;c}\,\mathbf{x}_c = \mathbf{n}_{ba}$$

Moreover we have by virtue of (1, 8*a*)

(1, 12*a*) $$\begin{cases} y^{a_r}D_{a_r}\mathbf{n}_{a_{r-1}\ldots a_1} = y^{a_r}[\partial_{a_r}\mathbf{n}_{a_{r-1}\ldots a_1} - \Gamma_{a_{r-1}a_r}^{\;\;\;\;\;\;\;\;h}\mathbf{n}_{ha_{r-2}\ldots a_1} - \ldots - \Gamma_{a_1a_r}^{\;\;\;\;\;h}\mathbf{n}_{a_{r-1}\ldots a_2h}] \\ \qquad = [N+1-(r-1)]\,\mathbf{n}_{a_{r-1}\ldots a_1} - Q(r-1)\,\mathbf{n}_{a_{r-1}\ldots a_1} = c_r\,\mathbf{n}_{a_{r-1}\ldots a_1}. \\ \qquad\qquad r = 3,\ldots,N \end{cases}$$

On the other hand if we use (2, 2) in *PII*, (1, 9*a*), (1, 1) and (1, 4) we obtain

$$y^{a_r}[\mathbf{n}_{a_r\ldots a_1} - D_{a_r}\mathbf{n}_{a_{r-1}\ldots a_1}] = y^{a_r}[\mathbf{n}_{(a_r\ldots a_1)} - D_{(a_r}\,\mathbf{n}_{a_{r-1}\ldots a_1]} =$$

(1, 12*b*) $$-\sum_{2}^{r-2}{}_s\, y^{a_r}\left\{\left\{K_{(a_r\ldots a_s}^{\;\;\;\;\;\;\;\;b_s}\,\delta_{a_{s-1}\ldots a_1)}^{b_{s-1}\;\;\;b_1}\right\}\right\}\mathbf{n}_{b_s\ldots b_1}{}^{3)} - y^{a_r}\,K_{(a_r\ldots a_1)}^{\;\;\;\;\;\;\;\;b}\,\mathbf{x}_b \quad r=3,\ldots,N.$$

The equations (1, 11*b*, *c*) follow at once from (1, 12*a*, *b*).

Lemma (1, 4). *If for each* $r = 2, \ldots, N$ *and* $q' = 3, \ldots, N$ *we have*

(1, 13) *a*) $c_r \neq 0$, (cf. (1, 10)) *b*) $\mathbf{n}_{a_{q'}\ldots a_1}\,y^{a_1}\ldots y^{a_p} \neq 0$, $(p = 2,\ldots,q')$

then the osculating space $\overset{s}{P}_{m_s}$ [4]) *is spanned by the points* $\mathbf{n}_{a_s\ldots a_1}$ $(s = 1, \ldots, N;$ $\mathbf{x}_a \equiv \mathbf{n}_a)$.

Proof. The equations (4, 2) in *PI* show that $\overset{s}{P}_{m_s}$ may be thought of as spanned by the points

(1, 14) $$\mathbf{x},\ \mathbf{x}_a \equiv \mathbf{n}_a, \ldots, \mathbf{n}_{a_{s-1}\ldots a_1}, \mathbf{n}_{a_s\ldots a_1}.$$

On the other hand $\mathbf{x}$ is a linear combination of $\mathbf{n}_a$, $(N+1)\,\mathbf{x} = y^a\,\mathbf{n}_a$ and if (1, 13) hold then by virtue of (1, 11) $\mathbf{n}_{a_{q-1}\ldots a_1}$ is a linear combination of the points

$$\mathbf{n}_{a_q\ldots a_1}, \mathbf{n}_{a_{q-2}\ldots a_1}, \ldots, \mathbf{n}_a \qquad (q = 2,\ldots,N).$$

Hence the space spanned by the points (1, 14) is identical with the space spanned by $\mathbf{n}_{a_s\ldots a_1}$.

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3) For $r=3$ one has to put $\sum_{2}^{r-2}{}_s \equiv 0$.

4) Cf. § 1 in *PI*.

In the next Lemma we use (1, 8b) and put

$$H_{a_u} \equiv \frac{1}{K} K_{\underline{a_{u+1}\ldots a_1}}{}^{a_{u+1}} y^{a_{u-1}} \ldots y^{a_1]}$$

so that we have

$$(1, 15) \qquad y^a H_a = 1.$$

Lemma (1, 5). *The equation*

$$(1, 16) \qquad H_a^c (\delta_b^a + y^a H_b) = \delta_b^c$$

admits only one solution H_a^c. *If we put*

$$(1, 17) \quad \begin{cases} a) & H_{ab}{}^c \equiv 2H_{(a}H_{b)}^c = H_{(ab)}{}^c \\ b) & H_{a_{r+1}\ldots a_1}{}^c \equiv H_{e(a_{r+1}}{}^c H_{a_r\ldots a_1)}{}^e = H_{(a_{r+1}\ldots a_1)}{}^c \qquad (r = 2,\ldots,N-1) \end{cases}$$

then we have

$$(1, 18) \qquad a) \quad y^a H_{ab}{}^c = \delta_b^c \qquad b) \quad y^{a_{r+1}} H_{a_{r+1}\ldots a_1}{}^c = H_{a_r\ldots a_1}{}^c.$$

Proof. The projective tensor $\delta_b^a + y^a H_b^a$ (homogeneous of degree 0) has obviously the rank $m+1$. Hence (1, 16) admits only one solution H_a^c. On the other hand we obtain from (1, 17a), (1, 17b) for $r = 2$, (1, 15) and (1, 16),

$$(1, 19a) \qquad y^a H_{ab}{}^c = H_b^c + H_b H_a^c y^a = H_a^c (\delta_b^a + y^a H_b) = \delta_b^c$$

$$(1, 19b) \qquad \begin{cases} y^a H_{abc}{}^d = \frac{1}{3} y^a (H_{ea}{}^d H_{bc}{}^e + H_{eb}{}^d H_{ca}{}^e + H_{ec}{}^d H_{ab}{}^e) \\ \qquad = \frac{1}{3} (\delta_e^d H_{bc}{}^e + H_{eb}{}^d \delta_c^e + H_{ec}{}^d \delta_b^e) = H_{bc}{}^d \end{cases}$$

and these equations prove (1, 18a) as well as (1, 18b) for $r = 2$. The remaining equations (1, 18b) may be obtained by usual induction.

§ 2. *Privileged points.*

Definition (2, 1). *An object* Ω *with the components* $\Omega_{a_q\ldots a_1}^{\ldots b \ldots}$ *will be termed a privileged object if the equation*

$$(2, 1) \qquad y^{a_s} \Omega_{a_q\ldots a_1}^{\ldots b\ldots} = 0 \qquad (s = 1, \ldots, q)$$

holds and is (x, y, g)*-invariant* (*cf. Definition* (1, 2) *in PI*).

Theorem (2, 1a). *Let* $N \geqq 2$. *Then*

$$(2, 2a) \qquad \underline{\mathbf{N}_{ab} = \mathbf{N}_{(ab)} = \mathbf{n}_{ab} - c_{21} H_{ab}{}^c \mathbf{x}_c}$$
$$(c_{21} = c_2)$$

are privileged points,

$$(2, 3a) \qquad y^a \mathbf{N}_{ab} = y^a \mathbf{N}_{ba} = 0$$

homogeneous of degree $N-1$

$$(2, 4a) \qquad \overset{*}{\mathbf{N}}_{ab} = g^{(N-1)} \mathbf{N}_{ab}.$$

Proof. We have from (2, 2a) by virtue of (1, 18a) and (1, 11a)

$$y^a \mathbf{N}_{ab} = y^a \mathbf{N}_{(ab)} = (c_2 - c_{21}) \mathbf{x}_b$$

and consequently if we put $c_{21} = c_2$ we have $(2, 3a)$. This equation is obviously (x, y)-invariant. Because $H_{ab}^{\;\;d}$ is homogeneous of degree -1, $\mathbf{x}_c$ homogeneous of degree N and $\mathbf{n}_{ab}$ homogeneous of degree $N-1$, we obtain $(2, 4a)$. Hence $(2, 3a)$ is also g-invariant.

Theorem (2, 1*b*). *Let* $N \geqq 3$. *Then* (2, 2*a*) *and*

$$(2, 2b) \qquad \mathbf{N}_{cba} = \mathbf{N}_{(cba)} = \mathbf{n}_{cba} - c_{32}\, H_{(bc}^{\;\;\;e}\, \delta_{a)}^{f}\, \mathbf{N}_{ef} - (c_{21}\, c_3\, H_{(cba)}^{\;\;\;\;\;e} - K_{(cba)}^{\;\;\;\;\;e})\, \mathbf{x}_e$$

$$c_{32} = \frac{3\, c_3}{2}$$

are privileged points

$$(2, 3b) \qquad y^{a_s}\, \mathbf{N}_{a_3 a_2 a_1} = 0 \qquad s = 1, 2, 3$$

and the points (2, 2*b*) *are homogeneous of degree* $N-2$

$$(2, 4b) \qquad \dot{\mathbf{N}}_{cba} = g^{(N-2)}\, \mathbf{N}_{cba}.$$

Proof. Let $N \geqq 3$ and consider the equation

$$(2, 5a) \qquad \mathbf{N}_{cba} = \mathbf{n}_{cba} - Q_{cba}^{\;\;\;\;de}\, \mathbf{N}_{de} - Q_{cba}^{\;\;\;\;d}\, \mathbf{x}_d$$

where the Q's are to be found. Using $(1, 11b)$ and $(2, 2a)$ we obtain

$$(2, 5b) \qquad \begin{cases} y^c\, \mathbf{N}_{cba} = c_3\, \mathbf{n}_{ba} - y^c\, K_{(cba)}^{\;\;\;\;\;d}\, \mathbf{x}_d - y^c\, [Q_{cba}^{\;\;\;\;de}\, \mathbf{N}_{de} + Q_{cba}^{\;\;\;\;d}\, \mathbf{x}_d] \\ \qquad = (c_3\, \delta_{(ba)}^{de} - y^c\, Q_{cba}^{\;\;\;\;de})\, \mathbf{N}_{de} + [c_3\, c_{21}\, H_{ba}^{\;\;d} - y^c\, K_{(cba)}^{\;\;\;\;\;d} - y^c\, Q_{cba}^{\;\;\;\;d}]\, \mathbf{x}_d. \end{cases}$$

Because

$$y^c\, H_{cba}^{\;\;\;\;d} = y^c\, H_{(cba)}^{\;\;\;\;\;d} = H_{ba}^{\;\;d}$$

the tensor

$$(2, 6a) \qquad Q_{cba}^{\;\;\;\;d} \equiv c_3\, c_{21}\, H_{cba}^{\;\;\;\;d} - K_{(cba)}^{\;\;\;\;\;d} = Q_{(cba)}^{\;\;\;\;\;d}$$

reduces the last member on the right hand side to zero. On the other hand we have by virtue of $(2, 3a)$

$$3\, y^c\, H_{(cb}^{\;\;\;d}\, \delta_{a)}^{e}\, \mathbf{N}_{de} = y^c\, (H_{cb}^{\;\;d}\, \delta_a^e + H_{ba}^{\;\;d}\, \delta_c^e + H_{ac}^{\;\;d}\, \delta_b^e)\, \mathbf{N}_{de}$$

$$= (\delta_{ba}^{de} + \delta_{ab}^{de})\, \mathbf{N}_{de} = 2\, \delta_{(ba)}^{dc}\, \mathbf{N}_{de}.$$

Hence the tensor

$$(2, 6b) \qquad Q_{cba}^{\;\;\;\;de} \equiv \frac{3\, c_3}{2}\, H_{(cb}^{\;\;\;d}\, \delta_{a)}^{e} = Q_{(cba)}^{\;\;\;\;\;de}$$

reduces the first member on the right hand side of $(2, 5b)$ to zero so that we have $y^c\, \mathbf{N}_{cba} = 0$. This equation together with $\mathbf{N}_{cba} = \mathbf{N}_{(cba)}$ (which we obtain from $(2, 5a)$ and $(2, 6)$) leads to $(2, 3b)$. The remaining part of the theorem is very easily proved.

Note. If $N > 3$ then

$$y^d\, H_{dcba}^{\;\;\;\;\;\;e} = H_{cba}^{\;\;\;\;e}$$

and by virtue of $(1, 8a)$ *and* $(1, 9b)$ (used for the first time) for $s = 3$, $p = 4$

$$(2, 7) \qquad y^d\, D_{(d}\, K_{cba)}^{\;\;\;\;\;e} = y^d\, D_d\, K_{(cba)}^{\;\;\;\;\;e} = -2\, (Q+1)\, K_{(cba)}^{\;\;\;\;\;e}.$$

Hence $Q_{cba}{}^{d}$ as defined by (2, 6a) satisfies the relation

$$Q_{cba}{}^{e} = y^{d}\, P_{dcba}{}^{e} \tag{2, 8a}$$

where

$$P_{dcba}{}^{e} = P_{(dcba)}{}^{e} \equiv c_3\, c_{21}\, H_{dcba}{}^{e} + \frac{1}{2\,(Q+1)}\, D_{(d} K_{cba)}{}^{e}\,. \tag{2, 8b}$$

In the next section we shall generalize this equation in order to be able to generalize the results of Theorems (2, 1).

§ 3. *Auxiliary Lemma.*

In the following lemma we use the abbreviations

$$(3,\,1)\quad \begin{aligned} &a)\quad H_{a_u\ldots a_s\ldots a_1}{}^{b_s\ldots b_1} \equiv H_{(a_u\ldots a_s}{}^{b_s}\, \delta_{a_{s-1}\ldots a_1)}^{b_{s-1}\ldots b_1}\ {}^{5)}\\ &b)\quad k_{a_{u+1}\ldots a_s\ldots a_1}{}^{b_s\ldots b_1} \equiv \left\{\!\left\{ K_{(a_{u+1}\ldots a_s}{}^{b_s}\, \delta_{a_{s-1}\ldots a_1)}^{b_{s-1}\ldots b_1} \right\}\!\right\}\\ &c)\quad k_{a_{u+1}\ldots a_1}{}^{b_1} \equiv K_{(a_{u+1}\ldots a_1)}{}^{b_1}\\ &\qquad (u = 2,\ldots, N;\ \ s = 2,\ldots, u-1)\end{aligned}$$

and

$$(3,\,1d)\qquad \left\{\begin{array}{ll} c_{r\,r-1} \equiv \dfrac{r c_r}{2}\,{}^{6)}, & c'_r \equiv c_r\, k_r,\\ r = 2,\ldots, N & r = 3,\ldots, N\end{array}\right.$$

where $k_3 = 1$ and k_r for $r > 3$ is the number taken from the equation $(r = 4, \ldots N)$

$$\left(k_r\, y^{a_r}\, H_{a_r a_{r-1} a_{r-2}\ldots a_1}{}^{b_{r-2}\ldots b_1} - H_{a_{r-1} a_{r-2}\ldots a_1}{}^{b_{r-2}\ldots b_1}\right) t_{(b_{r-2}\ldots b_1)} = 0$$

which holds for any privileged tensor $t_{(b_{r-2}\ldots b_1)}$ whatsoever. Moreover, if $A_{\ldots}^{\ldots a\ldots}$ and $B_{\ldots}^{\ldots a\ldots}$ are two tensors which satisfy the equation

$$(A_{\ldots}^{\ldots a\ldots} - B_{\ldots}^{\ldots a\ldots})\, t_{\ldots a\ldots} = 0 \tag{3, 1e}$$

for any privileged tensor $t_{\ldots a\ldots}$ whatsoever, then we write

$$A_{\ldots}^{\ldots a\ldots} \simeq B_{\ldots}^{\ldots a\ldots}\,. \tag{3, 1f}$$

Lemma (3, 1). *If $N \geq 4$ and if a set of tensors Q satisfies the following conditions*

$$(3,\,2)\quad \left\{\begin{aligned} &a)\quad Q_{a_u a_{u-1}\ldots a_1}{}^{c_{u-1}\ldots c_1} \equiv c_{u\,u-1}\, H_{a_u a_{u-1}\ldots a_1}{}^{c_{u-1}\ldots c_1} \qquad u = 2,\ldots, N\\ &b)\quad Q_{a_v a_{v-1} a_{v-2}\ldots a_1}{}^{c_{v-2}\ldots c_1} \equiv c_{v-1\,v-2}\, c'_v\, H_{a_v a_{v-1} a_{v-2}\ldots a_1}{}^{c_{v-2}\ldots c_1} - k_{a_v a_{v-1} a_{v-2}\ldots a_1}{}^{c_{v-2}\ldots c_1} \qquad v = 3,\ldots, N\ {}^{6)}\\ &c)\quad y^{a_w}\, Q_{a_w\ldots a_s\ldots a_1}{}^{c_s\ldots c_1} = c_w\, Q_{a_{w-1}\ldots a_s\ldots a_1}{}^{c_s\ldots c_1}\\ &\qquad - y^{a_w}\left[k_{a_w\ldots a_s\ldots a_1}{}^{c_s\ldots c_1} + \sum_{s+1}^{w-2}{}_{q}\, k_{a_w\ldots a_q\ldots a_1}{}^{b_q\ldots b_1}\, Q_{b_q\ldots b_s\ldots b_1}{}^{c_s\ldots c_1}\right]{}^{6)}, \qquad \begin{aligned} &w = 4,\ldots N\\ &s = 1, 2,\ldots, w-3\end{aligned}\end{aligned}\right.$$

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5) $H_{a_u\ldots a_1}{}^{b_1}$ is defined by (1, 17).

6) For c_r cf. (1, 10).

then the equation

$$(3,3)\quad y^{a_w} P_{a_w \dots a_s \dots a_1}^{\;c_s \dots c_1} \cong Q_{a_{w-1} \dots a_s \dots a_1}^{\;c_s \dots c_1} \qquad w = 4, \dots, N;\ s = 1, \dots, w-3$$

admits a solution $P_{a_w \dots a_s \dots a_1}^{\;c_s \dots c_1} = P_{(a_w \dots a_s \dots a_1)}^{\;c_s \dots c_1}$ *homogeneous of degree* $s - w$, *which is a function of the H's, K's as well as of the derivatives up to* $D_{a_w} \dots D_{a_{q+1}} k_{a_q \dots a_1}^{\;c}$ $(q = 3, \dots, w-1)$ [7] *and consequently* $Q_{a_w \dots a_s \dots a_1}^{\;c_s \dots c_1}$ *satisfies the equation*

$$(3,2d)\quad \begin{cases} Q_{a_w \dots a_s \dots a_1}^{\;c_s \dots c_1} \cong c_w P_{a_w \dots a_s \dots a_1}^{\;c_s \dots c_1} - [k_{a_w \dots a_s \dots a_1}^{\;c_s \dots c_1} + \\ + \sum_{s+1}^{w-2} {}_q\, k_{a_w \dots a_q \dots a_1}^{\;b_q \dots b_1} Q_{a_q \dots b_s \dots b_1}^{\;c_s \dots c_1}] \qquad w = 4, \dots, N,\ s = 1, 2, \dots w-3. \end{cases}$$

Proof. We see from Theorems (2, 1) that the conditions $(3, 2a, b)$ for $u = 2, 3$, $v = 3$ are satisfied [7a] while $(3, 2c)$ reduces for $w = 4$ to

$$(3.4a)\qquad y^d Q_{dcba}^{\;e} = c_4 Q_{cba}^{\;e} - y^d [K_{(dcba)}^{\;e} + c_{21} k_{dcba}^{\;ij} H_{ij}^{\;e}]$$

The equation (3, 3) (for $w = 4$) is equivalent to $(2, 8a)$, where P is given by $(2, 8b)$ so that we have

$$(3.5)\qquad Q_{dcba}^{\;e} = c_4 \left[c_3 c_{21} H_{dcba}^{\;e} + \frac{1}{2(Q+1)} D_{(d} K_{cba)}^{\;e} \right] - (K_{(dcba)}^{\;e} + c_{21} k_{dcba}^{\;ij} H_{ij}^{\;e})$$

which proves our lemma for $w = 4$ (and $s = 1 = w - 3$). For the case $w = 5 \leqq N$ we have to consider $Q_{a_5 \dots a_1}^{\;b_s \dots b_1}$, $s = 1, 2$. The tensor $Q_{a_4 \dots a_2 a_1}^{\;b_2 b_1}$ which appears in $(3, 2c)$ for $w = 5$, $s = 2$ is given by $(3, 2b)$ for $v = 4$. Because of (2, 7) and $(1, 9b)$ we have

$$(3.6a)\qquad \begin{cases} y^{a_5} D_{a_5} k_{a_4 a_3 a_2 a_1}^{\;b_2 b_1} = 3 y^{a_5} D_{a_5} K_{(a_4 a_3 a_2}^{\;b_2} \delta_{a_1)}^{b_1} = \\ \cong \frac{15}{4} y^{a_5} D_{(a_5} K_{a_4 a_3 a_2}^{\;b_2} \delta_{a_1)}^{b_1} = \frac{5}{4} y^{a_5} D_{(a_5} k_{a_4 a_3 a_2 a_1)}^{\;b_2 b_1} \end{cases}$$

$$(3.6b)\quad 3 y^{a_5} D_{a_5} K_{(a_4 a_3 a_2}^{\;b_2} \delta_{a_1)}^{b_1} = -\,3 \,.\, 2\,(1+Q)\, K_{(a_4 a_3 a_2}^{\;b_2} \delta_{a_1)}^{b_1} = -\,2\,(1+Q)\, k_{a_4 a_3 a_2 a_1}^{\;b_2 b_1}$$

and

$$(3.6c)\qquad y^{a_5} H_{a_5 a_4 a_3 a_2 a_1}^{\;b_2 b_1} \cong \tfrac{4}{5} H_{a_4 a_3 a_2 a_1}^{\;b_2 b_1}.$$

Consequently

$$(3.7a)\qquad Q_{a_5 a_4 a_3 a_2 a_1}^{\;b_2 b_1} \cong c_5 P_{a_5 a_4 a_3 a_2 a_1}^{\;b_2 b_1} - (k_{a_5 a_4 a_3 a_2 a_1}^{\;b_2 b_1} + k_{a_5 a_4 a_3 a_2 a_1}^{\;b_3 b_2 b_1} Q_{b_3 b_2 b_1}^{\;c_2 c_1})$$

where

$$(3.7b)\quad P_{a_5 a_4 a_3 a_2 a_1}^{\;b_2 b_1} = P_{(a_5 a_4 a_3 a_2 a_1)}^{\;b_2 b_1} = \tfrac{5}{4} \left[c_{32} c_4' H_{a_5 a_4 a_3 a_2 a_1}^{\;b_2 b_1} + \frac{1}{2(1+Q)} D_{(a_5} k_{a_4 a_3 a_2 a_1)}^{\;b_2 b_1} \right]$$

and these two equations prove our lemma for $w = 5 \leqq N$ and $s = 2$. In

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[7] Its construction will be given in the proof.

[7a] cf. the equations (2, 5) and (2, 6).

order to complete the proof for $w = 5$ and $s = 1$, we use the relationships deduced from (1, 8a) and (1, 9b)

$$y^{a_5} D_{(a_5} D_{a_4} K_{a_3 a_2 a_1)}^{\;\;b} = y^{a_5} D_{a_5} D_{(a_4} K_{a_3 a_2 a_1)}^{\;\;b_1} = -3\,(1+Q)\, D_{(a_4} K_{a_3 a_2 a_1)}^{\;\;b_1}$$
$$y^{a_5} D_{(a_5} K_{a_4 a_3 a_2 a_1)}^{\;\;b_1} = y^{a_5} D_{a_5} K_{(a_4 a_3 a_2 a_1)}^{\;\;b} = -3\,(1+Q)\, K_{a_4 a_3 a_2 a_1}^{\;\;b_1}$$

so that we have by virtue of (3, 5) and (3, 6)

$$(3,\ 8a) \qquad Q_{a_4 a_3 a_2 a_1}^{\;\;b_1} = y^{a_5} P_{a_5 \ldots a_1}^{\;\;b_1}$$

where

$$(3,\ 8b) \qquad \left\{ \begin{aligned} P_{a_5 \ldots a_1}^{\;\;c_1} = P_{(a_5 \ldots a_1)}^{\;\;c_1} \cong c_4 \Big[c_3\, c_{21}\, H_{a_5 \ldots a_1}^{\;\;c_1} - \frac{1}{2 \,.\, 3\,(1+Q)^2} D_{(a_5} D_{a_4} K_{a_3 a_2 a_1)}^{\;\;c_1} \Big] \\ + \Big[\frac{1}{3\,(Q+1)} D_{(a_5} K_{a_4 \ldots a_1)}^{\;\;b_1} + \frac{5\, c_{21}}{8\,(1+Q)} D_{(a_5} k_{a_4 a_3 a_2 a_1)}^{\;\;b_2 b_1} H_{c_2 b_1}^{\;\;b_1} \Big] \end{aligned} \right.$$

and consequently

$$(3,\ 8c) \qquad Q_{a_5 \ldots a_1}^{\;\;c_1} \cong c_5\, P_{a_5 \ldots a_1}^{\;\;c_1} - \Big(k_{a_5 \ldots a_1}^{\;\;c_1} + \sum_{2}^{3} {}_{q}\; k_{a_w \ldots a_q \ldots a_1}^{\;\;b_q \ldots b_1} Q_{b_q \ldots b_1}^{\;\;c_1} \Big).$$

The equations (3, 8) prove the lemma for $w = 5 \leqq N$, $s = 1$. Let us now suppose that we already proved the lemma for all $x = 4, 5, \ldots, w' < N$. Then we have in particular

$$(3,\ 9) \quad Q_{a_{w'} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} \cong c_{w'}\, P_{a_{w'} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} - \Big[k_{a_{w'} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} + \sum_{s-1}^{w'-2} {}_{q}\; k_{a_{w'} \ldots a_q \ldots a_1}^{\;\;b_q \ldots b_1} Q_{b_q \ldots b_s \ldots b_1}^{\;\;c_s \ldots c_1} \Big]$$
$$s = 1, \ldots, w' - 3$$

where $P_{a_{w'} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} = P_{(a_{w'} \ldots a_s \ldots a_1)}^{\;\;c_s \ldots c_1}$ is a function of the H's and K's as well as of the derivatives up to $D_{a_{w'}} \ldots D_{a_{q+1}} k_{a_q \ldots a_1}^{\;\;b_1}$, $(q = 3, \ldots, w'-1)$ and

$$(3,\ 10) \qquad \left\{ \begin{aligned} y^{a_{w'+1}} Q_{a_{w'+1} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} = c_{w'+1}\, Q_{a_{w'} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} \\ - y^{a_{w'+1}} \Big[k_{a_{w'+1} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} + \sum_{s+1}^{w'-1} {}_{q}\; k_{a_{w'} \ldots a_q \ldots a_1}^{\;\;b_q \ldots b_1} Q_{b_q \ldots b_s \ldots b_1}^{\;\;c_s \ldots c_1} \Big] \\ s = 1, 2, \ldots, w' - 2. \end{aligned} \right.$$

Using now the conditions (1, 8a) and (1, 9b) we prove by the same argument as before that

$$(3,\ 11) \qquad Q_{a_{w'} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} \cong y^{a_{w'+1}} P_{a_{w'+1} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} \qquad s = 1, \ldots, w' - 3$$

where $P_{a_{w'+1} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} = P_{(a_{w'+1} \ldots a_s \ldots a_1)}^{\;\;c_s \ldots c_1}$ is a function of the H's and k's as well as of the derivatives up to $D_{a_{w'+1}} \ldots D_{a_{q+1}} k_{a_q \ldots a_1}$ $(q = 3, \ldots, w')$. Hence we have from (3, 10) and (3, 11) the equation

$$(3,\ 12) \qquad \left\{ \begin{aligned} Q_{a_{w'+1} \ldots a_s \ldots a_1}^{\;\;b_s \ldots b_1} \cong c_{w'+1}\, P_{a_{w'+1} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} - \\ - \Big(k_{a_{w'+1} \ldots a_s \ldots a_1}^{\;\;c_s \ldots c_1} + \sum_{s+1}^{w'-1} {}_{q}\; k_{a_{w'+1} \ldots a_q \ldots a_1}^{\;\;b_q \ldots b_1} Q_{b_q \ldots b_s \ldots b_1}^{\;\;c_s \ldots c_1} \Big) \end{aligned} \right.$$

for $s = 1, \ldots, w' - 3$. The equation (3, 12) for $s = w' - 2$ may be obtained by a similar argument based on (3, 2*b*) for $v = w'$. The induction based on these results proves our Lemma but for the statement of the homogeneity of the P's. In order to prove this statement we observe first from (3, 2*a*, *b*) that $Q_{a_u a_{u-1} \ldots a_1}^{\;\;\;\;b_{u-1} \ldots b_1}$ resp. $Q_{a_u \ldots a_{u-2} \ldots a_1}^{\;\;\;\;b_{u-2} \ldots b_1}$ is homogeneous of degree -1 resp. -2. Hence from (3, 2*c*) for $s = w - 3$ we see that $Q_{a_w \ldots a_{w-3} \ldots a_1}^{\;\;\;\;b_{w-3} \ldots b_1}$ must be homogeneous of degree -3. Consequently, from the same equation for $w = x - 1$ we obtain by the same argument that $Q_{a_x \ldots a_{x-4} \ldots a_1}^{\;\;\;\;b_{x-4} \ldots b_1}$ is homogeneous of degree -4. Proceeding in the same way we arrive at the conclusion that $Q_{a_w \ldots a_s \ldots a_1}^{\;\;\;\;b_s \ldots b_1}$ is homogeneous of degree $s - w$, $(w = 4, \ldots, N;\ s = 1, \ldots w - 3)$. Hence $P_{a_w \ldots a_s \ldots a_1}^{\;\;\;\;b_s \ldots b_1}$ which satisfies (3, 3) must be by virtue of (3, 2*d*) and (3, 3) homogeneous of degree $s - w$.

§ 4. *Privileged points. Continuation.*

Lemma (3, 1) enables us to prove the following

Theorem (4, 1). *Let* $N \geqq 4$ *and let the tensors* Q *be defined by* (3, 2*a*,*b*,*c*,*d*). *Then the points*

$$(4,\ 1a)\qquad \begin{cases} \mathbf{N}_{a_r \ldots a_1} = \mathbf{N}_{(a_r \ldots a_1)} \equiv \mathbf{n}_{a_r \ldots a_1} - \sum_{1}^{r-1}{}_s\, Q_{a_r \ldots a_s \ldots a_1}^{\;\;\;\;b_s \ldots b_1}\, \mathbf{N}_{b_s \ldots b_1}, \\ \qquad (r = 2, \ldots, N;\ \mathbf{N}_a \equiv \mathbf{x}_a) \end{cases}$$

are privileged points

$$(4,\ 1b)\qquad y^{a_p}\, \mathbf{N}_{a_r \ldots a_1} = 0 \qquad p = 1, \ldots, r$$

homogeneous of degree $N + 1 - r$

$$(4,\ 1c)\qquad \dot{\mathbf{N}}_{a_r \ldots a_1} = g^{(N+1-r)}\, \mathbf{N}_{a_r \ldots a_1}.$$

The proof may be accomplished in four steps:

a) Theorems (2, 1) are particular cases of our theorem for $r = 2$ resp. $r = 3$ [8]). Let us assume that we proved our Theorem for all $x = 4, \ldots, r' - 1, r' \leqq N$.

b) Denote by $Q_{a_{r'} \ldots a_s \ldots a_1}^{\;\;\;\;b_s \ldots b_1}$ a set of unknown tensors and consider the points

$$(4,\ 2)\qquad \mathbf{N}_{a_{r'} \ldots a_1} \equiv \mathbf{n}_{a_{r'} \ldots a_1} - \sum_{1}^{r'-1}{}_s\, Q_{a_{r'} \ldots a_s \ldots a_1}^{\;\;\;\;b_s \ldots b_1}\, \mathbf{N}_{b_s \ldots b_1}.$$

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[8]) For $r = 2$ the equations (3, 2, *b*, *c*, *d*) do not exist. For $r = 3$ the equations (3, 2*c*, *d*) do not exist (cf. the equation (2, 8*a*)).

Using Lemma (1, 3) and the equations (4, 1a) (where instead of r we put $x = 2, 3, \ldots, r'-1$) we obtain from (4, 2)

$$(4, 3a)\quad \left\{\begin{aligned} y^{a_{r'}} \mathrm{N}_{a_{r'}\ldots a_1} \equiv & \Big(c_{r'}\, \delta^{\;b_{r'-1}\ldots b_1}_{(a_{r'-1}\ldots a_1)} - y^{a_{r'}} Q_{a_{r'}a_{r'-1}\ldots a_1)}^{\;\;\;\;\;\;b_{r'-1}\ldots b_1}\Big) \mathrm{N}_{b_{r'-1}\ldots b_1} \\ & + \Big[c_{r'} Q_{a_{r'-1}a_{r'-2}\ldots a_1}^{\;\;\;\;\;\;b_{r'-2}\ldots b_1} - y^{a_{r'}}\Big(k_{a_{r'}\ldots a_{r'-2}\ldots a_1}^{\;\;\;\;\;\;b_{r'-2}\ldots b_1} + Q_{a_{r'}\ldots a_{r'-2}\ldots a_1)}^{\;\;\;\;\;\;b_{r'-2}\ldots b_1}\Big)\Big] \mathrm{N}_{b_{r'-2}\ldots b} \\ & + \sum_{1}^{r'-3}{}_{s} \Big[c_{r'} Q_{a_{r'-1}\ldots a_s\ldots a_1}^{\;\;\;\;\;\;c_s\ldots c_1} - y^{a_{r'}}\Big(Q_{a_{r'}\ldots a_s\ldots a_1}^{\;\;\;\;\;\;c_s\ldots c_1} + k_{a_{r'}\ldots a_s\ldots a_1}^{\;\;\;\;\;\;c_s\ldots c_1} + \\ & + \sum_{s+1}^{r'-2}{}_{q}\, k_{a_{r'}\ldots a_q\ldots a_1}^{\;\;\;\;\;\;b_q\ldots b_1} Q_{b_q\ldots b_s\ldots b_1)}^{\;\;\;\;\;\;c_s\ldots b_1}\Big)\Big] \mathrm{N}_{c_s\ldots c_1}. \end{aligned}\right.$$

On the other hand we have according to (3, 1a), (1, 18b) and *by virtue of* (4, 1b) (for $r'-1$ instead of r)

$$(4, 4a)\quad \left\{\begin{aligned} y^{a_{r'}} H_{a_{r'}a_{r'-1}\ldots a_1}^{\;\;\;\;\;\;b_{r'-1}\ldots b_1} \mathrm{N}_{b_{r'-1}\ldots b_1} &= y^{a_{r'}} H_{(a_{r'}a_{r'-1}}^{\;\;\;\;\;\;b_{r'-1}} \delta_{a_{r'-2}\ldots a_1}^{\;\;b_{r'-2}\ldots b_1} \mathrm{N}_{b_{r'-1}\ldots b_1} \\ &= \frac{2}{r'}\, \delta^{\;b_{r'-1}\ldots b_1}_{(a_{r'-1}\ldots a_1)} \mathrm{N}_{b_{r'-1}\ldots b_1} \end{aligned}\right.$$

and moreover according to (3, 2a) for $u = r'-1$ [9])

$$(4, 4b)\quad \left\{\begin{aligned} c_{r'} Q_{a_{r'-1}a_{r'-2}\ldots a_1}^{\;\;\;\;\;\;b_{r'-2}\ldots b_1} \mathrm{N}_{b_{r'-2}\ldots b_1} &= c_{r'}\, c_{r'-1\, r'-2} H_{a_{r'-1}a_{r'-2}\ldots a_1}^{\;\;\;\;\;\;b_{r'-2}\ldots b_1} \mathrm{N}_{b_{r'-2}\ldots b_1} \\ &= c'_{r'}\, c_{r'-1\, r'-2}\, y^{a_{r'}} H_{a_{r'}\, a_{r'-1}a_{r'-2}\ldots a_1}^{\;\;\;\;\;\;b_{r'-2}\ldots b_1} \mathrm{N}_{b_{r'-2}\ldots b_1}. \end{aligned}\right.$$

Hence if we impose on $Q_{a_{r'}\ldots a_s\ldots a_1}^{\;\;\;\;\;\;b_s\ldots b_1}$ $(s = 1, \ldots, r'-1)$ the conditions (3, 2a,b,c) for $u = v = w = r'$ then we obtain

$$(4, 3b)\qquad y^{a_{r'}} \mathrm{N}_{a_{r'}\ldots a_1} = 0.$$

Moreover from (3, 2a, b, c) for $u = v = w = r'$ we obtain (3, 2d). Hence all tensors $Q_{a_{r'}\ldots a_s\ldots a_1}^{\;\;\;\;\;\;b_s\ldots b_1}$ are symmetric in their subscripts so that we have according to (4, 2)

$$\mathrm{N}_{a_{r'}\ldots a_1} = \mathrm{N}_{(a_{r'}\ldots a_1)}.$$

This equation together with (4, 3b) leads to

$$(4, 3c)\qquad y^{a_p} \mathrm{N}_{a_{r'}\ldots a_1} = 0 \qquad p = 1, \ldots, r'.$$

On the other hand the $Q_{a_{r'}\ldots a_s\ldots a_1}^{\;\;\;\;\;\;b_s\ldots b_1}$ $(s = 1, \ldots, r'-1)$ defined by (3, 2a,b,c,d) are homogeneous of degree $s - r'$. Hence $\mathrm{N}_{a_{r'}\ldots a_1}$ are homogeneous of degree $N + 1 - r'$.

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[9]) According to our assumption in the section a) of the proof, the equation (3, 2a), resp. (3, 2b), resp. (3, 2c, d) exist for $u = 2, \ldots, r'-1$, resp. $v = 3, \ldots, r'-1$, resp. $w = 4, \ldots, r'-1$.

c) Starting with Theorems (2, 1) and applying the same arguments as in b), we easily prove the statements of our theorems for all $x = 4, \ldots$ $\ldots r' - 1$. Hence the assumption of section a) is fulfilled.

d) The usual induction based on the assumption of the section a) and on the results of section b) proves our theorem for $r = 2, \ldots, N$.

§ 5. *Projective normal spaces.*

Definition (5, 1). *The space spanned by the points* $\mathbf{x}$, $\mathbf{N}_{a_r \ldots a_1}$ *will be denoted by* $\overset{r-1}{N}_{n_{r-1}}$ *and referred to as the* $(r-1)$*st projective normal space of our* $\mathfrak{P}_m$, $(r = 2, \ldots, N)$.

Theorem (5, 1). *The normal space* $\overset{r-1}{N}_{n_{r-1}}$ *has the following properties*

a) *It is* (x, y, g)*-invariant.*

b) *It is contained in the osculating space* $\overset{r}{P}_{m_r}$ *and intersects* $\overset{s}{P}_{m_s}$ $(s < r)$ *only in* $\mathbf{x}$.

c) *If*

$$(5, 1) \qquad a)\quad c_2 c_3 \ldots c_r \neq 0, \qquad b)\quad \mathbf{n}_{a_q \ldots a_1} y^{a_1} \ldots y^{a_p} \neq 0,$$
$$q = 3, \ldots, r;\ p = 2, \ldots, r$$

then its dimension is

$$(5, 2) \qquad n_{r-1} = m_r - m_{r-1}.$$

Proof. Because the points $\mathbf{x}$, $\mathbf{N}_{a_r \ldots a_1}$ are (x, y)-invariant, $\overset{r-1}{N}_{n_{r-1}}$ is obviously (x, y)-invariant. Because $\mathbf{x}$, $\mathbf{N}_{a_r \ldots a_1}$ are homogeneous (of degree $N+1$ resp. $N+1-r$) the space spanned by them must be g-invariant.

On the other hand using (4, 1a) as well as the equations (4, 2) in *PI* we see that $\mathbf{N}_{a_r \ldots a_1}$ may be expressed in the following way

$$(5, 3) \qquad \mathbf{N}_{a_r \ldots a_1} \equiv \mathbf{x}_{a_r \ldots a_1} + \sum_{1}^{r-1}{}_s\, \Omega_{a_r \ldots a_s \ldots a_1}^{\quad b_s \ldots b_1} \mathbf{x}_{b_s \ldots b_1}.$$

Hence all points $\mathbf{N}_{a_r \ldots a_1}$ are in $\overset{r}{P}_{m_r}$ and consequently $\overset{r-1}{N}_{n_{r-1}} \subset \overset{r}{P}_{m_r}$. Moreover if some point $\mathbf{P} \neq \mathbf{x}$ of $\overset{r-1}{N}_{n_{r-1}}$ is in $\overset{q}{P}_{m_q}$ $(q < r)$ then according to (5, 3) it must be a linear combination of points $y^{a_r} \ldots y^{a_{q+1}} \mathbf{N}_{a_r \ldots a_1}$. Because $\mathbf{N}_{a_r \ldots a_1} \equiv \mathbf{N}_{(a_r \ldots a_1)}$ are privileged points, we have

$$y^{a_r} \ldots y^{a_{q+1}} \mathbf{N}_{a_r \ldots a_1} = 0$$

and consequently there is no point $\mathbf{P} \neq \mathbf{x}$ of $\overset{r-1}{N}_{n_{r-1}}$ in $\overset{q}{P}_{m_q}$ $(q < r)$. Consider now the equations

$$(5, 4) \qquad \begin{cases} a) \quad \mathbf{n}_{a_r \ldots a_1} = \mathbf{N}_{a_r \ldots a_1} + \mathbf{M}_{a_r \ldots a_1} \\ b) \quad \mathbf{M}_{a_r \ldots a_1} \equiv \displaystyle\sum_{1}^{r-1}{}_s\, Q_{a_r \ldots a_s \ldots a_1}^{\quad b_s \ldots b_1} \mathbf{N}_{b_s \ldots b_1}, \qquad (\mathbf{N}_a \equiv \mathbf{x}_a) \end{cases}$$

(equivalent with (4, 1a)) and denote by $\overset{r-1}{P'}_{m'_{r-1}} \subset \overset{r-1}{P}_{m_{r-1}}$ the space spanned by the points $\mathbf{x}$, $\mathbf{M}_{a_r \ldots a_1}$. Suppose $m'_{r-1} < m_{r-1}$. Since $\overset{r-1}{N}_{n_{r-1}}$ intersects $\overset{q}{P}_{m_q}$ $(q < r)$ only in $\mathbf{x}$, the space spanned by the points $\mathbf{n}_{a_r \ldots a_1}$, which satisfy (5, 4$a$), can not be $\overset{r}{P}_{m_r}$ for $m'_{r-1} < m_{r-1}$ and this together with (5, 1) contradicts Lemma (1, 4). Because we can not have $m'_{r-1} > m_{r-1}$ and the assumption $m'_{r-1} < m_{r-1}$ is a contradictory one, we must have $m'_{r-1} = m_{r-1}$, so that the space spanned by the points $\mathbf{x}$, $\mathbf{M}_{a_r \ldots a_1}$ is the osculating space $\overset{r-1}{P}_{m_{r-1}}$. Hence we see from (5, 4a) and from the statement b) that (5, 2) holds.

Note: The points $\mathbf{N}_{(a_r \ldots a_1)}$ $(r = 2, \ldots, N)$ which (together with $\mathbf{x}$) span the normal space $\overset{r-1}{N}_{n_{r-1}}$ are linearly dependent even in the maximal case (cf. equation (4, 1b)):

Theorem (5, 2). *In the maximal case the points* $\mathbf{N}_{(a_r \ldots a_1)}$ *are linearly "interdependent", e.g. any of their linear combination which is equal to zero must be built up as a linear combination of* $y^{a_p}\, \mathbf{N}_{a_r \ldots a_1}$ $(p = 1, \ldots, r)$.

Proof. Introduce a special parameter system for which $y^a = \delta^a_o$ at P. Then (5, 3) reduces to

$$(5, 5) \qquad \mathbf{N}_{(a_r \ldots a_1)} = \mathbf{x}_{(a_r \ldots a_1)} + \sum_{1}^{r-1} {}_s\, \Omega_{(a_r \ldots a_s \ldots a_1)}^{\;b_s \ldots b_1}\, \mathbf{x}_{b_s \ldots b_1} \text{ at } P$$

$$(a_1, \ldots, a_r = 1, \ldots, m)$$

while the remaining equations (5, 3) reduce to identity $0 = 0$. In order to prove our theorem, it is sufficient to prove that $\mathbf{N}_{(a_r \ldots a_1)}$ are linearly independent: The points $\mathbf{x}_{(a_r \ldots a_1)}$ span $\overset{r}{P}_{m_r}$ while the points $\mathbf{x}_{(a_r o\, o \ldots o\, a_{s-1} \ldots a_1)}$ span $\overset{s}{P}_{m_s}$. Hence in the maximal case [where the points $\mathbf{x}_{(a_r \ldots a_1)}$ are linearly independent] the points $\mathbf{x}_{(a_r \ldots a_1)}$ are linearly independent and span (together with $\mathbf{x}$) a n_{r-1}-dimensional space [10]) $M_{n_{r-1}} \subset \overset{r}{P}_{m_r}$ not contained in $\overset{r-1}{P}_{m_{r-1}}$. Consequently by virtue of (5, 5) the points $\mathbf{N}_{(a_r \ldots a_1)}$ are linearly independent.

Note. Suppose $m = 1$. Because $y^a\, \mathbf{x}_a = (N + 1)\, \mathbf{x}$, the points $\mathbf{x}_0$, $\mathbf{x}_1$ are on the tangential line (the first osculating space $\overset{1}{P}_1$) of $\mathfrak{P}_1$ at $\mathbf{x}$. Introduce a parameter system y^a for which $y^a = \delta^a_0$ at $\mathbf{x}$. Then we have in this parameter system

$$0 = y^a\, \mathbf{N}_{ab} = \mathbf{N}_{0b} \text{ at } \mathbf{x}.$$

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[10]) $n_{r-1} = \binom{m+r-1}{r} = \binom{m+r}{r} - \binom{m+r-1}{r-1} = m_r - m_{r-1}$.

Hence the set of points $\mathbf{N}_{ab}$ reduces here to $\mathbf{N}_{11}$ and the space $\overset{1}{N}_1$ spanned by $\mathbf{x}$, $\mathbf{N}_{ab}$ is the line which joins the points $\mathbf{x}$ and $\mathbf{N}_{11}$ (the first projective normal). Because $\overset{1}{N}_1$ is y-invariant it does not depend on the choice of parameters. Hence if we chose again an arbitrary parameter system, we obtain the same straight line $\overset{1}{N}_1$ which contains the points $\mathbf{x}$, $\mathbf{N}_{ab}$. It may be easily proved by the same argument that the space $\overset{r-1}{N}_{n_{r-1}}$ is a straight line (the $(r-1)$st projective normal) which contains the points $\mathbf{x}$, $\mathbf{N}_{a_r \ldots a_1}$ $(r = 2, \ldots N)$.

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MATHEMATICS

PROJECTIVE GEOMETRIZATION OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS, IV: FRENET FORMULAE FOR PROJECTIVE NORMAL SPACES AND THEIR INTEGRABILITY CONDITIONS

BY

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In this paper we apply the results of three previous papers with the same title, which will be referred to as *PI*, *PII*, *PIII* (Kon. Ned. Akad. v. Wet., Proceeding Vol. **53**, Nos 3, 4 and 6, 1950). In *PIII* we introduced the projective normal spaces of a "symmetric" $\mathfrak{P}_m$. In this paper we shall deal with the corresponding Frenet formulae and their integrability conditions. Consequently, we shall consider only the case of a symmetric $\mathfrak{P}_m$, without stating it explicitly.

§ 1. *Consequences of projective normal spaces.*

Definition (1, 1). *A tensor* $T^{\cdots}_{\cdots}$ *will be termed y-free (and occasionally denoted by* $\overset{\circ}{T}{}^{\cdots}_{\cdots}$*) if it does not contain a y with a free (contravariant) index.*

Note. The tensor H^c_a introduced in Lemma (1, 5) *PIII* which satisfies the condition

$$H^c_a\,(\delta^a_b + y^a\,H_b) = \delta^c_b \tag{1, 1a}$$

may be written

$$H^c_a = \delta^c_a - \tfrac{1}{2}\,y^c\,H_a. \tag{1, 1b}$$

Hence

$$\overset{\circ}{H}{}^c_a = \delta^c_a \tag{1, 1c}$$

and

$$\overset{\circ}{H}{}^{\;c}_{ab} = 2\,H_{(a}\,\overset{\circ}{H}{}^c_{b)} = 2\,H_{(a}\,\delta^c_{b)}\,. \tag{1, 1d}$$

Lemma (1, 1). *Let our* $\mathfrak{P}_m$ *be a maximal one and let* $X^{c_u\cdots c_1}_{\cdots}$ $(u = 2, \ldots r)$ *be a set of y-free tensors which satisfy the condition*

$$\sum_{1}^{r}{}_{s}\; X^{c_s\cdots c_1}_{\cdots}\,\mathbf{n}_{c_s\cdots c_1} = 0, \qquad r \leqslant N. \tag{1, 2a}$$

Then

$$(1, 2b) \qquad X^{(c_s \ldots c_1)}_{\ldots} = 0 \qquad s = 1, \ldots, r.$$

Proof. The lemma is obvious for $r = 1$, since the points $\mathbf{n}_c$ are linearly independent. Let us assume $r > 1$. Using the definition of privileged points

$$(1, 3) \qquad \mathbf{N}_{c_s \ldots c_1} = \mathbf{n}_{c_s \ldots c_1} - \sum_{1}^{s-1} {}_q\, Q_{\underline{c}_s \ldots c_q \ldots c_1}^{\quad b_q \ldots b_1} \mathbf{N}_{b_q \ldots b_1} \;*) \qquad (s = 2, \ldots, r;\; \mathbf{N}_c \equiv \mathbf{n}_c)$$

we obtain

$$(1, 4) \qquad \sum_{2}^{r} {}_s\, X^{c_s \ldots c_1}_{\ldots} \Big[\mathbf{N}_{c_s \ldots c_1} - \sum_{1}^{s-1} {}_q\, Q_{c_s \ldots c_q \ldots c_1}^{\quad b_q \ldots b_1} \mathbf{N}_{b_q \ldots b_1}\Big] + X^c_{\ldots} N_c = 0$$

from (1, $2a$). On the left hand side there is only one set of points in $\overset{r-1}{N}_{n_{r-1}}$ namely $X^{c_r \ldots c_1}_{\ldots} \mathbf{N}_{c_r \ldots c_1}$. Consequently we must have

$$(1, 5) \qquad X^{c_r \ldots c_1}_{\ldots} \mathbf{N}_{c_r \ldots c_1} = 0.$$

Because $\mathbf{N}_{c_r \ldots c_1}$ are symmetric and linearly interdependent (for our $\mathfrak{P}_m$ is a maximal one) and because $X^{c_r \ldots c_1}_{\ldots}$ is y-free the equation (1, 5) yields (1, $2b$) for $s = r$. Hence (1, 4) simplifies to a similar equation for $s = 2, \ldots r - 1$. (If $r = 2$ we get at once $X^c_{\ldots} = 0$, and the lemma is proved.) If $r > 2$ we obtain by the previous argument from the simplified equation (1, 4) the equation (1, $2b$) for $s = r - 1$. Proceeding in this way we obtain all equations (1, $2b$).

We use this lemma in the proof of the following

Theorem (1, 1). *If our* $\mathfrak{P}_m$ *is a maximal one and the tensors* $K_{a_q \ldots a_1}^{\quad c}$ *and* $L_{a_{N+1} \ldots a_r \ldots a_1}^{\quad b_r \ldots b_1}$, $(q = 3, \ldots, N;\; r = 2, \ldots N)$ (*cf.* § 1 *in PII*) *are* y-*free, then*

$$(1, 6) \quad \begin{cases} a) & y^{a_{N+1}} L_{a_{N+1} a_N \ldots b_1}^{\quad b_N \ldots b_1} = (1 - NQ)\, \delta^{(b_N \ldots b_1)}_{a_N \ldots a_1} \\ b) & y^{a_{N+1}} L_{a_{N+1} + a_r \ldots a_1}^{\quad b_r \ldots b_1} = \Big\{ L_{a_N \ldots a_r}^{\quad (b_r} \delta^{b_{r-1} \ldots b_1)}_{a_{r-1} \ldots a_1} \Big\} \\ & \Big(r = 2, \ldots, N - 1;\; L_{a_N \ldots a_r}^{\quad b_r} \equiv - y^{a_{N+1}} K_{a_{N+1} \ldots a_r}^{\quad b_r}\Big) \\ c) & y^{a_{N+1}} K_{a_{N+1} \ldots a_1}^{\quad b} = 0. \end{cases}$$

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*) cf. Theorem (4, 1) in *PIII*.

Proof. First of all we have

$$y^{a_{N+1}} D_{a_{N+1}} \mathbf{n}_{a_N \cdots a_1} = (1-QN)\, \mathbf{n}_{a_N \cdots a_1}$$

and consequently we obtain from (2, 2*d*) in *PII*

$$\mathbf{n}_{b_N \cdots b_1} \left(y^{a_{N+1}} L_{a_{N+1} a_N \cdots a_1}^{\phantom{a_{N+1} a_N \cdots a_1} b_N \cdots b_1} - (1-NQ)\, \delta_{a_N \cdots a_1}^{b_N \cdots b_1} \right) +$$

$$\sum_2^{N-1}{}_r\, \mathbf{n}_{b_r \cdots b_1} y^{a_{N+1}} \left[\left\{\!\!\left\{ K_{a_{N+1} \cdots a_r}^{\phantom{a_{N+1} \cdots a_r} b_r}\, \delta_{a_{r-1} \cdots a_1}^{b_{r-1} \cdots b_1} \right\}\!\!\right\} + L_{a_{N+1} \cdots a_r \cdots a_1}^{\phantom{a_{N+1} \cdots a_r \cdots a_1} b_r \cdots b_1} \right] + y^{a_{N+1}} K_{a_{N+1} \cdots a_1}^{\phantom{a_{N+1} \cdots a_1} b} \mathbf{n}_b = 0$$

Applying now on this equation Lemma (1, 1) and remembering the relationship between the symbols $\{\{\ \ \}\}$ and $\{\ \}$ (cf. Note II at the end of § 1 in *PII*) one obtains at once (1, 6).

Note. For $1 \neq QN$ we could have used another scalar K in the definition (1, 1) *PIII*, namely

$$K \equiv y^{a_{N+1}} L_{a_{N+1} a_N \cdots a_1}^{\phantom{a_{N+1} a_N \cdots a_1} a_N \cdots a_1}.$$

§ 2. *Frenet formulae for projective normal spaces.*

Throughout this section we shall use the following abbreviations

$$(2,1)\quad \begin{cases} a)\quad K_{a_r \cdots a_s \cdots a_1}^{ b_s \cdots b_1} \equiv \left\{\!\!\left\{ K_{a_r \cdots a_s}^{ b_s}\, \delta_{a_{s-1} \cdots a_1}^{b_{s-1} \cdots b_1} \right\}\!\!\right\}, \qquad s = 2, \ldots, r-2 \\ b)\quad G_{a_{N+1} a_N \cdots a_1}^{\phantom{a_{N+1} a_N \cdots a_1} b_N \cdots b_1} \equiv L_{a_{N+1} a_N \cdots a_1}^{\phantom{a_{N+1} a_N \cdots a_1} b_N \cdots b_1}, \qquad c)\quad G_{a_{N+1} \cdots a_1}^{\phantom{a_{N+1} \cdots a_1} b} \equiv K_{a_{N+1} \cdots a_1}^{\phantom{a_{N+1} \cdots a_1} b} \\ d)\quad G_{a_{N+1} \cdots a_r \cdots a_1}^{\phantom{a_{N+1} \cdots a_r \cdots a_1} b_r \cdots b_1} \equiv L_{a_{N+1} \cdots a_r \cdots a_1}^{\phantom{a_{N+1} \cdots a_r \cdots a_1} b_r \cdots b_1} + K_{a_{N+1} \cdots a_r \cdots a_1}^{\phantom{a_{N+1} \cdots a_r \cdots a_1} b_r \cdots b} \\ \qquad\qquad r = 2, \ldots, N-1. \end{cases}$$

Theorem (2, 1). *The privileged points* $\mathbf{N}_{a_q \cdots a_1}$ *) $(q = 2, \ldots, N)$ *and the point* $\mathbf{N}_a = \mathbf{n}_a$ *satisfy the following equations (Frenet formulae for projective normal spaces)*

$$(2,2)\quad \begin{cases} a)\quad D_{a_{r+1}} \mathbf{N}_{a_r \cdots a_1} = \mathbf{N}_{a_{r+1} \cdots a_1} + \sum_1^r{}_q\, F_{a_{r+1} \cdots a_q \cdots a_1}^{\phantom{a_{r+1} \cdots a_q \cdots a_1} (b_q \cdots b_1)} \mathbf{N}_{b_q \cdots b_1}, \\ \qquad\qquad r = 1, \ldots, N-1 \\ b)\quad D_{a_{N+1}} \mathbf{N}_{a_N \cdots a_1} = \sum_1^N{}_q\, F_{a_{N+1} \cdots a_q \cdots a_1}^{\phantom{a_{N+1} \cdots a_q \cdots a_1} (b_q \cdots b_1)} \mathbf{N}_{b_q \cdots b_1} \end{cases}$$

where the F's are functions of the K's and their derivatives and $F_{a_{N+1} \cdots a_q \cdots a_1}^{\phantom{a_{N+1} \cdots a_q \cdots a_1} b_q \cdots b_1}$ *are also functions of the L's*

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*) cf., also for the Q's which appear in (2, 3), § 4 in *PIII*.

$$(2,3)\quad \begin{cases} a)\quad F_{a_2a_1}^{\ \ \ b} = Q_{a_2a_1}^{\ \ \ b} \\ b)\quad F_{a_{r+1}\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1} = Q_{a_{r+1}\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1} + (\delta_{1s}-1)\,\delta_{a_{r+1}}^{c_s} Q_{a_r\ldots a_{s-1}\ldots a_1}^{\qquad c_{s-1}\ldots c_1} \\ \quad + (\delta_{sr}-1)\Big[D_{a_{r+1}} Q_{a_r\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1} + \sum_s^{r-1}{}_q\, \delta_{a_{r+1}}^{b_{q+1}} Q_{a_r\ldots a_q\ldots a_1}^{\qquad b_q\ldots b_1} F_{b_{q+1}\ldots b_s\ldots b_1}^{\qquad c_s\ldots c_1} \\ \quad + K_{a_{r+1}\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1} - (\delta_{s\,r-1}-1)\sum_{s+1}^{r-1}{}_q\, K_{a_{r+1}\ldots a_q\ldots a_1}^{\qquad b_q\ldots b_1} Q_{b_q\ldots b_s\ldots b_1}^{\qquad c_s\ldots c_1}\Big] \\ \qquad (r=2,\ldots,N-1;\ s=1,\ldots,r) \\ c)\quad F_{a_{N+1}a_N\ldots a_1}^{\qquad b_N\ldots b_1} \equiv L_{a_{N+1}a_N\ldots a_1}^{\qquad b_N\ldots b_1} - \delta_{a_{N+1}}^{b_N} Q_{a_Na_{N-1}\ldots a_1}^{\qquad b_{N-1}\ldots b_1} \\ d)\quad F_{a_{N+1}\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1} \equiv G_{a_{N+1}\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1} + \sum_{s+1}^{N}{}_q\, G_{a_{N+1}\ldots a_q\ldots a_1}^{\qquad b_q\ldots b_1} Q_{b_q\ldots b_s\ldots b_1}^{\qquad c_s\ldots c_1} \\ \quad - D_{a_{N+1}} Q_{a_N\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1} - \sum_{s+1}^{N}{}_q\, \delta_{a_{N+1}}^{b_q} Q_{a_N\ldots a_{q-1}\ldots a_1}^{\qquad b_{q-1}\ldots b_1} F_{b_q\ldots b_s\ldots b_1}^{\qquad c_s\ldots c_1} \\ \quad + (\delta_{s1}-1)\,\delta_{a_{N+1}}^{c_s} Q_{a_N\ldots a_{s-1}\ldots a_1}^{\qquad c_{s-1}\ldots c_1} \\ \qquad (s=1,\ldots,N-1). \end{cases}$$

Proof. Our beginning equations are the Frenet formulae for the points $\mathbf{n}$ (cf. § 2 in *PII*)

$$(2,4)\quad \begin{cases} a)\quad D_{a_2}\mathbf{n}_{a_1} = \mathbf{n}_{a_2a_1} \\ b)\quad D_{a_{r+1}}\mathbf{n}_{a_r\ldots a_1} = \mathbf{n}_{a_{r+1}\ldots a_1} + \sum_1^{r-1}{}_s\, K_{a_{r+1}\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1}\mathbf{n}_{c_s\ldots c_1},\quad r=2,\ldots,N-1. \\ c)\quad D_{a_{N+1}}\mathbf{n}_{a_N\ldots a_1} = \sum_1^{N}{}_s\, G_{a_{N+1}\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1}\mathbf{n}_{c_s\ldots c_1} \end{cases}$$

If we substitute in (2, 4) from

$$(2,5)\quad \begin{cases} a)\quad \mathbf{n}_a = \mathbf{N}_a \\ b)\quad \mathbf{n}_{a_r\ldots a_1} = \mathbf{N}_{a_r\ldots a_1} + \sum_1^{r-1}{}_s\, Q_{a_r\ldots a_s\ldots a_1}^{\qquad c_s\ldots c_1}\mathbf{N}_{c_s\ldots c_1} \end{cases}$$

(cf. the equation (4, 1) in *PIII*) and collect the coefficients of $\mathbf{N}_{b_q\ldots b_1}$, $(q=1,\ldots,r)$ we obtain (2, 2) where the F's are defined by (2, 3).

Theorem (2, 2). *The equations* (2, 2) *are g-invariant.*

Proof. $\mathbf{N}_{a_u\ldots a_1}$ $(u=1,\ldots,N)$ is homogeneous of degree $N+1-u$ (cf. the equation (4, 1c) in *PIII*). Hence we have by virtue of Lemma (2, 1) in *PII*

$$\overset{*}{D}_{a_{u+1}}\overset{*}{\mathbf{N}}_{a_u\ldots a_1} = g^{(N-u)} D_{a_{u+1}}\mathbf{N}_{a_u\ldots a_1}.$$

On the other hand we know that $Q_{a_w \ldots a_p \ldots a_1}^{\; b_p \ldots b_1}$ is homogeneous of degree $p - w$ (cf. Lemma (3, 1) in *PIII*) while $K_{b_w \ldots a_1}^{\; c}$ is homogeneous of degree $1 - w$ (cf. Theorems (1, 2) in *PII*) and $L_{a_{N+1} \ldots a_s \ldots a_1}^{\; b_s \ldots b_1}$ is homogeneous of degree $s - N - 1$ (cf. Theorem (1, 1) in *PII*). Hence according to (2, 3) $F_{a_u \ldots a_q \ldots a_1}^{\; b_q \ldots b_1}$ is homogeneous of degree $q - u$ $(u = 2, \ldots, N+1;$ $q = 1, \ldots, u-1)$ and consequently the right hand member in (2, 2) for $D_{a_{u+1}} \mathrm{N}_{a_u \ldots a_1}$ is homogeneous of degree $N - u$, $(u = 1, \ldots N)$.

Note. The equations (2, 3) for $s = r$ may be slightly simplified. According to (1, 1*d*) and (3, 2*a*) in *PIII* we have

$$\begin{aligned} \mathring{Q}_{a_{r+1} a_r \ldots a_1}^{\; b_r \ldots b_1} &= c_{r+1\, r}\, \mathring{H}_{(a_{r+1} a_r}^{\; b_r}\, \delta_{a_{r-1} \ldots a_1)}^{b_{r-1} \ldots b_1} \\ &= 2\, c_{r+1\, r}\, H_{(a_{r+1}}\, \delta_{a_r \ldots a_1)}^{b_r \ldots b_1} \end{aligned}$$

and consequently *)

$$(2, 6) \quad \begin{cases} a) & \mathring{F}_{a_2 a_1}^{\; b} = 2\, c_{21}\, H_{(a_2}\, \delta_{a_1)}^{b_1} \\ b) & \mathring{F}_{a_{r+1} a_r \ldots a_1}^{\; b_r \ldots b_1} = 2 c_{r+1\, r}\, H_{(a_{r+1}}\, \delta_{a_r \ldots a_1)}^{b_r \ldots b_1} - 2 c_{r\, r-1}\, \delta_{a_{r+1}}^{b_r}\, H_{(a_r}\, \delta_{a_{r-1} \ldots a_1)}^{b_{r-1} \ldots b_1} \\ & \qquad = 2 \left[c_{r+1\, r}\, H_{(a_{r+1}}\, \delta_{a_r}^{b_r} - c_{r\, r-1}\, \delta_{a_{r+1}}^{b_r}\, H_{(a_r} \right] \delta_{a_{r-1} \ldots a_1)}^{b_{r-1} \ldots b_1}. \end{cases}$$

Because $\mathbf{N}_{a_r \ldots a_1}$ for $r > 1$ is a privileged point, we may insert the values (2, 6) in (2, 3*b*) for $s = r$. On the other hand if we put

$$P_{a_r \ldots a_1}^{b_r \ldots b_1} \equiv y^{a_{r+1}}\, F_{a_{r+1} a_r \ldots a_1}^{\; b_r \ldots b_1}$$

we obtain from (2, 6) *)

$$(2, 7) \quad \begin{cases} \mathring{P}_{a_r \ldots a_1}^{b_r \ldots b_1} = \dfrac{2\, c_{r+1\, r}}{r+1}\, \delta_{(a_r \ldots a_1)}^{b_r \ldots b_1} = [N + 1 - r\,(Q + 1)]\, \delta_{(a_r \ldots a_1)}^{b_r \ldots b_1} \\ \qquad r = 1, \ldots, N-1. \end{cases}$$

The equation (2, 7) may be generalized:

Theorem (2, 3). *Let our* $\mathfrak{P}_m$ *be a maximal one. Put*

$$(2, 8) \quad \begin{cases} P_{a_r \ldots a_r \ldots a_1}^{\; b_s \ldots a_1} \equiv y^{a_{r+1}}\, F_{a_{r+1} \ldots a_s \ldots a_1}^{\; b_s \ldots b_1} \\ \qquad r = 1, \ldots, N, \; s = 1, \ldots, r. \end{cases}$$

Then

$$(2, 9) \quad \begin{cases} \mathring{P}_{a_r \ldots a_s \ldots a_1}^{\; (b_s \ldots b_1)} = [N + 1 - r\,(Q + 1)]\, \delta_{a_r \ldots a_1}^{(b_r \ldots b_1)}\, \delta_{rs} \\ \qquad r = 2, \ldots, N, \; s = 2, \ldots, r. \end{cases}$$

*) cf. the equations (1, 10) in *PIII* and (3, 1*d*) in *PIII*.

Proof. First of all we have

$$y^{a_{r+1}} D_{a_{r+1}} \mathbf{N}_{a_r \ldots a_1} = [N+1-r(Q+1)] \mathbf{N}_{a_r \ldots a_1}, \quad r = 1, \ldots, N$$

and consequently by virtue of (2, 2) and (2, 8)

$$(2,10) \quad \begin{cases} [N+1-r(Q+1)] \mathbf{N}_{a_r \ldots a_1} = \sum_{1}^{r} {}_s\, P_{a_r \ldots a_s \ldots a_1}^{\quad (b_s \ldots b_1)} \mathbf{N}_{b_s \ldots b_1} \\ \qquad = \sum_{2}^{r} {}_s\, \mathring{P}_{a_r \ldots a_s \ldots a_1}^{\quad (b_s \ldots b_1)} \mathbf{N}_{b_s \ldots b_1} + \\ \qquad + P_{a_r \ldots a_1}^{\quad c} \mathbf{N}_c. \end{cases}$$

The proof follows at once from (2, 10).

Note. Cf. (2, 9) for $r = N$ and (1, 6a)

§ 3. *Integrability conditions of* (2, 2).

Lemma (3, 1). *The systems* (2, 2) *and* (2, 4) *are equivalent.*

Proof. One gets (2, 2) [resp. (2, 4)] from (2, 4) [resp. (2, 2)] by means of the substitution (2, 5).

In the next theorem we use the following abbreviations

$$(3,1) \qquad a) \quad R_{abc}^{\quad d} \equiv 2 \partial_{[b} \Gamma_{a]c}^{\ d} + 2 \Gamma_{e[b}^{\ d} \Gamma_{a]c}^{\ e}$$

(curvature tensor of Γ_{bc}^{a}) and

$$(3,1) \quad \begin{cases} b) \quad I_{eba_N \ldots a_s \ldots a_1}^{\qquad c_s \ldots c_1} \equiv D_{[e} G_{b]a_N \ldots a_s \ldots a_1}^{\qquad c_s \ldots c_1} \\ + \delta_{[e}^{d_{N+1}} L_{b]a_N \ldots a_1}^{\quad d_N \ldots d_1} G_{d_{N+1} \ldots d_s \ldots d_1}^{\qquad c_s \ldots c_1} + (1-\delta_{s1}) \delta_{[e}^{c_s} G_{b]a_N \ldots a_{s-1} \ldots a_1}^{\qquad c_{s-1} \ldots c_1} \\ + (1-\delta_{sN})(1-\delta_{s\,N-1}) \sum_{s+2}^{N} {}_q\, \delta_{[e}^{d_q} G_{b]a_N \ldots a_{q-1} \ldots a_1}^{\qquad d_{q-1} \ldots d_1} K_{d_q \ldots d_s \ldots d_1}^{\qquad c_s \ldots c_1} \\ - \delta_{sN} \sum_{1}^{N} {}_q\, K_{[eb]a_q}^{\quad c_q} \delta_{a_s \ldots a_{q+1} a_{q-1} \ldots a_1}^{c_s \ldots c_{q+1} c_{q-1} \ldots c_1} \\ \qquad s = 1, \ldots, N. \end{cases}$$

Theorem (3, 1). *The first* $N-1$ *sets of integrability conditions of* (2, 2) *are*

$$(3,2) \quad \begin{cases} a) \quad R_{a_3 a_2 a_1}^{\quad c} - 2 K_{[a_3 a_2] a_1}^{\quad c} = 0 \\ b) \quad K_{[a_4 a_3] a_2 a_1}^{\quad c} + D_{[a_4} K_{a_3] a_2 a_1}^{\quad c} = 0 \ ^{*)} \\ c) \quad K_{[a_{r+1} a_r] \ldots a_1}^{\quad c} + D_{[a_{r+1}} K_{a_r] \ldots a_1}^{\quad c} \\ \qquad + \sum_{2}^{r-2} {}_q\, \delta_{[a_{r+1}}^{b_{q+1}} K_{a_r] \ldots a_q \ldots a_1}^{\qquad b_q \ldots b_1} K_{b_{q+1} \ldots b_1}^{\quad c} = 0 \ ^{**)}\ ^{*)} \\ \qquad r = 4, \ldots, N. \end{cases}$$

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*) This equation does not exist for $N = 2$.

**) This equation does not exist for $N = 3$.

Moreover, if our $\mathfrak{P}_m$ *is a maximal one and the tensors* K*'s and* L*'s are* y*-free the last set of integrability conditions are*

$$I_{a_{N+2}\dots a_s\dots a_1}^{\quad (c_s\dots c_1)} = 0 \qquad s = 1, \dots, N. \tag{3, 2d}$$

The proof of this theorem will be accomplished in seven steps:

I) According to Lemma (3, 1) it is sufficient to prove that (3, 2) are integrability conditions of (2, 4). These conditions are obtained if one expresses $D_{[a}\, D_{b]}\, \mathbf{n}\dots$ by means of (2, 4) and by means of the curvature tensor and compares both results. On the other hand we have to remember that

$$\left\{\begin{aligned} &a)\quad K_{a_{r+1}\dots a_1}^{\quad c} = K_{a_{r+1}(a_r\dots a_1)}^{\quad c} && r = 2, \dots, N \\ &b)\quad L_{a_{N+1}\dots a_s\dots a_1}^{\quad b_s\dots b_1} = L_{(a_{N+1}\dots a_s\dots a_1)}^{\quad b_s\dots b_1} \ {}^{*)} && s = 1, \dots, N. \end{aligned}\right. \tag{3, 3}$$

II) Assume first $N > 3$ **) and consider only the sets (3, $2a\, b\, c$). From (2, $4a\, b$) we obtain

$$\left\{\begin{aligned} D_{[a_3}\, D_{a_2]}\, \mathbf{n}_{a_1} &= \tfrac{1}{2}\, R_{a_3 a_2 a_1}^{\quad c}\, \mathbf{n}_b = \\ &= \mathbf{n}_{[a_3 a_2] a_1} + K_{[a_3 a_2] a_1}^{\quad c}\, \mathbf{n}_c. \end{aligned}\right. \tag{3, 4}$$

Because $\mathbf{n}_{a_3 a_2 a_1}$ is symmetric in all indices (cf. Lemma (1, 1) in *PIII*) and $\mathbf{n}_c$ are linearly independent, the equations (3, 4) leads to (3, $2a$). From (2, $4b$) for $r = 2, 3$ we obtain

$$\begin{aligned} D_{[a_4}\, D_{a_3]}\, \mathbf{n}_{a_2 a_1} &= \tfrac{1}{2}\,(R_{a_4 a_3 a_2}^{\quad c_2}\, \delta_{a_1}^{c_1} + R_{a_4 a_3 a_1}^{\quad c_1}\, \delta_{a_2}^{c_2})\, \mathbf{n}_{c_2 c_1} = \\ &= D_{[a_4}\,(\mathbf{n}_{a_3] a_2 a_1} + K_{a_3] a_2 a_1}^{\quad c}\, \mathbf{n}_c) = \\ &= \mathbf{n}_{c_2 c_1}\,[K_{[a_4 a_3] a_2 a_1}^{\quad c_2 c_1} + \delta_{[a_4}^{c_2}\, K_{a_3] a_2 a_1}^{\quad c_1}] + \\ &\quad + \mathbf{n}_c\,[K_{[a_4 a_3] a_2 a_1}^{\quad c} + D_{[a_4}\, K_{a_3] a_2 a_1}^{\quad c}] \end{aligned}$$

or

$$\left\{\begin{aligned} &\mathbf{n}_{c_2 c_1}\,[K_{[a_4 a_3] a_2 a_1}^{\quad c_2 c_1} + \delta_{[a_4}^{c_2} K_{a_3] a_2 a_1}^{\quad c_1} - \tfrac{1}{2}\, R_{a_4 a_3 a_2}^{\quad c_2} \delta_{a_1}^{c_1} - \tfrac{1}{2}\, R_{a_4 a_3 a_1}^{\quad c_1} \delta_{a_2}^{c_2}] \\ &\qquad + \mathbf{n}_c\,[K_{[a_4 a_3] a_2 a_1}^{\quad c} + D_{[a_4} K_{a_3] a_2 a_1}^{\quad c}] = 0. \end{aligned}\right. \tag{3, 5a}$$

According to (3, $2a$) and (3, $3a$) the first member on the left hand side may be transcribed

$$\mathbf{n}_{c_2 c_1}\,[K_{[a_4 a_3] a_2 a_1}^{\quad c_2 c_1} - K_{[a_4 a_3] a_2 a_1}^{\quad c_2 c_1}] = 0. \tag{3, 5b}$$

The equation (3, $2b$) follows from (3, 5).

III) Let us now assume that we have already proved the equations (3, $2a\, b\, c$) for all $r = 4, 5, \dots, u - 1 < N - 1$ and let us consider the

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*) These equations follow from the constructions of the K's and the L's (cf. § 1 in *PII*).

**) The cases $N = 2, 3$ are dealt with in the last step.

expression $D_{[a_{u+1}} D_{a_u]} \mathbf{n}_{a_{u-1}\dots a_1}$. Applying the argument exposed in I) we obtain the equation

$$(3,\,6a) \qquad A + B + C + D = 0$$

where

$$(3,\,6)\begin{cases} b) & A \equiv \Big[K_{[a_{u+1} a_u] a_{u-1}\dots a_1}{}^{c_{u-1}\dots c_1} + \delta_{[a_{u+1}}^{c_{u-1}} K_{a_u] a_{u-1} a_{u-2}\dots a_1}{}^{c_{u-2}\dots c_1} \\ & \quad - \frac{1}{2} \sum_1^{u-1}{}_w R_{a_{u+1} a_u a_w}{}^{c_{u-1}} \delta_{a_{u-1}\dots a_{w+1} a_{w-1}\dots a_1}^{c_{u-2}\dots c_1} \Big] \mathbf{n}_{c_{u-1}\dots c_1} \\ c) & B \equiv \Big[D_{[a_{u+1}} K_{a_u] a_{u-1} a_{u-2}\dots a_1}{}^{c_{u-2}\dots c_1} + \delta_{[a_{u+1}}^{c_{u-2}} K_{a_u]\dots a_{u-3}\dots a_1}{}^{c_{u-3}\dots c_1} \\ & \quad + K_{[a_{u+1} a_u]\dots a_{u-2}\dots a_1}{}^{c_{u-2}\dots c_1} \Big] \mathbf{n}_{c_{u-2}\dots c_1} \\ d) & C \equiv \sum_2^{u-3}{}_s \Big[D_{[a_{u+1}} K_{a_u]\dots a_s\dots a_1}{}^{c_s\dots c_1} + \delta_{[a_{u+1}}^{c_s} K_{a_u]\dots a_{s-1}\dots a_1}{}^{c_{s-1}\dots c_1} \\ & \quad + K_{[a_{u+1} a_u]\dots a_s\dots a_1}{}^{c_s\dots c_1} + \sum_{s+2}^{u-1}{}_q \delta_{[a_{u+1}}^{b_q} K_{a_u]\dots a_{q-1}\dots a_1}{}^{b_{q-1}\dots b_1} K_{b_q\dots b_s\dots b_1}{}^{c_s\dots c_1} \Big] \mathbf{n}_{c_s\dots c_1} \\ e) & D \equiv \Big[D_{[a_{u+1}} K_{a_u]\dots a_1}{}^{c} \\ & \quad + \sum_2^{u-2}{}_q \delta_{[a_{u+1}}^{b_{q+1}} K_{a_u]\dots a_q\dots a_1}{}^{b_q\dots b_1} K_{b_{q+1}\dots b_1}{}^{c} + K_{[a_{u+1} a_u]\dots a_1}{}^{c} \Big] \mathbf{n}_c . \end{cases}$$

Using (3, 2a) and (3, 3a) we obtain first

$$(3,\,7a) \qquad A \equiv \Big[K_{[a_{u+1} a_u] a_{u-1}\dots a_1}{}^{c_{u-1}\dots c_1} - K_{[a_{u+1} a_u] a_{u-1}\dots a_1}{}^{c_{u-1}\dots c_1} \Big] \mathbf{n}_{c_{u-1}\dots c_1} \equiv 0.$$

On the other hand if we decompose $K_{abcd}{}^{e}$ and $K_{abc}{}^{e}$ into *ideal* factors

$$K_{abcd}{}^{e} \equiv k_{ab}\, k_{cd}{}^{e}$$
$$K_{abc}{}^{e} \equiv K_a\, K_{bc}{}^{e}$$

we obtain by virtue of (3, 2a b) and (3, 3)

$$(3,\,7b) \qquad \begin{cases} B \equiv {}^{*)} \Big[\Big(D_{[a_{u+1}} K_{a_u]} \Big\{ K_{a_{u-1} a_{u-2}}{}^{c_{u-2}} \Big) \delta_{a_{u-3}\dots a_1}^{c_{u-3}\dots c_1} \Big\} + \\ \qquad + k_{[a_{u+1} a_u]} \Big\{ k_{a_{u-1} a_{u-2}}{}^{c_{u-2}} \delta_{a_{u-3}\dots a_1}^{c_{u-3}\dots c_1} \Big\} \Big] \mathbf{n}_{c_{u-2}\dots c_1} = 0. \end{cases}$$

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*) One has to remember the relationship between the symbols $\{\{\,\}\}$ and $\{\,\}$. Thus for instance

$$\Big\{\Big\{ K_{a_u a_{u-1} a_{u-2}}{}^{c_{u-2}} \delta_{a_{u-3}\dots a_1}^{c_{u-3}\dots c_1} \Big\}\Big\} = K_{a_u} \Big\{ K_{a_{u-1} a_{u-2}}{}^{c_{u-2}} \delta_{a_{u-3}\dots a_1}^{c_{u-3}\dots c_1} \Big\}$$

and so forth.

In a similar manner we obtain

$$C = 0. \tag{3, 7c}$$

Consequently the equation (3, 6a) reduces to (3, 2c) for $u = r$.

IV) Applying the same method as used in III on $D_{[a_5} D_{a_4]} \mathbf{n}_{a_3 a_2 a_1}$ we obtain (3, 2c) for $r = 4 \leq N - 1$. This result together with the result of the third step leads by induction to (3, 2c) for $r = 4, \ldots, N - 1$.

V) The same argument applied on $D_{[a_{N+1}} D_{a_N]} \mathbf{n}_{a_{N-1} \ldots a_1}$ leads (together with (3, 3b)) to (3, 2c) for $r = N > 3$.

VI) If we now apply the same argument as used above on

$$D_{[a_{N+2}} D_{a_{N+1}]} \mathbf{n}_{a_N \ldots a_1}$$

we obtain for $N > 3$

$$\sum_{1}^{N} {}_s I_{a_{N+2} a_{N+1} \ldots a_s \ldots a_1}^{\;c_s \ldots c_1} \mathbf{n}_{c_s \ldots c_1} = 0 \tag{3, 8}$$

and this equation together with Lemma (1, 1) yields (3, 2d).

VII) If $N = 2$ then the method described above leads to (3, 2a) and (3, 2d). If $N = 3$ we obtain by the same method (3, 2a b) and (3, 2d).

Note. The equations (3, 2) are to be considered as beginning equations for existence theorems. One of the basic theorems for this theory is

Theorem (3, 2). *We have*

$$\left\{ \begin{aligned} &a) \quad y^{a_s} R_{a_3 a_2 a_1}^{\quad\;\; c} = 0 \\ &b) \quad y^{a_s} K_{[a_3 a_2] a_1}^{\quad\;\;\; c} = 0 \end{aligned} \right. \qquad s = 1, 2, 3 \tag{3, 9}$$

and consequently for $m = 1$

$$a) \quad R_{abc}^{\quad d} = 0, \qquad\qquad b) \quad K_{a_3 a_2 a_1}^{\quad\;\; c} = K_{(a_3 a_2 a_1)}^{\quad\;\;\; c}. \tag{3, 10}$$

Proof. From the equation (1, 4a) in *PIII* we have (3, 9b) for $s = 3$ and this equation together with (3, 2a) leads to (3, 9a) for $s = 3$. From (3, 3a) (for $r = 2$) and from (3, 9b) for $s = 3$ we have

$$y^{a_3} K_{a_3 a_2 a_1}^{\quad\;\; c} = y^{a_3} K_{a_2 a_3 a_1}^{\quad\;\; c} = y^{a_3} K_{a_2 a_1 a_3}^{\quad\;\; c}$$

while the first member equals $y^{a_3} K_{(a_3 a_2 a_1)}$ (cf. (1, 4a) in *PIII*). Hence (3, 9b) also holds for $s = 1, 2$. From (3, 9b) for $s = 1, 2$ we obtain by virtue of (3, 2a) the equations (3, 9a) for $s = 1, 2$ *). Hence (3, 9a) holds for $s = 1, 2, 3$. Consequently if $m = 1$ we must have (3, 10a), and this equation together with (3, 3a) (for $r = 2$) and (3, 2a) leads to (3, 10b).

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*) (3, 9a) for $s = 1$ is obviously the integrability condition for $D_a y^b = (1 + Q)\, \delta_a^b$. (Cf. also the paper by J. A. Schouten and J. Haantjes referred to in the Synopsis in *PI*).

TABLE OF THE CUMULATIVE SYMMETRIC BINOMIAL DISTRIBUTION

BY

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(*Report R 70 of the Computation Department of the Mathematical Centre at Amsterdam*)

(Communicated by Prof. D. VAN DANTZIG at the meeting of April 29, 1950)

1. *Introduction.* In this paper a table is given of the cumulative symmetric binomial distribution-function

$$P(n,c) = 1 - 2^{1-n} \sum_{s=0}^{c} \binom{n}{s}. \tag{1.1}$$

This function is of some mathematical interest, and plays especially a role in statistics. The immediate reason for the construction of the table was a recent paper by H. THEIL [1]), where $P(n, c)$, together with another distribution-function to be tabulated later on, occur.

By means of the incomplete Beta-function

$$I_x(p,q) = \frac{\int_0^x t^{p-1}(1-t)^{q-1}\,dt}{\int_0^1 t^{p-1}(1-t)^{q-1}\,dt} \tag{1.2}$$

the function $P(n, c)$ can be written in the form

$$P(n,c) = 1 - 2\, I_{\frac{1}{2}}(n-c, c+1). \tag{1.3}$$

This relation permits to read $P(n, c)$ from a table of $I_x(p, q)$. As these tables [2]), however, are restricted to $p \leqslant 50$, the complete distribution can be found if $n \leqslant 50$, and part of it only if $50 < n \leqslant 100$. This range is not always sufficient for practical purposes. This difficulty may be overcome by approximation of $P(n, c)$ by a normal distribution, if need be, with application of continuity-corrections, a procedure that may easily lead to computational errors, however. The present table gives $P(n, c)$ directly for $n \leqslant 200$, a range large enough for most statistical applications.

2. *Preparation of the table.* By the use of suitable recurrence relations a table of $P(n, c)$ multiplied by certain powers of 2 were produced on a National-accounting machine. The superfluous powers of 2 were eliminated

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[1]) H. THEIL, A rank-invariant method of linear and polynomial regression analysis, I, These Proceedings, 53, 386–392 (1950).

[2]) K. PEARSON, Tables of the incomplete Beta-function, Cambridge, (1934).

by division and the printers copy and proofsheets checked by means of the recurrence relations that exist between the sums with respect to n for consecutive values of c. For $n = 100$ and 200 the binomial coefficients were computed separately to provide a last independent check.

The accuracy is well under one unit of the last (fifth) decimal place. This is, of course, much better than is needed for statistical purposes, but it provides the possibility to find also the binomial coefficients with some accuracy in the table by means of

$$2^{1-n}\binom{n}{c} = P(n, c-1) - P(n, c). \tag{2.1}$$

Instead of n and c the arguments $n - 2c$ and c were chosen in the table. This will not cause much difficulty for the user but reduces the required space considerably.

The computations have been performed by Miss R. D. M. MULDER.

3. *Table of* $P(n, c)$.

n-$2c$	$c = 0$	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$	$c = 6$	$c = 7$	$c = 8$
1	0	0	0	0	0	0	0	0	0
2	0.50000	0.37500	0.31250	0.27344	0.24609	0.22559	0.20947	0.19638	0.18547
3	0.75000	0.62500	0.54688	0.49219	0.45117	0.41895	0.39276	0.37094	0.35239
4	0.87500	0.78125	0.71094	0.65625	0.61230	0.57605	0.54550	0.51932	0.49656
5	0.93750	0.87500	0.82031	0.77344	0.73315	0.69824	0.66769	0.64072	0.61669
6	0.96875	0.92969	0.89062	0.85400	0.82043	0.78989	0.76212	0.73682	0.71372
7	0.98438	0.96094	0.93457	0.90771	0.88153	0.85654	0.83293	0.81075	0.78996
8	0.99219	0.97852	0.96143	0.94263	0.92319	0.90375	0.88468	0.86620	0.84841
9	0.99609	0.98828	0.97754	0.96484	0.95096	0.93643	0.92165	0.90686	0.89225
10	0.99805	0.99365	0.98706	0.97873	0.96912	0.95861	0.94752	0.93609	0.92448
11	0.99902	0.99658	0.99261	0.98727	0.98079	0.97340	0.96531	0.95671	0.94776
12	0.99951	0.99817	0.99582	0.99246	0.98818	0.98310	0.97734	0.97104	0.96430
13	0.99976	0.99902	0.99765	0.99557	0.99280	0.98938	0.98537	0.98084	0.97588
14	0.99988	0.99948	0.99869	0.99742	0.99566	0.99339	0.99064	0.98746	0.98388
15	0.99994	0.99973	0.99927	0.99851	0.99740	0.99592	0.99408	0.99187	0.98933
16	0.99997	0.99986	0.99960	0.99914	0.99846	0.99751	0.99628	0.99478	0.99300
17	0.99998	0.99992	0.99978	0.99951	0.99909	0.99849	0.99768	0.99667	0.99545
18	0.99999	0.99996	0.99988	0.99972	0.99947	0.99909	0.99857	0.99790	0.99706
19	1.00000	0.99998	0.99993	0.99984	0.99969	0.99945	0.99912	0.99868	0.99812
20	1.00000	0.99999	0.99996	0.99991	0.99982	0.99968	0.99946	0.99918	0.99881
21	1.00000	0.99999	0.99998	0.99995	0.99990	0.99981	0.99968	0.99949	0.99925
22	1.00000	1.00000	0.99999	0.99997	0.99994	0.99989	0.99980	0.99969	0.99953
23	1.00000	1.00000	0.99999	0.99998	0.99997	0.99993	0.99988	0.99981	0.99971
24	1.00000	1.00000	1.00000	0.99999	0.99998	0.99996	0.99993	0.99988	0.99982
25	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998	0.99996	0.99993	0.99989
26	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99998	0.99996	0.99993
27	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99998	0.99996
28	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99998
29	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998
30	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.09999
31	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999
32	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

n-2c	c = 9	c = 10	c = 11	c = 12	c = 13	c = 14	c = 15	c = 16	c = 17
1	0	0	0	0	0	0	0	0	0
2	0.17620	0.16819	0.16118	0.15498	0.14945	0.14446	0.13995	0.13583	0.13206
3	0.33638	0.32236	0.30996	0.29889	0.28893	0.27990	0.27167	0.26412	0.25717
4	0.47653	0.45874	0.44280	0.42841	0.41534	0.40339	0.39241	0.38228	0.37290
5	0.59513	0.57564	0.55793	0.54174	0.52687	0.51315	0.50044	0.48862	0.47760
6	0.69254	0.67306	0.65507	0.63841	0.62291	0.60847	0.59497	0.58231	0.57041
7	0.77048	0.75221	0.73507	0.71896	0.70379	0.68950	0.67599	0.66322	0.65111
8	0.83136	0.81507	0.79951	0.78467	0.77052	0.75702	0.74412	0.73181	0.72004
9	0.87792	0.86395	0.85039	0.83724	0.82453	0.81226	0.80041	0.78898	0.77795
10	0.91284	0.90126	0.88982	0.87855	0.86750	0.85669	0.84614	0.83585	0.82583
11	0.93857	0.92924	0.91986	0.91047	0.90113	0.89187	0.88273	0.87371	0.86484
12	0.95723	0.94990	0.94239	0.93475	0.92705	0.91931	0.91157	0.90386	0.89619
13	0.97055	0.96492	0.95904	0.95297	0.94675	0.94042	0.93400	0.92754	0.92106
14	0.97994	0.97569	0.97118	0.96645	0.96152	0.95644	0.95123	0.94592	0.94054
15	0.98647	0.98333	0.97993	0.97630	0.97246	0.96846	0.96430	0.96001	0.95561
16	0.99096	0.98867	0.98615	0.98341	0.98048	0.97737	0.97410	0.97069	0.96716
17	0.99401	0.99237	0.99052	0.98849	0.98628	0.98390	0.98137	0.97870	0.97591
18	0.99607	0.99490	0.99357	0.99208	0.99044	0.98865	0.98672	0.98465	0.98247
19	0.99744	0.99662	0.99568	0.99460	0.99339	0.99206	0.99060	0.98903	0.98734
20	0.99834	0.99778	0.99711	0.99634	0.99547	0.99448	0.99340	0.99221	0.99093
21	0.99893	0.99855	0.99809	0.99754	0.99691	0.99620	0.99540	0.99452	0.99355
22	0.99932	0.99906	0.99874	0.99836	0.99791	0.99740	0.99682	0.99616	0.99544
23	0.99957	0.99940	0.99918	0.99891	0.99860	0.99823	0.99781	0.99734	0.99680
24	0.99973	0.99961	0.99947	0.99928	0.99906	0.99881	0.99851	0.99816	0.99777
25	0.99983	0.99975	0.99966	0.99953	0.99938	0.99920	0.99899	0.99874	0.99845
26	0.99990	0.99984	0.99978	0.99969	0.99959	0.99947	0.99932	0.99914	0.99893
27	0.99994	0.99990	0.99986	0.99980	0.99973	0.99965	0.99954	0.99942	0.99927
28	0.99996	0.99994	0.99991	0.99988	0.99983	0.99977	0.99969	0.99961	0.99950
29	0.99998	0.99996	0.99994	0.99992	0.99989	0.99985	0.99980	0.99974	0.99966
30	0.99999	0.99998	0.99997	0.99995	0.99993	0.99990	0.99987	0.99982	0.99977
31	0.99999	0.99999	0.99998	0.99997	0.99995	0.99994	0.99991	0.99988	0.99984
32	0.99999	0.99999	0.99999	0.99998	0.99997	0.99996	0.99994	0.99992	0.99990
33	1.00000	1.00000	0.99999	0.99999	0.99998	0.99997	0.99996	0.99995	0.99993
34	1.00000	1.00000	1.00000	0.99999	0.99999	0.99998	0.99998	0.99997	0.99996
35	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99998	0.99998	0.99997
36	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99998
37	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999
38	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999
39	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

n-2c	c = 18	c = 19	c = 20	c = 21	c = 22	c = 23	c = 24	c = 25	c = 26
1	0	0	0	0	0	0	0	0	0
2	0.12859	0.12537	0.12239	0.11960	0.11700	0.11457	0.11227	0.11012	0.10808
3	0.25074	0.24477	0.23921	0.23401	0.22913	0.22455	0.22023	0.21615	0.21229
4	0.36417	0.35603	0.34841	0.34126	0.33453	0.32819	0.32219	0.31651	0.31112
5	0.46729	0.45762	0.44851	0.43993	0.43183	0.42415	0.41687	0.40995	0.40336
6	0.55920	0.54862	0.53861	0.52912	0.52011	0.51154	0.50338	0.49560	0.48816
7	0.63962	0.62870	0.61831	0.60840	0.59894	0.58990	0.58125	0.57296	0.56501
8	0.70878	0.69800	0.68767	0.67776	0.66825	0.65911	0.65032	0.64186	0.63371
9	0.76731	0.75704	0.74713	0.73756	0.72832	0.71939	0.71076	0.70240	0.69432
10	0.81608	0.80659	0.79736	0.78839	0.77967	0.77120	0.76295	0.75494	0.74714
11	0.85613	0.84759	0.83922	0.83102	0.82300	0.81515	0.80747	0.79997	0.79263
12	0.88860	0.88108	0.87365	0.86633	0.85911	0.85199	0.84500	0.83812	0.83136

n-2c	c = 18	c = 19	c = 20	c = 21	c = 22	c = 23	c = 24	c = 25	c = 26
13	0.91457	0.90808	0.90163	0.89521	0.88884	0.88252	0.87627	0.87008	0.86396
14	0.93509	0.92961	0.92410	0.91857	0.91305	0.90754	0.90204	0.89658	0.89114
15	0.95112	0.94656	0.94194	0.93726	0.93255	0.92782	0.92307	0.91832	0.91356
16	0.96352	0.95978	0.95595	0.95206	0.94811	0.94410	0.94006	0.93598	0.93188
17	0.97299	0.96997	0.96686	0.96366	0.96038	0.95704	0.95364	0.95020	0.94671
18	0.98017	0.97776	0.97525	0.97266	0.96998	0.96723	0.96441	0.96154	0.95861
19	0.98555	0.98365	0.98166	0.97958	0.97742	0.97518	0.97288	0.97050	0.96807
20	0.98955	0.98807	0.98651	0.98487	0.98314	0.98134	0.97947	0.97754	0.97554
21	0.99249	0.99136	0.99015	0.98886	0.98750	0.98606	0.98457	0.98300	0.98138
22	0.99465	0.99378	0.99285	0.99186	0.99079	0.98966	0.98847	0.98722	0.98592
23	0.99621	0.99556	0.99485	0.99408	0.99326	0.99238	0.99144	0.99046	0.98942
24	0.99733	0.99684	0.99631	0.99573	0.99510	0.99442	0.99369	0.99291	0.99209
25	0.99813	0.99777	0.99737	0.99693	0.99645	0.99593	0.99537	0.99477	0.99413
26	0.99870	0.99844	0.99814	0.99781	0.99745	0.99706	0.99663	0.99616	0.99566
27	0.99910	0.99891	0.99869	0.99845	0.99818	0.99788	0.99755	0.99720	0.99682
28	0.99938	0.99924	0.99908	0.99890	0.99870	0.99848	0.99824	0.99797	0.99767
29	0.99958	0.99948	0.99936	0.99923	0.99908	0.99892	0.99873	0.99853	0.99831
30	0.99971	0.99964	0.99956	0.99946	0.99936	0.99923	0.99910	0.99895	0.99878
31	0.99980	0.99975	0.99969	0.99963	0.99955	0.99946	0.99936	0.99925	0.99912
32	0.99987	0.99983	0.99979	0.99974	0.99969	0.99962	0.99955	0.99946	0.99937
33	0.99991	0.99989	0.99986	0.99982	0.99978	0.99974	0.99968	0.99962	0.99955
34	0.99994	0.99992	0.99990	0.99988	0.99985	0.99982	0.99978	0.99973	0.99968
35	0.99996	0.99995	0.99993	0.99992	0.99990	0.99987	0.99985	0.99981	0.99978
36	0.99997	0.99997	0.99996	0.99994	0.99993	0.99991	0.99989	0.99987	0.99984
37	0.99998	0.99998	0.99997	0.99996	0.99995	0.99994	0.99993	0.99991	0.99989
38	0.99999	0.99999	0.99998	0.99998	0.99997	0.99996	0.99995	0.99994	0.99992
39	0.99999	0.99999	0.99999	0.99998	0.99998	0.99997	0.99996	0.99996	0.99995
40	1.00000	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99997	0.99996
41	1.00000	1.00000	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99997
42	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999	0.99998
43	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999
44	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999
45	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999
46	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

n-2c	c = 27	c = 28	c = 29	c = 30	c = 31	c = 32	c = 33	c = 34	c = 35
1	0	0	0	0	0	0	0	0	0
2	0.10615	0.10432	0.10258	0.10092	0.09934	0.09784	0.09640	0.09502	0.09371
3	0.20863	0.20516	0.20185	0.19869	0.19568	0.19280	0.19005	0.18741	0.18488
4	0.30600	0.30112	0.29646	0.29202	0.28777	0.28370	0.27980	0.27605	0.27245
5	0.39708	0.39108	0.38534	0.37985	0.37459	0.36954	0.36469	0.36003	0.35554
6	0.48104	0.47423	0.46769	0.46142	0.45539	0.44959	0.44400	0.43862	0.43343
7	0.55737	0.55004	0.54298	0.53618	0.52963	0.52331	0.51721	0.51132	0.50562
8	0.62585	0.61827	0.61095	0.60388	0.59704	0.59042	0.58401	0.57781	0.57179
9	0.68650	0.67892	0.67157	0.66444	0.65753	0.65082	0.64430	0.63797	0.63181
10	0.73957	0.73219	0.72501	0.71802	0.71122	0.70459	0.69813	0.69183	0.68569
11	0.78546	0.77845	0.77160	0.76490	0.75835	0.75195	0.74569	0.73957	0.73359
12	0.82471	0.81819	0.81178	0.80549	0.79932	0.79326	0.78732	0.78148	0.77575
13	0.85793	0.85197	0.84609	0.84029	0.83457	0.82894	0.82339	0.81792	0.81254
14	0.88574	0.88039	0.87508	0.86982	0.86461	0.85946	0.85436	0.84932	0.84434
15	0.90881	0.90407	0.89935	0.89466	0.88998	0.88534	0.88072	0.87614	0.87160
16	0.92776	0.92363	0.91949	0.91535	0.91122	0.90709	0.90297	0.89886	0.89477
17	0.94318	0.93963	0.93605	0.93245	0.92883	0.92521	0.92158	0.91794	0.91431

n-2c	c = 27	c = 28	c = 29	c = 30	c = 31	c = 32	c = 33	c = 34	c = 35
18	0.95563	0.95260	0.94955	0.94645	0.94333	0.94019	0.93703	0.93385	0.93065
19	0.96558	0.96304	0.96046	0.95783	0.95517	0.95247	0.94975	0.94700	0.94422
20	0.97348	0.97137	0.96921	0.96701	0.96476	0.96247	0.96014	0.95779	0.95540
21	0.97970	0.97797	0.97618	0.97434	0.97246	0.97054	0.96858	0.96658	0.96455
22	0.98456	0.98315	0.98168	0.98017	0.97862	0.97702	0.97538	0.97370	0.97198
23	0.98833	0.98719	0.98600	0.98477	0.98349	0.98218	0.98081	0.97942	0.97798
24	0.99123	0.99032	0.98936	0.98836	0.98733	0.98625	0.98513	0.98398	0.98279
25	0.99345	0.99272	0.99196	0.99116	0.99032	0.98945	0.98854	0.98760	0.98662
26	0.99513	0.99456	0.99396	0.99332	0.99266	0.99195	0.99122	0.99045	0.98965
27	0.99640	0.99596	0.99549	0.99498	0.99445	0.99389	0.99330	0.99269	0.99204
28	0.99736	0.99701	0.99665	0.99625	0.99583	0.99539	0.99492	0.99443	0.99391
29	0.99807	0.99780	0.99752	0.99721	0.99689	0.99654	0.99617	0.99577	0.99536
30	0.99859	0.99839	0.99818	0.99794	0.99768	0.99741	0.99712	0.99681	0.99648
31	0.99898	0.99883	0.99866	0.99848	0.99829	0.99807	0.99785	0.99760	0.99735
32	0.99927	0.99915	0.99903	0.99889	0.99874	0.99857	0.99840	0.99821	0.99801
33	0.99948	0.99939	0.99929	0.99919	0.99907	0.99895	0.99881	0.99867	0.99851
34	0.99962	0.99956	0.99949	0.99941	0.99932	0.99923	0.99913	0.99901	0.99889
35	0.99973	0.99969	0.99963	0.99957	0.99951	0.99944	0.99936	0.99927	0.99918
36	0.99981	0.99978	0.99974	0.99969	0.99964	0.99959	0.99953	0.99947	0.99939
37	0.99987	0.99984	0.99981	0.99978	0.99974	0.99970	0.99966	0.99961	0.99955
38	0.99991	0.99989	0.99987	0.99984	0.99982	0.99979	0.99975	0.99971	0.99967
39	0.99993	0.99992	0.99991	0.99989	0.99987	0.99985	0.99982	0.99979	0.99976
40	0.99995	0.99994	0.99993	0.99992	0.99991	0.99989	0.99987	0.99985	0.99982
41	0.99997	0.99996	0.99995	0.99994	0.99993	0.99992	0.99991	0.99989	0.99987
42	0.99998	0.99997	0.99997	0.99996	0.99995	0.99994	0.99993	0.99992	0.99991
43	0.99998	0.99998	0.99998	0.99997	0.99997	0.99996	0.99995	0.99994	0.99993
44	0.99999	0.99999	0.99998	0.99998	0.99998	0.99997	0.99996	0.99996	0.99995
45	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99997	0.99997
46	1.00000	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99998
47	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999	0.99999	0.99998
48	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999
49	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999
50	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999
51	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

n-2c	c = 36	c = 37	c = 38	c = 39	c = 40	c = 41	c = 42	c = 43	c = 44
1	0	0	0	0	0	0	0	0	0
2	0.09244	0.09122	0.09005	0.08893	0.08784	0.08680	0.08579	0.08482	0.08387
3	0.18245	0.18011	0.17786	0.17569	0.17359	0.17158	0.16963	0.16774	0.16592
4	0.26899	0.26566	0.26245	0.25935	0.25636	0.25347	0.25067	0.24797	0.24535
5	0.35121	0.34703	0.34301	0.33912	0.33535	0.33172	0.32819	0.32478	0.32147
6	0.42841	0.42357	0.41888	0.41435	0.40996	0.40571	0.40159	0.39759	0.39370
7	0.50010	0.49476	0.48959	0.48457	0.47970	0.47498	0.47040	0.46594	0.46160
8	0.56596	0.56029	0.55479	0.54945	0.54426	0.53920	0.53429	0.52951	0.52485
9	0.62582	0.62000	0.61433	0.60880	0.60343	0.59818	0.59307	0.58809	0.58322
10	0.67971	0.67386	0.66816	0.66260	0.65716	0.65186	0.64667	0.64160	0.63664
11	0.72773	0.72200	0.71639	0.71090	0.70553	0.70027	0.69511	0.69006	0.68512
12	0.77013	0.76462	0.75920	0.75389	0.74868	0.74356	0.73853	0.73359	0.72874
13	0.80724	0.80202	0.79688	0.79182	0.78684	0.78194	0.77712	0.77237	0.76770
14	0.83942	0.83456	0.82976	0.82501	0.82033	0.81571	0.81115	0.80665	0.80221
15	0.86710	0.86263	0.85820	0.85382	0.84948	0.84518	0.84093	0.83672	0.83255
16	0.89070	0.88665	0.88263	0.87863	0.87465	0.87071	0.86679	0.86290	0.85904
17	0.91068	0.90705	0.90343	0.89982	0.89623	0.89265	0.88908	0.88553	0.88200

n-$2c$	$c = 36$	$c = 37$	$c = 38$	$c = 39$	$c = 40$	$c = 41$	$c = 42$	$c = 43$	$c = 44$
18	0.92745	0.92424	0.92102	0.91781	0.91459	0.91137	0.90816	0.90495	0.90176
19	0.94142	0.93861	0.93579	0.93295	0.93010	0.92724	0.92438	0.92151	0.91865
20	0.95299	0.95055	0.94809	0.94561	0.94311	0.94060	0.93807	0.93554	0.93300
21	0.96248	0.96039	0.95827	0.95612	0.95395	0.95177	0.94956	0.94733	0.94509
22	0.97023	0.96845	0.96664	0.96480	0.96293	0.96104	0.95913	0.95719	0.95524
23	0.97651	0.97501	0.97347	0.97191	0.97032	0.96870	0.96705	0.96538	0.96369
24	0.98157	0.98031	0.97902	0.97770	0.97635	0.97498	0.97358	0.97214	0.97070
25	0.98561	0.98457	0.98349	0.98239	0.98126	0.98010	0.97891	0.97770	0.97646
26	0.98882	0.98797	0.98708	0.98616	0.98522	0.98424	0.98326	0.98223	0.98119
27	0.99137	0.99066	0.98994	0.98918	0.98840	0.98760	0.98677	0.98591	0.98504
28	0.99336	0.99279	0.99220	0.99158	0.99094	0.99028	0.98959	0.98888	0.98815
29	0.99492	0.99446	0.99398	0.99348	0.99296	0.99242	0.99185	0.99127	0.99066
30	0.99613	0.99577	0.99538	0.99498	0.99456	0.99411	0.99365	0.99317	0.99267
31	0.99707	0.99678	0.99647	0.99615	0.99581	0.99545	0.99507	0.99468	0.99428
32	0.99779	0.99756	0.99731	0.99706	0.99679	0.99650	0.99620	0.99588	0.99555
33	0.99834	0.99816	0.99797	0.99776	0.99755	0.99732	0.99707	0.99682	0.99655
34	0.99876	0.99862	0.99847	0.99831	0.99814	0.99795	0.99776	0.99755	0.99734
35	0.99908	0.99897	0.99885	0.99872	0.99859	0.99844	0.99829	0.99813	0.99796
36	0.99932	0.99923	0.99914	0.99904	0.99893	0.99882	0.99870	0.99857	0.99843
37	0.99949	0.99943	0.99936	0.99928	0.99920	0.99911	0.99902	0.99891	0.99881
38	0.99962	0.99958	0.99952	0.99946	0.99940	0.99933	0.99926	0.99918	0.99909
39	0.99972	0.99969	0.99965	0.99960	0.99955	0.99950	0.99944	0.99938	0.99931
40	0.99980	0.99977	0.99974	0.99970	0.99967	0.99963	0.99958	0.99953	0.99948
41	0.99985	0.99983	0.99981	0.99978	0.99975	0.99972	0.99969	0.99965	0.99961
42	0.99989	0.99988	0.99986	0.99984	0.99982	0.99979	0.99977	0.99974	0.99971
43	0.99992	0.99991	0.99990	0.99988	0.99987	0.99985	0.99983	0.99981	0.99978
44	0.99994	0.99994	0.99993	0.99991	0.99990	0.99989	0.99987	0.99986	0.99984
45	0.99996	0.99995	0.99995	0.99994	0.99993	0.99992	0.99991	0.99989	0.99988
46	0.99997	0.99997	0.99996	0.99995	0.99995	0.99994	0.99993	0.99992	0.99991
47	0.99998	0.99998	0.99997	0.99997	0.99996	0.99996	0.99995	0.99994	0.99993
48	0.99999	0.99998	0.99998	0.99998	0.99997	0.99997	0.99996	0.99996	0.99995
49	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99997	0.99997	0.99996
50	0.99999	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99997
51	1.00000	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99998
52	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
53	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999
54	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999
55	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

n-$2c$	$c = 45$	$c = 46$	$c = 47$	$c = 48$	$c = 49$	$c = 50$	$c = 51$	$c = 52$	$c = 53$
1	0	0	0	0	0	0	0	0	0
2	0.08296	0.08208	0.08122	0.08039	0.07959	0.07881	0.07805	0.07731	0.07660
3	0.16415	0.16244	0.16079	0.15918	0.15762	0.15610	0.15463	0.15320	0.15180
4	0.24281	0.24035	0.23796	0.23565	0.23340	0.23121	0.22908	0.22701	0.22500
5	0.31826	0.31514	0.31211	0.30917	0.30631	0.30353	0.30082	0.29819	0.29562
6	0.38993	0.38626	0.38270	0.37923	0.37586	0.37257	0.36936	0.36624	0.36320
7	0.45739	0.45328	0.44929	0.44540	0.44160	0.43791	0.43430	0.43078	0.42734
8	0.52031	0.51588	0.51156	0.50735	0.50324	0.49923	0.49532	0.49149	0.48774
9	0.57847	0.57384	0.56931	0.56488	0.56056	0.55633	0.55219	0.54815	0.54418
10	0.63180	0.62705	0.62241	0.61787	0.61343	0.60907	0.60480	0.60063	0.59653
11	0.68027	0.67552	0.67086	0.66629	0.66181	0.65742	0.65311	0.64887	0.64472
12	0.72398	0.71931	0.71471	0.71020	0.70576	0.70141	0.69712	0.69291	0.68877
13	0.76310	0.75857	0.75411	0.74972	0.74540	0.74114	0.73695	0.73282	0.72875

n-2c	c = 45	c = 46	c = 47	c = 48	c = 49	c = 50	c = 51	c = 52	c = 53
14	0.79782	0.79350	0.78923	0.78503	0.78087	0.77677	0.77272	0.76873	0.76480
15	0.82843	0.82436	0.82033	0.81634	0.81240	0.80850	0.80465	0.80084	0.79707
16	0.85522	0.85142	0.84766	0.84393	0.84024	0.83658	0.83295	0.82935	0.82579
17	0.87849	0.87499	0.87152	0.86807	0.86465	0.86125	0.85787	0.85451	0.85118
18	0.89857	0.89538	0.89222	0.88906	0.88592	0.88279	0.87967	0.87657	0.87349
19	0.91578	0.91291	0.91004	0.90718	0.90433	0.90147	0.89863	0.89575	0.89297
20	0.93044	0.92787	0.92531	0.92274	0.92016	0.91759	0.91502	0.91245	0.90988
21	0.94284	0.94057	0.93829	0.93600	0.93371	0.93140	0.92910	0.92678	0.92447
22	0.95326	0.95127	0.94927	0.94725	0.94522	0.94318	0.94112	0.93906	0.93699
23	0.96198	0.96025	0.95850	0.95673	0.95495	0.95315	0.95134	0.94951	0.94768
24	0.96922	0.96773	0.96621	0.96468	0.96312	0.96155	0.95996	0.95836	0.95674
25	0.97521	0.97392	0.97262	0.97130	0.96995	0.96859	0.96721	0.96582	0.96440
26	0.98012	0.97903	0.97792	0.97678	0.97563	0.97446	0.97326	0.97206	0.97083
27	0.98414	0.98321	0.98227	0.98131	0.98032	0.97932	0.97830	0.97726	0.97620
28	0.98740	0.98662	0.98583	0.98502	0.98418	0.98333	0.98246	0.98157	0.98066
29	0.99004	0.98939	0.98873	0.98804	0.98734	0.98662	0.98588	0.98513	0.98436
30	0.99215	0.99162	0.99107	0.99050	0.98991	0.98931	0.98869	0.98805	0.98740
31	0.99385	0.99341	0.99295	0.99248	0.99199	0.99149	0.99097	0.99044	0.98989
32	0.99520	0.99484	0.99447	0.99408	0.99367	0.99325	0.99282	0.99238	0.99192
33	0.99627	0.99598	0.99567	0.99535	0.99502	0.99468	0.99432	0.99395	0.99357
34	0.99711	0.99688	0.99663	0.99637	0.99610	0.99582	0.99552	0.99522	0.99490
35	0.99777	0.99758	0.99736	0.99717	0.99696	0.99672	0.99648	0.99623	0.99597
36	0.99829	0.99814	0.99798	0.99781	0.99763	0.99744	0.99725	0.99704	0.99683
37	0.99869	0.99857	0.99844	0.99830	0.99816	0.99801	0.99785	0.99769	0.99751
38	0.99900	0.99891	0.99880	0.99870	0.99858	0.99846	0.99833	0.99820	0.99806
39	0.99924	0.99917	0.99908	0.99900	0.99891	0.99881	0.99871	0.99860	0.99849
40	0.99943	0.99937	0.99930	0.99924	0.99916	0.99909	0.99901	0.99892	0.99883
41	0.99957	0.99952	0.99947	0.99942	0.99936	0.99930	0.99924	0.99917	0.99909
42	0.99968	0.99964	0.99960	0.99956	0.99951	0.99947	0.99941	0.99936	0.99930
43	0.99976	0.99973	0.99970	0.99967	0.99963	0.99959	0.99955	0.99951	0.99946
44	0.99982	0.99980	0.99977	0.99975	0.99972	0.99969	0.99966	0.99963	0.99959
45	0.99986	0.99985	0.99983	0.99981	0.99979	0.99977	0.99974	0.99971	0.99969
46	0.99990	0.99989	0.99987	0.99986	0.99984	0.99982	0.99980	0.99978	0.99976
47	0.99992	0.99991	0.99990	0.99989	0.99988	0.99987	0.99985	0.99984	0.99982
48	0.99994	0.99994	0.99993	0.99992	0.99991	0.99990	0.99989	0.99988	0.99986
49	0.99996	0.99995	0.99995	0.99994	0.99993	0.99992	0.99992	0.99991	0.99990
50	0.99997	0.99996	0.99996	0.99996	0.99995	0.99994	0.99994	0.99993	0.99992
51	0.99998	0.99997	0.99997	0.99997	0.99996	0.99996	0.99995	0.99995	0.99994
52	0.99998	0.99998	0.99998	0.99998	0.99997	0.99997	0.99997	0.99996	0.99996
53	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99997	0.99997	0.99997
54	0.99999	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99998
55	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998
56	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
57	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999	0.99999
58	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
59	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999
60	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

n-2c	c = 54	c = 55	c = 56	c = 57	c = 58	c = 59	c = 60	c = 61	c = 62
1	0	0	0	0	0	0	0	0	0
2	0.07590	0.07522	0.07456	0.07392	0.07330	0.07268	0.07209	0.07151	0.07094
3	0.15045	0.14913	0.14784	0.14659	0.14537	0.14418	0.14302	0.14188	0.14078
4	0.22303	0.22112	0.21926	0.21744	0.21567	0.21394	0.21225	0.21060	0.20899

n-$2c$	$c = 54$	$c = 55$	$c = 56$	$c = 57$	$c = 58$	$c = 59$	$c = 60$	$c = 61$	$c = 62$
5	0.29312	0.29068	0.28830	0.28597	0.28371	0.28149	0.27933	0.27721	0.27515
6	0.36023	0.35733	0.35450	0.35174	0.34904	0.34640	0.34382	0.34130	0.33883
7	0.42399	0.42071	0.41751	0.41437	0.41131	0.40832	0.40538	0.40251	0.39970
8	0.48409	0.48051	0.47701	0.47358	0.47023	0.46695	0.46373	0.46058	0.45749
9	0.54031	0.53651	0.53279	0.52915	0.52558	0.52208	0.51865	0.51528	0.51198
10	0.59251	0.58858	0.58471	0.58093	0.57721	0.57356	0.56998	0.56646	0.56301
11	0.64064	0.63664	0.63270	0.62884	0.62504	0.62131	0.61764	0.61404	0.61050
12	0.68470	0.68069	0.67675	0.67288	0.66906	0.66531	0.66162	0.65798	0.65441
13	0.72475	0.72080	0.71691	0.71308	0.70931	0.70559	0.70193	0.69832	0.69476
14	0.76091	0.75708	0.75329	0.74956	0.74587	0.74224	0.73865	0.73511	0.73161
15	0.79335	0.78967	0.78603	0.78244	0.77888	0.77537	0.77190	0.76846	0.76507
16	0.82227	0.81877	0.81532	0.81189	0.80850	0.80514	0.80182	0.79853	0.79527
17	0.84788	0.84460	0.84135	0.83812	0.83492	0.83174	0.82859	0.82547	0.82237
18	0.87042	0.86737	0.86434	0.86133	0.85834	0.85536	0.85241	0.84947	0.84656
19	0.89015	0.88734	0.88454	0.88176	0.87899	0.87622	0.87348	0.87074	0.86802
20	0.90731	0.90475	0.90219	0.89964	0.89709	0.89455	0.89201	0.88949	0.88697
21	0.92215	0.91983	0.91751	0.91519	0.91287	0.91055	0.90823	0.90592	0.90361
22	0.93491	0.93283	0.93074	0.92865	0.92655	0.92445	0.92235	0.92024	0.91814
23	0.94583	0.94397	0.94210	0.94023	0.93835	0.93646	0.93457	0.93267	0.93077
24	0.95511	0.95347	0.95182	0.95015	0.94848	0.94679	0.94510	0.94340	0.94169
25	0.96297	0.96153	0.96007	0.95860	0.95712	0.95563	0.95412	0.95261	0.95108
26	0.96958	0.96833	0.96705	0.96576	0.96446	0.96315	0.96182	0.96048	0.95913
27	0.97512	0.97403	0.97293	0.97181	0.97067	0.96952	0.96836	0.96718	0.96599
28	0.97974	0.97880	0.97785	0.97688	0.97589	0.97489	0.97388	0.97285	0.97181
29	0.98357	0.98276	0.98194	0.98111	0.98026	0.97940	0.97852	0.97763	0.97672
30	0.98673	0.98604	0.98534	0.98463	0.98390	0.98316	0.98240	0.98163	0.98085
31	0.98932	0.98874	0.98815	0.98754	0.98692	0.98629	0.98564	0.98498	0.98431
32	0.99144	0.99096	0.99046	0.98994	0.98942	0.98888	0.98833	0.98776	0.98719
33	0.99317	0.99276	0.99235	0.99191	0.99147	0.99101	0.99055	0.99007	0.98958
34	0.99457	0.99423	0.99388	0.99352	0.99315	0.99277	0.99237	0.99197	0.99156
35	0.99570	0.99542	0.99513	0.99483	0.99452	0.99420	0.99387	0.99353	0.99318
36	0.99661	0.99638	0.99614	0.99589	0.99563	0.99536	0.99509	0.99480	0.99451
37	0.99733	0.99714	0.99695	0.99674	0.99653	0.99631	0.99608	0.99584	0.99560
38	0.99791	0.99776	0.99760	0.99743	0.99725	0.99707	0.99688	0.99669	0.99648
39	0.99837	0.99824	0.99811	0.99798	0.99783	0.99768	0.99753	0.99737	0.99720
40	0.99873	0.99863	0.99852	0.99841	0.99830	0.99817	0.99805	0.99791	0.99778
41	0.99902	0.99894	0.99885	0.99876	0.99866	0.99857	0.99846	0.99835	0.99824
42	0.99924	0.99918	0.99911	0.99904	0.99896	0.99888	0.99879	0.99870	0.99861
43	0.99941	0.99936	0.99931	0.99925	0.99919	0.99912	0.99906	0.99898	0.99891
44	0.99955	0.99951	0.99946	0.99942	0.99937	0.99932	0.99926	0.99921	0.99914
45	0.99966	0.99962	0.99959	0.99955	0.99951	0.99947	0.99943	0.99938	0.99933
46	0.99974	0.99971	0.99968	0.99966	0.99962	0.99959	0.99956	0.99952	0.99948
47	0.99980	0.99978	0.99976	0.99974	0.99971	0.99968	0.99966	0.99963	0.99960
48	0.99985	0.99983	0.99982	0.99980	0.99978	0.99976	0.99974	0.99971	0.99969
49	0.99989	0.99987	0.99986	0.99985	0.99983	0.99981	0.99980	0.99978	0.99976
50	0.99992	0.99990	0.99989	0.99988	0.99987	0.99986	0.99984	0.99983	0.99981
51	0.99993	0.99993	0.99992	0.99991	0.99990	0.99989	0.99988	0.99987	0.99986
52	0.99995	0.99995	0.99994	0.99993	0.99993	0.99992	0.99991	0.99990	0.99989
53	0.99996	0.99996	0.99995	0.99995	0.99994	0.99994	0.99993	0.99992	0.99992
54	0.99997	0.99997	0.99997	0.99996	0.99996	0.99995	0.99995	0.99994	0.99994
55	0.99998	0.99998	0.99997	0.99997	0.99997	0.99996	0.99996	0.99996	0.99996
56	0.99998	0.99998	0.99998	0.99998	0.99998	0.99997	0.99997	0.99997	0.99996
57	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998	0.99998	0.99998	0.99997

n-2c	c = 54	c = 55	c = 56	c = 57	c = 58	c = 59	c = 60	c = 61	c = 62
58	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998
59	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998	0.99998
60	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
61	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999	0.99999
62	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999
63	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

n-2c	c = 63	c = 64	c = 65	c = 66	c = 67	c = 68	c = 69	c = 70	c = 71
1	0	0	0	0	0	0	0	0	0
2	0.07039	0.06984	0.06932	0.06880	0.06829	0.06780	0.06731	0.06684	0.06637
3	0.13969	0.13863	0.13760	0.13659	0.13560	0.13463	0.13368	0.13275	0.13184
4	0.20742	0.20588	0.20437	0.20290	0.20146	0.20004	0.19866	0.19731	0.19598
5	0.27312	0.27114	0.26921	0.26732	0.26546	0.26364	0.26186	0.26012	0.25841
6	0.33641	0.33405	0.33173	0.32946	0.32724	0.32506	0.32293	0.32084	0.31878
7	0.39695	0.39426	0.39161	0.38902	0.38649	0.38400	0.38155	0.37916	0.37680
8	0.45446	0.45150	0.44858	0.44573	0.44293	0.44018	0.43748	0.43483	0.43222
9	0.50874	0.50556	0.50243	0.49937	0.49636	0.49340	0.49050	0.48764	0.48484
10	0.55962	0.55628	0.55301	0.54979	0.54663	0.54352	0.54046	0.53745	0.53449
11	0.60701	0.60358	0.60021	0.59689	0.59363	0.59042	0.58726	0.58414	0.58108
12	0.65088	0.64742	0.64400	0.64063	0.63732	0.63405	0.63084	0.62767	0.62454
13	0.69125	0.68779	0.68437	0.68101	0.67769	0.67442	0.67119	0.66801	0.66486
14	0.72816	0.72475	0.72138	0.71806	0.71478	0.71154	0.70834	0.70519	0.70206
15	0.76171	0.75840	0.75512	0.75187	0.74867	0.74550	0.74237	0.73927	0.73621
16	0.79204	0.78885	0.78568	0.78255	0.77945	0.77639	0.77335	0.77034	0.76737
17	0.81930	0.81625	0.81323	0.81024	0.80727	0.80433	0.80142	0.79853	0.79566
18	0.84366	0.84078	0.83793	0.83509	0.83228	0.82948	0.82671	0.82396	0.82122
19	0.86531	0.86262	0.85994	0.85728	0.85463	0.85200	0.84938	0.84678	0.84420
20	0.88446	0.88196	0.87947	0.87699	0.87452	0.87206	0.86961	0.86718	0.86475
21	0.90130	0.89900	0.89670	0.89441	0.89212	0.88984	0.88757	0.88530	0.88304
22	0.91603	0.91393	0.91183	0.90972	0.90762	0.90552	0.90343	0.90134	0.89925
23	0.92886	0.92695	0.92504	0.92312	0.92120	0.91929	0.91737	0.91545	0.91354
24	0.93997	0.93825	0.93652	0.93479	0.93306	0.93132	0.92957	0.92783	0.92608
25	0.94955	0.94801	0.94646	0.94490	0.94334	0.94177	0.94020	0.93862	0.93704
26	0.95777	0.95640	0.95502	0.95363	0.95223	0.95083	0.94941	0.94799	0.94657
27	0.96479	0.96358	0.96236	0.96112	0.95988	0.95863	0.95737	0.95610	0.95482
28	0.97076	0.96969	0.96861	0.96753	0.96643	0.96532	0.96420	0.96307	0.96194
29	0.97580	0.97487	0.97393	0.97298	0.97201	0.97104	0.97005	0.96905	0.96805
30	0.98006	0.97925	0.97843	0.97760	0.97675	0.97590	0.97503	0.97416	0.97327
31	0.98362	0.98292	0.98221	0.98149	0.98076	0.98002	0.97926	0.97849	0.97772
32	0.98660	0.98600	0.98539	0.98477	0.98413	0.98349	0.98283	0.98217	0.98149
33	0.98908	0.98857	0.98804	0.98751	0.98696	0.98641	0.98584	0.98526	0.98468
34	0.99113	0.99069	0.99025	0.98979	0.98932	0.98885	0.98836	0.98786	0.98736
35	0.99282	0.99245	0.99207	0.99168	0.99129	0.99088	0.99046	0.99004	0.98960
36	0.99421	0.99390	0.99358	0.99325	0.99291	0.99257	0.99221	0.99185	0.99148
37	0.99535	0.99509	0.99482	0.99454	0.99426	0.99396	0.99366	0.99336	0.99304
38	0.99627	0.99605	0.99583	0.99560	0.99536	0.99511	0.99486	0.99460	0.99433
39	0.99702	0.99684	0.99666	0.99646	0.99626	0.99606	0.99584	0.99562	0.99540
40	0.99763	0.99748	0.99733	0.99717	0.99700	0.99683	0.99665	0.99647	0.99628
41	0.99812	0.99800	0.99787	0.99774	0.99760	0.99746	0.99731	0.99716	0.99700
42	0.99852	0.99841	0.99831	0.99820	0.99809	0.99797	0.99784	0.99772	0.99759
43	0.99883	0.99875	0.99866	0.99857	0.99848	0.99838	0.99828	0.99817	0.99806
44	0.99908	0.99901	0.99894	0.99887	0.99879	0.99871	0.99863	0.99854	0.99845
45	0.99928	0.99923	0.99917	0.99911	0.99904	0.99898	0.99891	0.99884	0.99877

n-$2c$	$c = 63$	$c = 64$	$c = 65$	$c = 66$	$c = 67$	$c = 68$	$c = 69$	$c = 70$	$c = 71$
46	0.99944	0.99939	0.99935	0.99930	0.99925	0.99919	0.99914	0.99908	0.99902
47	0.99956	0.99953	0.99949	0.99945	0.99941	0.99937	0.99932	0.99927	0.99922
48	0.99966	0.99963	0.99960	0.99957	0.99954	0.99950	0.99947	0.99943	0.99939
49	0.99974	0.99971	0.99969	0.99966	0.99964	0.99961	0.99958	0.99955	0.99952
50	0.99980	0.99978	0.99976	0.99974	0.99972	0.99970	0.99967	0.99965	0.99962
51	0.99984	0.99983	0.99982	0.99980	0.99978	0.99976	0.99974	0.99972	0.99970
52	0.99988	0.99987	0.99986	0.99985	0.99983	0.99982	0.99980	0.99979	0.99977
53	0.99991	0.99990	0.99989	0.99988	0.99987	0.99986	0.99985	0.99983	0.99982
54	0.99993	0.99992	0.99992	0.99991	0.99990	0.99989	0.99988	0.99987	0.99986
55	0.99995	0.99994	0.99994	0.99993	0.99992	0.99992	0.99991	0.99990	0.99989
56	0.99996	0.99996	0.99995	0.99995	0.99994	0.99994	0.99993	0.99992	0.99992
57	0.99997	0.99997	0.99996	0.99996	0.99996	0.99995	0.99995	0.99994	0.99994
58	0.99998	0.99997	0.99997	0.99997	0.99997	0.99996	0.99996	0.99995	0.99995
59	0.99998	0.99998	0.99998	0.99998	0.99997	0.99997	0.99997	0.99997	
60	0.99999	0.99999	0.99998	0.99998	0.99998	0.99998	0.99998	0.99997	
61	0.99999	0.99999	0.99999	0.99999	0.99999	0.99998	0.99998		
62	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999		
63	1.00000	0.99999	0.99999	0.99999	0.99999	0.99999			
64	1.00000	1.00000	1.00000	1.00000	0.99999	0.99999			
65	1.00000	1.00000	1.00000	1.00000	1.00000				

n-$2c$	$c = 72$	$c = 73$	$c = 74$	$c = 75$	$c = 76$	$c = 77$	$c = 78$	$c = 79$	$c = 80$
1	0	0	0	0	0	0	0	0	0
2	0.06592	0.06547	0.06504	0.06461	0.06419	0.06378	0.06338	0.06298	0.06259
3	0.13095	0.13008	0.12922	0.12838	0.12756	0.12675	0.12596	0.12518	0.12442
4	0.19468	0.19340	0.19215	0.19093	0.18972	0.18854	0.18738	0.18625	0.18513
5	0.25673	0.25509	0.25347	0.25189	0.25034	0.24881	0.24731	0.24584	0.24439
6	0.31677	0.31479	0.31285	0.31094	0.30908	0.30724	0.30543	0.30366	0.30192
7	0.37450	0.37223	0.37000	0.36782	0.36567	0.36355	0.36148	0.35944	0.35743
8	0.42967	0.42715	0.42468	0.42226	0.41987	0.41752	0.41522	0.41295	0.41072
9	0.48208	0.47937	0.47670	0.47408	0.47149	0.46895	0.46645	0.46399	0.46157
10	0.53158	0.52872	0.52589	0.52312	0.52038	0.51769	0.51504	0.51244	0.50986
11	0.57806	0.57509	0.57216	0.56927	0.56643	0.56363	0.56087	0.55815	0.55547
12	0.62146	0.61842	0.61543	0.61248	0.60957	0.60670	0.60388	0.60107	0.59832
13	0.66176	0.65870	0.65569	0.65271	0.64977	0.64686	0.64400	0.64117	0.63837
14	0.69898	0.69594	0.69293	0.68996	0.68703	0.68412	0.68126	0.67843	0.67563
15	0.73318	0.73018	0.72722	0.72429	0.72139	0.71852	0.71569	0.71288	0.71011
16	0.76442	0.76150	0.75861	0.75575	0.75292	0.75011	0.74734	0.74459	0.74187
17	0.79282	0.79000	0.78721	0.78445	0.78171	0.77899	0.77629	0.77362	0.77098
18	0.81851	0.81582	0.81315	0.81049	0.80786	0.80525	0.80266	0.80009	0.79754
19	0.84163	0.83908	0.83654	0.83402	0.83152	0.82903	0.82656	0.82410	0.82166
20	0.86234	0.85994	0.85755	0.85517	0.85280	0.85045	0.84811	0.84578	0.84347
21	0.88079	0.87855	0.87632	0.87409	0.87187	0.86966	0.86746	0.86528	0.86310
22	0.89717	0.89509	0.89301	0.89094	0.88888	0.88682	0.88477	0.88272	0.88068
23	0.91162	0.90971	0.90780	0.90588	0.90398	0.90207	0.90017	0.89827	0.89637
24	0.92433	0.92285	0.92083	0.91907	0.91732	0.91557	0.91382	0.91206	0.91031
25	0.93545	0.93386	0.93226	0.93066	0.92906	0.92746	0.92586	0.92425	0.92264
26	0.94513	0.94370	0.94225	0.94080	0.93935	0.93790	0.93643	0.93497	0.93350
27	0.95354	0.95224	0.95094	0.94964	0.94833	0.94701	0.94569	0.94436	0.94303
28	0.96079	0.95964	0.95848	0.95731	0.95613	0.95495	0.95376	0.95256	0.95136
29	0.96703	0.96601	0.96497	0.96393	0.96288	0.96182	0.96076	0.95968	0.95860
30	0.97237	0.97147	0.97055	0.96963	0.96870	0.96776	0.96681	0.96585	0.96489
31	0.97693	0.97614	0.97533	0.97452	0.97369	0.97286	0.97202	0.97117	0.97032

n-$2c$	$c = 72$	$c = 73$	$c = 74$	$c = 75$	$c = 76$	$c = 77$	$c = 78$	$c = 79$	$c = 80$
32	0.98081	0.98011	0.97940	0.97869	0.97796	0.97723	0.97649	0.97574	0.97498
33	0.98408	0.98348	0.98286	0.98224	0.98160	0.98096	0.98031	0.97965	0.97899
34	0.98684	0.98632	0.98578	0.98524	0.98469	0.98413	0.98356	0.98299	0.98240
35	0.98916	0.98871	0.98825	0.98778	0.98730	0.98682	0.98632	0.98582	0.98531
36	0.99110	0.99071	0.99032	0.98991	0.98950	0.98908	0.98865	0.98822	0.98778
37	0.99272	0.99239	0.99205	0.99170	0.99135	0.99099	0.99062	0.99024	0.98986
38	0.99406	0.99378	0.99349	0.99319	0.99289	0.99258	0.99227	0.99194	0.99161
39	0.99517	0.99493	0.99469	0.99443	0.99418	0.99391	0.99364	0.99337	0.99308
40	0.99608	0.99588	0.99568	0.99546	0.99524	0.99502	0.99479	0.99456	0.99431
41	0.99683	0.99667	0.99649	0.99631	0.99613	0.99594	0.99574	0.99554	
42	0.99745	0.99731	0.99716	0.99701	0.99686	0.99670	0.99653	0.99637	
43	0.99795	0.99783	0.99771	0.99759	0.99746	0.99732	0.99718		
44	0.99836	0.99826	0.99816	0.99806	0.99795	0.99784	0.99772		
45	0.99869	0.99861	0.99853	0.99844	0.99835	0.99826			
46	0.99896	0.99889	0.99882	0.99875	0.99867	0.99860			
47	0.99917	0.99912	0.99906	0.99900	0.99894				
48	0.99934	0.99930	0.99925	0.99920	0.99915				
49	0.99948	0.99945	0.99941	0.99937					
50	0.99959	0.99956	0.99953	0.99950					
51	0.99968	0.99966	0.99963						
52	0.99975	0.99973	0.99971						
53	0.99980	0.99979							
54	0.99985	0.99984							
55	0.99988								
56	0.99991								

n-$2c$	$c = 81$	$c = 82$	$c = 83$	$c = 84$	$c = 85$	$c = 86$	$c = 87$	$c = 88$	$c = 89$
1	0	0	0	0	0	0	0	0	0
2	0.06221	0.06183	0.06147	0.06111	0.06075	0.06040	0.06006	0.05972	0.05939
3	0.12367	0.12293	0.12221	0.12150	0.12080	0.12011	0.11944	0.11878	0.11812
4	0.18403	0.18296	0.18189	0.18085	0.17983	0.17882	0.17783	0.17686	0.17590
5	0.24297	0.24158	0.24021	0.23886	0.23753	0.23623	0.23494	0.23368	0.23244
6	0.30021	0.29852	0.29686	0.29523	0.29363	0.29205	0.29050	0.28897	0.28747
7	0.35546	0.35352	0.35161	0.34973	0.34788	0.34606	0.34426	0.34250	0.34076
8	0.40852	0.40635	0.40422	0.40213	0.40006	0.39803	0.39603	0.39406	0.39211
9	0.45919	0.45684	0.45453	0.45225	0.45000	0.44779	0.44561	0.44346	0.44135
10	0.50733	0.50483	0.50237	0.49995	0.49756	0.49520	0.49287	0.49058	0.48832
11	0.55282	0.55021	0.54764	0.54510	0.54260	0.54013	0.53770	0.53529	0.53292
12	0.59560	0.59291	0.59026	0.58765	0.58507	0.58252	0.58000	0.57752	0.57507
13	0.63561	0.63289	0.63019	0.62753	0.62491	0.62231	0.61975	0.61721	0.61471
14	0.67286	0.67012	0.66742	0.66475	0.66210	0.65949	0.65691	0.65435	0.65183
15	0.70736	0.70465	0.70196	0.69930	0.69667	0.69407	0.69149	0.68894	0.68642
16	0.73917	0.73650	0.73386	0.73124	0.72865	0.72608	0.72354	0.72102	0.71852
17	0.76836	0.76575	0.76318	0.76062	0.75809	0.75558	0.75309	0.75062	0.74818
18	0.79501	0.79250	0.79000	0.78753	0.78508	0.78264	0.78023	0.77783	0.77545
19	0.81924	0.81683	0.81444	0.81207	0.80971	0.80737	0.80504	0.80273	0.80044
20	0.84117	0.83888	0.83660	0.83434	0.83209	0.82985	0.82763	0.82542	0.82322
21	0.86092	0.85876	0.85661	0.85447	0.85234	0.85022	0.84811	0.84600	0.84391
22	0.87865	0.87662	0.87460	0.87259	0.87058	0.86858	0.86659	0.86461	0.86263
23	0.89448	0.89259	0.89071	0.88883	0.88695	0.88508	0.88321	0.88135	
24	0.90856	0.90681	0.90506	0.90331	0.90157	0.89983	0.89810	0.89636	
25	0.92103	0.91942	0.91780	0.91620	0.91459	0.91298	0.91137		
26	0.93203	0.93056	0.92908	0.92760	0.92613	0.92464	0.92316		

n-2c	c = 81	c = 82	c = 83	c = 84	c = 85	c = 86
27	0.94169	0.94035	0.93901	0.93766	0.93631	0.93496
28	0.95015	0.94894	0.94772	0.94649	0.94527	0.94403
29	0.95752	0.95643	0.95533	0.95422	0.95311	
30	0.96392	0.96294	0.96195	0.96096	0.95996	
31	0.96945	0.96858	0.96770	0.96681		
32	0.97422	0.97344	0.97266	0.97187		
33	0.97831	0.97763	0.97694			
34	0.98182	0.98121	0.98060			
35	0.98479	0.98427				
36	0.98733	0.98687				
37	0.98947					
38	0.99128					

n-2c	c = 90	c = 91	c = 92	c = 93	c = 94	c = 95	c = 96	c = 97	c = 98
1	0	0	0	0	0	0	0	0	0
2	0.05906	0.05874	0.05842	0.05811	0.05781	0.05751	0.05721	0.05692	0.05663
3	0.11748	0.11685	0.11623	0.11562	0.11501	0.11442	0.11384	0.11326	0.11270
4	0.17496	0.17403	0.17312	0.17222	0.17134	0.17046	0.16961	0.16876	0.16793
5	0.23121	0.23001	0.22882	0.22766	0.22651	0.22538	0.22426	0.22316	
6	0.28599	0.28453	0.28309	0.28168	0.28029	0.27891	0.27756	0.27623	
7	0.33905	0.33736	0.33570	0.33407	0.33245	0.33086	0.32929		
8	0.39020	0.38831	0.38645	0.38462	0.38281	0.38103	0.37927		
9	0.43926	0.43720	0.43517	0.43317	0.43119	0.42925			
10	0.48609	0.48389	0.48172	0.47958	0.47746	0.47538			
11	0.53058	0.52827	0.52598	0.52373	0.52151				
12	0.57264	0.57025	0.56789	0.56555	0.56325				
13	0.61224	0.60979	0.60737	0.60498					
14	0.64933	0.64686	0.64441	0.64200					
15	0.68393	0.68145	0.67901						
16	0.71605	0.71360	0.71118						
17	0.74575	0.74335							
18	0.77309	0.77075							
19	0.79816								
20	0.82104								

n-2c	c = 99
1	0
2	0.05635

A TABLE OF PARTITIONS INTO TWO SQUARES WITH AN APPLICATION TO RATIONAL TRIANGLES.

BY

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(*Report R* 68 *of the Computation Department of the Mathematical Centre at Amsterdam*)

(Communicated by Prof. D. VAN DANTZIG at the meeting of April 29, 1950)

1. *Introduction.* In order to solve a specific problem on rational triangles, mentioned in section 3, a table was needed of all partitions of a number n into two squares according to

$$n = p^2 + q^2 \tag{1. 1}$$

for n up to many thousands. C. E. BICKMORE and O. WESTERN [1]) gave such a table but only for n up to 1000, whereas M. RIGNAUX [2]) announced a manuscript table up to $n = 10000$ that is, however, not published to our knowledge. Therefore, such a table was prepared that is reproduced here in full.

2. *Preparation of the table.* It is well known, which numbers n can be written in the form (1. 1). In fact they are exactly the numbers that have no prime factors of the form $4k + 3$ to an odd power. Also the number of partitions can be determined without difficulty for each n whose factors are known but the determination of the actual partitions, though straightforward, is rather tedious. In constructing a table of some size, one better starts, therefore, from the right hand side of (1. 1).

On a National accounting-machine an auxiliary double-entry table of n was made by building up from a constant second difference for $p^2 + q^2 \leq 10000$. Twelve columns were produced simultaneously. From this auxiliary table the wanted one was constructed by rewriting the partitions p, q as function of n. This table was first checked against omissions by counting the number of partitions in both tables between round values of n, and next against reproduction errors by verifying that (1. 1) holds for all entries. The same checks were applied to the manuscript for the printer (that had to be arranged in another way) and to the proofsheets.

The table is arranged in six lines of three columns each. The first column gives n, the second and third p and q. In case of double and multiple partitions the argument n is not repeated in order to facilitate their recognition.

1) C. E. BICKMORE and O. WESTERN, Messenger Math. 41, 52—64 (1911).

2) M. RIGNAUX, L'intermédiaire des math., 25, 143 (1918), 26, 54—55 (1919).

3. *An application of the table in the field of rational triangles.* An application of this table, which was in fact the immediate cause for its construction, is the finding of the "smallest" triangle (a, b, c) with integer sides a, b and c and three rational medians. Here, by definition, a triangle (a, b, c) is smaller than another triangle (A, B, C), where $a \leqslant b \leqslant c$, and $A \leqslant B \leqslant C$, if either $c < C$ or $c = C$ and $b < B$ or $c = C$, $b = B$ and $a < A$. L. EULER[3]) has given already some examples of these triangles, the smallest of which is (68, 85, 87). Several others have been found later on and by following the method of J. H. J. ALMERING[4]) arbitrarily many can be constructed without difficulty. This method, however, does not permit to state what is the smallest solution, but the fact that nobody ever found a smaller triangle than EULER's one, mentioned above, induces the hypothesis, that it is indeed the smallest one.

Of course, for the prove of this hypothesis, it has only to be verified that no smaller triangle than (68, 85, 87) has three rational medians, but the amount of smaller triangles is excessive. It shall be shown, however, that by the use of the table this verification is only a minor computing job.

Be A, B and C the double medians on the sides a, b and c respectively (so that c and C are diagonals of the parallelogram with sides a and b. etc.). They follow from

$$2a^2 + 2b^2 - c^2 = C^2, \tag{3. 1}$$

$$2a^2 - b^2 + 2c^2 = B^2, \tag{3. 2}$$

$$-a^2 + 2b^2 + 2c^2 = A^2. \tag{3. 3}$$

If a, b and c are all odd, or if a and b are even, and c is odd, then $C^2 = 3 \pmod 4$, so C is irrational. If a, b and c are all even, $a/2$, $b/2$, $c/2$ are integers and define a smaller triangle. Without loss of generality it can, therefore, be assumed that a and b are odd and c is even. Then (3. 1) can be written in the form

$$\left(\frac{-a+b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 = \left(\frac{c}{2}\right)^2 + \left(\frac{C}{2}\right)^2 = n, \tag{3. 4}$$

where all terms between brackets are integers. Moreover, the inequalities of the triangle require $(-a+b)/2 < c/2 < (a+b)/2$. Hence, n is a number permitting a double partition, and possible values of n are found directly by inspection of the table. As for the smallest triangle certainly holds $a \leqslant 85$, $b \leqslant 87$, $n = (a^2 + b^2)/2 \leqslant 7397$.

If a number n to be investigated has the partitions $n = p^2 + q^2 = r^2 + s^2$, with $p < q$, $r < s$ and $p < r < q$ then is also $p < s < q$. Therefore, each double partition with $r \neq s$ gives rise to two triangles, viz. $(-p+q,\ p+q,\ 2r)$ and $(-p+q,\ p+q,\ 2s)$.

[3]) L. EULER, Novi Comm. Acad. Petrop., **18**, 171 (1773).

[4]) J. H. J. ALMERING, "Rationaliteitseigenschappen in de vlakke meetkunde", Thesis (1950), Amsterdam.

Not all double partitions come into consideration, however. As a and b are odd, n is also odd. In particular, the case $r = s$ cannot occur. Also cases where p, q, r, s have a factor in common may be rejected.

In this way only a rather limited amount of values of n need a closer investigation. For those n the corresponding a, b and c were computed and inserted into (3. 2). In the few cases that B^2 proved to be the square of an integer, they were also inserted into (3. 3), and in only one case A^2 was the square of an integer, viz. in that of EULER's triangle (68, 85, 87). This verification was checked by complete duplication. Hence, the smallest triangle with integer sides and three rational medians is (68, 85, 87).

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0	0	0
1	0	1
2	1	1
4	0	2
5	1	2
8	2	2
9	0	3
10	1	3
13	2	3
16	0	4
17	1	4
18	3	3
20	2	4
25	0	5
	3	4
26	1	5
29	2	5
32	4	4
34	3	5
36	0	6
37	1	6
40	2	6
41	4	5
45	3	6
49	0	7
50	1	7
	5	5
52	4	6
53	2	7
58	3	7
61	5	6
64	0	8
65	1	8
	4	7
68	2	8
72	6	6
73	3	8
74	5	7
80	4	8
81	0	9
82	1	9
85	2	9
	6	7
89	5	8
90	3	9
97	4	9
98	7	7
100	0	10
	6	8
101	1	10
104	2	10
106	5	9
109	3	10
113	7	8
116	4	10
117	6	9
121	0	11
122	1	11
125	2	11
	5	10
128	8	8
130	3	11
	7	9
136	6	10
137	4	11
144	0	12
145	1	12
	8	9
146	5	11
148	2	12
149	7	10
153	3	12
157	6	11
160	4	12
162	9	9
164	8	10
169	0	13
	5	12
170	1	13
	7	11
173	2	13
178	3	13
180	6	12
181	9	10
185	4	13
	8	11
193	7	12
194	5	13
196	0	14
197	1	14
200	2	14
	10	10
202	9	11
205	3	14
	6	13
208	8	12
212	4	14
218	7	13
221	5	14
	10	11
225	0	15
	9	12
226	1	15
229	2	15
232	6	14
233	8	13
234	3	15
241	4	15
242	11	11
244	10	12
245	7	14
250	5	15
	9	13
256	0	16
257	1	16
260	2	16
	8	14
261	6	15
265	3	16
	11	12
269	10	13
272	4	16
274	7	15
277	9	14
281	5	16
288	12	12
289	0	17
	8	15
290	1	17
	11	13
292	6	16
293	2	17
296	10	14
298	3	17
305	4	17
	7	16
306	9	15
313	12	13
314	5	17
317	11	14
320	8	16
324	0	18
325	1	18
	6	17
	10	15
328	2	18
333	3	18
337	9	16
338	7	17
	13	13
340	4	18
	12	14
346	11	15
349	5	18
353	8	17
356	10	16
360	6	18
361	0	19
362	1	19
365	2	19
	13	14
369	12	15
370	3	19
	9	17
373	7	18
377	4	19
	11	16
386	5	19
388	8	18
389	10	17
392	14	14
394	13	15
397	6	19
400	0	20
	12	16
401	1	20
404	2	20
405	9	18
409	3	20
410	7	19
	11	17
416	4	20
421	14	15
424	10	18
425	5	20
	8	19
	13	16
433	12	17
436	6	20
441	0	21
442	1	21
	9	19
445	2	21
	11	18
449	7	20
450	3	21
	15	15
452	14	16
457	4	21
458	13	17
461	10	19
464	8	20
466	5	21
468	12	18
477	6	21
481	9	20
	15	16
482	11	19
484	0	22
485	1	22
	14	17
488	2	22
490	7	21
493	3	22
	13	18
500	4	22
	10	20
505	8	21
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8884	28	90
8885	7	94
	62	71
8893	53	78
8900	8	94
	34	88
	50	80
8905	16	93
	21	92
	43	84
	61	72
8906	25	91
	41	85
8912	56	76
8914	45	83
8917	9	94
	39	86
8929	60	73
8933	47	82
8936	10	94
8938	17	93
	37	87
8941	29	90
8945	32	89
	52	79
8948	22	92
8954	55	77
8957	11	94
	26	91
	59	74
8962	49	81
8969	35	88
8973	18	93
8978	67	67
8980	12	94
	66	68
8986	65	69
8989	42	85
	58	75
8992	44	84
8993	23	92
8996	40	86
	64	70
9000	30	90
	54	78
9001	51	80
9005	13	94
	46	83

9010	19	93
	27	91
	33	89
	63	71
9013	38	87
9025	0	95
	57	76
9026	1	95
9028	48	82
	62	72
9029	2	95
9032	14	94
9034	3	95
9040	24	92
	36	88
9041	4	95
9049	20	93
9050	5	95
	53	79
	61	73
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	15	94
	31	90
	50	81
9065	28	91
	56	77
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9077	34	89
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	65	70
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9137	64	71
9140	26	92
	58	76
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	35	89
9153	63	72
9157	54	79
9160	18	94
	42	86
9161	44	85
9162	51	81
9169	12	95
	40	87
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9173	62	73
9178	23	93
	57	77
9181	30	91
9188	38	88
9189	33	90
9193	27	92
	48	83
9194	13	95
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	64	72
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	54	80
9320	22	94
	62	74
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	35	90
	51	82
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9333	42	87
	57	78
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	48	84
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10000	0	100
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	60	80

INFLUENCE OF ORGANIC COMPOUNDS ON SOAP AND PHOSPHATIDE COACERVATES — XIII [1])

THE ACTION OF ALIPHATIC, AROMATIC AND MIXED HYDROCARBONS

BY

H. L. BOOIJ, C. J. VOGELSANG AND J. C. LYCKLAMA *), [2]) [3])

(Communicated by Prof. H. G. BUNGENBERG DE JONG at the meeting of April 29, '50)

1. *A comparison between benzene and cyclohexane.*

It has already been shown (BOOIJ, 1949, 2) that there exists a difference between aromatic and aliphatic hydrocarbons as regards their action on the oleate coacervate. Naphtalene had a stronger action than decahydronaphtalene and it was thought that this resulted from the fact that decahydronaphtalene has a stronger tendency to enter the micelle in the plane between the CH_3-end groups of the carbon chains.

Before starting with a description of our experiments with benzene and cyclohexane we would like to say a few words about some terms we have used frequently in our publications, viz. "opening" and "condensing" action. Formerly it was thought that the experiments on oleate coacervates gave some information on the mean distance between the carbon chains of the soap molecules. Then these experiments would be of great value for the problem of biological permeability. An "opening" action of an organic substance would mean an increase of the distance between the carbon chains and — it was hoped — an increase of the pore width of the protoplasmic membrane. It became clear, however, that the relations were not so simple and subsequently we tried to explain our experiments from the hypothesis that the mass of the soap micelles is changed under the influence of the added substance (BOOIJ, 1949, 1). From this it is clear that the words "opening" and "condensing" have lost their original sense. Moreover, it was found that an "opening" action may presumably be the result of two completely different mechanisms. Both the actions of fatty anions (BOOIJ and BUNGENBERG DE JONG, 1949) and of long alkanes (BOOIJ, LYCKLAMA and VOGELSANG, 1950)

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[1]) Publication no. XII of this series will be found in Proc. Kon. Ned. Akad. Wetensch. Amst. 53, 407 (1950).

[2]) We are much indebted to The Rockefeller Foundation for a gift of Na-oleate and to the Koninklijke/Shell-Laboratorium, Amsterdam, for most of the hydrocarbons investigated.

[3]) Publication no. 5 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

must be described as "opening" actions, but it is supposed that in the former case the action takes place between the parallel carbon chains (more or less an "opening" action in the original sense), while in the latter case we think that the alkanes have a tendency to go in between the CH_3-end groups of the carbon chains.

Bungenberg de Jong too has severe objections against the terms mentioned and in a discussion on this topic he proposed the words "KCl-sparing" (in stead of "condensing") and "KCl-demanding" (in stead of "opening") action — *S* and *D* for short. It seems advisable indeed to use these terms in the future as they are more neutral than the original ones and do not suggest relations with the problem of biological permeability which have to be proven yet. It seems likely that in some cases such a relation exists indeed (see e.g. the influence of fatty acid anions on the oleate coacervate, the pea test and the beet test, Veldstra and Booij, 1949), while in other cases a relation seems very doubtful.

Benzene had a lower KCl-sparing (condensing) action than cyclohexane. This is an exception to the rule that aromatic substances without polar groups have a higher KCl-sparing activity than aliphatic ones (see Booij, 1949, 2). It might be possible that the action of benzene is only seemingly less, because part of the added molecules are not taken up in the micelles. We have seen already (Booij et al. 1950, 5) that undecylate has a much higher activity (in this case KCl-demanding or "opening") than nonylate. Per molecule taken up in the micelles, however,

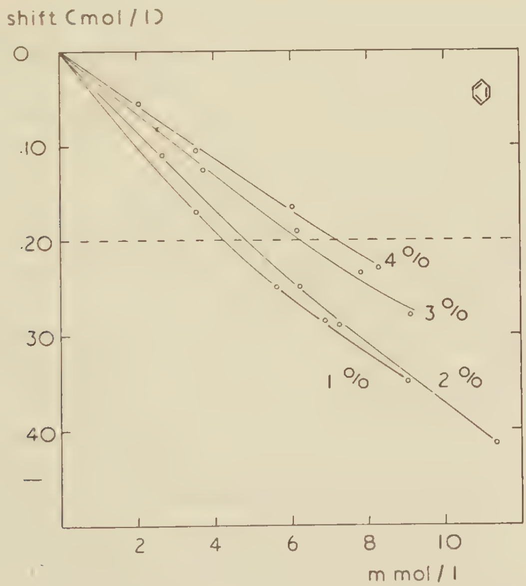

Fig. 1. Influence of benzene on coacervates of oleate (starting from 1, 2, 3 and 4 % solutions of Na-oleate). The activity is given as the shift of the KCl-concentration needed for a coacervate volume of 50 % (— = KCl-sparing activity). Abscissa = concentration of benzene.

the action of nonylate is higher. This does not come to the fore as a large part of the nonylate ions stay outside the micelles. Therefore we decided to examine whether in the case of benzene too many molecules will be found in the medium.

For this purpose we use the following method (Booij and Bungenberg de Jong, 1949). We investigate the action of the added substance at several concentrations of oleate. Presumably the action will be inversely proportional to the concentration of the oleate when all molecules added go into the micelles. If this is not the case this relation will not be found. Thus the influence of benzene and cyclohexane was measured starting from 1, 2, 3 and 4 % oleate solutions. To each oleate solution an extra amount of KOH was added to minimise the influence of carbondioxyde from the air. In order to make the substances dissolve more easily, they were added in a small amount of *n*-propylalcohol (exactly enough to give an unchanged volume of the blank coacervate — 150 mmol/1). See for this method Booij et al., 1950, 5. The results of these experiments are plotted in figs. 1 and 2. It is obvious that cyclohexane has a stronger

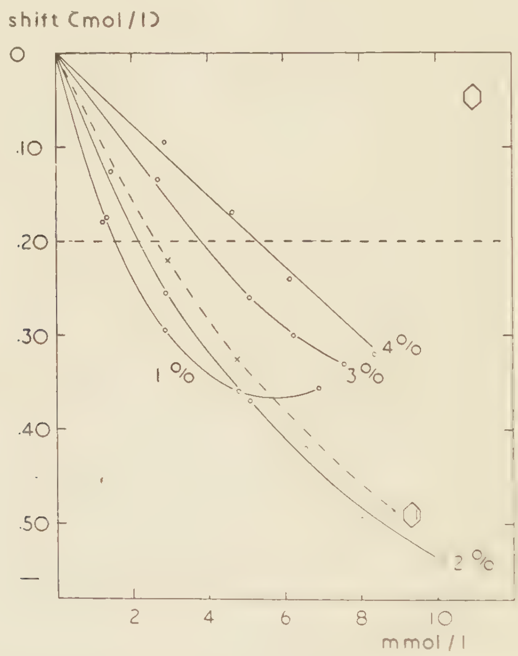

Fig. 2. Influence of cyclohexane (see fig. 1) and cyclohexene (starting from a 2 % solution of Na-oleate).

KCl-sparing action than benzene. In low oleate concentration (1 % in the original solution) cyclohexane shows a curve which resembles that of many aliphatic alkanes. (Booij et al., 1950, 4.) Another interesting point is the curve of cyclohexene. In all cases investigated till now the

introduction of a double bond in an aliphatic molecule resulted in a much stronger KCl-sparing action. This does not hold in the case of cyclohexene.

We will now examine what concentrations of benzene and cyclohexane are needed to give a certain shift of the coacervation curve (e.g. a shift of 0.20 mol/l). These data are plotted against the oleate concentration (fig. 3). The result is clear; whereas for cyclohexane the

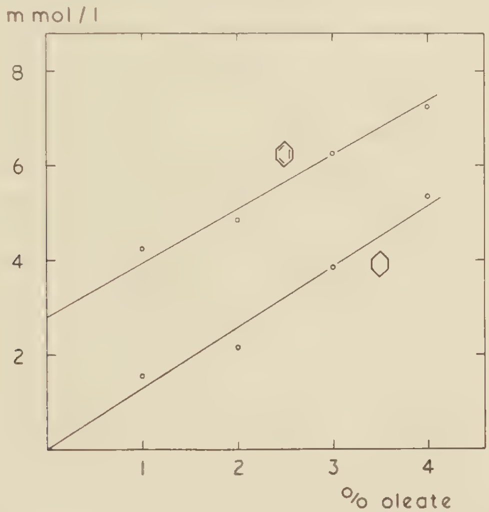

Fig. 3. Ordinate = concentration of added substance needed for a shift of 0.20 mol/l KCl (compare figs. 1 and 2). Abscissa = concentration of the original soap solution. Extrapolation to 0 % gives the equilibrium concentration (for benzene 2.8 mmol/l, for cyclohexane zero).

amount needed is proportional to the oleate concentration (the equilibrium concentration is zero, all molecules are taken up in the micelles), benzene, on the other hand, shows a definite equilibrium concentration (± 2.8 mmol/l). The slope of the two curves is approximately equal and this means that the activity of the molecules taken up in the micelles, is comparable for benzene and cyclohexane (and presumably for cyclohexene, see fig. 2). This would be understandable if these small molecules would be found in the micelles at the same place. For small aliphatic hydrocarbons we have already supposed that these molecules would be concentrated in the annular parts of the micelles (Booij et al., 1950, 4). From the foregoing it seems likely that small nuclei too prefer this place, where differences in polarity of the molecules — introduction of double bonds — do not play an important part.

2. *Condensed aromatic ring systems.*

In fig. 4 we have assembled some data on the action of aromatic compounds on the oleate coacervate. The method used was the same

which has been described in the preceding section; the substances were dissolved in a small amount of n-propylalcohol on which the measured quantity of oleate solution (2 %) was poured. For many compounds our method cannot be used as these substances (stilbene, anthraceen, naphtacene lin., naphtacene ang. and chrysene) do not dissolve readily in n-propylalcohol at room temperature. By heating a solution is obtained

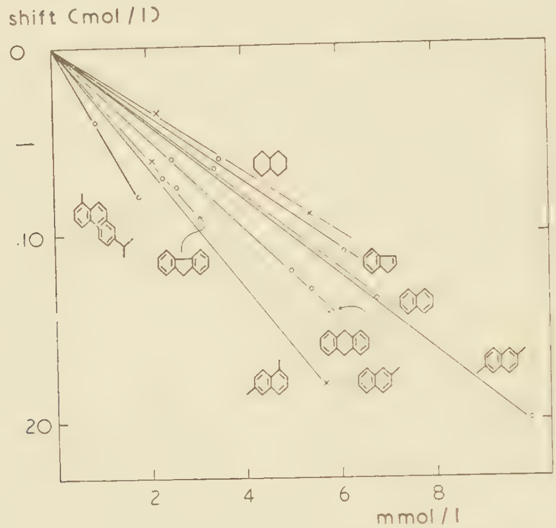

Fig. 4. The action of some condensed aromatic ring systems.

in some cases and after adding an oleate solution a clear solution of the aromatic compound may result. After some time, however, crystallisation sets in. It is obvious that these compounds cannot be studied with our method.

Koets and Bungenberg de Jong (1938) had already studied the influence of some related compounds and it is worth while to compare their results with ours. As their method was completely different, this comparison is not easily performed. They dissolved a certain amount of the aromatic substances in oleic acid, after which an oleate solution containing the substance was made with the aid of KOH. They compared the volume of a coacervate layer of the blank (V_0) with that of the oleate with the aromatic compound (V) at a constant concentration of KCl. Then log V/V_0 was plotted against log C (where C = concentration of the added substance). From their data we see that naphtalene has an action which is 4.4 times as strong as that of benzene. When we plot from our data the shift in KCl concentration against log C we find that naphtalene acts 5.0 times as strong as benzene. Considering that a) the concentrations of the oleate were not equal in both cases, which influences the curve of benzene and b) the presence of n-propylalcohol influences the curve of naphtalene, the agreement between the experiments

is good. With the above mentioned remarks in mind we might give the following series of aromatic compounds, in which the KCl-sparing action increases to the right: benzene < indene < naphtalene < 9, 10-dihydroanthracene < fluorene < anthracene < pyrene < retene — dibenzanthracene < phenanthrene (a combination of the series found by KOETS and BUNGENBERG DE JONG: benzene < naphtalene < anthracene < pyrene < dibenzanthracene < phenanthrene and the series examined by us).

It will not be easy to find correlations between the molecular structure and the activity. The background of this difficulty is presumably that these molecules may go to different places in the micelles. We gave the hypothesis that the presence of such molecules between the parallel carbon chains would result in a KCl-sparing action, when they are found between the planes of the CH_3-end groups a KCl-demanding action comes to the fore (BOOIJ et al. 1950, 4). The action as measured must be the difference between these two factors, in other words must be the result of the equilibrium of distribution of the molecules between the two places mentioned (a third place — see preceding section — might play an important part for short hydrocarbons). Seen from this point of view it must be clear that small alterations in a molecule of this kind may give large (and unpredictable) results. It is not yet understandable why the KCl-sparing action diminishes in the series:
1 : 6-dimethylnaphtalene > 2-methylnaphtalene > 2 : 6-dimethylnaphtalene — naphtalene.

The distribution between the two places in the micelles may be affected in several ways (BOOIJ, 1949, 2). Introduction of an OH-group results in a preference to the situation between the parallel carbon chains — resulting in a stronger KCl-sparing action (naphtol > naphtalene),

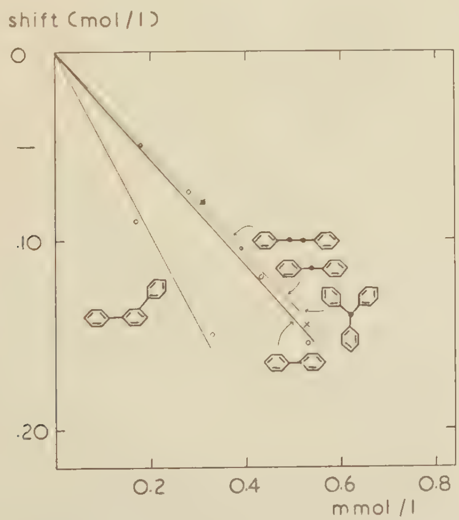

Fig. 5. The action of some non-condensed ring systems.

aliphatic rings show a preference to the place within the micelles (naphtalene > decahydronaphtalene).

Besides, we have observed that the addition of *n*-propylalcohol (the method used throughout this investigation) influences the action of the added compound. With added *n*-propylalcohol the difference in action between naphtalene and decahydronaphtalene is smaller than without propylalcohol (compare Booij, 1949, 2). In the latter case the KCl-sparing action of naphtalene has been found to be stronger, while that of decahydronaphtalene was smaller. Thus the addition of *n*-propylalcohol influences the action of both substances in different directions.

3. *Non-condensed ring systems.*

Here too a clear relation between chemical structure and action on the soap system has not been found (fig. 5). We do not yet understand why diphenylbenzene shows an exceptional behaviour. On the whole the differences between the substances investigated are very slight.

4. *Mixed aromatic/aliphatic hydrocarbons.*

We investigated a number of derivatives of benzene with alkyl groups introduced at different places of the benzene nucleus. Fig. 6 shows some of the results. As has been mentioned already — section 2 — these results are not clear cut. While in a "real" KCl-sparing substance (e.g. an alcohol) the introduction of a methyl group gives an increase of this action (see Booij et al. 1950, 2) the reverse is true for a hydrocarbon (Booij et al. 1950, 4). From this we may conclude that if the action of a substance will be the result of a strong KCl-sparing minus a strong KCl-demanding

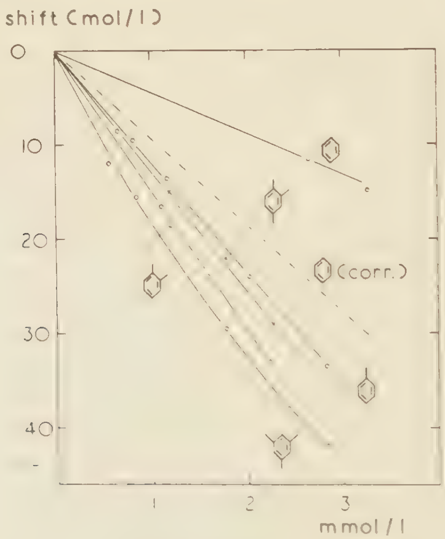

Fig. 6. The influence of the introduction of small alkyl groups into the benzene nucleus.

factor, we cannot foretell the influence of the introduction of a small alkyl group. Both the KCl-sparing or the KCl-demanding factor might be increased. It is clear that we must first investigate these two factors separately (which will be very difficult) before we may say something on their joint action.

We must expect that the KCl-demanding action will grow when we introduce in a benzene nucleus an ever larger alkyl group. This was found indeed (fig. 7). The introduction of small alkyl groups gives as yet unpredictable results; if the aliphatic part of the molecule becomes larger the action will get the same character as that of an aliphatic hydrocarbon. Cetylbenzene shows already at low concentrations a KCl-demanding action. The action is, however, not so high as that of hexade-

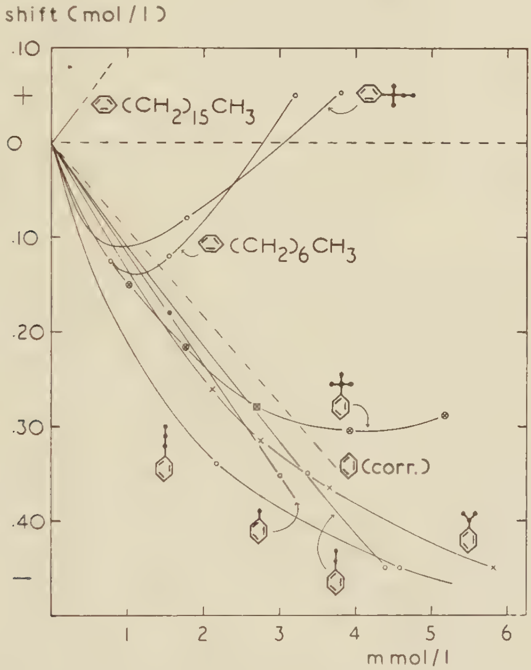

Fig. 7. The influence of an ever increasing alkyl group into a benzene ring.

cane, and this fact is undoubtedly due to the KCl-sparing factor of the benzene nucleus. Concluding we might say that the action of hydrocarbons is the result of two opposing factors, a KCl-demanding factor (long alkyl groups) and a KCl-sparing factor (aromatic nuclei).

Summary.

1. The difference in action of benzene and cyclohexane on the oleate coacervate must be ascribed to the fact that only a part of the benzene molecules are taken up in the soap micelles. The actions of the molecules

taken up in the micelles are practically the same for benzene and cyclohexane.

2. This suggests that these small molecules are taken up in the micelle preferently at the same place as small aliphatic hydrocarbons. It is suggested that this place will be found in the curved annular parts of the micelles.

3. Condensed and uncondensed aromatic ring systems do not yet show a clear relation between chemical structure and activity, presumably because the action of these substances must be seen as the result of two opposing factors. Introduction of e.g. a methyl group will influence both factors and we cannot foretell which influence will be the larger one.

4. In a series of alkylbenzenes an ever stronger KCl-demanding action is found when the alkyl groups grow larger. This was expected because long aliphatic hydrocarbons have a strong KCl-demanding activity.

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LA COUVERTURE ANALE DE HYPOCRINUS

PAR

E. TEN HAAF

(Communicated by Prof. H. A. BROUWER at the meeting of April 29, 1950)

La détermination d'un nombre de crinoïdes du Permien de Timor, dans la collection de l'Institut de Géologie à Amsterdam, fit rencontrer un exemplaire de *Hypocrinus schneideri* BEYR. où la couverture anale, inconnue jusqu' à présent, est complètement conservée.

Avec gratitude, l'auteur reconnaît son obligation à Prof. Dr H. A. BROUWER, Directeur de l'Institut, d'avoir mis la belle collection de fossiles à sa disposition; à Prof. Dr M. G. RUTTEN, de l'intérêt pris à ces travaux et de son aide précieuse dans la préparation de cette publication; et surtout à Prof. Dr J. WANNER, qui l'a fait si généreusement profiter de sa vaste connaissance du sujet.

Le fossile se trouve dans la collection de l'Institut de Géologie à Amsterdam (No. C 2319); provenance Basleo, Timor.

L'orifice anal est entièrement recouvert de 6 plaquettes triangulaires formant une pyramide hexagonale convexe. La porosité superficielle des plaques du calice se continue sur la partie inférieure de ces plaquettes de couverture.

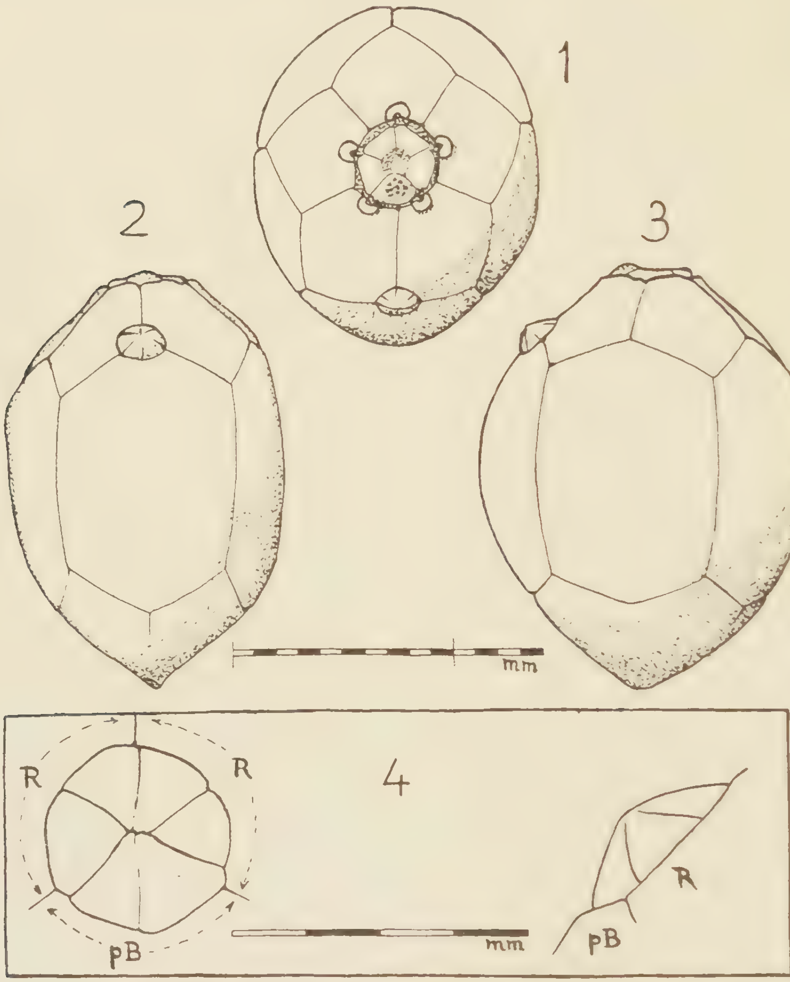
Dans sa correspondance, Prof. WANNER a relevé qu'une couverture anale est connue de quelques autres genres de la famille des *Hypocrinidae* WANN., comprenant les Cyathocrinidés aux bras réduits; or, elle est alors formée d'une seule plaque (*Acariaiocrinus* WANN.) ou d'un grand nombre de plaquettes non différenciées (*Embryocrinus* WANN.). Au contraire, il était frappé de la ressemblance entre la pyramide anale de *Hypocrinus* et celle d'un nombre de Cystidés, dont e.a. *Pyrocystis*, *Amygdalocystis*, *Glyptosphaerites*, *Protocrinus*, *Cryptocrinus* ont une pyramide formée de 6 plaquettes.

On remarquera que *Hypocrinus*, qui est un fossile assez commun dans les dépôts permiens fossilifères de Timor, a été classé pendant longtemps avec les Cystidés, et notamment avec *Cryptocrinus* de l'Ordovicien des environs de Leningrad. Ces deux genres ne diffèrent que dans les plaques de la région péristomale: chez *Cryptocrinus*, elles sont arrangées d'une façon assez irrégulière et variable; les 10 facettes articulaires se trouvent sur des petites plaquettes spéciales et superficielles, au bout de 5 fosses ambulacraires courtes et bifurquées.

Au contraire, les 5 radiales de *Hypocrinus*, pareilles et portant chacune

une facette brachiale complète, bien que minuscule, et ses orales avec un madréporite sur *pO*, sont tout à fait homologues à celles des autres Cyathocrinidés.

Aussi, *Hypocrinus* est à présent reconnu universellement comme un crinoïde, et la découverte d'une pyramide anale de forme cystoïde est



Hypocrinus schneideri BEYR. Permien de Basleo (Timor).

1. Aspect ventral, montrant les orales, les radiales avec leurs facettes minuscules, et la pyramide anale;
2. Aspect postérieur;
3. Aspect latéral;
4. La pyramide anale grossie.

bien surprenante. Cependant, on ne peut plus ramener *Hypocrinus* aux Cystidés. Si sa grande ressemblance à *Cryptocrinus* quant à la pyramide anale et la partie inférieure du calice indiquerait une affinité génétique,

elle supporterait plutôt la thèse que *Cryptocrinus* doit être considéré comme un crinoïde aberrant (e.a. JAEKEL 1921, p. 21, 25) ou comme une transition des *Cystoidea rhombifera* aux crinoïdes (YAKOVLEV 1917, p. 23); mais, compte tenu du grand intervalle stratigraphique entre *Cryptocrinus* de l'Ordovicien et *Hypocrinus* du Permien, l'auteur est d'avis que leur ressemblance ne réprésente probablement qu'une convergence remarquable.

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EMBRYOLOGY

ANALYSIS OF THE DEVELOPMENT OF THE EYE-LENS IN CHICKEN AND FROG EMBRYOS BY MEANS OF THE PRECIPITIN REACTION

BY

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(Communicated by Prof. M. W. WOERDEMAN at the meeting of April 29, 1950)

Introduction.

The developmental physiology of the eye-lens appears to be a very suitable subject for a biochemical analysis of some aspects of the general problem of organ-differentiation, because this part of the eye has a relative simple structure and a remarkable chemical specificity.

This organ-specificity, probably due to its protein components, has been detected by UHLENHUTH (1903) by means of the precipitin reaction, and several authors since then have substantiated his result (e.g. KRAUS, c.s., 1908, KRUSIUS 1910, WOLLMAN, a.o. 1938 *a, b.*). The precipitin test enables us to differentiate simple aqueous extracts of the lens from similar preparations from any other organ of the body, even other parts of the eye itself. Moreover, especially when carried out on a micro scale, this test is an extremely sensitive reaction, so that a hundredth of a microgram of protein substance can be detected. Eventually it seems to become possible to combine this test with the biochemical technique of isolating and concentrating lens proteins, in order to find the slightest traces of these substances in very young embryos.

This possibility opens perspectives for an analysis of lens induction and regeneration but, although these may be very attractive objects to direct the course of investigations, biochemical work on lens proteins in our laboratory has only been started this year and results cannot be awaited within short time.

Meanwhile, applying the precipitin reaction on simple extracts, we carried out an investigation about the presence of adult lens antigens in embryonic lens vesicles before the beginning of morphological differentiation. The reason for this study has been the question, discussed among embryologists, whether the substances, constituting the adult organism, are present already in the youngest stages of ontogenetic development or whether these materials originate only in the course of the differentiation of the embryo.

The old controversy between preformationists and epigeneticists seems to be continued on this chemical level.

Recognizing the important role of the genes, the preformationist standpoint supposes that, at least qualitatively, all of the building substances of the adult organism will be present at the onset of embryonic development, either bound in the genes of the nucleus, in plasmagenes, or free in the cytoplasm. Perhaps the great mass of yolk may conceal the minute amounts of these specialised substances, so that only in the course of development will they become detectable because of their increasing quantity by assimilation of the food from yolk; nevertheless these materials will be present already at the beginning.

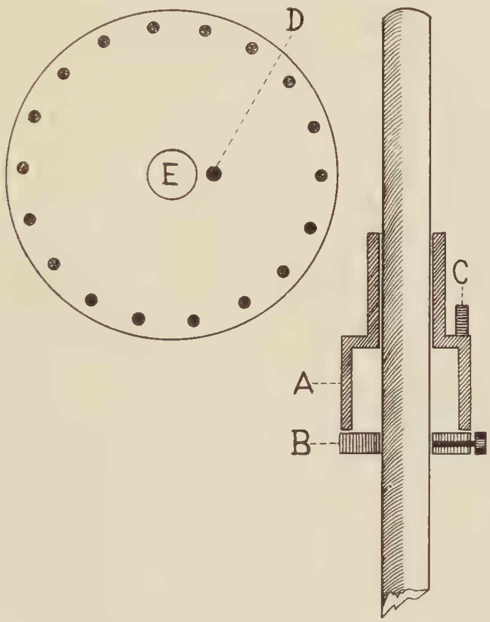

Fig. 1

The epigenetic view supposes that, may it be under the influence of the genes, most of the adult materials will become newly created qualitatively in different stages of development. Perhaps under the impression of the developing complicated form out of a simple egg, many embryologists adhere to the epigenetic viewpoint, which in a biochemical sense has been expressed by Needham (1942) in the following words: "Antigens come into being at successive points in time" (l.c., p. 349). Such opinion, however, based upon negative evidence, has to be changed as soon as positive experimental results are favoring an opposite view. A short survey of the literature concerning the antigenetic properties of the embryonic organism will demonstrate this.

In 1906, Braus arrived to the conclusion that embryos are devoid of any antigenetic action. This was a result of experiments in which he

tried to immunize rabbits by three intraperitoneal injections of a small quantity of an extract, made from Bombinator larvae.

In the same decade, however, other investigators were able to immunise rabbits by eggs of fishes and amphibia: they found a difference between embryos and adult tissues (DUNBAR 1910, UHLENHUTH and HAENDEL 1910). Also KRITCHEVSKI (1914) showed that amphibian larvae have antigenetic capacities, using larvae of Rana esculenta. For his conclusion that, corresponding with HAECKEL's biogenetic law, there is a difference between the chemical structure of embryos and that of adult frogs, the results of the complement fixation tests were not giving very strong evidence; in two out of seven tests the larval antigen reacted with the heterologous (adult) antiserum almost as strongly as with the homologous one and with heterologous antiserum a very great number of plus minus reactions was noted (in 33 out of 40 dilutions).

In 1923, KRITCHEVSKI studied the same problem in chick embryos and from his results inferred the following statement: "The facts presented in this article suggest the conclusion that the biochemical properties of animal cells are subject to transformations during the ontogenetic development" (l.c. p. 194). The author presented the following facts: he had been able to demonstrate the existence of FORSSMAN antigen in embryos of four days and older by the results of immunisation experiments in rabbits, whereas he failed to obtain the same result, using two days old embryos and unincubated eggs. Having indicated that this antigen is contained in the cell nucleus (1916), KRITCHEVSKI also stated that the chemical structure of the nucleus changes during ontogenesis, an opinion that probably will be refuted by most embryologists.

IWAE (1915), using the same method as KRITCHEVSKI, however had demonstrated the presence of the FORSSMAN antigen in yolk of chicken eggs and IDZUMI (1924), GUGGENHEIM (1929), WITEBSKY and SZEPSENWOL (1934) confirmed this. The antigen has also been detected in one day old chick embryos by the last authors, who found its concentration in young embryos to be about the same as in adult tissues.

IDZUMI (1924), injecting the whole content of chicken eggs in the peritoneal cavity of rabbits for immunisation, found an increase in quantity of both FORSSMAN and serum antigen during the incubation period and a decrease in that of the egg-white, determined after the titer of the antisera he could obtain. Although it may be very comprehensible that these quantitative changes are occurring during development, the results do not indicate any alteration in quality. The method of determining the amount of an antigen by the effect of immunisation is not very quantitative, because great differences exist in reaction of the injected rabbits. Therefore, exceptions in the rather regular quantitative changes, noted by IDZUMI, might be expected; so the serum antigenicity in chickens, four days after hatching, was found to be six times weaker than in unincubated eggs.

The results of ABE (1931) led this author to believe that the species specificity of organ lipoids is stronger in human adults than in fetuses and new-borns, but this belief is based upon rather subtile and only quantitative differences in the results of complement fixation tests.

PERLMAN and GUSTAFSON (1948) demonstrated the presence of certain antigens in plutei of Paracentrotus lividus and found them missing in young embryos, from which result the authors concluded that these antigens "are not present at all, or perhaps in only undetectable quantities" in the younger stages. Certainly this careful statement is likely to express the true situation; it corresponds with the opinion, expressed by COOPER (1948): "the failure so far to demonstrate the presence of some complex molecules in early embryos does not necessarily indicate that they are all absent" (l.c., p. 430).

The article of AVRECH and HERONIMUS (1937) has not been available. Results of recent biochemical investigations about the blood proteins are indicating that some differences do exist between embryonic proteins and those of adults. It has been found that the amino-acid composition of the fetal hemoglobin in mammals differs from that of the adult type, but still the possibility remains that both types of the globin molecules may be present in both life periods but that one type is predominating quantitatively.

The different composition of the blood serum too may be caused by differences in quantity of the constituent proteins. Even the special embryonic protein fetuin, discovered by PEDERSEN in fetuses of the cow perhaps may be present in minor quantities in adults. However, it is questionable if such functionally inactive proteins will continue their existence in the cytoplasm, but perhaps they remain preserved in the genome.

Besides these differences, some similarity between the embryonic and adult organism has been noted, e.g. by RÖSSLE (1905) and WILKOEWITZ a.o. (1928), who could not find any seizable difference in antigenicity, but like many authors in their time, both used a rather obsolete technique, causing denaturation of the antigens, so that their results cannot be useful in the present discussion.

Furthermore, evidence has been given that in the yolk several substances, found in the blood of the mothers may be present, e.g. serum proteins and antigens. (SCHECHTMAN 1947, 1948 and COOPER 1946, 1948). The article of COOPER (1948) contains an extensive review of literature on this subject to which may be referred.

Finally, the presence of many non-antigenetic enzymes in the unfertilized egg may be taken as an evidence of similarity.

The problem about the presence of more specialised materials, building up the organs, is more interesting however, and has been attracting our attention in connection with the development of the eye-lens. There is one report already, given by BURKE, and collaborators (1944), de-

scribing the presence of organ specific lens substances in chick and frog embryos. In embryos of the chicken, however, younger than 146 hours, the adult antigens could not be found by means of the complement fixation test, and using the precipitin reaction only embryos of 250 hours of age and older reacted positively.

Although they used these reactions "... to determine the age at which adult lens antigen appears in sufficient amount to be detected by these methods" (l.c., p. 229), the authors, in accordance with the epigenetic theory of changing chemical structure during morphological development, arrive at the conclusion that: "Adult organ specificity does not arise in the chick and frog until the organs are well differentiated morphologically" (l.c., p. 232). This conclusion is very interesting because it may be related to the problem of the organ-forming substances.

According to a theory which has been accepted by many embryologists, morphological differentiation is preceded by a chemo-differentiation (Huxley, 1924), by which the fate of organ-forming regions is determined. There is a common opinion that the determination of the organ-development is due to the presence of certain organ-forming or perhaps organ-determining substances. It has, however, not yet been possible to demonstrate irrefutably the existence of such an organ-determining or organ-forming substance. Histochemical evidence for the action of certain enzymes in this respect could not become substantiated by the results of microchemical investigations, and the enzymes which have been indicated by the histochemical pictures can be found in several organs.

Now there seems to exist the possibility that the organ specific substances may be responsible for the determination of organ-differentiation and especially in the development of the eye-lens it might be feasible to try and analyse this problem of the relation between organ-specific, organ-forming and organ-determining substances.

Two questions may be asked:

First, are organ specific substances present before morphological differentiation or do they arise only after or during this differentiation? And second:

If the organ specific substances might be detected before the stage of morphological differentiation, may they be indicated as organ-forming (organ-building) substances or do they determine the fate of the presumptive lens-region in still younger stages where no indication of a beginning lens-formation can be found?

The former question has already been answered by Burke c.s., who only found the specific adult lens antigen present after morphological differentiation, but, using a very careful technique and isolating a greater number of young lens vesicles by means of a micro surgical method we made an attempt to improve the sensitivity of the reaction. Indeed this has been successful, so that by demonstrating the presence

of adult antigens in the embryonic lens before its morphological differentiation, our answer to the first question has become contrary to that, given by BURKE c.s.

Methods.

1. *Immunisation.*

The adult antigen was prepared from fresh lenses, carefully dissected out and weighed. Then the material was thoroughly ground in a mortar and a suspension prepared, adding nine volumes of saline ($p_H = 7$). After centrifuging at 4000 r.p.m., the supernatant opalescent solution was used for immunisation. Strong rabbits, about three kilograms of body weight, were immunised by seven to eight injections of 2 to 3 ml. of the 10 % antigen solution in the marginal vein of the ear, once in two or three days. Beginning with the fourth injection the rabbits received a subcutaneous injection of the same dose one hour before the intravenous injection in order to prevent shock. (In the last year this injection was given on the night before). One week after the last injection the antiserum was tested and if it had a sufficient high titer (1 : 20000 to 1 : 50000), the animal was bled to death from the carotid artery or the blood was obtained by puncture of the heart. After clotting, the serum was removed, sterilized by filtration through a Seitz filter, distributed over a series of ampullae and stored in the refrigerator.

2. *Serological test.*

In 1948 we used the complement fixation test, as it is a most sensitive reaction. The results, however, were unsatisfactory for several reasons. First, specificity was low, probably caused by the presence of non-specific lipoids in the eye-lens; KRAUSE (1935) estimated the lipoid content to be 1 % of dry weight. We tried to improve specificity by absorbing the antisera with other antigens or by eliminating the lipoids with ether in the cold, but these methods reduced the strength of the sera too much. Furthermore, especially in extracts from whole parts of frog larvae, a sensitizing action was met with, so that more embryonic material had to be used for control titrations of complement, and finally these extracts sometimes caused a spontaneous hemolysis.

Therefore, several experiments were made for comparing the specificity of the complement fixation test with that of the precipitin reaction. The results of these experiments demonstrated a very great difference between the two reactions, the precipitin test being almost completely specific. This phenomenon perhaps may be explained by the possibility that the precipitin test in our experiments reacted with the proteins more than with lipoids or lipoid complexes (cf. WITEBSKY, 1928). In our definitive experiments we thus made use of the precipitin reaction.

Some details of the micro-technique we have been using may be described here:

a. Boyd's micro-method.

A series of antigen dilutions was prepared in micro test-tubes (25×2.5 mm). Each tube received 20 microliters of antigen solution and then an equal amount of antiserum was layered carefully beneath the antigen. After a 20 minutes period of incubation at 37° C. the presence of a ring at the interface between the two layers could be noted in case of a positive reaction. The result was expressed by recording the last positive dilution. This maximum dilution could be taken as an index for the strength of the antiserum as well as for estimating the amount of antigen present in a solution of unknown concentration.

It must be remarked that in mixtures of several antigens it is rather difficult to find the exact concentration of one antigen because one protein may change the reactivity of another, so that, if the concentration of the separate components is not known, any quantitative determination will be impossible. This could be demonstrated by the reaction between antisera prepared to a complete extract of the lens and different lens proteins. The α-cristallin fraction reacts more intensively than the other components and after mixing this fraction with the β-cristallin the reactivity of the α-fraction is decreased. This phenomenon must be taken into account if a comparison has to be made between antigens from different species or different stages of embryonic development, because the relative concentrations of the fractions may be different in different objects.

For filling the micro test-tubes with antigen and serum and for the preparation of dilutions, microburettes are used, calibrated in microliters, in which the fluid can be moved by air-pressure changes. For this purpose the microburette, by means of a narrow and thick-walled rubber tube, is connected with a small reservoir, made from a piece of rubber tubing (internal diameter about 10 mm). This reservoir and a great part of the narrow tube are filled with water, so that between the water and the fluid in the burette only a small air volume remains. By changing the volume of the reservoir with a screw-clamp the burette can be filled or emptied (fig. 2).

The tubes are placed in a circular rack (fig. 1) which fits on a stand (hole *E*) and rests upon a ring (*A*); this ring is freely movable along the rod and is resting itself on a second ring (*B*). The hole (*D*) in the bottom plate of the rack is fitted for a pin (*C*) on ring *A*. By taking ring *A* between thumb and fingers the rack may be turned and lifted and so each of the tubes can be brought around the tip of the microburette. For filling the burette, eventually one or two of the holes, designed for the test-tubes may be widened in order to contain a larger tube, supplying the saline or serum.

This design (fig. 2) has been found to be very practicable for more rapid work. The rack was placed in the incubator, after the incubating period the tubes taken out and the result read against a dark back ground.

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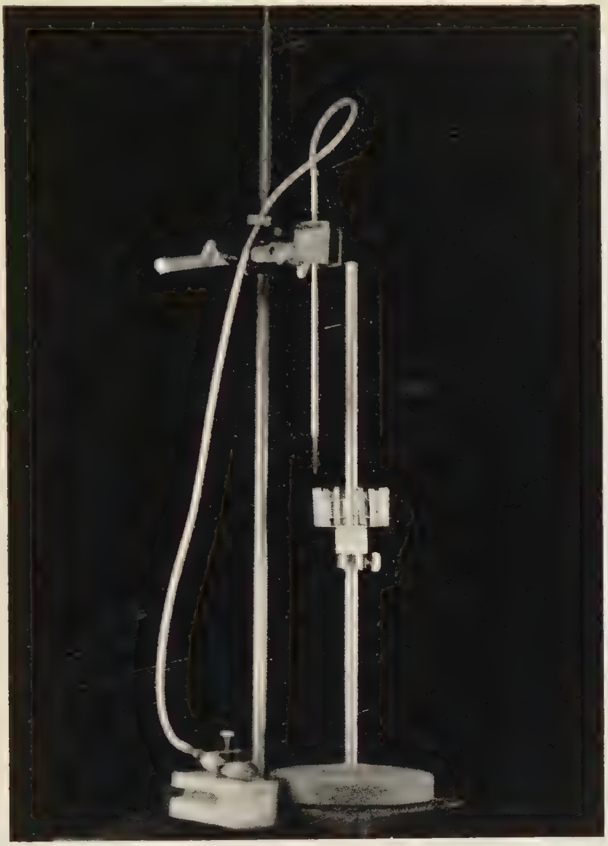

Fig. 2

Fig. 3

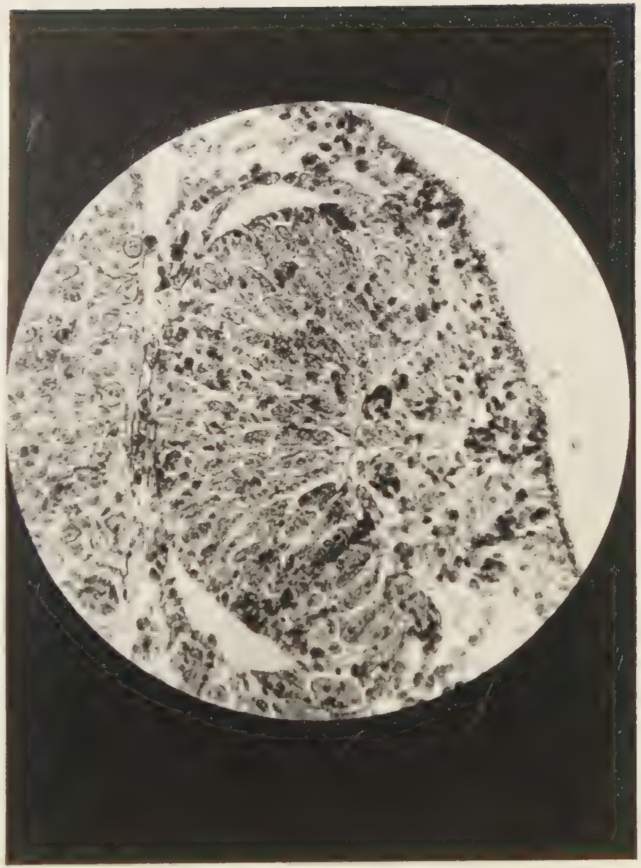

Fig. 4

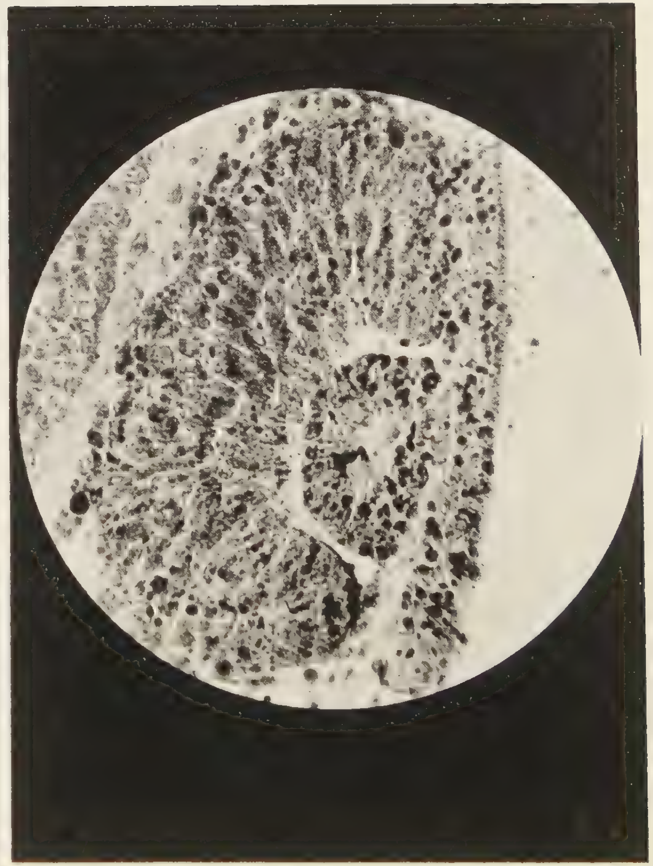

Fig. 5

b. Capillary method.

In experiments where only small amounts of antigen are present, it is useless to prepare a series of dilutions, and the initial concentration has to be taken as high as possible. Therefore, a method was developed, using narrow, thin-walled glass capillaries, so that for each test only one microliter was needed. From the extract this quantity was sucked up in the narrow capillary tube and then about the same volume of antiserum (or control fluid) followed. Both were blown out on a glass slide, mixed by stirring with the tube and the mixture again sucked up into the capillary. By repeating this a suitable mixing could be achieved and finally the mixture was brought in the middle of the tube, the ends of which were sealed by melting the glass in a micro flame. The tubes were put with their lower ends into plasticene and incubated during one hour, whereafter they were left standing over night at room temperature. Next day the result was read under a dissection microscope in dark-field illumination. Of course there were several control tubes, corresponding with the ordinary method.

The mixing of the reacting fluids is known to decrease the sensitivity as compared with the ring test. We found, by comparing both methods, the mean difference, expressed by the quotient of the last positive dilutions, to be about 5.

The increase in sensitivity by using the capillary method thus became about fourfold, using one microliter in stead of twenty in the ring test. (Of course, in both methods still smaller volumes may be used).

In some experiments only, a third method was used. The antigen extract was distributed over three watch-glasses and then frozen in dry ice and dried. For the reaction the antigen was dissolved directly in the antiserum or control fluid and with use of the capillary method sensitivity again was increased.

3. *Preparation of antigen from embryos.*

The young lens-vesicle was cut out from the embryo by means of the ordinary glass-needle technique. In this way as little yolk or other materials as possible have been taken together with the lens antigen itself.

For focussing the microscope during dissection we used a modification of the device, published by La Rue (1932).

To the focussing wheel a lever of variable length was attached which could be moved by means of a flexible cable (choke cable from an automobile). In stead of moving the pedal, to which this cable was connected, up and down with the foot, the displacement of the cable was obtained by moving the foot in a horizontal direction. The cable was attached at the fore-end of the pedal and the axis of motion placed near to the heel, so that by rotating the leg in the knee the microscope could be focussed. This modification of the original design, invented by our instrument maker de Vries, has become very useful, being less

tiresome because in every position the foot is resting on the floor

The material was assembled on the top of a small cylindrical glass pestle, cooled in ice. The extract was prepared by grinding the material in a small glass-mortar, made from a thick-walled glass capillary (pyrex) of 2 mm internal diameter, about 25 mm long, and widened at the upper end. After grinding and adding a few microliters of saline the "mortar" was centrifuged and the supernatant fluid used.

Results.

A. Control experiments.

1. *Comparative experiments with* BOYD'*s technique and capillary technique.*

The well-known fact that the precipitin reaction in a mixture is somewhat less sensitive than the ring test was also demonstrated by the results of our experiments (table 1). From each of the tubes, in which

TABLE 1

Relation (Q) between last dilution of antigen, reacting positive in Boyd's test (B) and in the capillary method (C)

	B	C	Q
Chicken lens (embryo) + frog lens antiserum	1600	400	4
Chicken lens (adult) + chicken lens antiserum	25600	3200	8
Frog lens + frog lens antiserum	51200	12800	4
Frog lens + frog lens antiserum	25600	12800	2
Frog lens + frog lens antiserum ($p_H = 9$) . .	6400	3200	2
			Mean $Q = 5$

an antigen dilution had been prepared for the ring test, one microliter had been taken out into a capillary tube, mixed with the same volume of antiserum and the results of both tests compared.

The difference in sensitivity has been expressed in the table by the quotient between the factors of the last positive dilutions. It may be concluded that the ring test is about five times more sensitive than the capillary test.

2. *Organ specificity of the antiserum.*

In table 2 the results have been given of precipitation reactions of lens-antiserum and extracts of other organs.

The number of experiments and of the negative and positive reactions as well as the last dilution index in case of positive tests has been presented. One of the antisera had to be rejected because of its weak specificity, but in all other sera this quality has been satisfactory, as may be seen from the facts in the table. Only the vitreous body has been reacting positively in three out of six experiments. Probably this can be explained by the possible presence of small parts of the lens, remaining adhered to the vitreous after this had been dissected out.

TABLE 2

Reactions between eye-lens antiserum and 5 % extracts (in saline) from other organs

Organ (part)	adult frog		adult chicken		chick embryo	
Results:	neg.	pos.	neg.	pos.	neg.	pos.
muscle (leg of embryo)	3	—	1	—	15	—
liver	1	—	1	—	3	—
skin	5	—	3	—	4	—
brain	3	—	3	—	4	—
retina	5	—	1	—	4	—
iris	4	—	1	—		
vitreous corpse	—	1 (160)	1	1 (80)	3	1 (40)
eye cup (complete)					12	—

(The number of experiments has been given. In case of a positive reaction the last positive dilution has been indicated in parentheses).

3. *Species specificity.*

By combining several portions of one antigen solution with different antisera and by combinations between one antiserum and different antigens an impression could be obtained about this quality. The results

TABLE 3

Species specificity

a. Reactions between one solution of antigen, combined with homologous and heterologous antisera.

Antigen from lens of:	Chicken lens antiserum	Frog lens antiserum
adult chicken	51200	25600
chick embryo (15 days)	25600	3200
chick embryo (9 days)	1600	800
adult frog	12800	51200

(indication of last positive dilution)

b. Reactions between one (adult frog lens) antiserum and lens antigens from different species.

antiserum no. 2		antiserum no. 7	
antigen	last dilution	antigen	last dilution
frog	32000	frog	16000
chicken	16000	chicken	8000

of the first method are presented in table 3*a* and those of the second in table 3*b*, the last positive dilution having been indicated. Certainly a small difference is present, caused by species specificity, although it is possible that in the second combination the exact concentration of the active antigen is varying with the species. Such a variation also may be present in the first experiments, if the antisera are gained by injections of quantitatively differently constituted antigens, but the effect may be less intense.

Obviously the lens antigen only is weakly species specific and strongly organ specific.

B. The presence of adult lens antigen in embryos.

1. *Chick embryos.*

By means of the BOYD technique the presence of adult lens antigen in embryos 74 hours of age and older clearly could be demonstrated. Probably this result was due to the careful dissection and assembling of the lens material.

TABLE 4

Adult lens antigen in chick embryos

Age of embryo (hours)	Number of expts.	Number of lenses	Last. dil. BOYD's method.
192	1		6400
168	1		3200
144	1		3200
120	1		1600
96	1		800
84	1		400
74	1		400
72	1	11	+
66	1	40	+
60	1	40	—
capillary method:			
60	3	40	+ ; + ; ++.
58	1	60	+
54	3	50; 30; 30.	± ; ± ; —.
51	1	50	—
48	1	50	—

In a 72-hour embryo the reaction still was positive when 11 lens vesicles were dissolved in 10 microliters of saline, and in the 66 hours stage a positive result could be got by preparing an extract from 40 lens vesicles in 20 microliters saline. In the 60-hour embryo this method failed, however, and henceforth the capillary method was used, extracting the material, assembled from 20 to 30 embryos with about 4 microliters of saline.

By this procedure positive results were obtained in embryos of 60 and 58 hours (four experiments) and weakly positive reactions in material from 54-hour embryos. In 51- and 48-hour embryos only negative results could be noted.

Taking into account the possible individual variations, we thus safely may draw the conclusion that the 60-hour embryo already possesses the adult antigen (or one of the lens antigens) and in a sufficient amount to be detectable by this method.

The change in quantity of the antigen is apparent from the fact that in younger stages the detection of it is becoming increasingly difficult. Also the results of our quantitative tests in older embryos demonstrate this phenomenon. The series of decreasing values of the last positive dilutions from 1/6400 in 8-day embryos to approximatively 1/400 in 3-day embryos makes this clear.

In nearly all experiments the results have been checked by comparing the lens material with extracts from other parts of the same embryos. In all these controls the reaction gave negative results, even in extracts, prepared from eye-cups. Other control experiments, using normal serum, muscle antiserum as well as saline have been negative.

Because the time, required for operating so great a number of embryos became rather long, extending to about 6 hours, it was possible that, if the embryos were stored in this period at room temperature, they were changing too much. We, therefore, carried out some experiments in which the total number of eggs was divided into portions of ten and these were incubated one after another with intervals of one hour. So, during the dissection period, at each hour a portion of ten embryos became available.

The individual variation in rate of development remained, but the few exceptional cases were excluded from the experiments. Ordinarily, about 30 embryos out of 50 incubated eggs could be used, because of mortality or of low fertilisation percentage.

The results of these experiments in chick embryos have been presented in table 4.

2. *Frog embryos.*

In these experiments, material from embryos of Rana esculenta has been investigated. In most cases the embryos were reared before the experiment at 12° C. for retarding the development. There was no opportunity to compare these results with the properties of embryos, reared at higher temperatures.

In the eldest stages the antigen material was assembled from about 50 to 100 embryos, in the younger stages some 200 embryos have been used for each experiment.

The results have been reported in table 5 in which also the control experiments have been presented, because of the small number of these experiments.

In stage 18—19 (according to SHUMWAY, 1940, 1942) one experiment was a failure, because of interference by a vast amount of yolk, resulting in a cloudiness in the extract that could not be removed by centrifuging at 4000 r.p.m. during a long period. In stage 19 two experiments gave negative results, whereas at the same stage a positive reaction could be obtained by drying the extract and dissolving the antigen directly in the sera, as has been described.

In stage 19—20 the result was positive, but no control experiments could be done.

From these facts it may be concluded that the appearance of the first positive reaction has been found at about stage 19 to 20.

TABLE 5

Adult lens antigen in frog embryos

Stage (SHUMWAY)	Antigen	Lens antiserum	Muscle antiserum	Normal serum
25	lens + eye-cup	+		
23	lens + eye-cup	+	—	
23	rest of embryo	—	—	
21—22	lens + eye-cup	+	—	
21—22	ventral epidermis . . .	—	± (adhering mesoderm?)	
19—20	lens + eye-cup	+		
19	lens	—		
19	lens	—		
	Antigen material dried and dissolved in the sera:			
19	lens	+	—	—
17—18	lensectoderm	—	—	—

3. *Morphology.*

According to the normal tables of development of the *chick embryo* (KEIBEL a.o. 1900) the formation of the lens-plate and its invagination takes place at the age of 42 to 50 hours. The separation of the lens vesicle from the epidermis can be found at the 60 to 63 hours' stage and thickening of its wall as a beginning of fiber differentiation takes place at about 70-hours stages. The microphotograph (fig. 3) shows a section through the lens vesicle of a 60-hour embryo, where it is still opening to the surface.

According to our experience, at the 60-hour stage this connection between ectoderm and lens vesicle is constantly present.

In embryos of *Rana esculenta*, stage 19, according to SHUMWAY, is characterized by the presence of a lens bud only, no vesicle having been formed yet (fig. 4). In stage 20 the formation of a lens vesicle has taken place, but the vesicle is still very primitive (fig. 5). Because of the presence of abundant yolk, the structure of the lens bud in fig. 4 seems to be more irregular than it is.

Consequently the conclusion may be drawn, that the adult lens antigen in both species could be detected in stages before the complete formation of the lens-vesicle in which no specific morphological differentiation could be found.

Discussion.

The experimental results have demonstrated that organ-specific antigen(s) is(are) present in developmental stages where no specific differentiation of the lens-vesicle is visible, and even practically no

difference can be found between the morphology of this primitive lens and the rudiments of nose and ear.

It therefore seems justifiable to draw the conclusion that chemical differentiation is preceding morphological differentiation. This answer to the question about the priority of these two modes of differentiation is contradicting the conclusion of Burke c.s. (1944), but the conclusion of these authors has not been based upon positive results.

The early appearance of chemical differentiation is substantiating the theory of chemo-differentiation which, since Huxley, has been discussed in embryological literature. It seems very unlikely that these important constituents of the adult lens would not have any significance in the period of construction of this organ-part. At least it may be supposed that the adult antigens, being organ-specific substances, also may represent organ-forming materials, enabling the embryo to build up its eye-lens. Whether these antigens also may be representing organ-determining substances cannot yet be stated, because they have not yet been detected in the period before the first beginning of the morphogenesis of the lens.

It is clear that preformationists have strong trumps in hand, if they state that qualitatively all specialised materials of the adult organism will be present at the onset of ontogenesis, because a theory of epigenetic creation of such substances during embryonic development only can get its evidence from negative results of experiments and these always may be explained by deficiency of technique.

Another question is that of the presence of other than adult antigens in the lens. Burke and collaborators are of the opinion that embryonic lens antigen, differing from that of the adult lens, may be present in young stages of development. These authors immunised rabbits by the 300-hour embryonic lens antigen and the immune sera reacted with younger stages than the adult antiserum. The explanation for this phenomenon seems to be at hand: there must be another antigen common to the 120-hour and 300-hour lens but not present in the adult lens. But now that we demonstrated the presence of adult antigen in still younger embryonic lenses the possibility remains that the different antisera, used by Burke, have been reacting with different intensity; as no control experiments have been reported, solution of this problem is difficult. It also may be remarked that these results only have been obtained by the complement fixation test, and we were necessitated to leave this test for the precipitin reaction. The fact, that Burke did not mention any difficulty when using this technique whereas we found a very weak specificity, is difficult to explain. Perhaps the mode of preparation of the antigen for immunization has been different. Burke reports the use of formalin in some cases for preserving this antigen, while in our experiments only fresh preparations were used. Denaturation may be expected by the treatment with formalin, and perhaps the antigen

preparation of Burke contained less lipoids than ours, because they centrifuged it during 20 to 30 minutes at high speed; no data about the centrifugal force have been given, however.

In the experiments with 160-hour lens antiserum the authors themselves remarked that these sera were rather unspecific. They have based their opinion on the general damage, inflicted to embryos, and no control experiments have been reported. It seems possible that the presence of yolk in the material, used for immunising the rabbits has resulted in the production of antiserum against yolk, more than against the very small quantity of lens antigen that has been present. The reactions with yolk also may have disturbed some of the precipitin or complement fixation tests. Whether the embryonic antigen, met in the 300-hour lens, also may be a yolk antigen cannot be excluded nor can it be proved.

The possibility remains that a special embryonic lens antigen may be present. Moreover, it is possible that the ratio between the concentration of the different protein fractions of the lens will be changing during ontogeny. Such changes have been reported to occur in the lens during post-natal life, where a shift is found in the relative concentration of α- and β-cristallin. Such a change in quantitative relation between the different protein components of the lens would correspond with the differences in protein composition of the blood that have been found.

The increasing indexes of the last positive dilutions during the development from the 3-day to the 8-day embryo suggest a gradual increase during this period of development. Whether periods of more rapid increase do occur cannot yet be stated. Perhaps the period of lens induction is characterised by a large increase in quantity of the lens antigen, caused by the stimulus.

Summary.

It has been possible to demonstrate the presence of adult lens antigen in young embryonic lenses by means of the precipitin technique. In the chick embryo the youngest stages where the adult antigen could be detected were about 60 hours old and in the embryo of Rana esculenta the adult antigen was present from stage 19 to 20 (Shumway) on.

In the 60-hour chick embryo the young lens-vesicle still opens to the surface and no specific morphological differentiation can be observed. In the frog embryo the lens vesicle is absent in stage 19 and just has been formed in stage 20; in stage 19 only a lens bud is found.

Therefore it can be concluded that the chemical differentiation of the lens is preceding the morphological differentiation. This conclusion is contradictory to the results of Burke and collaborators (1944), but their opinion, like that of many others, has been based upon negative experimental results.

Our results thus do not permit the conclusion that in still younger stages the characteristic adult substances may be lacking. Improving

the technique probably will result in positive reactions in these stages.

The suggestion has been made that the organ-specific lens substances really may be called organ-forming substances, whereas it still remains more doubtful whether they also can be ranged among the organ-determining substances.

The complement fixation test in our experiments has failed to give reliable and specific results. Some additional notes about the technique and the organ- and species-specificity of the antisera, used in these experiments, have been presented.

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ZOOLOGY

ON THE INFLUENCE OF TEMPERATURE AND OF LITHIUM CHLORIDE ON THE AMOEBOID MOBILITY OF UNSEGMENTED EGGS OF LIMNAEA STAGNALIS L.

BY

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1. *Introduction.*

Unsegmented eggs of *Limnaea stagnalis* show two periods of amoeboid mobility; the first period falls at the time of the extrusion of the first polar body, the second immediately after the extrusion of the second polar body. This mobility is apparent from the irregularity of the outline of the egg.

DE GROOT (1948), M. GRASVELD (1949) and RAVEN and ROBORGH (1949) observed abnormally strong movements in eggs, which had been treated with hypertonic, isotonic and hypotonic solutions of LiCl. Therefore further investigations were made on the influence of the temperature and of solutions of LiCl on the mobility of these eggs.

2. *Material and methods.*

Snails were stimulated to oviposit in the usual way (RAVEN and BRETSCHNEIDER 1942). With the experiments on the influence of temperature part of the capsulated eggs were kept at 29—33° C, part at 17—22° C and at 13—15° C in a water bath. About 350 eggs were observed. In the LiCl-experiments the eggs were decapsulated and transferred to solutions of LiCl of different concentrations, varying from 1 % to 0,006 %. Control eggs developed in distilled water and in their own capsules. About 1000 eggs have been studied in these experiments.

3. *Results of experiments.*

a. Influence of temperature.

The temperature did not perceptibly influence the degree of mobility of each egg individually. The movements of the eggs, developing at 29—33° C, were as strong as those of the eggs kept at 17—22° C and at 13—15° C. However, the number of eggs in which mobility occurred depended on the temperature. At 29—33° C only 4 out of 100 eggs, at 17—22° C 70 out of 130 eggs and at 13—15° C 80 out of 125 eggs showed amoeboid movements. The stage at which mobility occurred did not depend on the temperature.

b. Influence of LiCl.

LiCl influenced the degree of mobility of the eggs. In 1 % and 0,8 % LiCl no polar bodies were formed. No amoeboid movements were observed.

In 0,5 % LiCl development was inhibited after the first or second maturation division dependent on the circumstances. The formation of the second polar body was suppressed when freshly-laid eggs were transferred to LiCl immediately after oviposition, but the second polar body was formed, when the transfer to LiCl was delayed till the extrusion of the first polar body. If only one polar body was formed, there were yet two periods of mobility, corresponding with the periods of mobility of the controls. During the first period the amoeboid movements were somewhat stronger than those in the controls, during the second period they were much stronger. If the second polar body was extruded too, this was attended with strong amoeboid movements.

In 0,3 % and 0,19 % LiCl usually both polar bodies were formed. The mobility during the first period was a little increased as compared with the mobility of the controls; during the second period the mobility was very much increased. There were some differences between the effect of 0,3 % LiCl and that of 0,19 % LiCl. 0,3 % LiCl retarded the development of the eggs; LiCl 0,19 % did not. The eggs showed very small processes, when transferred to LiCl 0,3 %. Their outline appeared crenated. In 0,19 % LiCl, on the contrary, the outline was sinuate, the processes were much larger (fig. 1).

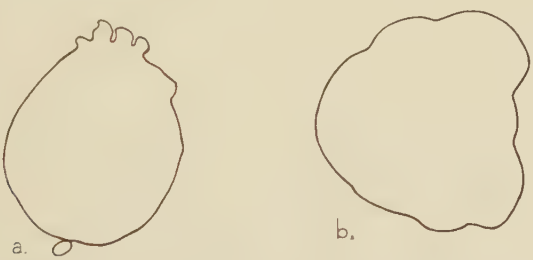

Fig. 1*a*. Egg, treated with 0.3 % LiCl. Second period of mobility.
Fig. 1*b*. Egg, treated with 0.19 % LiCl. Second period of mobility.

Solutions of LiCl more diluted than 0,1 % had no perceptible influence.

Capsulated and decapsulated eggs in 0,5 % and 0,3 % LiCl were compared. No difference between the mobility of these eggs was observed.

The moment at which amoeboid mobility occurred was not changed by LiCl, when it is related to the time the polar bodies were formed.

4. *Summary.*

The number of eggs which show amoeboid movements before the first cleavage depends on temperature. Low temperatures favour the

occurrence of amoeboid movements. The degree of mobility is influenced by LiCl. In hypertonic and isotonic solutions of LiCl strong amoeboid movements were observed, especially during the extrusion of the second polar body. Solutions more diluted than 0,1 % had no influence on amoeboid mobility.

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FUNDAMENTELE BESCHOUWING VAN PROFIELCONSTRUCTIES

DOOR

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Onder geologisch profiel verstaat men het beeld van de doorsnijding van een deel van de aardkorst met een meestal vertikaal vlak. Evenals bij de geologische kaart is het speciaal de configuratie van de geologische discontinuiteitsvlakken, dwz. van grensvlakken en breuken, welke de essentie van het geologisch profiel uitmaakt.

Het profiel geeft een meer direct beeld van de tektoniek van een gebied, dan de geologische kaart. Onder geologische (tektonische) strukturen verstaat men verstoringen van de horizontale ligging; deze verstoringen komen in een vertikale doorsnede het meest direct tot uiting.

Zelden en dan nog maar partieel is de geologische struktuur rechtstreeks waarneembaar; bijna steeds moet deze gereconstrueerd worden uit elementaire strekkings- en hellingswaarnemingen aan het oppervlak. Het doel, dat men zich met de constructie stelt, is om van minstens één laagvlak de doorsnijding met het gekozen profielvlak te bepalen. Daardoor kunnen dan ook de doorsnijdingen van andere laag- of discontinuiteitsvlakken bepaald zijn.

Van de beschikbare hellingsmetingen behoren in het algemeen maar enkele tot het af te beelden laagvlak; het merendeel behoort tot daarboven of daaronder gelegen vlakken. Bij de gebruikelijke constructies neemt men aan, dat deze gegevens toch de vorm van een bepaald laagvlak determineren; men *vòòronderstelt, dat de gemeten hellingen onveranderlijk en in dezelfde volgorde in alle laagvlakken terugkeren.*

Deze praesuppositie alléén leidt echter nog niet tot een eenduidige oplossing, er is een eindeloze variatie mogelijk van krommen, die dezelfde tangenten-richtingen in dezelfde volgorde vertonen of anders gezegd: die in dezelfde volgorde achtereenvolgens evenwijdig zijn aan een aantal in volgorde gegeven richtingen. Denkt men zich de constructie van een tangentenveelhoek van de kromme, en begint men door een gegeven punt de tangent te trekken evenwijdig aan de eerste gegeven richting, dan is het vooralsnog geheel onbepaald in welk punt van de eerste tangent men moet overgaan tot de tweede in de tweede gegeven richting. Zijn hieromtrent geen gegevens beschikbaar — en deze ontbreken gewoonlijk — dan eist constructie verdere vòòronderstellingen.

Men onderstelt — als uitersten — afstandsgetrouwe en vormgetrouwe

plooiing. Bij de eerste is de loodrechte afstand tussen elke twee laagvlakken constant, een loodlijn op een der laagvlakken staat loodrecht op alle andere, de laagvlakelementen, behorende bij zulk een loodlijn hebben hetzelfde kromte-middelpunt, de laagvlakken zijn concentrisch.

Profielconstructies, welke gebaseerd zijn op het principe van afstandsgetrouwheid voeren dus een tweede *vòòronderstelling* in nl. de *aanname van concentriciteit.*

Op het eerste gezicht zou men kunnen menen, dat concentrische figuren gelijkvormig zijn. Dit is echter alleen het geval, wanneer men zich beperkt tot cirkelbogen of rechtlijnige figuren, welke in cirkels kunnen worden beschreven. Twee „afstandsgetrouwe'', willekeurige 5-hoeken, hebben wel gelijke hoeken, maar de zijden zijn niet evenredig. Een kromme, welke met een ellips afstandsgetrouw is, is daarmee niet gelijkvormig, het is zelfs geen ellips meer. In het algemeen zijn gelijkvormige figuren niet afstandsgetrouw. Bij de meest gebruikelijke afstandsgetrouwe constructie (H. G. BUSK, Earth Flexures, 1929), wordt de te construeren kromme benaderd door een aantal cirkelbogen, waarvan telkens de twee aaneensluitende een gemeenschappelijke raaklijn hebben. Deze bogen keren elk voor zich gelijkvormig in een volgend laagvlak terug, de vergroting of verkleining is echter voor elke boog verschillend.

Onder vormgetrouwe plooiing verstaat men een plooiing, waarbij alle laagvlakken congruent zijn. Uiteraard bestaat er dan in het algemeen geen loodrechte afstand tussen de laagvlakken meer en deze kan dus ook niet constant zijn, met uitzondering van het geval, dat men met platte vlakken te doen heeft. Congruentie van de laagvlakken, waarbij tevens dezelfde hellingen in dezelfde volgorde terugkeren sluit gelijkstandigheid in. Dit wil echter zeggen, dat men zich een laagvlak uit een ander meetkundig ontstaan kan denken door dit laatste over een bepaalde (rechte) afstand evenwijdig aan zich zelf te verschuiven. Daar hierbij elk punt van het laagvlak dezelfde translatie ondergaat, zal in de richting van de translatie de afstand tussen twee laagvlakken constant zijn. Men kan dus ook zeggen, dat een congruente (vormgetrouwe) plooiing een zodanige is, waarbij de afstand van de laagvlakken, gemeten in een bepaalde richting constant is.

Congruente constructiemethoden moeten eveneens van een verdere praesuppositie uitgaan nl. juist *de veronderstelling, dat de plooiing congruent en gelijkstandig is.*

Beide constructiemethoden komen hierin overeen, dat in hun praesuppositie opgesloten ligt, dat in bepaalde richtingen de afstand tussen twee vlakken constant is; twee punten, welke in de bepaalde richtingen de gegeven afstand hebben, hebben *ook dezelfde helling.* De translatierichting bij de congruente plooiing en de loodlijnen op de laagvlakken bij de concentrische plooiing zijn dus de richtingen, waarin men veronderstelt, dat de hellingen constant zijn. Lijnen, die deze richtingen hebben zijn isoklinen en deze isoklinen zijn rechte lijnen.

De mening, dat men congruente of concentrische profielconstructies kan baseren op een aantal aan het aardoppervlak verrichte metingen, waarvan de onderlinge situatie uiteraard bekend moet zijn, berust dus op de inhaerente praesuppositie, dat men door de waarnemingspunten rechtlijnige isoklinen kan trekken, waarvan men het verloop zou kunnen vaststellen òf op grond van feitelijke gegevens, maar meestal op grond van verdere vòòronderstellingen. Door de aldus gegeven schaar van isoklinen zou de profielkromme eenduidig bepaald zijn. Dit laatste is een vraagstuk, dat in deze verhandeling buiten beschouwing zal blijven.

In het voorgaande zijn twee bijzondere plooiingsvormen beschouwd, waarvoor profielconstructie mogelijk geacht wordt. De allereerste vòòronderstelling nl. de aanname, dat alle gemeten hellingen onveranderlijk en in dezelfde volgorde in alle laagvlakken terugkeren eist geen rechtlijnigheid van de isoklinen, dit kunnen evengoed kromme lijnen zijn, mits zij elkaar niet snijden, daar dan de opeenvolging van de hellingen aan weerszijden van het snijpunt niet gelijk zou blijven. Men zou in het laatste geval de betrokken metingen aan de andere zijde van het snijpunt, dan die waar de metingen gedaan zijn, kunnen elimineren. Inderdaad gebeurt dit bij toepassing van de constructiemethode van Busk, doch men zal inzien, dat een dergelijke eliminatie niet steunt op geologische gronden, maar een willekeurig gevolg is van de toegepaste constructiemethode. Dit heeft dan ook het verbazingwekkend resultaat, dat de plooiingsintensiteit bij aldus geconstrueerde profielen naar boven en beneden uitsterft en steeds zijn maximum vertoont in het niveau van het huidige — toevallige — landoppervlak.

Van de kromlijnige isoklinen zijn de rechtlijnige slechts bijzondere gevallen, waarvan op hun beurt afstandsgetrouwheid en congruentie weer verbijzonderingen zijn.

Tenslotte karakteriseert ook de praesuppositie, dat de hellingen onveranderlijk en in dezelfde volgorde in alle laagvlakken terugkeren — en deze praesuppositie stelt men, wanneer men aanneemt, dat metingen, welke gedaan zijn aan verschillende laagvlakken, vormbepalend zouden zijn voor één bepaald of voor alle laagvlakken — een bijzonder geval. Het is duidelijk, dat hiermede verandering in de plooiingsintensiteit — zoals bv. aanwezig is in door inklinking ontstane strukturen, of ontstaat tengevolge van plooiing tijdens de sedimentatie —, evenals ongelijkstandigheid — zoals die bv. optreedt bij gebogen assenvlakken, ook al is de plooiing overigens congruent — uitgeschakeld worden. Disharmonische plooiing moet hier uiteraard geheel buiten beschouwing blijven.

Uit het voorgaande volgt, dat een serie oppervlakte-waarnemingen in het algemeen onvoldoende is om zonder een betrekkelijk groot aantal vòòronderstellingen een profielconstructie uit te voeren. Men dient in feite het verloop van de isoklinen te kennen en hiervoor zijn metingen aan een min of meer horizontaal oppervlak onvoldoende. Het probleem

is in het profielvlak twee-dimensionaal, men zou voor de oplossing over horizontale en vertikale reeksen van metingen moeten beschikken.

In alle voorgaande beschouwingen ligt nog één vòòronderstelling opgesloten. Wij zagen, dat de mogelijkheid van profielconstructie de mogelijkheid veronderstelt om het verloop van de isoklinen vast te stellen. De aanname echter, dat men op grond van in het profielvlak gelegen — of op grond van daarin geëxtrapoleerde of geïntrapoleerde — gegevens de constructie kan uitvoeren, reduceert het vraagstuk tot een twee-dimensionaal probleem en dit impliceert, dat de isoklinen *in* het profielvlak zijn gelegen. In het algemeen is dit niet het geval: de isoklinen zijn ruimte-krommen, geen vlakke krommen en zelfs, wanneer het rechte lijnen zijn, zullen ze slechts in bijzondere gevallen in het profielvlak liggen. De vorm van de doorsnijdingskromme van een laagvlak met het profielvlak, wordt in het algemeen niet bepaald door de in het profielvlak gelegen metingen, maar door daar buiten gelegene. Verwaarlozing van dit principe is inhaerent aan alle constructiemethoden, welke het probleem twee-dimensionaal opvatten, maar het is het meest spectaculair bij de afstandsgetrouwe constructie. Uitdrukkelijk houdt het principe van de afstandsgetrouwheid in, dat de isoklinen rechte lijnen zijn, die in de waarnemingspunten loodrecht op de laagvlakken staan. Een in het profielvlak gelegen meting, waarvan de strekking niet loodrecht op het profielvlak staat determineert een isokline, die niet in het profielvlak ligt, maar die dit vlak snijdt in het waarnemingspunt. Deze isokline heeft dus slechts een bepalende waarde voor de vorm van de doorsnijdingskromme van het door het waarnemingspunt gaande laagvlak en dan nog uitsluitend in dit punt. In het algemeen heeft elke isokline, die het profielvlak snijdt slechts bepalende waarde in dat snijpunt en uitsluitend voor een laagvlak door dat punt. Men meent deze moeilijkheid te kunnen ondervangen door de meting zogenaamd te reduceren, dat wil zeggen door slechts rekening te houden met de in het profielvlak gelegen hellingscomponent van het gemeten laagvlak en men trekt dan een isokline loodrecht op de gereduceerde helling *in* het laagvlak. Dit is een fictie, de hellingen van de doorsnijdingskrommen in elk punt van de aldus gevormde lijn zijn niet die welke bij deze pseudo-isokline behoren, maar de gereduceerde hellingen behorende bij de isoklinen, welke het profielvlak in de betrokken punten snijden, resp., welke in het profielvlak liggen en door het betrokken punt gaan. Slechts in één geval is de gewraakte methode toelaatbaar nl. wanneer de laagvlakken cylindervormig zijn en een horizontale beschrijvende lijn hebben, dwz. als alle gemeten strekkingen evenwijdig zijn.

De gegevens waaruit het geologisch profiel geconstrueerd moet worden zijn dus de doorsnijdingspunten van de isoklinen met het profielvlak en de in die punten door de betrokken isoklinen bepaalde (gereduceerde) hellingen. Zijn deze gegevens in voldoende aantal beschikbaar, dan kan men in het profielvlak de isoklinen van de doorsnijdingskrommen van de

laagvlakken tekenen en met behulp daarvan deze krommen construeren. Het zal dus duidelijk zijn, dat deze doorsnijdingskrommen bij vòòronderstelde afstandsgetrouwe, dan wel congruente plooiing nièt afstandsgetrouw, dan wel congruent zullen zijn. Hier ligt de tweede misvatting van de gebruikelijke methoden. Zij pretenderen afstandsgetrouwe of congruente plooiingen af te beelden, doch in feite streven zij naar afstandsgetrouwheid of congruentie in de vlakke figuur van het profiel. Het is echter gemakkelijk in te zien, dat een doorsnede door een afstandsgetrouwe plooi in het algemeen niet afstandsgetrouw kàn zijn; de breedte van de doorsnijding met het profielvlak van een gelijkmatig dikke laag hangt af van de hoek waaronder de laag het profielvlak snijdt en deze is in het algemeen veranderlijk.

In wezen is de profielconstructie een drie-dimensionaal probleem, dat aan de hand van de gebruikelijke oppervlakte-gegevens in de regel niet oplosbaar is zonder invoering van een aantal vòòronderstellingen, welke dus niet uit de feitelijke gegevens afgeleid kunnen worden, maar eventueel gebaseerd kunnen zijn op regionale kennis en op persoonlijke ervaring en inzichten. Als zodanig hebben ze uiteraard een subjectief karakter.

In hoeverre bij voldoende gegevens of onder aanvoering van bepaalde vòòronderstellingen betreffende het verloop van de isoklinen een geometrisch (mathematisch) verantwoorde constructie mogelijk is, zal elders in beschouwing worden genomen. Uit het voorgaande volgt reeds, dat wegens de contingentie van de gegevens hiervoor grafische methoden in aanmerking komen en dat deze neer zullen komen op integratie of het bepalen van trajectorieën.

Summary.

Construction of geologic structure sections is based on the supposition that dips measured at the earth' surface on any bedding plane are determinative for the form of other strata. This means that space curves can be determined along which the dip and strike of layers cutting these curves are constant. Such space curves are named isoclines. In general isoclines can not be determined by means of the available measurements at a more or less horizontal surface, the determination is a space problem. Actual construction requires further suppositions. Usual assumptions are the suppositions of parallel and of similar folding. The first assumes the isoclines to be perpendicular to the strata, in the second the isoclines are mutually parallel, moreover, in both cases the isoclines are supposed to be straight lines. In general the isoclines do not lie in the vertical plane of the structure section, in other words the isoclines of the bedding planes are in general not isoclines of the structure section. The bedding plane isoclines only determine the apparant dip in their points of intersection with the section plane. In general the section isoclines will have to be defined from these separate apparent dips. In case of cylindrical bedding

planes the isoclines are lying in planes through the descriptive lines of the cylinders. In this case the lines of intersection of these isocline planes with the section plane are isoclines of the section, however, the dips defined by these section isoclines are not identical with those of the bedding plane isoclines. From this it is concluded, that the usual methods of construction of structure sections are not valid, e.g. the section isoclines in parallel folding are not perpendicular to the bedding plane intersections, hence the usual construction is not valid. A section through a parallel fold can not be aequidistant unless all strikes are perpendicular to the section plane.

Section construction is a mathematical problem that will be treated elsewhere.

CHEMISTRY

ION ADSORPTION BY SOIL-MINERALS AS A DONNAN-EQUILIBRIUM

BY

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1. *Introduction.*

The theory of the Donnan-equilibrium has received an increasing attention from investigators in the field of ion adsorption and rightly so. The Donnan-distribution of ions is of a fundamental nature and occurs whenever ions or ionized groups cannot spread evenly through a system. It is true that when adsorption measurements are carried out over a wide range of conditions the results seldom agree with the Donnan laws in their simplest form. However, this only means that the quantities characterizing the Donnan-distribution are no constants, but vary with changing conditions. This variation may be of a complicated nature and thus no simple equation can be expected to express the data. Several simple empirical formulae have been proposed for certain quantitative and mostly practical aspects of ion adsorption, but it is not surprising that they only apply under a limited set of conditions and do not clarify anything about the mechanism of ion adsorption. It is illustrative that one of the best of these, Vageler's extrapolation formula, can be derived with minor approximations from a more fundamental formula based on the law of mass action, as was shown by Damsgaard-Sørensen (2).

Mattson (7*a*, 7*b*), being the first to apply the Donnan principle to base exchange phenomena in soils, together with his pupils has shown that the Donnan laws directly explain such phenomena as negative adsorption, swelling and — upon dilution of clay suspensions in salt mixtures — the displacement of monovalent ions by divalent or trivalent ions. Among the facts not following directly from the Donnan laws these two are very important:

a. the replacing power of hydrogen-ions under some conditions seems to be out of all proportion, compared with that of other cations.

b. different metal-ions are adsorbed with different intensities, even if they have the same valency.

The modifications of the Donnan-theory necessary to explain these facts will be examined in this paper, partly from an angle somewhat different from MATTSON's point of view.

2. *The quantities characterizing the Donnan-distribution.*

Two distinct parts are recognized in a Donnan-system, an "inside" solution containing the non-diffusing ions or ionized groups and an "outside" solution with less or none of these ions or groups. In soil chemistry we find these parts referred to as micellar solution and equilibrium solution. The freely diffusing ions, attracted by the non-diffusing ions and accumulated in the micellar solution, are identical with the adsorbed or exchangeable ions. Assigning the indices i and o to the "inside" and "outside" part of the Donnan-system, we may express the distribution of the diffusing cations B_a, B_b, ... and anions A_α, A_β, ... over the two parts of the system by the following equations:

$$\text{(I)}\quad \left(\frac{f_{ao}[B_a]_o}{f_{ai}[B_a]_i}\right)^{1/a} = \left(\frac{f_{bo}[B_b]_o}{f_{bi}[B_b]_i}\right)^{1/b} = \ldots = \left(\frac{f_{\alpha i}[A_\alpha]_i}{f_{\alpha o}[A_\alpha]_o}\right)^{1/\alpha} = \left(\frac{f_{\beta i}[A_\beta]_i}{f_{\beta o}[A_\beta]_o}\right)^{1/\beta} = \ldots = q$$

and

$$\text{(II)}\qquad a[B_a]_i + b[B_b]_i + \ldots = \alpha[A_\alpha]_i + \beta[A_\beta]_i + \ldots + A.$$

In these equations [ ] denotes concentration; $a, b, \ldots, \alpha, \beta, \ldots$ represent the valencies of the cations and anions and $f_a, f_b, \ldots, f_\alpha, f_\beta, \ldots$ the activity coefficients. The quotient q is characteristic for the distribution both of cations and anions (for monovalent ions it is the ratio of the activities in the outside solution and in the micellar solution) and A is the concentration of the non-diffusing ion (if it is an anion, A will be positive, if it is a cation A will be negative). Obviously all the quantities having the index o can be determined directly by analysis of the equilibrium solution. As the micellar solution cannot be separated for analysis, all the other quantities must be found by specially designed experiments.

At least three principles have been described to estimate the unknown quantities in the equations (I) and (II). The first two of these principles have been applied by the Swedish workers DU RIETZ (13) and SAMUELSON (14), the third one was proposed by us when elaborating the results of potential measurements on plant roots (15). If we assume all activity coefficients to be unity, the three methods are schematically as follows:

a. quantitative determination of the adsorption of two competing cations of unequal valency. If these cations are K^+ and Ca^{++}, say, and their concentrations in the outside solution equal to c_1 and c_2 respectively, equation (I) gives

$$\text{(III)}\qquad \frac{[K^+]_i}{c_1} = \sqrt{\frac{[Ca^{++}]_i}{c_2}}.$$

Remembering that the total amount of potassium adsorbed, denoted by (K^+), equals $[K^+]_i \times V$, where V is the volume of the micellar solution, and likewise $(Ca^{++}) = [Ca^{++}]_i \times V$, we may write equation (III) as follows

$$\text{(IV)} \qquad \frac{(K^+)}{c_1 V} = \sqrt{\frac{(Ca^{++})}{c_2 V}}.$$

As (K^+) and (Ca^{++}) can be measured, V can be solved from equation (IV). A similar procedure for two cations of the same valency would have been useless, because the quantity V would have been eliminated from equation (IV) in that case. Once the quantity V is known, all the values found experimentally for the total amounts adsorbed of each of the partaking ions can be converted into concentrations by dividing these values by V. Equation (I) now provides a value for q and equation (II) for A.

b. quantitative determination of the adsorption of a cation and of the (negative) adsorption of an accompanying anion, say K^+ and Cl^-. Taking $[K^+]_o = c_1$ and $[Cl^-]_o = c_3$, we get from equation (I):

$$\text{(V)} \qquad \frac{[K^+]_i}{c_1} = \frac{c_3}{[Cl^-]_i}.$$

As before we have $[K^+]_i = \frac{(K^+)}{V}$ and $[Cl^-]_i = \frac{(Cl^-)}{V}$, so that equation (V) becomes

$$\text{(VI)} \qquad \frac{(K^+)}{c_1 V} = \frac{c_3 V}{(Cl^-)}.$$

From this equation V can be solved and q and A calculated in the same way as under a.

c. measurement of the membrane potential, having a value, which according to the NERNST equation, equals

$$\text{(VII)} \qquad E = \frac{RT}{F} \ln q,$$

where q is the characteristic quotient of equation (I).

For room temperature the equation becomes

$$E \text{ (in Volts)} = 0{,}0577 \log q$$

By measuring E we get a value for q and this leads by means of equation (I), disregarding the activity coefficients, to values for the concentrations of the ions in the micellar solution. Now A can be found from equation (II). The possibility of measuring membrane potentials on soil constituents requires some explanation that will be given in a later paragraph.

It would seem as if these three methods could serve to check one another, as they all provide values for the same unknown quantities. If, however, account is taken of the fact that most likely the activity

coefficients in the micellar solution are different from unity, the picture changes somewhat. In that case, with the aid of the value q found by method c, and the ion activities in the outside solution, we can calculate the activities in the micellar solution from equation (I). These activities can also be written

$$\frac{f_{ai}(B_a)}{V}, \frac{f_{bi}(B_b)}{V}, \ldots, \frac{f_{\alpha i}(A_\alpha)}{V}, \frac{f_{\beta i}(A_\beta)}{V}, \ldots$$

and since the values (B_a), (B_b), $\ldots$, (A_α), (A_β), $\ldots$ can be found experimentally, we can calculate values for

$$\frac{f_{ai}}{V}, \frac{f_{bi}}{V}, \ldots, \frac{f_{\alpha i}}{V}, \frac{f_{\beta i}}{V}, \ldots$$

but we do not obtain separate values for either the quantity V or the activity coefficients in the micellar solution. Obviously, the more these activity coefficients differ from unity, the less will be the reliability of the values found for V by the methods a and b, where the activity coefficients were neglected. The error will probably be greater for polyvalent ions than for monovalent ions. Calculations based on experiments with monovalent ions may be suited to give at least approximative values. In any case method (a) seems to be far less reliable than method b, though in the latter the determination of the negative adsorption requires a high degree of experimental accuracy.

An independent determination of the quantity V would be needed to calculate the missing values for the activity coefficients. WIKLANDER (19) used approximative values for V, obtained by weighing the amount of water held by a certain amount of the adsorbing material after it had been subjected to a definite suction for a certain length of time.

Summarizing this paragraph, we may conclude that only a combination of adsorption measurements, potential measurements and determinations of the volume of the micellar solution can give us the data needed for a complete description of the behaviour of soil constituents as a Donnan-system.

3. *The inconstancy of the quantities governing the Donnan-distribution.* Though no complete analysis, as proposed in the previous paragraph, has been carried out on any adsorbing substance, there is ample evidence to show, that none of the quantities governing the Donnan-system is constant. This is also the reason for the deviations from the simple Donnan-laws already referred to in the introduction.

It follows from equation (II) that A represents the net cation binding capacity of the adsorbing material. Now the base exchange capacity, which has nearly the same value, varies considerably, as was shown by investigations of MATTSON and HESTER (8), BÄR and TENDELOO (1) and GORTIKOV (4). The base exchange capacity depends on several factors, pH being by far the most important. Thus, if the hydrogen activity in the outside solution is brought down to values as low as 10^{-8} N, a further

lowering will still cause the replacement of a considerable amount of H-ions, the adsorption of metal-ions increasing by an equivalent amount. The presence of considerable quantities of replaceable H-ions in a medium where the concentration of H-ions is so small that they could hardly have any competitive effect, might suggest a very high replacing power of the H-ions, probably caused by their very low activity coefficient in the micellar solution. The assumption of a weak acid character of the adsorbing material affords a much better explanation, however. As the pH-value of the outside solution increases, the (relatively small) number of H-ions in the micellar solution decreases accordingly. As a consequence there is a further dissociation of the weak acid, which means an increase in the concentration A of the non-diffusing ions. So the problem is not only to find the value of A corresponding to any particular pH-value under one set of conditions, but also to find the relation between A and the changing pH-values of the micellar solution, when other conditions prevail.

Several facts, leading to the conclusion that also the volume V of the micellar solution and the activity coefficients of the adsorbed ions have variable values, have been known for a long time before an attempt to interpret adsorption phenomena in terms of these quantities had been undertaken. Thus the difference in replacing power between ions of the same valency could be the consequence of differences in activity coefficients of these ions. Such differences are known to occur in homogeneous solutions and analyses as outlined in the previous paragraph could probably show whether the activity coefficients in the micellar solution behave similarly. The volume V of the micellar solution, as determined by WIKLANDER (19) for a resin, is dependent on the nature of the adsorbed ions. It seems probable, that this volume V is also closely related to the degree of swelling of the adsorbing material and in that case one would also expect a marked influence of the concentration of the outside solution. Here again a complete analysis of Donnan-systems under diverse conditions should be able to shed more light.

4. *The measurement of membrane potentials on soil-constituents.*

For the measurement of membrane potentials in Donnan-systems in general, both the o-space and the i-space are brought into contact with unpolarisable electrodes and the e.m.f. of the element formed thus is determined. When the i-space consists of a mineral or another solid substance, it will not be possible to bring the electrode inside the i-space. It would seem possible however, to use a glasselectrode-like construction, whereby a thin plate of the material under test is brought between two solutions, these making contact with the two electrodes. There are two membrane potentials in such a chain, one at each junction between the solid substance and the solutions. As a consequence the absolute value of the e.m.f. measured cannot be used for drawing conclusions

about the properties of the solid material. However, if the conditions on one side of the plate are kept constant, we may consider the effect of a changing composition of the other solution upon the e.m.f. as a pure reflection of the change of the membrane potential, provided the influence of both solutions does not extend through the solid material so as to affect the membrane potential on the opposite side. With a glasselectrode this condition is satisfied without any doubt, but this seems uncertain in the case of the clay membrane electrodes used by MARSHALL (6) and his collaborators for the purpose of measuring ion-activities in solutions and suspensions. A careful study would be necessary to ascertain whether there are possibilities of using the data supplied by the MARSHALL method for conclusions about the adsorptive properties of the clay minerals in the membrane.

Potential measurements on different kinds of glass have been performed by several investigators, but unfortunately not in combination with quantitative adsorption measurements. So for the present no complete analysis for any kind of adsorbing material can be presented.

To give some idea about the possibilities of potential measurements, we chose a few data collected by ZWART VOORSPUY (20) in a study on the electrode properties of several kinds of glass. Figure 1 gives the values for the e.m.f. measured by him on electrode 22. The composition of the glass in question was 6 % CaO, 14 % Li_2O, 10 % Al_2O_3 and 70 % SiO_2. The measurements were done in N/500, N/100 and N/10 NaCl(+NaOH)-solutions of different pH-values, obtained by adding acetic and hydrochloric acid. To demonstrate that the potentials found are essentially membrane potentials in a Donnan-system with a weak acid as the non-diffusing part, we had to start from a definite assumption about the dissociation of the weak acid in dependence of pH. The assumption that we had to do with one monobasic acid proved to be too simple, but potential measurements do not allow to discriminate between the individual dissociation constants in the case of a mixture of polybasic acids. Thus, for simplicity we assumed the concentration of the dissociated part of the acid to change proportionally to pH, in other words

$$A = K_1 \ln [H^+]_i + K_2. \tag{VIII}$$

For small pH-ranges this certainly is a logical assumption, but even for a broad pH-range it may be quite near the truth if there is a sufficient diversity of the dissociation constants (17). A method to calculate the unknown constants K_1 and K_2 from the potential measurements (disregarding activity coefficients) has been given in a previous paper (17). As was mentioned above, the measured potentials must be corrected by a constant voltage to obtain the pure membrane potential. A correction term of — 377 mV proved to be most suitable. The best fitting values for K_1 and K_2 were — 0,00687 and 0,0242. In figure 1 the theoretical curves for the membrane potential, calculated by using the values found

for K_1 and K_2, have been drawn to show that there is a very satisfactory agreement between theoretical and experimental values, in spite of the approximations used in the calculation and the neglect of the activity coefficients.

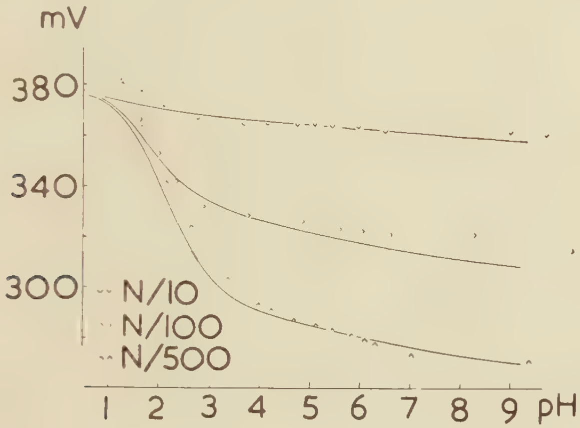

Fig. 1. Influence of pH and Na^+-concentration of a solution of NaCl on the potential of a glasselectrode. Measurements by ZWART VOORSPUY.

A further calculation was made with the aid of the two constants found. By means of the equations (I) and (II) we computed the sodium concentrations in the *i*-phase (the glass phase) of the Donnan system for a number of different pH-values and different sodium concentrations in the equilibrium solution. The results are shown in figure 2. Remem-

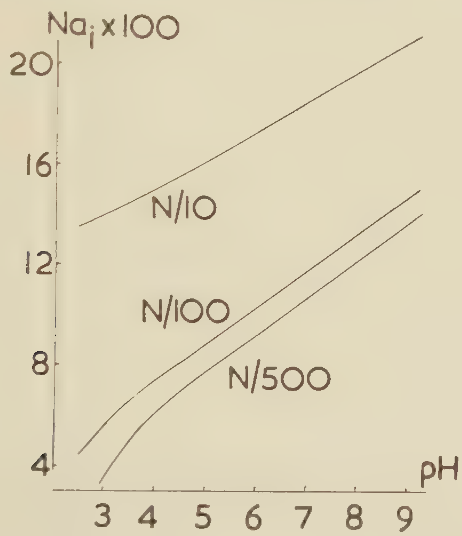

Fig. 2. The influence of pH and Na^+-concentrations of an equilibrium solution of NaCl on the concentration of Na-ions inside glass. Curves calculated on the basis of the formula $A = -0{,}00687 \ln [H^+]_i + 0{,}0242$ for the dissociated part of the glass-acidoid.

bering that the concentrations $[Na^+]_i$ can be converted into the total amounts (Na^+) adsorbed, simply by multiplying by the volume V of the micellar solution, we see that figure 2 also shows how the exchange capacity changes for different pH and $[Na^+]_o$ values, in so far as the volume V may be taken to have a constant value. Though no experimental values of the exchange capacity of this special kind of glass are available for comparison, we find a general agreement between these curves and those found by BÄR and TENDELOO (1) for the relation between exchange capacity and pH, determined for different clays. There is also a qualitative agreement with the data published by GORTIKOV (4) and NICHOLSKY and PARAMANOVA (11) on the influence of the concentration of the metal-ion on the base exchange capacity, but the difference in exchange capacity between N/100 and N/10 seems unusually high compared with the differences actually found by these investigators. A summary of these differences is given in the following table.

Difference in exchange capacity with different salt concentrations

Author	Material	pH	Ion	Difference in concentration	Difference in exchange capacity
GORTIKOV (4)	tschernosem	6,5	Ba	N/100 — N/1	20 %
NICHOLSKY (11)	glauconite	7,0	Na	N/20 — N/5	13 %
,,	,,	7,0	Ba	N/100 — N/10	7,5 %

This discrepancy is most probably caused by the fact that figure 2 does not quite give a true picture of the exchange capacity, because the volume V of the micellar solution is inconstant. DU RIETZ (13) found, that in the case of cellulose fibers V decreases when the salt concentration increases and for soil minerals it is known, that they also hold more water in diluted solutions. Almost certainly the same would be true for glass.

One other interesting comparison can be made. If in equation (VIII) the empirical values for K_1 and K_2 are substituted, we find the following values for A: 0,10327 (pH = 5) and 0,13490 (pH = 7). Both values are of the same order of magnitude as the value $A = 12/100$ N, calculated by DU RIETZ (13) for a permutite investigated by MOLLER (10), which seems the more promising as permutites and glasses are very closely related materials.

5. *The spatial distribution of the exchangeable ions.*

For a Donnan-distribution to establish itself for any ion, it is necessary that this ion can freely move through the whole system. If the exchange phenomena of soil constituents were restricted to the surface of the particles, as schematically outlined in one of MATTSON's earlier papers (7*a*), there should be no reason to doubt whether the micellar solution is accessible to all diffusing ions. However, from the studies of WIEGNER

and his pupils it has become clear, that at least some of the mineral adsorbing substances in the soil are built in such a way, that a very considerable part of the exchange phenomena takes place in the interior of the particles. In a survey, given by PALLMANN (12), a distinction is made between compact exchanging substances and exchanging substances with inner surfaces. The latter can be subdivided into rigid materials, when the dimensions of the inner pores are not variable (permutites, zeolithes) and elastic materials, when the dimensions of the inner pores are variable (montmorillonite, beidelite, notronite). In the non-compact substances we may expect deviations from the normal laws of ion exchange if ions are present which due to their size cannot penetrate into the interior of the substance. WIEGNER (18) performed experiments which gave a striking illustration of such a "blockade" action. In terms of the Donnan-distribution this would mean that part of the micellar solution is only available for the smaller and not for the bigger ions. Thus there would not be one single volume of the micellar solution, but the kinds of adsorbed ions would take different volumes. If, in addition, the proportion of non-available micellar solution is variable, as will probably be the case for the elastic exchanging substances, it would certainly become extremely difficult to unravel the complex of these and other factors responsible for the variation in replacing power of the different ions. Yet, this phenomenon calls for further study, as will also be seen from the following examples in which ion-blockade seems to play an important part. Some of these examples also lend some additional support to the conception that glass-electrode potentials are essentially membrane potentials.

The establishment of a Donnan-equilibrium may take rather a long time, if the size of the pores is so small, that the exchanging ions can only pass with great difficulty. The fixation of potassium in soils, as found in different parts of the world and theoretically examined by DOMINGO (3), may be an example of such a retarded diffusion. It is an exchange reaction which hardly becomes apparent in normal exchange experiments, which last only a short time. Only under extreme conditions, like heating and drying, both increasing the rate of diffusion, and in long-duration experiments a substantial proportion of the potassium ions taken up may be converted into a practically non-exchangeable form.

The hydrogen-ion is the smallest ion and the last to be affected by too small a dimension of the pores. An interesting consequence is that it is possible to make electrodes with hydrogen as the only potential determining ion. When only H-ions pass through the boundary, equation (VII) will hold for H-ions only and the potential of the electrode will be

$$E = \frac{RT}{F} \ln \frac{[H^+]_o}{[H^+]_i}.$$

As no other ions are able to penetrate into the material and to displace H-ions, $[H^+]_i$ is constant and we get

$$E = \frac{RT}{F} \ln [H^+]_o + \text{const.},$$

the well-known equation for the glass-electrode. Some kinds of glass have these ideal hydrogen-electrode properties. Quartz also behaves as an ideal hydrogen-electrode, implying that no metal-ions penetrate into quartz particles.

Any kind of metal-ion that is able to penetrate into the material will affect the electrode potential of the material, but since hydrogen is the smallest ion, it will be able to pass too and will also affect the potential. The electrode will be a mixed electrode and not specific for the metal-ion. Only in very special cases will the entrance of H-ions be prevented, e.g. by a precipitate of AgCl, a circumstance which enables us to make Ag- or Cl-electrodes.

The influence which the incorporation of alluminium into a silicate seems to have both on the electrode behaviour of the silicate and on its adsorptive properties affords another example of the connection between the two. As found by some investigators and confirmed by Tendeloo and Zwart Voorspuy (16), the potential of glass-electrodes, otherwise indifferent to metal-ions, becomes highly sensitive to variations in the concentration of metal-ions if the glass contains Al_2O_3. In the light of the conception developed above, this would mean that alluminium makes the glass structure "open", so that metal-ions can move in together with H-ions. A parallel can be found in the fact that artificially prepared Al-silicates, called permutites, have a high adsorption capacity and in the observation of van der Meulen (9) that Al in the lattice of minerals is connected with a high adsorption capacity. Kelley and Jenny (5) found exceptions to this rule, it is true, when they showed that some Al-containing minerals have a low adsorption capacity, but Zwart Voorspuy [1]) occasionally obtained Al-containing glasses behaving as an ideal glass-electrode. Apparently the presence of Al_2O_3 does not with certainty lead to an "open" structure.

This paragraph thus leads to the conclusion that the exchange laws may become very complicated by partial or complete inhibition of ion-penetration due to "blockade". Some special cases of this phenomenon provide instructive examples of the connection between electrode properties and adsorptive behaviour of silicates.

6. *Conclusions and summary.*

Adsorption phenomena in the soil can best be described by the laws of the Donnan-equilibrium. It should be remembered that the quantities governing the Donnan-distribution, such as the concentration of the

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[1]) Not published.

non-diffusing ion, the volume of the micellar solution and the activity coefficients of the adsorbed ions may have variable values. A complete picture of the variation of these values can only be obtained by the coordinated application of several methods, that have been described in literature for the examination of individual aspects of Donnan-systems. Among these methods the measurement of the membrane potential of solid materials deserves a more thorough investigation, because it seems to be the only method supplying direct information about the activities of the adsorbed ions. The measurement could be performed by using a glass-electrode-like construction, but the difficulties of transforming soil-constituents into suitable electrodes have not yet been solved. Some data are presented in paragraph 4 to show that the potentials measured with glass-electrodes are essentially membrane potentials.

A very serious complication of the exchange laws is caused by the fact that in some important soil minerals adsorption is not only a two-dimensional, a surface phenomenon, but also occurs in the interior of the particles and has a three-dimensional nature. This may lead to a retarded diffusion or even a complete "blockade" effect for the bigger ions. It would seem extremely difficult to find a mathematical expression for these mechanical effects, the more so, since the dimensions of the inner pores may change with varying conditions.

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MATHEMATICS

ON THE THEORY OF SPHEROIDAL WAVE FUNCTIONS OF ORDER ZERO

BY

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In this paper, the standard form of the differential equation of spheroidal wave functions [1, 2, 3, 4, 5] of order zero is understood to be

(1) $$\Omega_z\, y = (1-z^2)\, y'' - 2zy' + (\lambda + k^2 z^2)\, y = 0,$$

in which λ and k are independent of z. Those values $\lambda = \lambda_\nu(k)$ for which equation (1) admits a solution $y = y_\nu(z; k)$ that is finite at $z = \pm 1$ are called the characteristic values or eigenvalues of the differential equation (1). The corresponding characteristic solutions or eigenfunctions $y_\nu(z; k)$ are spheroidal wave functions (of order zero) of the first kind and of degree ν, where ν runs through the set of non-negative integers.

In the limiting case $k = 0$ equation (1) reduces to the familiar differential equation of Legendre functions, so that $\lambda_\nu(0) = \nu(\nu+1)$ while $y_\nu(z; 0)$ is proportional to $P_\nu(z)$.

It is convenient [2, 4] to normalize the eigenfunctions so that their norm is independent of k, viz.

(2) $$\int_{-1}^{1} \{y_\nu(z; k)\}^2\, dz = \int_{-1}^{1} \{P_\nu(z)\}^2\, dz = \frac{2}{2\nu+1}.$$

The remaining ambiguity in algebraic sign is removed by the auxiliary condition

(3) $$\lim_{k\to 0} y_\nu(z; k) = P_\nu(z),$$

which in addition to (2) suffices to fix the eigenfunctions completely, since they are uniform functions of k if k^2 is real. As in the analogous case of Mathieu functions [6] this property ceases to hold if k is unrestricted complex [7]. In this paper, however, k^2 will be taken real so that the question of normalization does not lead to difficulties.

It is known that the set of eigenfunctions of the differential equation (1) is identical with the set of eigenfunctions of the integral equation

(4) $$y(z) = \mu \int_{-1}^{1} e^{kzt}\, y(t)\, dt.$$

To verify this statement [2], let us substitute in (1)

$$y(z) = \int e^{kzt}\, h(t)\, \mathrm{d}t.$$

Then, by the familiar procedure of differentiation with respect to z and subsequent integration by parts with respect to t, it is easy to see that

$$\Omega_z\{y(z)\} = (1-t^2)\{kz\, h(t) - h'(t)\}\, e^{kzt} + \int e^{kzt}\, \Omega_t\{h(t)\}\, \mathrm{d}t.$$

Now the integral vanishes if $h(t)$ is a characteristic solution of the differential equation (1), at least if the, as yet unspecified, contour of integration lies wholly inside the regularity domain of $h(t)$. If $h(t)$ is an eigenfunction of (1), which implies that $y(t)$ and $y'(t)$ are finite at $t = \pm 1$, then the integrated term vanishes identically in z at the points $t = \pm 1$. Consequently,

$$\int_{1}^{1} e^{kzt}\, y_\nu(t; k)\, \mathrm{d}t$$

is a solution of (1) corresponding to the eigenvalue $\lambda_\nu(k)$ of (1). This solution is clearly analytic at $z = \pm 1$, so that it is simply proportional to $y_\nu(z; k)$. Therefore, any eigenfunction of (1) is an eigenfunction of (4). The converse is also true, which completes the proof of our statement.

In passing it may be noted that there does not seem to exist a simple relationship between the eigenvalues $\lambda_\nu(k)$ of the differential equation and the eigenvalues $\mu_\nu(k)$ of the integral equation. As a matter of fact, equation (4) becomes nugatory in the limiting case $k = 0$, in the sense that μ tends to infinity if k tends to zero (unless $\nu = 0$). Generally, it can be proved that

$$(5) \qquad \mu_\nu(k) = \frac{(2\nu)!\,(2\nu+1)!}{2^{2\nu+1}(\nu!)^3}\,\frac{1}{k^\nu}\{1 + O(k^2)\} \qquad (k \to 0),$$

as is seen by ν-times differentiation of (4) followed by application of $k \to 0$, which gives

$$\mu \sim \frac{\mathrm{d}^\nu P_\nu(z)/\mathrm{d}z^\nu}{k^\nu \int_{-1}^{1} t^\nu P_\nu(t)\, \mathrm{d}t}.$$

In the conventional approach to spheroidal wave functions [1, 2, 3, 4, 5], the eigenfunctions of equation (1) are expanded in a series of Legendre polynomials, viz.

$$(6) \qquad y(z) = \sum_{n=0}^{\infty} a_n P_n(z),$$

where for simplicity in notation we have omitted any explicit indication as to the degree ν and the parameter k of the eigenfunction $y(z)$ and its coefficients a_n, which actually depend on ν and k. The coefficients a_n can be determined by the use of infinite continued fractions, as in the analogous theory of Mathieu functions. At the same time the technique

provides a transcendental equation for the eigenvalues $\lambda=\lambda_\nu(k)$. In accordance with (2), the coefficients are normalized so that

$$\sum_{n=0}^{\infty} \frac{a_n^2}{2n+1}=\frac{1}{2\nu+1}\,, \quad \lim a_\nu=1 \quad (k\to 0). \tag{7}$$

For details the reader is referred to an earlier paper [4]. Let it suffice to mention that the eigenfunctions are entire functions of z, with an irregular singularity at infinity; and they are either even ($a_{2n+1}=0$) or odd ($a_{2n}=0$) with respect to z. Thus

$$y_\nu(-z;k)=(-1)^\nu\, y_\nu(z;k). \tag{8}$$

For a given value of k and a corresponding eigenfunction $y_\nu(z;k)$ with characteristic value $\lambda_\nu(k)$, the complete solution of equation (1) is often required in physical applications. Spheroidal wave functions of the second, third, etc., kinds are necessarily singular at $z=\pm 1$.

It has been suggested [1, 8] that a spheroidal wave function of the second kind would be given by

$$\bar{y}(z)=\sum_{n=0}^{\infty} a_n Q_n(z), \tag{9}$$

in which Q_n is Legendre's function of the second kind and the coefficients a_n are the same as in the expansion (6). In fact, it was argued that these coefficients were to be found simply by application of the familiar recurrence relations for Legendre functions, which were said to be the same for P_n and Q_n. This, however, is erroneous because the two types of Legendre function behave differently in the beginning, that is, for $n=0$ and $n=1$. Among others, in deriving (6) the term $n\,P_{n-1}$ is rightly equated to zero for $n=0$, but in the analogous case of (9) the corresponding term $n\,Q_{n-1}$ must be interpreted as being unity for $n=0$ owing to the fact that Q_n, considered as a function of n, has a simple pole at $n=-1$ with residue 1. As a consequence,

$$\Omega_z\{\bar{y}(z)\}=k^2 a_0 P_1(z) \quad \text{or} \quad \tfrac{1}{3} k^2 a_1 P_0(z),$$

depending on whether the relevant eigenfunction (6) is even or odd in z. Thus the function $\bar{y}(z)$ does not solve equation (1) if $k\neq 0$, since it may be shown that a_0 and a_1 are never zero for the even and odd eigenfunctions of (1) unless $k=0$. This peculiarity has been overlooked by Hanson [8] who apparently committed the error for the first time, though the correct expansion was known long before the publication of his paper. The error mentioned has been noted by Stratton and co-workers [3] and by the author [2] almost simultaneously.

According to Herzfeld [9], the series (9) should be supplemented by one of the type (6), viz.

$$Y(z)=\sum_{n=0}^{\infty} a_n Q_n(z)+\sum_{n=0}^{\infty} b_n P_n(z), \tag{10}$$

in which the coefficients b_n satisfy a similar three-term recurrence relation as do the coefficients a_n. From the conditions $b_{n+2}/b_n \sim k^2/4n^2$ $(n \to \infty)$ and

$$\Omega_z \{ \sum_{n=0}^{\infty} b_n P_n(z) \} = -k^2 a_0 P_1(z) \quad \text{or} \quad -\tfrac{1}{3} k^2 a_1 P_0(z)$$

for $\lambda = \lambda_\nu(k)$, the coefficients b_n are uniquely determined. In particular, the link between the two systems of coefficients is provided by an infinite continued fraction for the ratio of the first non-vanishing coefficients a and b. For an even eigenfunction $(a_{2n+1} = 0)$ the additional series $\sum b_n P_n$ is odd in z $(b_{2n} = 0)$, and we have

$$(11) \qquad \frac{b_1}{a_0} = \frac{k^2 |}{|a_1 - \lambda} - \frac{\beta_3 |}{|a_3 - \lambda} - \frac{\beta_5 |}{|a_5 - \lambda} - \dots$$

where λ is the corresponding eigenvalue of the even function, while

$$a_n = n(n+1) - \tfrac{1}{2} k^2 \left\{ 1 + \frac{1}{(2n-1)(2n+3)} \right\},$$

$$\beta_n = \frac{n^2 (n-1)^2}{(2n-1)^2} \frac{k^4}{(2n-3)(2n-1)}.$$

On the other hand, for an odd eigenfuntion $(a_{2n} = 0)$, the series $\sum b_n P_n$ is even in z $(b_{2n+1} = 0)$ and

$$(12) \qquad \frac{b_0}{a_1} = \frac{\tfrac{1}{3} k^2 |}{|a_0 - \lambda} - \frac{\beta_2 |}{|a_2 - \lambda} - \frac{\beta_4 |}{|a_4 - \lambda} - \dots,$$

where λ now is the eigenvalue of the odd function.

It may be remarked that the right-hand member of (11), where λ is meant to denote an eigenvalue belonging to an *even* eigenfunction, becomes infinitely large for λ equal to any of the eigenvalues of the *odd* eigenfunctions. The reverse applies to (12). This should be so, since in either case we must take all coefficients a_n identically zero, to have finite values for b_n, so that then the series (10) reduces to one of the type (6), that is, represents an eigenfunction itself. In other words, the "even eigenvalues" are the roots of

$$0 = a_0 - \lambda - \frac{\beta_2 |}{|a_2 - \lambda} - \frac{\beta_4 |}{|a_4 - \lambda} - \dots$$

and the "odd eigenvalues" are the roots of

$$0 = a_1 - \lambda - \frac{\beta_3 |}{|a_3 - \lambda} - \frac{\beta_5 |}{|a_5 - \lambda} - \dots$$

in conformity with known results [2, 4].

As follows from their construction, the series $\sum a_n P_n$ and $\sum b_n P_n$ both converge for any finite value of z, and in fact they converge to an entire function of z. On the other hand, the series $\sum a_n Q_n$ diverges at $z = \pm 1$, where it leads to a logarithmic singularity of $Y(z)$ because the individual terms are singular themselves. Since the singular part of

$Q_n(z)$ is $Q_0(z)\,P_n(z)$, it may be anticipated that $Y(z) - Q_0(z) \sum a_n P_n(z)$ remains finite at $z = \pm 1$. As a matter of fact, the function

$$Y(z) - \tfrac{1}{2}\, y(z) \log \frac{z+1}{z-1}$$

is an entire function of z, as will be seen later on.

In order to make $Y(z)$ one-valued, it is convenient to introduce a cut in the complex plane of z in the same way as is usually done for $Q_n(z)$. This cut is the straight line segment drawn from $z = -1$ to $z = 1$ along the real axis. For real values of z between $z = -1$ and $z = 1$ a spheroidal function of the second kind can be defined in a similar manner as this is done for Q_n in the theory of Legendre functions.

A disadvantage in accepting the expansions (6) and (10) as a canonic system of spheroidal wave functions of order zero is that they require two different sets of coefficients, a_n and b_n, though it is true that their respective recurrence relations are closely related to each other. In this connection it may be emphasized that Stratton and co-workers [3] derive a_n and b_n, as limiting cases, from one and the same difference equation, valid for spheroidal wave functions of unrestricted degree and order, though at the cost of simplicity in presentation.

From the point of view of numerical computation as well as of analysis, it would be profitable to have a single set of coefficients. In fact, such types of solution have been known for a long time. They may conveniently be obtained in the following manner. If in the integral equation (4) we substitute for $y(t)$ the corresponding expansion (6) we get, for all finite values of z,

$$y(z) = 2\,\mu \sqrt{\frac{\pi}{2}}\,(kz)^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n\, I_{n+\frac{1}{2}}(kz), \tag{13}$$

where I denotes the modified Bessel function of the first kind. In deriving (13) we have used the known identity

$$\int_{-1}^{1} e^{kzt}\, P_n(t)\, \mathrm{d}t = \sqrt{\frac{2\pi}{kz}}\, I_{n+\frac{1}{2}}(kz). \tag{14}$$

An alternative way to derive the solution (13) is by direct substitution in equation (1) and employing the recurrence relations for Bessel functions. It then appears that the resulting difference equation for the coefficients in (13) is precisely that for the coefficients in the expansion (6) in Legendre functions. However, equation (13) gives more in that it fixes the constant of proportion in terms of μ, the eigenvalue of the integral equation (4), which can be calculated as soon as the coefficients a_n are known. For example, if the eigenfunction under consideration is even in z (i.e. if ν is even), we substitute $z = 0$ in (4), so as to obtain

$$\mu_\nu = \frac{y(0)}{2\,a_0} = \frac{1}{2\,a_0} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1)}\, a_{2n}\,, \ (\nu \text{ even}). \tag{15}$$

Similarly, for the odd eigenfunctions,

$$(16) \qquad \mu_\nu = \frac{3y'(0)}{2ka_1} = \frac{3}{ka_1} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1)}\, a_{2n+1}\,, \ (\nu \text{ odd}),$$

which is found by first differentiating (4) with respect to z and then taking z equal to zero.

Now, unlike the Legendre function Q_n, the modified Bessel function of the second kind, $K_{n+\frac{1}{2}}$, is an analytic function of n for all n. Further, $K_{n+\frac{1}{2}}$ and $I_{n+\frac{1}{2}}$ obey the same set of recurrence relations. It is thus evident that a second solution of equation (1) is given by

$$(17) \qquad Y^*(z) = \frac{k}{2\mu}\sqrt{\frac{2}{\pi}}\,(kz)^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n\, K_{n+\frac{1}{2}}(kz).$$

in so far as this series is convergent. This in fact is true if $|z| > 1$, as may be inferred from the asymptotic behaviour of the general term of (17) as $n \to \infty$.

This asymptotic behaviour follows at once from the recurrence relations for a_n and $K_{n+\frac{1}{2}}$, viz.

$$\frac{a_{n+2}}{a_n} \sim \frac{k^2}{4n^2}\,, \quad \frac{K_{n+5/2}(kz)}{K_{n+1/2}(kz)} \sim \frac{4n^2}{k^2z^2}\ (n \to \infty).$$

Therefore, the ratio of successive terms of (17) tends to z^{-2} so that (17) is absolutely convergent if $|z| > 1$ and divergent if $|z| < 1$. Furthermore, the degree of convergence of (17) for $|z| > 1$ is comparable to that of the series $\sum_n z^{-2n}$, which implies that (17) is useful only if $|z|$ is much greater than unity.

By a similar reasoning it may be shown that the degree of convergence of the analogous series (13) is comparable to that of the series

$$\sum_n (k^2z/16)^{2n}/(n!)^4,$$

so that (13), besides being convergent for arbitrary z, is much stronger convergent than (17).

In this connection it may be mentioned that the degree of convergence of the expansions (6) and (10) is intermediate between those of (13) and (17). For points outside the cut $(-1 \leqslant z \leqslant 1)$ the former series converge like

$$\sum_n \{\tfrac{1}{4}k\,(z \pm \sqrt{z^2-1})\}^{2n}/(n!)^2.$$

The divergence inside the unit circle of the series (17) for the spheroidal wave function of the third kind, $Y^*(z)$, is a serious disadvantage in physical applications as, e.g., the diffraction of a plane wave by a circular disk or aperture [2]. On the other hand the asymptotic behaviour as $|z| \to \infty$ of (17) is easily recognized (see below) in contradistinction to the case of (10), and it is the function of the third kind $Y^*(z)$ which

is required for the representation of travelling waves diverging from the origin of coordinates.

Therefore we have to continue the series expansion (17) analytically inside the unit circle. This can be accomplished as follows [2]. The analogue of (14) is, if $\mathrm{Re}(kz) > 0$,

(18) $$\int_1^\infty e^{-kzt} P_n(t)\, \mathrm{d}t = \sqrt{\frac{2}{\pi kz}}\, K_{n+\frac{1}{2}}(kz).$$

Assume for a while that $k > 0$ and $z > 1$. Then we may substitute equation (18) in (17) and change the order of summation and integration. Hence, by using (6),

(19) $$Y^*(z) = \frac{k}{2\mu} \int_1^\infty e^{-kzt}\, y(t)\, \mathrm{d}t.$$

Now, according to the integral equation (4),

$$\frac{1}{\mu} y(t) = \int_{-1}^{1} e^{kt\tau}\, y(\tau)\, \mathrm{d}\tau,$$

which after substitution in (19) and reversing the order of integration yields

$$Y^*(z) = \tfrac{1}{2} k \int_{-1}^{1} y(\tau)\, \mathrm{d}\tau \int_1^\infty e^{-kt(z-\tau)}\, \mathrm{d}t.$$

The second integral is elementary, so that the final result becomes

(20) $$Y^*(z) = \tfrac{1}{2} e^{-kz} \int_{-1}^{1} \frac{e^{kt} y(t)}{z-t}\, \mathrm{d}t.$$

This identity has been proved for $z > 1$ and $k > 0$. By the principle of analytic continuation it is obvious that the integral (20) continues the function (17) analytically for all values of z outside the cut ($-1 \leqslant z \leqslant 1$). In addition, the restriction $k > 0$ made during the proof is immaterial for the general validity of (20) for all values of k. For a different proof of (20), see [2].

In passing it may be mentioned that the character of the singularities of $Y^*(z)$ follows at once from (20). Obviously,

$$Y^*(z) - \tfrac{1}{2} y(z) \log \frac{z+1}{z-1} = \tfrac{1}{2} \int_{-1}^{1} \frac{e^{-k(z-t)} y(t) - y(z)}{z-t}\, \mathrm{d}t,$$

in which the remaining integral is an analytic function of z for all z.

Clearly, the function $Y(z)$ is a linear combination of the functions $Y^*(z)$ and $y(z)$. As a matter of fact it will be shown that

(21) $$Y(z) - Y^*(z) = (-1)^\nu (k/4\mu^2)\, y(z),$$

which implies that also

$$Y(z) - \tfrac{1}{2} y(z) \log \frac{z+1}{z-1}$$

is an analytic function of z for all finite z, as stated before.

To prove (21) we make use of the following integral, valid if $\mathrm{Re}(z) > 1$:

$$Q_n(z) = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zt} t^{-\frac{1}{2}} I_{n+\frac{1}{2}}(t)\, dt. \tag{22}$$

In view of (13) we thus find that

$$\sum_{n=0}^\infty a_n Q_n(z) = \frac{k}{2\mu} \int_0^\infty e^{-kzt} y(t)\, dt, \tag{23}$$

valid, e.g., if $z > 1$ and $k > 0$ (see below). Further, by using (19), we arrive at

$$Y^*(z) = \sum_{n=0}^\infty a_n Q_n(z) - \frac{k}{2\mu} \int_0^1 e^{-kzt} y(t)\, dt, \tag{24}$$

which apparently holds for unrestricted complex values of z outside the cut (the integral in (24) is an entire function of z). For a different proof of (24), see [2]. Now, if equation (24) is compared with equation (10), in which $\sum b_n P_n(z)$ is an entire function of z, it is evident that $Y(z) - Y^*(z)$ is an entire function of z, too. Since, moreover, this difference must be a solution of the differential equation (1) for the same eigenvalue, it is simply proportional to $y(z)$. Thus

$$\begin{cases} [Y(z) - Y^*(z) =] \\ \dfrac{k}{2\mu} \displaystyle\int_0^1 e^{-kzt} y(t)\, dt + \sum_{n=0}^\infty b_n P_n(z) = a\, y(z), \end{cases} \tag{25}$$

in which a does not depend on z. Let us now apply to (25) the substitution $z \to -z$. If ν is, as before, the degree of the eigenfunction $y(z)$ one has, in addition to (8), that

$$\sum_{n=0}^\infty b_n P_n(-z) = (-1)^{\nu+1} \sum_{n=0}^\infty b_n P_n(z).$$

so that

$$\frac{k}{2\mu} \int_0^1 e^{kzt} y(t)\, dt + (-1)^{\nu+1} \sum_{n=0}^\infty b_n P_n(z) = (-1)^\nu a\, y(z).$$

By multiplication of this identity by $(-1)^\nu$ and subsequent addition to (25), the series $\sum b_n P_n$ is eliminated and we are left with

$$\begin{aligned} 2a\, y(z) &= \frac{k}{2\mu} \Big[\int_0^1 e^{-kzt} y(t)\, dt + (-1)^\nu \int_0^1 e^{kzt} y(t)\, dt\Big] \\ &= \frac{k}{2\mu} (-1)^\nu \int_{-1}^1 e^{kzt} y(t)\, dt = (-1)^\nu \frac{k}{2\mu^2} y(z), \end{aligned}$$

which leads to $a = (-1)^\nu k/4\mu^2$. This completes the proof of (21).

It will be obvious that neither of the series (6) and (10) is convenient for an investigation of the functions $y(z)$ and $Y(z)$ for large values of z. On the other hand, the asymptotic behaviour of $Y^*(z)$ readily follows from equation (20). To that end, let us suppose that $|z| > 1$. Then the

integrand of (20) can be expanded in a series uniformly convergent with regard to t $(-1 \leqslant t \leqslant 1)$, so that

$$Y^*(z) = \tfrac{1}{2}\, e^{-kz} \sum_{n=0}^{\infty} z^{-n-1} \int_{-1}^{1} e^{kt}\, y(t)\, t^n\, dt.$$

The remaining integral is expressible in terms of the nth derivative, $y^{(n)}(1)$, of $y(z)$ at $z = 1$, as follows from the integral equation (4) by differentiation with respect to z and substitution of $z = 1$. We thus obtain for the function of the third kind:

$$Y^*(z) = \frac{k}{2\mu}\, e^{-kz} \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{(kz)^{n+1}}, \tag{26}$$

which is *absolutely convergent* when $|z| > 1$. The coefficients $y^{(n)}(1)$ in this expansion satisfy a third-order difference equation, viz.

$$\left\{\begin{aligned} 2(n+1)\, y^{(n+1)}(1) + [n(n+1) - \lambda - k^2]\, y^{(n)}(1) - 2nk^2\, y^{(n-1)}(1) \\ - n(n-1)\, k^2\, y^{(n-2)}(1) = 0, \end{aligned}\right. \tag{27}$$

which is most easily obtained by differentiating equation (1) n times and substituting $z = 1$. This difference equation cannot be solved by the technique of continued fractions. It can be verified, however, that (27) admits a solution that makes the expansion (26) convergent outside the unit circle. For numerical purposes the successive derivatives of $y(z)$ at $z = 1$ can be computed from

$$y^{(n)}(1) = \frac{1}{2^n\, n!} \sum_{m=n}^{\infty} \frac{(m+n)!}{(m-n)!}\, a_m, \tag{28}$$

which is easy to derive from (6).

It should be noted that the expansion (26) is also obtained when the integral in equation (19) is evaluated by repeated integration by parts. A third way of deriving (26) is to replace the functions $K_{n+\frac{1}{2}}$ in the series (17) by their explicit expressions, which are elementary functions, and then arrange the terms according to powers of z^{-1}.

The last-mentioned procedure can also be applied to the Bessel functions $I_{n+\frac{1}{2}}$ in equation (13), and for the function of the first kind it is then found that

$$y(z) = \mu \left[e^{kz} \sum_{n=0}^{\infty} \frac{(-1)^n\, y^{(n)}(1)}{(kz)^{n+1}} + (-1)^{\nu+1}\, e^{-kz} \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{(kz)^{n+1}} \right], \tag{29}$$

which also follows from repeated integration by parts of equation (4). Either series in (29) converges absolutely when $|z| > 1$, and is divergent inside the unit circle. On the unit circle they have two singularities at $z = \pm 1$. These singularities, however, cancel in the sum in accordance with the fact that $y(z)$ is everywhere analytic.

The order of growth of the eigenfunctions as $|z| \to \infty$ is immediately given by (29). In particular, for positive values of t,

$$y(t) = o\{\exp(|\mathrm{Re}\, k|\, t)\} \qquad (t \to \infty),$$

so that the integral representations (19) and (23) are valid at least if $|\mathrm{Re}(k)| < \mathrm{Re}\,(kz)$.

An expansion for $Y(z)$ analogous to (26) and (29) is obtained from (21) by simple substitution; thus, for the function of the second kind,

$$Y(z) = \frac{k}{4\mu}\left[e^{-kz}\sum_{n=0}^{\infty}\frac{y^{(n)}(1)}{(kz)^{n+1}} + (-1)^{\nu}\, e^{kz}\sum_{n=0}^{\infty}\frac{(-1)^n\, y^{(n)}(1)}{(kz)^{n+1}}\right], \tag{30}$$

valid outside the unit circle.

There exist many other interesting relations between the spheroidal wave functions y, Y and Y^*. For example it follows from (26) and (30) that

$$Y(z) = \tfrac{1}{2}\{Y^*(z) + (-1)^{\nu+1}\, Y^*(-z)\}, \tag{31}$$

and from (26) and (29) that

$$y(z) = -\frac{2\mu^2}{k}\{Y^*(-z) + (-1)^{\nu}\, Y^*(z)\}, \tag{32}$$

which indicate that $Y^*(z)$ and $Y^*(-z)$ form a fundamental system of spheroidal wave functions. Moreover, if in equation (31) the functions Y^* are replaced by their respective integral representations according to (20), we obtain an integral representation for the function of the second kind, viz.

$$Y(z) = \tfrac{1}{2}\int_{-1}^{1}\frac{\cosh(kz-kt)}{z-t}\, y(t)\, \mathrm{d}t, \tag{33}$$

valid for all values of z outside the cut $-1 \leqslant z \leqslant 1$.

Similarly, the identity (32) leads to a new integral equation for the functions of the first kind, viz.

$$y(z) = \frac{2\mu^2}{k}\int_{-1}^{1}\frac{\sinh(kz+kt)}{z+t}\, y(t)\, \mathrm{d}t, \tag{34}$$

which, however, may be obtained directly from (4) by iteration.

In addition to the expansions (6), (10), (13) and (17), there exist numerous other types of solution of the differential equation (1) with a three-term recurrence relation between the coefficients, which thus are open to numerical computations by the technique of continued fractions. As already mentioned, a disadvantage of the set formed by (6) and (10) is that they require two different systems of coefficients, a_n and b_n. Though this does not hold for the series (13) and (17), the latter fails to represent the function of the third kind inside the unit circle.

Some years ago the author succeeded in deriving a complete solution of (1) that possesses all desired features simultaneously. That is to say, it is possible to choose two linearly independent series solutions of the differential equation (1) that are expressible in terms of a single set of coefficients and converge for all values of z outside the cut. This new system of spheroidal wave functions will be now discussed.

According to Baber and Hassé [10], the eigenfunctions of equation (1) can be expanded as follows:

$$y(z) = e^{-kz} \sum_{n=0}^{\infty} c_n P_n(z), \tag{35}$$

in which c_n is a solution of the difference equation

$$\frac{2k(n+1)^2}{2n+3} c_{n+1} + \{n(n+1) - \lambda - k^2\} c_n - \frac{2kn^2}{2n-1} c_{n-1} = 0, \tag{36}$$

satisfying the boundary conditions

$$c_n = 0 \quad (n < 0), \qquad c_{n+1}/c_n \sim k/n \quad (n \to \infty). \tag{37}$$

Except for a constant of proportion, the solution c_n is unique for any fixed characteristic value λ. This constant is uniquely determined if the normalization (2) is assumed.

The characteristic values now correspond to the roots of

$$0 = -\Lambda + \frac{p_0\,|}{|2-\Lambda} + \frac{p_1\,|}{|6-\Lambda} + \dots + \frac{p_{n-1}\,|}{|n(n+1)-\Lambda} + \dots, \tag{38}$$

in which

$$\Lambda = \lambda + k^2 \quad , \quad p_n = \frac{(4n+1)^4}{(2n+1)(2n+3)} k^2. \tag{39}$$

As in the former expansions, the coefficients c_n are readily calculated by a process of iteration. For example, in the direction of decreasing values of n, the ratios of successive coefficients follow from

$$\frac{c_n}{c_{n-1}} = \left[\frac{A_n}{k}\{n(n+1) - \Lambda\} + B_n \frac{c_{n+1}}{c_n}\right]^{-1}, \tag{40}$$

in which

$$A_n = (2n-1)/2n^2 \quad , \quad B_n = (n+1)^2(2n-1)/n^2(2n+3). \tag{41}$$

Using the asymptotic behaviour of the coefficients c_n as $n \to \infty$ it is further seen that the degree of convergence of (35) is comparable to that of the series

$$\sum_n \{k(z + \sqrt{z^2-1})\}^n/n!$$

if z is outside the cut. Hence, the expansion (35) is not as rapidly convergent as the series (6).

A further disadvantage is that (35) does not indicate whether the corresponding eigenfunction is even or odd in z. This property, of course, is implicit in the coefficients c_n. In virtue of (8) it necessarily follows that also

$$y(z) = (-1)^\nu e^{kz} \sum_{n=0}^{\infty} (-1)^n c_n P_n(z). \tag{42}$$

This symmetry property induces many interesting relations between the coefficients c_n; for example,

$$\frac{\sum\limits_0^\infty c_{2n}}{\sum\limits_0^\infty c_{2n+1}} = \begin{cases} \coth k & (\nu \text{ even}), \\ \tanh k & (\nu \text{ odd}), \end{cases} \tag{43}$$

which easily follows from (35) and (42) upon the substitution $z = 1$. In particular, the sum of the coefficients c_n with even (odd) subscripts is zero for $k = \frac{1}{2}\, m\, \pi\, i$, where m is an odd (even) integer if ν is even and an even (odd) integer if ν is odd.

According to Meixner [7], the same symmetry property is useful in the actual normalization of the coefficients c_n. In fact one has

$$\begin{aligned}\int_{-1}^{1} \{y(z)\}^2\, dz &= \int_{-1}^{1} \{e^{-kz} \sum c_n P_n(z)\} \{(-1)^\nu e^{kz} \sum (-1)^n c_n P_n(z)\}\, dz \\ &= (-1)^\nu \int_{-1}^{1} \{\sum c_n P_n(z)\} \{\sum (-1)^n c_n P_n(z)\}\, dz \\ &= (-1)^\nu \sum (-1)^n \frac{2}{2n+1}\, c_n^2.\end{aligned}$$

In other words, the coefficients c_n of (35) and a_n of (6) are to be normalized in essentially the same way. In accordance with (2) and (7) we thus require for the νth eigenfunction

$$\frac{1}{2\nu+1} = \sum_{n=0}^{\infty} (-1)^{n+\nu} \frac{c_n^2}{2n+1}, \quad \lim c_\nu = 1 \ (k \to 0). \tag{44}$$

Let us now substitute Baber and Hassé's series (35) in the integrand of (20). If use is made of Neumann's integral for the Legendre function of the second kind, viz.

$$Q_n(z) = \tfrac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{z-t}\, dt,$$

we obtain the simple result

$$Y^*(z) = e^{-kz} \sum_{n=0}^{\infty} c_n Q_n(z). \tag{45}$$

Therefore, the series (35) and (45) represent a fundamental system of spheroidal wave functions. They are expressed in terms of a single set of coefficients c_n satisfying a three-term recurrence relation, and either series converges in the whole regularity domain of the corresponding function. As emphasized before, it is the function of the third kind (45) which is required for the construction of travelling waves in physical problems. For example, in the problem of diffraction of a plane wave by a circular disk or aperture [2] the function $Y^*(z)$ is required for $k > 0$ and $z = i\,\zeta\ (0 \leqslant \zeta < \infty)$, which range of values is completely covered by the uniform representation (45).

At first sight it is surprising that the series (35) and (45) are both solutions of the differential equation (1), if it is remembered that the functions (6) and (9) do not both solve this equation. Of course, by direct substitution of the series (45) in (1), and using with care the recurrence relations for the Q-function, it can be verified independently that the series (45) does solve the differential equation. The difference with the case of (9) is that in the beginning we have to evaluate n^2Q_{n-1}, instead of nQ_{n-1}, when $n=0$. On account of the extra factor n we may take n^2Q_{n-1} equal to zero for $n=0$, like n^2P_{n-1}. In terms of the difference equations underlying the coefficients c_n and a_n, it may be noted that in (36) the factor of c_{n-1} shows a double zero at $n=0$, while in the corresponding difference equation for the system a_n [4, eq. 3] the factor of a_{n-2} has a single zero at $n=0$ and a single zero at $n=1$.

The eigenvalues μ of the integral equation (4) are expressible in terms of c_n in a very simple way. To that end we take $z=1$ in equation (4) and substitute for $y(t)$ the corresponding series (35). Thus

$$\mu = \frac{y(1)}{2c_0} = \frac{1}{2c_0} e^{-k} \sum_{n=0}^{\infty} c_n. \tag{46}$$

A further question of interest is how the set of coefficients a_n can be expressed in terms of the set c_n and vice versa. The answer is provided by the transformations

$$c_n = (n+\tfrac{1}{2}) \sum_{m=0}^{\infty} A_{n,m}\, a_m, \tag{47}$$

$$a_n = (n+\tfrac{1}{2}) \sum_{m=0}^{\infty} (-1)^{\nu+m} A_{n,m}\, c_m, \tag{48}$$

in which the coefficients $A_{n,m}$ depend only on k, not on the degree ν of the eigenfunction, viz.

$$A_{n,m} = \int_{-1}^{1} e^{kt} P_n(t)\, P_m(t)\, \mathrm{d}t. \tag{49}$$

To prove (48), we observe that in view of (6) one has

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} y(t)\, P_n(t)\, \mathrm{d}t,$$

which at once leads to (48) and (49) if $y(t)$ is replaced by the corresponding expansion (42). Similarly (47) follows from (35) and (6). The coefficient $A_{n,m}$ is an elementary function of k. It can be shown that

$$A_{n,m} = \sqrt{\frac{2}{\pi k}} \sum_{s=0}^{\min(n,m)} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \frac{\Gamma(n-s+\frac{1}{2})}{\Gamma(n-s+1)} \frac{\Gamma(m-s+\frac{1}{2})}{\Gamma(m-s+1)} \frac{\Gamma(n+m-s+1)}{\Gamma(n+m-s+\frac{1}{2})} I_{n+m-2s+\frac{1}{2}}(k). \tag{50}$$

In concluding this paper, the author wants to emphasize that a similar canonic system as (35) and (45) is possible for spheroidal wave functions

of non-negative integral order. Meixner [7] has generalized them so as to be applicable to spheroidal wave functions of unrestricted complex order and degree.

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A FAMILY OF PARAMETERFREE TESTS FOR SYMMETRY WITH RESPECT TO A GIVEN POINT. I

BY

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1. *Introduction.*

In this paper a family of tests will be described for the hypothesis H_0, that a number of random variables $\mathbf{z}_i$ $(i = 1, \ldots, n)$ [1]) are distributed independently, each having a probability distribution, which is symmetrical with respect to a given point $z_i = a$, which is the same for every i. The tests will be based on the information supplied by n observations $z_1, \ldots, z_n$, where z_i denotes an observation of the random variable $\mathbf{z}_i$. A special case of H_0 is the hypothesis, that all $\mathbf{z}_i$ have the same symmetrical probability distribution with a given point of symmetry, z_i $(i = 1, \ldots, n)$ being a random sample from this distribution. No additional assumptions will be made about the probability distributions of $\mathbf{z}_i$, such as normality or even continuity.

It will be tacitly understood, if not mentioned otherwise, that $a = 0$; this does not imply a loss of generality.

In mathematical statistics these tests may be applied to many problems. An important practical application to a problem, occurring often in connection with medical and biological experiments, will be described in section 2.

The sign test, introduced by R. A. FISHER (1925) (cf. also W. J. DIXON and A. M. MOOD (1946)), may be regarded as a partial solution of our problem; this test, however, is a test for the common median of the $\mathbf{z}_i$ only. It can therefore not be a powerful test for the hypothesis H_0. Other tests for the common median of a number of random variables $\mathbf{z}_i$ have been developed (cf. J. E. WALSH (1949)) on the assumption of continuity of the probability distributions of $\mathbf{z}_i$. Apart from the fact, that we want to avoid this assumption of continuity, we shall try to exploit the additional assumption of symmetry contained in H_0, which cannot be taken into account in a parameterfree test for the median only.

The test, derived in this paper is an application of the randomization method, introduced by R. A. FISHER (1925) and described extensively e.g. by E. L. LEHMANN and C. STEIN (1949).

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[1]) The random character of a variable is denoted by the use of a bold type symbol; a special value, assumed by a random variable is denoted by the same symbol in normal type.

2. *Applications.*

2. 1. Let us suppose, that a number of patients, (say n), suffering from a certain disease, have been treated with a drug, which has to be tested, and that the effect of the drug can only be ascertained by measuring the value, before and after the treatment, of a random quantity $\boldsymbol{x}$, e.g. the blood pressure. The hypothesis to be tested is, in such a case, that the drug does not have any effect on the distribution of $\boldsymbol{x}$.

Two sets of observed values of $\boldsymbol{x}$ are available: for every patient we have one value before and one value after the drug has been administered. In general, however, the application of a two-sample test on the two sets of values mentioned is not justified since the probability distribution of $\boldsymbol{x}$ is usually different for every patient. If, on the other hand, a unique probability distribution of $\boldsymbol{x}$ is assumed to exist on a population, containing all patients, the two sets of observed values are correlated, because two observations have been made for every patient, one for each of the two samples. If $\boldsymbol{x}$ is assumed to have the *same* distribution for every patient, a two sample test can be applied, but this condition is rarely satisfied.

2. 2. In order to overcome these difficulties the observations are paired and the test is based on these pairs, consisting of the value of $\boldsymbol{x}$ for each patient before and after the treatment.

Let $\boldsymbol{x}_i$ $(i = 1, \ldots, n)$ be the random variable $\boldsymbol{x}$ for the i^{th} patient before the treatment and $\boldsymbol{x}_i'$ the corresponding variable after.

Then, if the hypothesis, that the drug has no effect, is true, $\boldsymbol{x}_i$ and $\boldsymbol{x}_i'$ have the same probability distribution [2]) and the random variables

$$\boldsymbol{z}_i = \boldsymbol{x}_i - \boldsymbol{x}_i' \tag{1}$$

are distributed symmetrically with respect to zero. Moreover, if the random variables $\boldsymbol{x}_i$ $(i = 1, \ldots, n)$, as well as the $\boldsymbol{x}_i'$ are independently distributed, the same applies to the $\boldsymbol{z}_i$. In this case the hypothesis to be tested is H_0.

Applications of a similar type may be found in other fields of experiment. We confine ourselves, however, to this example, indicating the kind of problem, which may be solved by testing the hypothesis H_0 in question.

3. *The general principle of the tests.*

3. 1. Let us denote an n-dimensional Euclidean space, with coordinates $z_1, \ldots, z_n$, by S. The random point $\boldsymbol{E} \equiv (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n)$, representing the set of random variables $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$, has a probability distribution on S, which, if H_0 is true, is *symmetrical in every coordinate of S* (with the origin of S as point of symmetry.)

Let further the symbol Z denote the conditions

$$Z: \quad |\boldsymbol{z}_i| = |z_i| \qquad i = 1,\ldots,n \tag{2}$$

[2]) That is, if no other factors than the drug affect the probability distribution of $\boldsymbol{x}_i$ systematically. This assumption is inevitable, if the influence of the drug only is to be ascertained.

where the z_i are any given numbers, and let $F(Z)$ denote the (unknown) simultaneous cumulative distribution function $F(|z_1|, \ldots, |z_n|)$ of the random variables $|\mathbf{z}_i|$ $(i = 1, \ldots, n)$. Let $M = M(Z)$ be the subset of S, consisting of those points, which satisfy Z. If m of the values z_i are equal to zero, M consists of 2^{n-m} different points.

According to the above mentioned symmetry of the probability distribution of $\mathbf{E}$, all points of M have, if H_0 is true, the same conditional probability, under the condition Z, if we exclude from S a set of probability zero, where this conditional probability is undetermined. Moreover, each point of M corresponds with one of the 2^{n-m} different ways, in which signs may be attributed to $n-m$ numbers $\neq 0$. If $x_1, \ldots, x_{n_1}$ are the positive coordinates and $-y_1, \ldots, -y_{n_2}$ the negative coordinates of $E \in S$ (all x_j and y_k thus being positive; $n_1 + n_2 = n - m$), then there is a one to one correspondence between the points of M and the possible partitions of those of the values $|z_i|$ which are $\neq 0$, into a group of values x_j and a group of values y_k.

We thus have

Lemma 1: *If H_0 is true, and condition Z is satisfied, all 2^{n-m} partitions of the $n-m$ positive values among $|z_1|, \ldots, |z_n|$ into a group of values x_j and a group of y_k are equally probable.*

3. 2. A uniquely determined function

$$s = s(E) = s(z_1, \ldots, z_n) \tag{3}$$

of $z_1, \ldots, z_n$, defined on S, is called a *statistic*; $\mathbf{s} = s(\mathbf{E}) = s(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ has a probability distribution, which follows from the probability distribution of $\mathbf{E}$ on S; the latter distribution being unknown, the same will usually be the case with the former. Assuming H_0 to be true, however, we may derive properties of the distribution of certain statistics and, sometimes, the distribution itself.

Two types of statistics will be used in the tests for H_0:

A. A number ν_1 of statistics, the values of which are uniquely determined by the values $|z_i|$ $(i = 1, \ldots, n)$. The simultaneous distribution function of these statistics need not be known, even if H_0 is supposed to be true. We need these statistics, which might be called ,,nuisance-statistics'', to overcome difficulties with discontinuities of the distributions of $\mathbf{z}_i$. The elimination of their unknown distribution function is described in theorem I, at the end of this section.

One of these statistics will be $\mathbf{m}$, the number of those among the variables $\mathbf{z}_i$ which assume the value zero.

B. A number ν_2 of statistics, of which the conditional simultaneous distribution function, under the condition Z, will be derived, assuming H_0 to be true. Let us denote these statistics simultaneously by a random point $\mathbf{Q}$ in a subset of a ν_2-dimensional Euclidean space.

One of these statistics will be $\mathbf{n}_1$, the number of positive coordinates of $\mathbf{E}$. For this statistic we have [3])

Lemma 2: *If H_0 is true, the conditional probability distribution of the number of positive coordinates $\mathbf{n}_1$, under the condition Z, is given by*

$$\mathrm{P}\,[\mathbf{n}_1 = n_1 \,|\, Z;\, H_0] = 2^{-n+m}\binom{n-m}{n_1} \tag{4}$$

with $0 \leqq n_1 \leqq n - m$, where m is the number of values $|z_i|$, which are equal to zero.

Proof: The number of partitions of $n-m$ values into two groups of n_1 and $n_2 = n-m-n_1$ values respectively, is equal to $\binom{n-m}{n_1}$. From this and lemma 1, lemma 2 follows.

Remark: Lemma 2 is also true, if the condition Z is replaced by the condition $\mathbf{m} = m$. The sign test (cf. section 1) is based on lemma 2 with this latter condition.

We further have the following lemma:

Lemma 3: *If H_0 is true, and the conditions Z and $\mathbf{n}_1 = n_1$ are satisfied, all $\binom{n-m}{n_1}$ partitions of the $n-m$ positive values among $|z_i|$ into a group of n_1 values x_j and a group of $n_2 = n - m - n_1$ values y_k, are equally probable.*

This lemma follows at once from lemma 1; the condition $\mathbf{n}_1 = n_1$ selects a number of equally probable partitions from the 2^{n-m} partitions, which are possible if only Z is imposed.

3. 3. Given Z, i.e. the values $|z_i|$ $(i = 1, \ldots, n)$, the statistics mentioned in 3. 2. *B* are represented simultaneously by a random point $\mathbf{Q}$ in a ν_2-dimensional set of points V. For every Z we shall, in later sections, indicate a *critical region* $R = R(Z)$ in V ,with the property

$$\mathrm{P}\,[\mathbf{Q} \in R(Z) \,|\, Z;\, H_0] \leqq \varepsilon \tag{5}$$

with given $\varepsilon > 0$. Then the following theorem is easy to prove:

Theorem I: *If we reject H_0 if and only if*

$$Q \in R(Z) \tag{6}$$

where Z represents the observed values $|z_i|$ $(i = 1, \ldots, n)$, then the probability, that H_0, being true, will be rejected, is $\leqq \varepsilon$.

Proof: The probability, that H_0, being true, will be rejected, is

$$\int \mathrm{P}\,[\mathbf{Q} \in R(Z) \,|\, Z;\, H_0]\, \mathrm{d}F(Z) \leqq \varepsilon \int \mathrm{d}F(Z) = \varepsilon,$$

where the integral-sign denotes integration over the n-dimensional space S.

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[3]) The symbol $\mathrm{P}\,[A \,|\, B;\, H]$ denotes the conditional probability of the event A, under the condition B, and the hypothesis H.

4. *An exact test for H_0.*

4. 1. In this and the following section an exact test for H_0 will be given, which in a way is nothing more than an application of the test for independence in a 2×2 table. The family of tests mentioned in the title of this paper, which is a generalisation of this one, will be described later.

We shall use the following statistics (cf. section 3. 2):

A 1. The number $\boldsymbol{m}$ of values $\mathbf{z}_i$, which are equal to zero. Since the probabilities $\mathrm{P}[\mathbf{z}_i = 0]$ are unknown, the probability distribution of $\boldsymbol{m}$ is unknown too and H_0 does not specify any assumption about it.

A 2. A statistic $\boldsymbol{r} = r(\mathbf{E})$ which is defined as follows:

If there are no equal values among $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$ (as defined in section 3. 1), we put

$$r = \left[\frac{n_1 + n_2 + 1}{2}\right]$$

$[x]$ denoting the integral part of x.

If some of the values x_j and y_k are equal, we define r by dividing the set of values $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$ into two sets A and B, containing a and b of those values respectively (with $b - a \geqq 0$), and with the property, that every member of A is *smaller* than every member of B (thus no member of A being equal to any member of B); among all possible divisions, which satisfy these conditions, that one is chosen, which minimizes $b - a$. This division is uniquely determined and we now define $r = r(E) = b$. The above definition of r, for the case, that there are no equal values among $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$ is a special case of this general definition.

The distribution of $\boldsymbol{r}$ depends on the discontinuities of the distributions of $\mathbf{z}_i$. H_0 does not contain any assumption about these discontinuities; therefore the distribution of $\boldsymbol{r}$ remains unknown if H_0 is assumed to be true.

B 1. The number $\boldsymbol{n}_1$ of positive values among the $\mathbf{z}_i$.

B 2. A statistic $\boldsymbol{u}$, defined in the following way:

For every $E \in S$ the values $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ and the number r having been defined in such a way that the r largest among these values are uniquely determined, we now define $u(E)$ as the number of values x_i among these r largest values. Thus $u(E)$ is uniquely determined for every $E \in S$.

Remark: If, instead of the statistic $\boldsymbol{r}$, as defined above, we had chosen any statistic satisfying the conditions:

a. To be uniquely determined by $|z_1|, \ldots, |z_n|$;

b. To determine uniquely the r largest elements among $|z_1|, \ldots, |z_n|$, the method would otherwise have remained unchanged. Our choice of

the definition of $\mathbf{r}$, which might seem rather arbitrary, has been made on the consideration, that for $r = n_1 + n_2$ or $r = 0$ the test reduces to the sign test, which is not very sensitive. An optimum choice for $\mathbf{r}$ should be based on some knowledge of the power function of the test. This problem has not yet been solved, the determination of power functions being generally a rather difficult one if parameter-free methods are concerned.

4. 2. We now proceed to prove the second theorem, on which the test will be based:

Theorem II: *If H_0 is true, the conditional simultaneous probability distribution of $\mathbf{n}_1$ and $\mathbf{u}$, under the condition Z, is given by*

$$\text{(7)} \qquad \mathrm{P}\,[\mathbf{n} = n_1;\, \mathbf{u} = u \,|\, Z;\, H_0] = 2^{-n+m} \binom{r}{u} \binom{n-m-r}{n_1-u}$$

with $0 \leqq n_1 \leqq n-m$ and Max $(0,\, r+m+n_1-n) \leqq u \leqq$ Min $(n_1,\, r)$.

Proof: According to lemma 1 all partitions of the $n-m$ positive values among $|z_i|$, are equally probable under the condition Z. There are 2^{n-m} such partitions, among which

$$\binom{r}{u} \binom{n-m-r}{n_1-u}$$

have u values x_j among the r largest and $n_1 - u$ among the n-m-r smallest values.

This proves the theorem.

Remarks: 1. It may be proved in a way analogous to the proof of theorem I, that the condition Z may be replaced by the conditions $\mathbf{m} = m$ and $\mathbf{r} = r$.

2. According to theorem II, $\mathbf{u}$ and $\mathbf{n}_1 - \mathbf{u}$ are, under the conditions Z and H_0, independently distributed, each with a binomial distribution, with $0 \leqq \mathbf{u} \leqq r$ and

$$\text{(8)} \qquad \mathcal{E}\,(\mathbf{u} \,|\, Z;\, H_0) = \tfrac{1}{2}\, r;\quad \sigma_{\mathbf{u}\,|\,Z;\,H_0} = \tfrac{1}{2}\sqrt{r}$$

and with $0 \leqq \mathbf{n}_1 - \mathbf{u} \leqq n$-$m$-$r$ and

$$\text{(9)} \qquad \mathcal{E}\,(\mathbf{n}_1 - \mathbf{u} \,|\, Z;\, H_0) = \frac{n-m-r}{2};\quad \sigma_{\mathbf{n}_1-\mathbf{u}\,|\,Z;\,H_0} = \tfrac{1}{2}\sqrt{n-m-r}\,.$$

3. The conditional distribution of $\mathbf{u}$, given m, r and n_1, which follows easily from theorem II and remark 1, is a hypergeometrical distribution, given by

$$\text{(10)} \qquad \begin{cases} \mathrm{P}\,[\mathbf{u} = u \,|\, \mathbf{m} = m;\; \mathbf{r} = r;\; \mathbf{n}_1 = n_1;\, H_0] = \\[1ex] = \dfrac{\binom{r}{u}\binom{n-m-r}{n_1-u}}{\binom{n-m}{n_1}} = \dfrac{\binom{n_1}{u}\binom{n_2}{r-u}}{\binom{n_1+n_2}{r}} \end{cases}$$

with $n_1 + n_2 = n$-m and Max $(0,\, r - n_2) \leqq u \leqq$ Min (n_1, r).

This result may also be obtained directly from lemma 3.

5. *The critical region.*

5. 1. The choice of a critical region depends on the alternative hypothesis or hypotheses against which H_0 is to be tested. To give an impression of the possibilities for this choice, a special case of the conditional distribution of $\boldsymbol{n}_1 - \boldsymbol{u}$ and $\boldsymbol{u}$ has been given in table 1, where n–$m = 20$ and $r = 11$; u has only been tabulated up to $u = 5$ and $n_1 - u$ up to 4, because of the symmetry of the distribution. The values, given in the table, are to be multiplied by 2^{-20} in order to get the probabilities wanted.

TABLE 1

$\uparrow u$ \ $n_1 - u \rightarrow$	0	1	2	3	4
5	462	4158	16632	38808	58212
4	330	2970	11880	27720	41580
3	165	1485	5940	13860	20790
2	55	495	1980	4620	6930
1	11	99	396	924	1386
0	1	9	36	84	126

2^{20}. P $[\boldsymbol{n}_1 = n_1;\ \boldsymbol{u} = u \mid n - \boldsymbol{m} = 20;\ \boldsymbol{r} = 11;\ H_0]$.

5. 2. If no alternative hypothesis is specified, we may adopt the system of G. A. Barnard (1947) to find a critical region. According to this system, the critical region is a set of points $(n_1 - u, u)$, all of which have probabilities smaller than or equal to those of the points not contained in the critical region. In our case, the largest symmetrical critical region R_1 with size $\leqq 0{,}05$ is then the region, indicated in fig. 1.

On intuitive grounds this region may be expected to be a rather good one.

The computations are not very numerous and of a simple kind. The marginal distributions may be found in tables of the binomial coefficients, cf. e.g. T. C. Fry (1928) or A. van Wijngaarden (1950). The other probabilities follow from the marginal distributions by multiplication. The number of computations, moreover, can be reduced considerably. If a certain point $(n_1 - u, u)$ has been found as the result of the experiment, only those products have to be computed, which are $\leqq \binom{r}{u}\binom{n-m-r}{n_1-u}$. If the sum of these products divided by 2^{n-m} is smaller than the significance level chosen, H_0 is to be rejected.

If n–m and r are large, the probability distribution of $\boldsymbol{n}_1 - \boldsymbol{u}$ and $\boldsymbol{u}$ may be represented approximately by a two-dimensional normal distribution. This approximation is, however, not a very good one for values of n–m and r of the order of magnitude used for table 1. Another approximative method, which is more satisfactory, will be given in a later section, in connection with the generalisation to be described there.

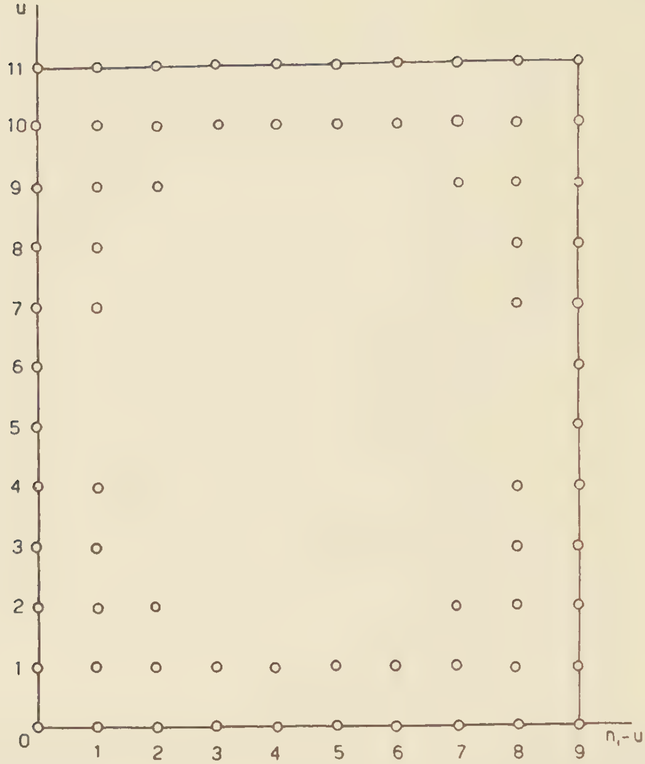

Fig. 1. Critical region R_1 when no alternative hypothesis is specified. Significance level 0,042.

5. 3. We shall now consider a special alternative hypothesis H, against which H_0 may be tested more profitably with another critical region than R_1. H is the hypothesis, that the $\mathbf{z}_i$ $(i = 1, \ldots, n)$ are distributed independently and symmetrically with respect to points $z_i = a_i$, satisfying the following conditions:

1. $a_i \neq 0$ for at least one value of i;
2. $a_i \geqq 0$ for all i, or $a_i \leqq 0$ for all i.

This situation will often arise. In the example of section 2 for instance, although it cannot be said, that H is exactly the alternative hypothesis, it will certainly often be more important to detect a displacement of the distributions of $\mathbf{z}_i$ along the z-axis than asymmetry of these distributions.

Assuming H to be true, little can be said about the distribution of $\mathbf{n}_1$, since H does not specify any assumption about the amount of the displacements. It is, however, reasonable, to exclude for n_1 the value $n_1 = \frac{n-m}{2}$, if it is an integer, from the critical region, since the probability of this value is asymptotically equal to zero, if the number of distributions, shifted along the z-axis, increases indefinitely. If the distributions of some

of the variables z_i have been shifted to the left, i.e. if

$$P\left[n_1 > \frac{n-m}{2} \,\middle|\, H\right] < P\left[n_1 < \frac{n-m}{2} \,\middle|\, H\right],$$

it is evident, that small values of u are more probable than large values. An analogous statement may be made for a displacement to the right.

We therefore propose the following construction of critical region R_2, when H is the alternative hypothesis:

We divide the lattice of possible points $(n_1 - u, u)$ into two parts by the line $n_1 = \frac{n-m}{2}$, excluding the lattice points on this line. In both resulting parts we build up a critical region of half the size wanted according to the system of Barnard; we only have to describe this region for one of these parts, (e.g. the lower part; cf. figure 2) since the

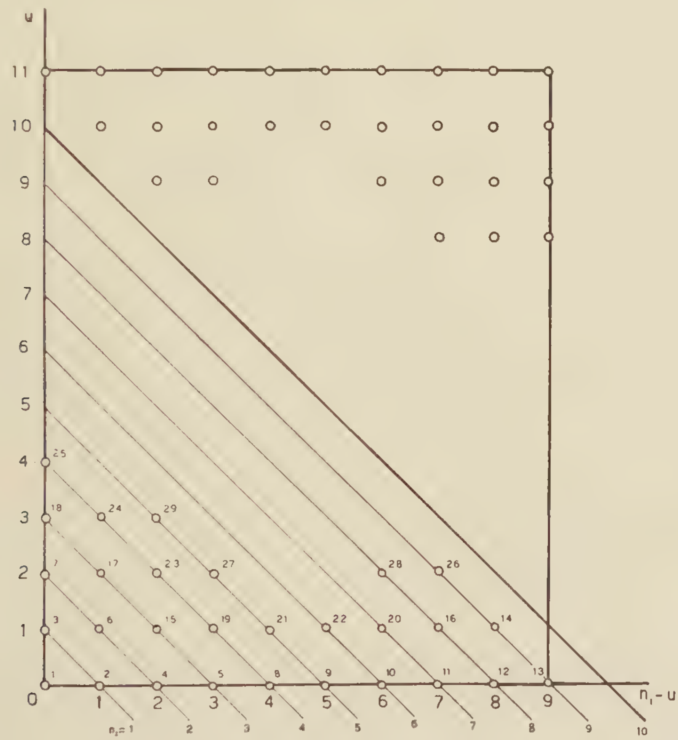

Fig. 2. Critical region R_2, when the alternative hypothesis is a displacement of some of the distributions in one direction along the z-axis. Significance level 0,053.

critical region as a whole ought to be symmetrical with respect to the centre of the lattice, if no information is available about the direction of the displacement.

We start by taking the point (0, 0).

If ν points $P_1, \ldots, P_\nu$ have been chosen, the $(\nu + 1)^{\text{th}}$ point is the

point with smallest probability (under hypothesis H_0) among all points P satisfying the following conditions:

a. All points with the same n_1 as P, but with smaller value of u, are contained already in $P_1, \ldots, P_\nu$.

b. If $u = 0$ for P, the point directly to the left of P is already among $P_1, \ldots, P_\nu$.

In this way, the $(\nu + 1)^{\text{th}}$ point may not yet be determined uniquely, since two or more of the points P may have equal probabilities, all other points P having a larger probability; in that case we choose the point with smallest n_1 among the points with this (minimal) probability.

Applying this principle to our example, we get the critical region indicated in figure 2. The numbers inside the rectangle denote the order, in which the points have been added to the critical region.

It is clear, that a one-sided critical region may be constructed in one of the two parts of the lattice only, if H specifies the direction of the displacement.

5. 4. It is of some interest to compare the critical regions R_1 and R_2 of figure 1 and 2 with the corresponding critical region belonging to the sign test for $n-m = 20$. This region consists of the lattice points on the lines $n_1 = 0, \ldots, n_1 = 5$ and $n_1 = 15, \ldots, n_1 = 20$. The significance level is 0,041. If the points of the lines $n_1 = 6$ and $n_1 = 14$ are added, the significance level jumps to the value 0,115. Since the sign test is a test for the median only, it should be compared especially with the second critical region. The critical region of the sign test then contains four points, which are not contained in the region R_2 of figure 2, while 20 points of R_2 are not contained in the critical region of the sign test.

5. 5. *Example.*

Consider the following sample of 22 values z_i:

+ 7,4; + 6,3; + 3,6; + 3,5; + 3,4; + 2,9; + 2,5; + 1,1; 0; 0;
— 1,3; — 2,5; — 3,2; — 4,6; — 4,6; — 4,6; — 4,8; — 6,3; — 7,0; — 7,9;
— 8,0; — 8,7.

We have: $n-m = 20$; $n_1 = 8$; $r = 11$; $u = 2$. The point P_0 with coordinates $n_1 - u = 6$, $u = 2$ is contained in R_2, but not in R_1 (cf. fig. 2 and 1 resp.). The result is not significant, if the sign test is applied. If we compute the sizes of the smallest critical regions, which contain the point P_0, we find for the three methods:

Type of region	size
R_1	0,076
R_2	0,042
sign test	0,503

(To be continued).

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Publication of the Statistical Dept. of the "Mathematisch Centrum", Amsterdam

NOTE ON THE REPRESENTATION OF THE VALUES OF POLYNOMIALS WITH REAL COEFFICIENTS FOR COMPLEX VALUES OF THE VARIABLE

BY

B. MEULENBELD

(Communicated by Prof. J. G. VAN DER CORPUT at the meeting of May 20, 1950)

§ 1.

In a previous paper [1] an extension was given of LILL's method, in which a polynomial with real coefficients was represented by polygonparts with angle φ. According to that, the value of the polynomial $f(z)$ can be determined graphically for each real value of z. For $\varphi = \pi/2$ we obtain the orthogones, which LILL used in resolving the real roots of a numerical equation. He also indicated [2] a method (without proof) to resolve graphically the complex roots of a numerical equation.

In the present note it is shown how this latter method can be extended to that of the polygonparts with angle φ, and how the values of a polynomial $f(z)$ for *complex* z can be represented.

§ 2.

We consider a polynomial of degree n

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n \qquad (a_0 \neq 0), \tag{1}$$

where a_i $(i = 0, 1, \dots, n)$ are real numbers.

We construct the polygonpart with angle φ of $f(z)$. As the directions of the a_i-sides are very important, it will be useful to give the rule for the construction of the polygonpart here again, but in another form. Each pair of consecutive sides include a fixed angle φ. The positive direction of the a_{i+1} side is obtained by a rotation of the positive a_i-direction in the point of intersection over an angle φ in the positive sense (that is counterclockwise). The positive a_0-direction is arbitrary. The a_{i+1}-side is drawn in the positive or negative direction according to the sign of the coefficient a_{i+1}.

(In fig. 1 the polynomial $f(z) = z^4 - 2z^3 - z^2 + 3z - 2$ is represented by the polygonpart $P_0P_1P_2P_3P_4P_5$).

Let the polygonpart with angle φ $P_0P_1P_2 \dots P_{n+1}$ (in fig. 1 with $n = 4$) represent the polynomial (1).

Without loss of generality we again may assume $a_0 > 0$.

If A_1 is an arbitrarily chosen point, then we consider $\overrightarrow{P_1A_1}$ as a complex

vector with respect to P_1 as origin, and with the positive a_1-direction as positive axis of reals (shortly: *with respect to the a_1-side*).

In fig. 1 is $a_1 < 0$, so $\overrightarrow{P_1A_1} = \overline{P_1B_1} + i\,\overline{B_1A_1}$[1]), and $\overrightarrow{A_1P_1} = -\overline{P_1B_1} - i\,\overline{B_1A_1}$.

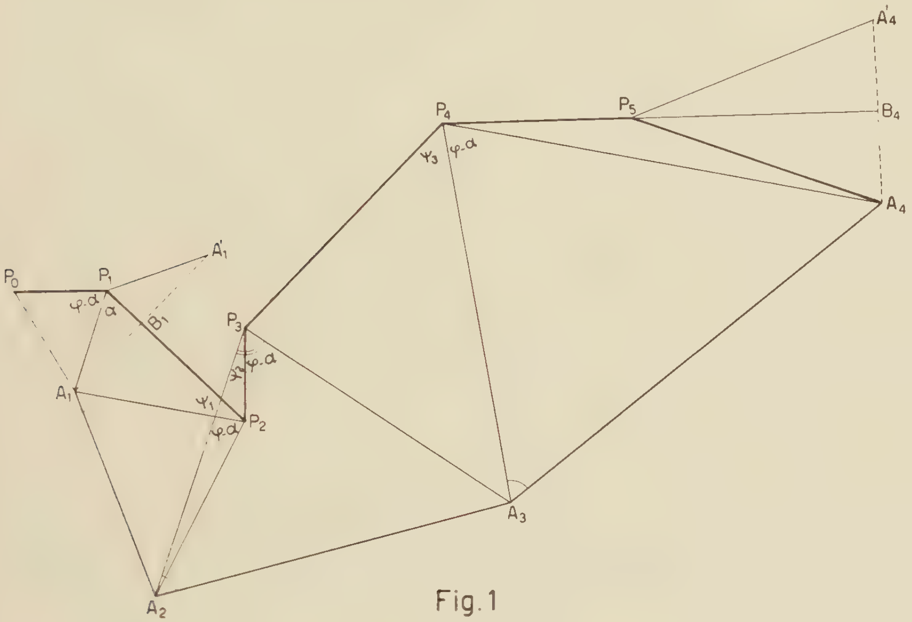

Fig. 1

Now we put

$$\overrightarrow{A_1P_1} = z\,a_0. \tag{2}$$

With each vector $\overrightarrow{A_1P_1}$ a complex number z corresponds, and conversely with each complex number z a vector $\overrightarrow{A_1P_1}$ with (2) can be found.

We construct the points $A_2, A_3, \ldots, A_n$, such that the triangles $P_0P_1A_1$, $A_1P_2A_2$, $A_2P_3A_3$, $\ldots$, $A_{n-1}P_nA_n$ are conformable.

We shall prove that *the vector $\overrightarrow{A_nP_{n+1}}$ represents the value of the polynomial $f(z)$ with respect to the a_n-side.*

Proof. *With respect to the a_1-side:*

$$\arg{}^{2)}\, z = \arg \overrightarrow{A_1P_1} = \angle A_1P_1P_2 = \alpha,\ |\overrightarrow{A_1P_1}| = a_0|z|.$$

Hence

$$\overrightarrow{A_1P_2} = a_1 + a_0z.$$

$\arg \overrightarrow{A_1P_2} = \angle A_1P_2P_1 - \pi = \psi_1 - \pi.$

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[1]) We remark that $\overrightarrow{XY} = -\overrightarrow{YX}$, and that $\overline{XY} = -\overline{YX}$.

[2]) With arg z we mean here the main argument of z.

With respect to the a_2-side:

$$\arg \overrightarrow{A_2P_2} = \psi_1 - \pi - \alpha = \arg z(a_1 + a_0 z),\ |\overrightarrow{A_2P_2}| = |z|\ |a_1 + a_0 z|.$$

Hence

$$\overrightarrow{A_2P_2} = z\,(a_1 + a_0 z),\ \text{and}\ \overrightarrow{A_3P_3} = a_2 + a_1 z + a_0 z^2.$$

$$\arg \overrightarrow{A_2P_3} = \pi - \angle A_2P_3P_2 = \pi - \psi_2.$$

With respect to the a_3-side:

$$\arg \overrightarrow{A_3P_3} = \pi - \psi_2 - \alpha = \arg z\,(a_2 + a_1 z + a_0 z^2),$$

$$|\overrightarrow{A_3P_3}| = |z|\ |a_2 + a_1 z + a_0 z^2|.$$

Hence

$$\overrightarrow{A_3P_3} = a_2 z + a_1 z^2 + a_0 z^3,\ \text{and}\ \overrightarrow{A_3P_4} = a_3 + a_2 z + a_1 z^2 + a_0 z^3.$$

Proceeding this process we find *with respect to the a_n-side:*

$$\arg \overrightarrow{A_nP_n} = \arg z\,(a_{n-1} + a_{n-1}\,z + \ldots + a_0\,z^{n-1}).$$

$$|\overrightarrow{A_nP_n}| = |z\,(a_{n-1} + a_{n-2}\,z + \ldots + a_0\,z^{n-1})|.$$

Hence

$$\overrightarrow{A_nP_n} = z\,(a_{n-1} + a_{n-2}\,z + \ldots + a_0\,z^{n-1})$$

and

$$\overrightarrow{A_nP_{n+1}} = a_n + a_{n-1}\,z + a_{n-2}\,z^2 + \ldots + a_0\,z^n = f(z).$$

In fig. 1 is $\overrightarrow{A_1P_1} = z = \frac{1}{2} - i$ and $\overrightarrow{A_4P_5} = f(\frac{1}{2} - i) = 2\frac{9}{16} - i$.

Remarks.

1. If the point A_1 (and so z) can be chosen such that $\overrightarrow{A_nP_{n+1}} = 0$, which means that the point A_n coincides with P_{n+1}, then z is found to be a root of the equation $f(z) = 0$.

2. It is obvious that with a point A_1', lying symmetrically to A_1 with respect to the a_1-side a point A_n' corresponds, symmetrically to A_n with respect to the a_n-side (in fig. 1 the point A_4'). So $\overrightarrow{A_n'P_{n+1}}$ represents the value $f(\bar{z})$, where $\bar{z}$ is the conjugate of z. This illustrates again the correctness of the wellknown theorem: If $f(z)$ is a polynomial with real coefficients, and z is a root of the equation $f(z) = 0$, then $\bar{z}$ is also a root.

Bandung, May 1950. *University of Indonesia.*

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MATHEMATICS

AN ARITHMETICAL PROPERTY OF SOME SUMMABLE FUNCTIONS

BY

J. F. KOKSMA

(Communicated by Prof. J. A. SCHOUTEN at the meeting of May 20, 1950)

§ 1. *Introduction.*

1. 1. If x denotes an irrational number, it is well known, that the sequence

$$x,\ 2x,\ 3x, \dots$$

is uniformly distributed mod 1; from this it follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(nx) = \int_0^1 g(t)\,dt, \tag{1}$$

if $g(t)$ denotes any bounded R-integrable function with period 1.

It is clear that (1) generally becomes false, if in stead of supposing that $g(t)$ is Riemann-integrable, we only assume that $g(t)$ is Lebesgue-integrable, since we can arbitrarily change the value of $g(t)$ at all points nx (mod 1), without changing the integral.

In 1922 A. KHINTCHINE [1]), considering the special case that the periodic function $g(t)$ for $0 \leq t < 1$ denotes the characteristic function of a measurable set $\mathcal{E}$ in (0, 1), introduced the question whether in this case (1) is true for almost all x. He proved this to be the fact under certain conditions as to the nature of $\mathcal{E}$. One may generalise KHINTCHINE's problem by replacing the sequence (nx) by $(\lambda_n x)$, where

$$\lambda_1 < \lambda_2 < \dots \tag{2}$$

denotes a sequence of increasing integers. In the special case $\lambda_n = a^n$, where a is a fixed integer ≥ 2, RAIKOFF proved that the formula

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\lambda_n x) = \int_0^1 g(t)\,dt \tag{3}$$

holds almost everywhere in x for any (periodic) function which is Lebesgue-integrable. [2])

F. RIESS [3]) pointed out that RAIKOFF's theorem is actually an instance

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[1]) A. KHINTCHINE, Ein Satz über Kettenbrüche mit arithmetischen Anwendungen. Math. Zeitschr. 18, 289—306 (1923).

[2]) D. RAIKOFF, On some arithmetical properties of summable functions. Rec. Math. Moscou, N.s. 1, 377—384 (1936) (Russian with engl. summary).

[3]) F. RIESS, Sur la théorie ergodique. Comment. Math. Helvet. 17, 221—239 (1945).

of the classical ergodic theorem. This may be the cause that RAIKOFF's special case $\lambda_n = a^n$ is easier to deal with than KHINTCHINE's case $\lambda_n = n$.

KAC, SALEM and ZYGMUND [4]) proved a generalisation of RAIKOFF's result, considering lacunary sequences (2), where λ_n need not be an integer, but where

$$\lambda_{n+1} \geqq q\,\lambda_n\,,$$

q denoting an arbitrarily given constant > 1. They proved that (3) holds almost everywhere in x, if $g(t)$ denotes a function of the class L^2, the Fourier coefficients of which satisfy a certain condition.

As far as I know, further results concerning the above problem are unknown, [5]) even in the case $\lambda_n = n$.

1. 2. In this paper I shall consider a general class of sequences $(f(n, x))$ $(n = 1, 2, \ldots;\ 0 \leqq x \leqq 1)$ and a general class of periodic functions $g(x) \in L^2$ and I shall prove that the relation

$$\text{(4)} \qquad \operatorname*{Lim}_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(f(n, x)) = \int_0^1 g(t)\,dt$$

holds almost everywhere in x $(0 \leqq x \leqq 1)$ (Theorem 2).

The class of sequences $(f(n, x))$ includes e.g.

a. the case $f(n, x) = xn$

b. the case $f(n, x) = \lambda_n x$
if (λ_n) is any sequence (2) of real numbers satisfying
$\lambda_{n+1} - \lambda_n \geqq \delta > 0$, where δ is a constant,

c. the case $f(n, x) = (1 + x)^n$.

Obviously all these cases are included in the following theorem 1, which itself, as we shall prove in 1. 4, is a special case of the main theorem 2.

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[4]) M. KAC, R. SALEM and A. ZYGMUND, A gap theorem. Trans. Amer. Math. Soc. **63**, 235—243. (1948).

[5]) After the finishing of the manuscript, I discovered the interesting recent paper of P. ERDÖS, On the strong law of large numbers. Trans Amer. Math. Soc. **67** 51—56 (1950).

Mr ERDÖS deals with the special lacunary case (2) which also has been treated by KAC-SALEM-ZYGMUND (l.c. [4])) and gives an interesting improvement of their result. But still more interesting is his Theorem 1, which states that there exists a function $g(x) \in L^2$, and a sequence of integers

$$\lambda_1 < \lambda_2 < \ldots$$

such that for almost all x

$$\frac{1}{N} \sum_{1}^{N} g(\lambda_n x) \to \infty\,.$$

It follows from ERDÖS's theorem that in my above Theorems 1, 2 in any case some condition on the Fourier coefficients, like (13), is necessary.

Theorem 1. *Let δ denote a positive constant. Let $f(1, x)$, $f(2, x)$, ... be a sequence of real numbers defined for every value of x $(0 \leqq x \leqq 1)$, such that, for each $n = 1, 2, \ldots$, the function $f(n, x)$ of x has a continuous derivative $f'_x \geqq \delta$ which is either a non-increasing or a non-decreasing function of x, whereas the expression $f'_x(n_1, x) - f'_x(n_2, x)$ for each couple of positive integers $n_1 \neq n_2$ is either a non-increasing or a non-decreasing function of x for $0 \leqq x \leqq 1$, the absolute value of which is $\geqq \delta$.*

Let $g(x) \in L^2$ denote a periodic function of period 1, and putting

$$g(x) \approx c_0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} c_k e^{2\pi i k x} \qquad (c_{-k} = \bar{c}_k),$$

let

$$\sum_{n=N+1}^{\infty} |c_n|^2 = O(\eta(N)),$$

where $\eta(N)$ denotes a positive non-increasing function, such that $\sum \frac{\eta(N}{N}$ converges. Then (4) holds almost everywhere in $0 \leqq x \leqq 1$.

1. 3. We now state the main result of the paper.

Definition 1. *Let $f(n, x)$ for $n = 1, 2, \ldots$, be a real differentiable function of x for $0 \leqq x \leqq 1$, the derivative of which is either a non-increasing or a non-decreasing, positive function of x. Then put*

$$A_n = \operatorname{Max}\left(\frac{1}{f'(n, 0)}, \frac{1}{f'(n, 1)}\right). \tag{5}$$

Further, let for each couple of integers $n_1 \geqq 1$, $n_2 \geqq 1$, $n_1 \neq n_2$ the derivate Φ'_x of the function

$$\Phi = \Phi(n_1, n_2, x) = f(n_1, x) - f(n_2, x) \quad (n_1 \neq n_2) \tag{6}$$

be $\neq 0$ and either a non-increasing, or a non-decreasing function of x in $0 \leqq x \leqq 1$.

Then put

$$A(M, N) = \frac{1}{N^2} \sum_{n_1=M+2}^{M+N} \sum_{n_2=M+1}^{n_1-1} \operatorname{Max}\left(\frac{1}{\Phi'_x(n_1, n_2, 0)}, \frac{1}{\Phi'_x(n_1, n_2, 1)}\right) \tag{7}$$

for all integers $M \geqq 0$ and $N \geqq 1$.

Definition 2. *Let $g(x)$ denote a periodic function of the class L^2 in $0 \leqq x \leqq 1$ with period 1 and mean value 0, i.e.* [6])

$$g(x) \approx \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} \qquad (c_0 = 0;\ c_{-k} = \bar{c}_k\ (k \geqq 1)), \tag{8}$$

and put

$$R_m = \sum_{k=m+1}^{\infty} |c_k|^2 \qquad (m \geqq 0). \tag{9}$$

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[6]) As the assumption $c_0 = \int_0^1 g(x)\, dx = 0$ may be made without loss of generality, we shall put $c_0 = 0$ for the rest of the paper.

Theorem 2.

1. *Let* C_1, C_2, C_3 *denote suitably chosen absolute constants. Let* $\eta_1(n)$ *and* $\eta_2(n)$ *denote two positive non-increasing functions of* $n = 1, 2, \ldots$ *such that*

$$(10) \qquad \sum_{n=1}^{\infty} \frac{\eta_1(n)}{n} < \infty; \quad \sum_{n=1}^{\infty} \frac{\eta_2(n)}{n} < \infty.$$

2. *Let* $f(n, x)$ $(n = 1, 2, \ldots)$ *denote the functions of Definition* 1 *and let*

$$(11) \qquad \sum_{n=M+1}^{M+N} A_n \leqq C_1 N \qquad (N \geqq 1)$$

$$(12) \qquad A(M, N) \leqq C_2 \eta_1^2(N) \qquad (M \geqq 0,\ N \geqq 1).$$

3. *Let* $g(x)$ *denote the function of Definition* 2 *and let*

$$(13) \qquad R_m \leqq C_3 \eta_2(m) \qquad (m \geqq 1).$$

Then

$$(14) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(f(n, x)) = 0.$$

1.4. It is clear that the functions $f(n, x)$ of theorem 1 satisfy the conditions of theorem 2, for if one ranges the N numbers

$$f'_x(M+1, x),\ f'_x(M+2, x), \ldots,\ f'_x(M+N, x)$$

in increasing order, the difference between each two consecutive ones is at least δ, and we find

$$\sum_{n=M+1}^{M+N} \frac{1}{f'_x(n, x)} \leqq \sum_{\mu=1}^{N} \frac{1}{\mu\delta} < \frac{1}{\delta} \log 3N.$$

$$\sum_{n_2=M+1}^{n_1-1} \frac{1}{|f'_x(n_1, x) - f'_x(n_2, x)|} \leqq 2 \sum_{\mu=1}^{N} \frac{1}{\mu\delta} < \frac{2}{\delta} \log 3N.$$

Hence

$$\sum_{n=M+1}^{M+N} A_n \leqq \sum_{n=M+1}^{M+N} \frac{1}{f'_x(n, 0)} + \sum_{n=M+1}^{M+N} \frac{1}{f'_x(n, 1)} \leqq \frac{2}{\delta} \log 3N$$

and

$$A(M, N) \leqq \frac{4}{\delta} \frac{\log 3N}{N}.$$

1.5. With a different method, which only holds for the case $f(n, x) = \lambda_n x$, where the λ_n are integers satisfying (2), I have proved a theorem, which is somewhat sharper than Theorem 1, as the inequality for $\sum_{n=N+1}^{\infty} |c_n|^2$ is replaced by a weaker one. I shall publish those results elsewhere.

§ 2. *Some Lemma's.*

2. 1. Lemma 1. *Let the function*

$$(15)\qquad g(x) \approx \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} \qquad (c_0 = 0,\ c_{-k} = c_k;\ k \geqq 1)$$

belong to the class L^2 *and put*

$$(16)\qquad R_m = \sum_{k=m+1}^{\infty} |c_k|^2 \qquad (m \geqq 0)$$

and

$$(17)\qquad G(x, a) = \frac{1}{2a} \int_{-a}^{a} g(x+t)\, dt \qquad (a > 0).$$

Then

$$(18)\qquad G(x, a) \approx \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k x},$$

where

$$(19)\qquad C_k = \frac{\sin 2\pi k a}{2\pi k a} c_k \qquad (k \lesseqgtr 0),$$

$$(20)\qquad \sum_{k=1}^{\infty} |C_k|^2 \leqq R_0,$$

$$(21)\qquad \sum_{k=p-1}^{\infty} |C_k| \leqq \frac{1}{2\pi a} \sqrt{\frac{R_p}{p}} \quad (p \geqq 1), \qquad \sum_{k=1}^{\infty} |C_k| \leqq \frac{\sqrt{R_0}}{4a},$$

whereas the relation

$$(22)\qquad G(x, a) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k x}$$

holds uniformly in x.

Remark. The condition $g(x) \in L^2$ is not required for the proof of all the statements of the lemma, but a refinement in that direction would be of no use for our purpose.

Proof. As is proved in textbooks on Fourier-series, [7]) any Fourier-series, whether convergent or not, may be integrated term by term between any limits; i.e. the sum of the integrals of the separate terms, is the integral of the function of which the series is the Fourier series. Now applying this process to the function

$$g(x+t) \approx \sum_{k=-\infty}^{\infty} (c_k e^{2\pi i k x}) \cdot e^{2\pi i k t},$$

integrating with respect to t between the limits $-a$ and a, we immediately find (22) and (19). We now shall prove the formula (21), from which it is immediately clear that (22) holds uniformly in x.

[7]) E.g. cf. E. C. Titchmarsh, The theory of functions, Oxford Univ. Press, (1939).

From (19) we have for $p \geqq 0,\ q \geqq 1$

$$\sum_{k=p+1}^{p+q} |C_k| \leqq \frac{1}{2\pi a} \sum_{k=p+1}^{p+q} |c_k| \cdot \frac{1}{k} \leqq \frac{1}{2\pi a} \Big(\sum_{k=p+1}^{p+q} |c_k|^2\Big)^{1/2} \Big(\sum_{k=p+1}^{p+q} \frac{1}{k^2}\Big)^{1/2},$$

using the CAUCHY–SCHWARZ–RIEMANN-inequality; hence

$$\sum_{k=1}^{\infty} |C_k| \leqq \frac{1}{2\pi a} \frac{\pi}{\sqrt{6}} \sqrt{R_0} < \frac{\sqrt{R_0}}{4a}$$

and, if $p \geqq 1$:

$$\sum_{k=p+1}^{p+q} |C_k| \leqq \frac{1}{2\pi a} R_p^{1/2} \Big(\int_p^{\infty} \frac{du}{u^2}\Big)^{1/2} = \frac{1}{2\pi a} \sqrt{\frac{R_p}{p}},$$

which proves (21).

Now from the fact that (22) holds uniformly in x we deduce immediately that the right hand side is the Fourier series of its sum, which proves (18). Finally we note that (20) is an immediate consequence of (16) and (19) because of

$$\left|\frac{\sin 2\pi k a}{2\pi k a}\right| \leqq 1.$$

Lemma 2. *If $g(x) \in L^2$ and $G(x, a)$ $(a > 0)$ denote the functions of* (15) *and* (17) *and if R_m is defined by* (16) $(m \geqq 0)$, *we have*

$$\int_0^1 |g(x) - G(x, a)|^2\, dx \leqq 100\, R_0\, m^4 a^4 + 8\, R_m, \tag{23}$$

for each integer $m \geqq 1$ which satisfies

$$2\pi m a \leqq 1.$$

Proof. By Lemma 1 we have (see (15) and (18))

$$g(x) - G(x, a) \approx \sum_{k=-\infty}^{\infty} (c_k - C_k)\, e^{2\pi i k x},$$

hence

$$g(x) - G(x, a) \approx \sum_{k=-\infty}^{\infty} c_k \Big(1 - \frac{\sin 2\pi k a}{2\pi k a}\Big) e^{2\pi i k x},$$

by (19). Now, since $g(x)$ and $G(x, a)$ belong to L^2 ($G(x, a)$ being a continuous function of x), we have, by PARSEVAL's theorem,

$$\begin{cases} \displaystyle\int_0^1 |g(x) - G(x, a)|^2\, dx = 2 \sum_{k=1}^{\infty} |c_k|^2 \Big(1 - \frac{\sin 2\pi k a}{2\pi k a}\Big)^2 = \\ \displaystyle = 2 \sum_{k=1}^{m} |c_k|^2 \Big(1 - \frac{\sin 2\pi k a}{2\pi k a}\Big)^2 + 2 \sum_{k=m+1}^{\infty} |c_k|^2 \Big(1 - \frac{\sin 2\pi k a}{2\pi k a}\Big)^2. \end{cases} \tag{24}$$

Now we note that

$$\Big(1 - \frac{\sin 2\pi k a}{2\pi k a}\Big)^2 \leqq 4$$

and also that

$$\left(1-\frac{\sin 2\pi ka}{2\pi ka}\right)^2 \leqq \frac{(2\pi ka)^4}{(3!)^2} < \frac{(6{,}5)^4 m^4 a^4}{6^2} < 100\, m^4 a^4,$$

if $1 \leqq k \leqq m$, and $2\pi m a \leqq 1$.

Therefore we have by (24)

$$\int_0^1 |g(x)-G(x,a)|^2\, dx \leqq 100\, m^4 a^4 \sum_{k=1}^{m} |c_k|^2 + 8 \sum_{k=m+1}^{\infty} |c_k|^2 \leqq$$

$$\leqq 100\, m^4 a^4 R_0 + 8\, R_m.$$

Q. e. d.

Lemma 3. *Let $f(n,x)$ for $n=1,2,\ldots$ denote a real continuous function of x for $a \leqq x \leqq b$, and let*

$$\Phi(n_1, n_2, x) = f(n_1, x) - f(n_2, x) \text{ for } n_1 \neq n_2$$

have a continuous derivative Φ'_x, which is $\neq 0$ and either non-decreasing or non-increasing for $a \leqq x \leqq b$.

Finally, put

$$A_N = \frac{1}{N^2} \sum_{n_1=2}^{N} \sum_{n_2=1}^{n_1-1} \operatorname{Max}\left(\frac{1}{\Phi'_x(n_1,n_2,a)},\ \frac{1}{\Phi'_x(n_1,n_2,b)}\right).$$

Then we have for $N \geqq 2$, $h > 0$ (h not depending on n and x)

$$\int_a^b \Big|\sum_{n=1}^{N} e^{2\pi i h f(n,x)}\Big|^2 dx \ \leqq (b-a)\, N + \frac{A_N}{h} N^2.$$

This lemma I have proved in a previous paper. [8]) I deduce from it the following

Lemma 4. *Let $f(n,x)$ for $n=1,2,\ldots$ denote the functions of Definition 1, and let $A(M,N)$ be defined by (7). Then for each integer $h \neq 0$ which does not depend on n and x we have*

$$\int_0^1 \Big|\sum_{n=M+1}^{M+N} e^{2\pi i h f(n,x)}\Big|^2 dx \leqq N + \frac{A(M,N)}{h} N^2. \qquad (M \geqq 0,\ N \geqq 1).$$

Proof. For $M \geqq 0$ consider the sequence

$$f(M+1, x),\ f(M+2, x), \ldots$$

These functions satisfy the conditions of Lemma 3 with $f(M+n, x)$ in stead of $f(n,x)$ and with $a=0$, $b=1$. The corresponding number A_N defined in Lemma 3 ($a=0$, $b=1$) is identical with the number $A(M,N)$ which was defined by (7). Therefore Lemma 4 is an immediate consequence of Lemma 3.

Lemma 5. *Let $f(n,x)$ for $n=1,2,\ldots$ denote the functions of*

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[8]) J. F. Koksma, Ein mengentheoretischer Satz über die Gleichverteilung modulo Eins. Comp. Math. 2, 250—258 (1935).

Definition 1, *let* $g(x)$ *denote the function of Definition* 2, *let* $G(x, \alpha)$ $(\alpha > 0)$ *denote the function which is defined by* (17) *and finally, let*

$$S(M, N, x; \alpha) = \sum_{n=M+1}^{M+N} G(f(n, x), \alpha). \tag{25}$$

Then for each integer $p \geqq 1$

$$\left\{\begin{aligned} &\int_0^1 |S(M, N, x; \alpha|^2\, dx \leqq 4\, R_0\, Np + 8\, R_0\, N^2 A(M, N) \log 3p + \\ &+ \frac{1}{\pi\alpha^2} N^{3/2} \sqrt{\frac{R_0\, R_p}{p}} + \frac{8}{\pi\alpha} N^2 \sqrt{\frac{R_0\, R_p}{p} A(M, N) \log 3p} + \frac{1}{\pi^2\alpha^2} N^2 \frac{R_p}{p}, \end{aligned}\right. \tag{26}$$

where $A(M, N)$ *and* R_p *are defined by* (7) *and* (9) (*Definition* 1 *and* 2).

Proof. Using Lemma 1, we deduce from (22) and (25)

$$S(M, N, x; \alpha) = \sum_{n=M+1}^{M+N} \sum_{k=-\infty}^{\infty} C_k\, e^{2\pi i k f(n,x)} = \sum_{k=-\infty}^{\infty} C_k \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}.$$

Hence

$$|S(M, N, x; \alpha)| \leqq 2 \sum_{k=1}^{\infty} |C_k| \left| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \right| \leqq$$

$$\leqq 2 \sum_{k=1}^{p} |C_k| \left| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \right| + 2N \sum_{k=p+1}^{\infty} |C_k|$$

$(p \geqq 1)$, as the inner sum is in absolute value $\leqq N$.

Therefore we have

$$|S(M, N, x; \alpha|^2 \leqq 4 \left(\sum_{k=1}^{p} |C_k| \left| \sum_{n=M=1}^{M+N} e^{2\pi i k f(n,x)} \right| \right)^2 +$$

$$+ 8N \left(\sum_{k=p+1}^{\infty} |C_k| \right) \left(\sum_{k=1}^{p} |C_k| \left| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \right| \right) + 4N^2 \left(\sum_{k=p+1}^{\infty} |C_k| \right)^2$$

and applying the Cauchy–Schwarz–Riemann-inequality

$$|S(M, N, x; \alpha)|^2 \leqq 4 \left(\sum_{k=1}^{p} |C_k|^2 \right) \left(\sum_{k=1}^{p} \left| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \right|^2 \right) +$$

$$+ 8N \left(\sum_{k=p+1}^{\infty} |C_k| \right) \left(\sum_{k=1}^{p} |C_k| \left| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \right| \right) + 4N^2 \left(\sum_{k=p+1}^{\infty} |C_k| \right)^2.$$

Integrating this and applying (20) and (21) we find:

$$\left\{\begin{aligned} &\int_0^1 |S(M, N, x; \alpha)|^2\, dx \leqq 4\, R_0 \sum_{k=1}^{p} \int_0^1 \left| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \right|^2 dx + \\ &+ \frac{4N}{\pi\alpha} \sqrt{\frac{R_p}{p}} \sum_{k=1}^{p} |C_k| \int_0^1 \left| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \right| dx + \frac{N^2}{\pi^2\alpha^2} \cdot \frac{R_p}{p}. \end{aligned}\right. \tag{27}$$

Now we have, by Lemma 4,

$$\int_0^1 \left| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \right|^2 dx \leqq N + \frac{A(M, N)}{k} N^2.$$

Moreover applying the CAUCHY–SCHWARZ–RIEMANN-inequality for integrals, we find

$$\int_0^1 \Big| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \Big| \, dx \leqq \Big(\int_0^1 1^2 \, dx\Big)^{1/2} \cdot \Big(\int_0^1 \Big| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \Big|^2 dx\Big)^{1/2}$$

$$\leqq \sqrt{N + \frac{A(M,N)}{k} N^2} \leqq \sqrt{N} + N \sqrt{\frac{A(M,N)}{k}},$$

by MINKOWSKI's inequality. Therefore it follows from (27)

$$\int_0^1 |S(M,N,x;a)|^2 \, dx \leqq 4 R_0 \sum_{k=1}^{p} \Big(N + \frac{A(M,N)}{k} N^2\Big) +$$

$$+ \frac{4N}{\pi a} \sqrt{\frac{R_p}{p}} \sum_{k=1}^{p} |C_k| \Big(\sqrt{N} + N \sqrt{\frac{A(M,N)}{k}}\Big) + \frac{N^2}{\pi^2 a^2} \cdot \frac{R_p}{p}$$

and hence, by (21), since

$$\sum_{k=1}^{p} |C_k| \cdot \frac{1}{\sqrt{k}} \leqq \Big(\sum_{k=1}^{p} |C_k|^2\Big)^{1/2} \Big(\sum_{k=1}^{p} \frac{1}{k}\Big)^{1/2} < 2 \sqrt{R_0} \sqrt{\log 3p},$$

we have

$$\int_0^1 |S(M,N,x;a)|^2 \, dx \leqq 4 R_0 N p + 8 R_0 N^2 A(M,N) \log 3p +$$

$$+ \frac{N^{3/2} \sqrt{R_0}}{\pi a^2} \sqrt{\frac{R_p}{p}} + \frac{8 N^2}{\pi a} \sqrt{R_0} \sqrt{\frac{R_p}{p}} \sqrt{A(M,N)} \sqrt{\log 3p} + \frac{N^2}{\pi^2 a^2} \frac{R_p}{p},$$

which proves (26).

Lemma 6. *Let $\Omega(x) \geqq 0$ denote a periodic function with period* 1, *which is Lebesgue-integrable. Let $\psi(x)$ denote a differentiable function in* $(0,1)$, *such that $\psi'(x)$ is a positive and either a non-increasing, or a non-decreasing function of x in* $(0,1)$. *Put*

$$\Lambda = \operatorname{Max}\Big(\frac{1}{\psi'(0)}, \frac{1}{\psi'(1)}, 1\Big). \tag{28}$$

Then we have

$$\int_0^1 \Omega(\psi(x)) \, dx \leqq 8 \Lambda \int_0^1 \Omega(u) \, du. \tag{29}$$

Proof. If $\Psi(u)$ denotes the inverse function of $u = \psi(x)$, we have

$$\int_0^1 \Omega(\psi(x)) \, dx = \int_{\psi(0)}^{\psi(1)} \Omega(u) \, \Psi'(u) \, du. \tag{30}$$

We now distinguish two cases.

A. If

$$\psi(1) - \psi(0) \leqq 8,$$

then the right hand side of (30) is

$$\leqq \Lambda \int_{\psi(0)}^{\psi(1)} \Omega(u) \, du \leqq 8 \Lambda \int_0^1 \Omega(u) \, du,$$

because of (28) and the fact that $\Omega \geqq 0$ is periodic. In this case therefore (29) has been proved already.

B. If

$$\psi(1) - \psi(0) > 8,$$

then we write (30) in the form

$$(31)\qquad \left\{\begin{aligned} \int_0^1 \Omega(\psi(x))\,dx = \sum_{v=A}^{B} \int_v^{v+1} \Omega(u)\,\Psi'(u)\,du + \\ + \int_{\psi(0)}^{A} \Omega(u)\,\Psi'(u)\,du - \int_B^{\psi(1)} \Omega(u)\,\Psi'(u)\,du, \end{aligned}\right.$$

where we have put

$$A = [\psi(0)] + 2 \quad , \quad B = [\psi(1)] - 3.$$

Now by the same argument as we have used in *A*, we have

$$(32)\qquad \int_{\psi(0)}^{A} \Omega(u)\,\Psi'(u)\,du \leqq 3\,\Lambda \int_0^1 \Omega(u)\,du,$$

$$(33)\qquad \int_B^{\psi(1)} \Omega(u)\,\Psi'(u)\,dn \leqq 4\,\Lambda \int_0^1 \Omega(u)\,du.$$

We now deal with the sum on the right hand side of (31) and write

$$\sum_{v=A}^{B} \int_v^{v+1} \Omega(u)\,\Psi'(u)\,du = \sum_{v=A}^{B} \int_0^1 \Omega(v+v)\,\Psi'(v+v)\,dv =$$

$$= \int_0^1 \Omega(v)\,\{\sum_{v=A}^{B} \Psi'(v+v)\}\,dv \leqq \{\sum_{v=A}^{B+1} \Psi'(v)\} \int_0^1 \Omega(v)\,dv,$$

since $\Psi'(u)$ is either non-increasing of non-decreasing. Now

$$\sum_{v=A}^{B+1} \Psi'(v) \leqq \int_{A-1}^{B+2} \Psi'(u)\,du < \int_{\psi(0)}^{\psi(1)} \Psi'(u)\,du = [\Psi(u)]_{\psi(0)}^{\psi(1)} = 1,$$

and therefore

$$(34)\qquad \sum_{v=A}^{B} \int_v^{v+1} \Omega(u)\,\Psi'(u)\,du \leqq \int_0^1 \Omega(u)\,du.$$

Combining (31), (32), (33), (34), we find

$$\int_0^1 \Omega(\psi(x))\,dx \leqq 8\,\Lambda \int_0^1 \Omega(u)\,du.$$

because of $\Lambda \geqq 1$. Q. e. d.

Lemma 7. *If $f(n, x)$ $(n = 1, 2, \ldots)$ and $g(x)$ denote the functions of Definition* 1 *and Definition* 2, *and if we put*

$$(35)\qquad S^*(M, N, x) = \sum_{n=M+1}^{M+N} g(f(n, x)),$$

where $M \geqq 0$, $N \geqq 1$ are integers, we have for every positive a, and every positive integer m, satisfying $2\pi m a \leqq 1$ and for every positive integer p the inequality

$$(36)\quad \left\{\begin{aligned} &\{\int_0^1 |S^*(M,N,x)|^2\,dx\}^{1/2} \leqq \{(100\,R_0 m^4 a^4 + 8\,R_m)\,N \sum_{n=M+1}^{M+N} 8\,(\Lambda_n+1)\}^{1/2} + \\ &+ \{4\,R_0 N p + 8\,R_0 N^2 A(M,N)\log 3p + \\ &\qquad + \frac{N^{3/2}}{\pi a^2}\sqrt{\frac{R_0 R_p}{p}} + \frac{8N^2}{\pi a}\sqrt{\frac{R_0R_p}{p}A(M,N)\log 3p} + \frac{N^2}{\pi^2 a^2}\frac{R_p}{p}\Big\}^{1/2}, \end{aligned}\right.$$

using the notations of Definitions 1, 2.

Proof. If $G(x,a)$ $(a>0)$ is defined by (17) and if $S(M,N,x,a)$ denotes the sum (25), we find from (35) by MINKOWSKI's inequality

$$(37)\quad \left\{\begin{aligned} &\{\int_0^1 |S^*(M,N,x)|^2\,dx\}^{1/2} \leqq \{\int_0^1 |S^*(M,N,x) - S(M,N,x;a)|^2\,dx\}^{1/2} + \\ &\qquad\qquad + \{\int_0^1 |S(M,N,x;a)|^2\,dx\}^{1/2}. \end{aligned}\right.$$

Now by (25) and (35)

$$(38)\quad \left\{\begin{aligned} &\int_0^1 |S^*(M,N,x) - S(M,N,x;a)|^2\,dx = \\ &= \int_0^1 \Big|\sum_{n=M+1}^{M+N} (g(f(n,x)) - G(f(n,x),a))\Big|^2\,dx \leqq \\ &\qquad \leqq N \sum_{n=M+N}^{M+N} \int_0^1 |g(f(n,x)) - G(f(n,x),a)|^2\,dx \end{aligned}\right.$$

using the CAUCHY–SCHWARZ–RIEMANN-inequality for sums. Applying Lemma 6 with

$$\Omega(x) = |g(x) - G(x,a)|^2\ ,\ \ \psi(x) = f(n,x),$$

we find

$$\int_0^1 |g(f(n,x)) - G(f(n,x),a)|^2\,dx \leqq 8\,(\Lambda_n+1)\int_0^1 |g(u) - G(u,a)|^2\,du,$$

where Λ_n is defined by (5). Therefore we have by Lemma 2

$$\int_0^1 |g(f(n,x)) - G(f(n,x),a)|^2\,dx \leqq 8\,(\Lambda_n+1)\,(100\,R_0 m^4 a^4 + 8\,R_m)$$

for every integer $m \geqq 1$ which satisfies $2\pi m a \leqq 1$. Therefore we have by (38)

$$\int_0^1 |S^*(M,N,x) - S(M,N,x;a)|^2\,dx \leqq (100\,R_0 m^4 a^4 + 8R_m)\,N \sum_{n=M+1}^{M+N} 8\,(\Lambda_n+1)$$

and thus we find from (37)

$$(39)\quad \left\{\begin{aligned} &\{\int_0^1 |S^*(M,N,x)|^2\,dx\}^{1/2} \leqq \{(100\,R_0 m^4 a^4 + 8R_m)\,N \sum_{n=M+1}^{M+N} 8\,(\Lambda_n+1)\}^{1/2} + \\ &\qquad\qquad + \{\int_0^1 |S(M,N,x;a)|^2\,dx\}^{1/2}. \end{aligned}\right.$$

Now applying Lemma 5 ($p \geqq 1$), we find from (39) and (26) immediately (36), which proves Lemma 7.

Lemma 8. *Be $\eta(x) > 0$ and non-increasing for $x \geqq 1$, such that*

(40) $$\sum_{N=1}^{\infty} \frac{\eta(N)}{N} < \infty.$$

Then we have for any positive ε

(41) $$\sum_{N=1}^{\infty} \frac{\eta(N^{\varepsilon})}{N} < \infty$$

and also

(42) $$\eta(N) \log N \to 0, \text{ as } N \to \infty.$$

Proof. *A*. It is clear that $\frac{\eta(N^{\varepsilon})}{N}$ is a non-increasing function of N and therefore

$$\sum_{N=1}^{\infty} \frac{\eta(N^{\varepsilon})}{N} < \infty, \text{ if } \int_1^{\infty} \frac{\eta(t^{\varepsilon})}{t}\, dt \text{ converges.}$$

Now putting $t^{\varepsilon} = u$, we find $\frac{dt}{t} = \frac{1}{\varepsilon}\frac{du}{u}$, hence

$$\int_1^{T} \frac{\eta(t^{\varepsilon})}{t}\, dt = \frac{1}{\varepsilon} \int_1^{T^{\varepsilon}} \frac{\eta(u)}{u}\, du,$$

which proves the first assertion, since $\sum_{N=1}^{\infty} \frac{\eta(N)}{N} < \infty$.

B. As $\eta(N)/N$ is non-increasing and as $\sum_{N=1}^{\infty} \frac{\eta(N)}{N} < \infty$, we conclude from a well known theorem that also the series

$$\sum_{n=1}^{\infty} \eta(2^n) < \infty.$$

Since the general term of this series is a non-increasing function of n, we have by another well known theorem

$$n\, \eta(2^n) \to 0, \text{ as } n \to \infty.$$

Now let N denote a positive integer > 2 and be n the integer for which

$$2^n \leq N < 2^{n+1}.$$

Then we have

$$\eta(N) \log N \leqq \eta(2^n)(n+1) \log 2 \to 0 \text{ as } N \to \infty.$$

Q. e. d.

Lemma 9. *Let $\eta_1(N)$ and $\eta_2(N)$ be positive non-increasing functions such that*

$$\sum_{N=1}^{\infty} \frac{\eta_1(N)}{N} < \infty, \quad \sum_{N=1}^{\infty} \frac{\eta_2(N)}{N} < \infty.$$

Then $\sqrt{\eta_1(N)\,\eta_2(N)}$ is also a non-increasing function of N, such that

$$\sum_{N=1}^{\infty} \frac{\sqrt{\eta_1(N)\,\eta_2(N)}}{N} < \infty.$$

Proof. It is trivial that $\sqrt{\eta_1(N)\,\eta_2(N)}$ is a non-increasing function. Further by the CAUCHY–SCHWARZ–RIEMANN-inequality

$$\sum_{N=1}^{N^*} \frac{\sqrt{\eta_1(N)\,\eta_2(N)}}{N} \leqq \sqrt{\sum_{N=1}^{N^*} \frac{\eta_1(N)}{N}} \cdot \sqrt{\sum_{N=1}^{N^*} \frac{\eta_2(N)}{N}},$$

which proves the lemma.

Finally we use a lemma, which is a special case of a theorem due to I. S. GÁL and the author [9]), and which has also been proved in a joint paper by R. SALEM and the author: [10])

Lemma 10. Let $f_n(x) \in L^p\,(0, 1)$ $(n = 1, 2, \ldots)$ $(p > 1)$ *be a sequence of functions such that*

$$\int_0^1 \Big| \sum_{n=M+1}^{M+N} f_n(x) \Big|^p dx < C\,(M+N)^{p-\sigma} N^{\sigma} \eta(N),$$

where $C > 0$, $\sigma > 1$ are constants and where $\eta(N)$ denotes a positive non-increasing function of N such that $\sum \eta(N)/N < \infty$.

Then

$$\frac{1}{N} \sum_{n=1}^{N} f_n(x) \to 0, \text{ as } N \to (\infty),$$

almost everywhere in $(0, 1)$.

§ 3. *Proof of Theorem* 2.

We put $(N \geqq 1)$

$$a = \frac{1}{2\pi N^{1/4}}, \quad m = [N^{1/8}], \quad p = [N^{3/4}]. \tag{43}$$

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[9]) I. S. GÁL and J. F. KOKSMA, Sur l'ordre de grandeur des fonctions sommables. These Proc. 53, 638–653 (1950) = Indag. Math. 12, Fasc. 3 (1950).

[10]) J. F. KOKSMA and R. SALEM, Uniform distribution and Lebesgue integration, Acta Sci. Math. Szeged. 12B, 87–96 (1950).

The main theorem of this joint paper is closely related with the above theorems, as it deals with the problem, whether from the uniform distribution modulo 1 of a sequence $u_1, u_2, \ldots,$ may be concluded that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(u_n + x) = \int_0^1 g(x)\,dx$$

for a given $g(x) \in L^2$ and for almost all x.

Then $m \geqq 1$, $p \geqq 1$, $2\pi m \alpha \leqq 1$.

Further we have by (13)

$$R_m = O(\eta_2(m)) = O(\eta_2(N^{1/9})), \tag{44}$$

$$R_p = O(\eta_2(p)) = O(\eta_2(N^{1/9})), \tag{45}$$

as $\eta_2(n)$ is non-increasing.

We now apply Lemma 7. Substituting (43), (44), (45), (11), (12) in (36) we find

$$\{\int_0^1 |S^*(M, N, x)|^2 \, dx\}^{1/2} = O(\{N^{3/2} + N^2 \eta_2(N^{1/9})\}^{1/2}) +$$

$$+ O(\{N^{7/4} + N^2 \eta_1^2(N) \log 3N + N^{13/8} \sqrt{\eta_2(N^{1/9})} + N^{15/8} \sqrt{\eta_2(N^{1/9}) \eta_1^2(N) \log 3N}$$
$$+ N^{7/4} \eta_2(N^{1/9})\}^{1/2}).$$

Hence using MINKOWSKI's inequality and applying Lemma 8, i.e.

$$\eta_1(N) \log 3N \to 0 \quad \text{and} \quad \eta_1(N) \to 0, \ \eta_2(N^{1/9}) \to 0,$$

we find

$$\{\int_0^1 |S^*(M, N, x)|^2 \, dx\}^{1/2} = O(N^{15/16} + N\sqrt{\eta_2(N^{1/9})} + N\sqrt{\eta_1(N)}).$$

Hence by Lemma 9

$$\int_0^1 |S^*(M, N, x)|^2 \, dx = O(N^2 \eta(N)),$$

where $\eta(N)$ is a positive non-increasing function such that

$$\sum_{N=1}^{\infty} \frac{\eta(N)}{N} < \infty.$$

Therefore by Lemma 10 and (35)

$$S^*(0, N, x) = o(N)$$

almost everywhere in $0 \leqq x \leqq 1$.

Q. e. d.

A REMARKABLE FEATURE OF THE EARTH'S TOPOGRAPHY

BY

F. A. VENING MEINESZ

preliminary paper

(Communicated at the meeting of June 24, 1950)

When studying the well-known development in spherical harmonics of the Earth's topography made by PREY in 1922 [1]), the writer found a remarkable feature which, as far as he knows, has not yet been noticed. This preliminary communication may give a short summary of it.

As the equations for convection in the Earth as well as of several other physical phenomena connected with our globe, if expressed in spherical harmonics, depend mainly on the order n of the spherical harmonic but less on the form and coefficients of its $2n + 1$ underterms, it seemed worth while to determine for each order a value representing the size of all the terms of this order combined. It appeared indicated for this purpose to derive the mean over the Earth's surface of the squares of the total part of the topographic elevation belonging to that order.

The writer has done so for all the orders 1 — 16 for which PREY deduced the coefficients and the result was remarkable. Instead of obtaining a more or less irregularly varying row of figures diminishing, as may be expected, for increasing order, the figures show a predominating feature besides the well-known strong first order term corresponding to the land and sea hemispheres; the figures for the orders 3, 4 and 5 are much larger than seems to correspond to the curve shown by the other figures. It seems, therefore, that the terms belonging to these three orders have a special significance for the topography as no doubt the first order term has likewise.

In a future larger paper the writer shall give the deductions, equations and figures leading to this conclusion and he shall investigate its meaning and importance. He may give here a few short remarks and ideas about it.

In the first place he may point out that this mathematical result comprises two notions already found long ago, viz. the fact that most of the continental parts of the Earth's surface are antipodal to oceanic parts; only 6 % forms an exception to this rule; and secondly that the continental shields seem to show a distribution of a more or less tetrahedal character.

[1]) A. PREY, Darstellung der Höhen und Tiefenverhältnisse der Erde, Abh. Ges. der Wiss., Göttingen, Math. Phys. Kl., N.F., XI, 1.

In the second place he may emphasize its importance for the problem about the origin of the continents and oceans: it indicates the probability that they have been brought about by phenomena at least partly governed by equations in spherical harmonics in which especially the orders 1 as well as 3, 4 and 5 play a dominating part. As the writer will show in a later paper this could be explained by a hypothesis already advanced by him in 1944 [2]) that convection currents have been responsible for the distribution of continents and oceans; the horizontal drag exerted by such currents on the sialic crust would have swept it together towards those areas where the currents descended. According to this hypothesis the prominence of the 3rd, 4th and 5th order spherical harmonics would be connected with the thickness of 2900 km of the mantle in which the convection took place, while the presence of the large 1st order term seems to point to a still older stage when the differentiation between core and mantle had not yet taken place and the current developed in the whole Earth's interior. This last current could have been responsible for the forming of the core. Other possibilities for explaining the 1st order term can also be considered as e.g. the moon-release hypothesis.

In the third place the writer may remark that the result here published provides a strong argument against WEGENER's hypothesis of continental drift. Movements of continents over large distances as supposed in that theory should have destroyed a spherical harmonic distribution of the type here mentioned if it had been present before. That they could have shifted in such a way that such a distribution would have originated by the movement seems highly improbable.

In the last place the writer may explain the fact that the systematic character of the big topographic features on the Earth's surface as revealed by the here mentioned mathematical result, has not yet been discovered long ago, by the complicated character of a complete spherical harmonic term and by the mixing in this instance of three of such terms together.

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[2]) De Verdeeling van Continenten en Oceanen over het Aardoppervlak, Versl. Ned. Akad. v. Wetensch. afd. Nat., 53, 151—159 (1944).

Major tectonic Phenomena and the Hypothesis of Convection Currents in the Earth, Quart. Journ. Geol. Soc. of London, Jan. 1948, pp. 204—206.

In these papers the writer supposed that the Earth's topography led to assuming a convection current corresponding to a third order spherical harmonic of a special kind which would explain a distribution of the sial of a tetrahedal character and he tried to make plausible such an assumption. It appears now that in reality 3rd, 4th and 5th order terms of a general type are prominent in the topography.

CHEMISTRY

ELASTIC VISCOUS OLEATE SYSTEMS CONTAINING KCl. XII [1])

1) *Influence of the terms of the homologous series of the alkylbenzenes on the elastic behaviour of the* 1.2 % *oleate system*
2) *Complications in the use of* $G - n$ *diagrams for the discussion of the results obtained with the terms of homologous series of alkanes and of alkylbenzenes*

BY

H. G. BUNGENBERG DE JONG, W. A. LOEVEN AND H. J. VERHAGEN * [2])

(Communicated at the meeting of May 20, 1950)

1. *Introduction.*

In part XI of this series the influence of a number of hydrocarbons (e.g. alkanes, benzene and alkylbenzenes) on the elastic behaviour of the KCl containing oleate system has been investigated.

The results have been discussed from the point of view that the total action is the resultant of two oppositely directed component actions, a KCl-demanding one (D) and a KCl-sparing one (S). We found that D is relatively enforced with regard to S by:

1. increase of the hydrocarbon concentration,
2. increase of the length of the carbon chain (or side chain),
3. lowering of the KCl concentration.

For the action of the first hydrocarbon additions there are accordingly two possibilities: a KCl-sparing total action ($D < S$) and a KCl-demanding total action ($D > S$).

In the homologous series of the n-alkanes these two possibilities do occur (at 1.2 N KCl pentane and hexane exert a KCl-sparing total action, heptane and the higher homologues exert a KCl-demanding total action). The few terms of the homologous series of the n-alkylbenzenes which had been investigated, (toluene, ethylbenzene, n-propylbenzene) all exerted a KCl-sparing total action at the first additions (even at 0.6 N KCl, which normality according to the above point 3, is favourable for a preponderance of D above S).

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*) Aided by grants from the "Netherlands Organisation for Basic Research (Z.W.O.)".

1) Part I has appeared in these Proceedings **51**, 1197 (1948), Parts II-VI in these Proceedings **52**, 15, 99, 363, 377, 465 (1949), Parts VII-XI in these Proceedings **53**, 7, 109, 233, 743, 759 (1950).

2) Publication no. 7 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

The occurrence of a KCl-demanding total action ($D > S$) of an alkylbenzene at the first additions, must be expected if the carbon side chain is sufficiently lengthened (see the above point 2). The present communication deals with the action of the terms, ethylbenzene up to and including *n*-heptylbenzene [3]), at three different KCl concentrations, with the aim to control that the above three points also hold for the higher homologues and to investigate if there exist terms which, at the first addition, exert a KCl-demanding total action.

Besides the present communication deals with $G - n$ diagrams for the plotting of the results obtained with the homologous series of the *n*-alkanes and the *n*-alkylbenzenes. These kind of diagrams proved useful at the discussion of the results obtained with the homologous series of the *n*-primary alcohols (part VIII) and the *n*-fatty acid anions (part IX), but in the case of the *n*-alkanes they will appear to lead to quite erroneous conclusions. The aim of the further experiments will be to explain this failure of the method of $G - n$ diagrams.

2. *Influence of ethylbenzene up to and including n-heptylbenzene at three KCl concentrations.*

The experiments with the above-mentioned alkylbenzenes were performed in quite the same way (rotational oscillation, 15°, exactly half filled spheres of approximately 500 ml capacity, H_2 marking) as is described in detail in part XI of this series. The mixture of the contents of five flasks of Na-oleate from BAKER [4]) which was used for the experiments showed a minimum damping at 1.08 N KCl. The experiments were therefore performed at this KCl concentration and at a concentration lower (0.6 N) and higher (1.5 N) than 1.08 N. The three elastic systems were made from the same stock oleate solution and were investigated simultaneously to make the results as much comparable as possible (thus 18 vessels were used at the same time, 6 for the system at 0.6 N, 6 for the system at 1.08 N and 6 for the system at 1.5 N).

The obtained results have been represented in the figures 1, 2 and 3. For some unknown reason the experimental errors are larger than usual. Compare the somewhat irregular position of the experimentally determined points. This is for instance very pronounced with the *n* curves at 0.6 N KCl for ethylbenzene and *n*-propylbenzene. Here the experimentally determined points suggest the presence of an additional minimum in the *n* curves. As nothing of this kind had been found for ethylbenzene and *n*-propylbenzene at 0.6 N in the experiments described in part XI of this series, we believe that the irregular position is only due to experi-

[3]) We are much indebted to the Koninklijke Shell Laboratorium, Amsterdam, for a gift of *n*-heptylbenzene; to Prof. HAVINGA, Leiden, for a gift of *n*-butylbenzene and to mr. J. H. F. BAAK who synthesized for us *n*-amylbenzene and *n*-hexylbenzene.

[4]) A generous gift of Na-oleate from The Rockefeller Foundation provided the means for the experiments of this communication.

mental errors. For this reason the n curves for ethylbenzene and n-propylbenzene have been drawn as simple as possible in fig. 1. The position of the G point in fig. 2 for the highest n-propylbenzene concentration, which would suggest an intersection of the curves for n-propylbenzene and n-butylbenzene is also to be mistrusted, because here, the large damping necessitated a relative large and therefore uncertain correction of the period.

The results of the experiments could first be discussed on the same lines (using the expressions: "actions according to type A, type B or type C") as are given in detail for the series of the n-alkanes in part XI (section 2) of this series, but we will at once switch over to a discussion

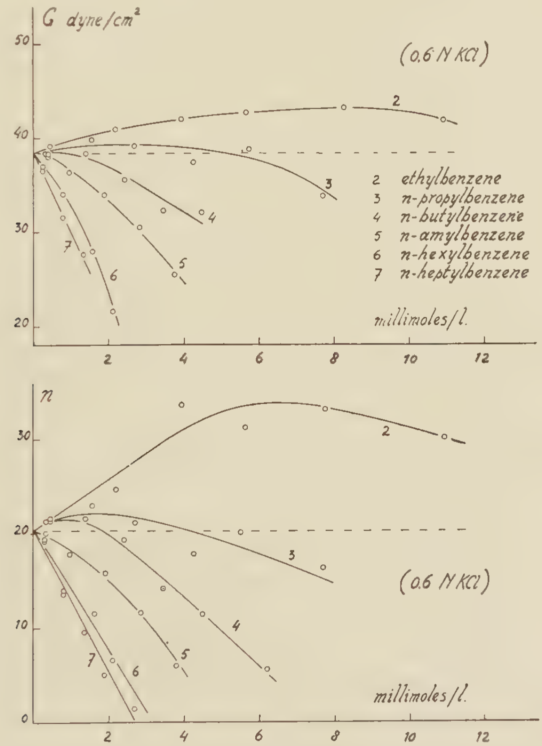

Fig. 1

from the point of view that the total action is the resultant of a KCl-sparing component action (S) and a KCl-demanding component action (D).

From this point of view type A means $D < S$ and type C means $D > S$, whereas type B means exact compensation of the two component actions (thus $D = S$). It has already been discussed in part X (section 6) that it is extremely unlikely that type B will be realized at a whole range of concentrations of the added substance and at a whole range of KCl concentrations.

When we now consider in fig. 1 the influence of the first amounts which were added, we perceive that both G and n increase with ethylbenzene

and n-propylbenzene and both decrease with n-amylbenzene, n-hexylbenzene and n-heptylbenzene.

At the KCl concentration which is used (0.6 N), a concentration lower than that corresponding to the minimum damping of the blank, this means that for ethylbenzene and n-propylbenzene $D < S$, whereas for n-amylbenzene, n-hexylbenzene and n-heptylbenzene it means either $D = S$ or $D > S$. But already n-butylbenzene shows an action (at the first additions G is not increased) which forbids the conclusion $D < S$. Therefore here too we must conclude $D = S$ or $D > S$.

From the distinct spreading of the curves for n-butylbenzene, n-amylbenzene, n-hexylbenzene and n-heptylbenzene we must on the other hand conclude that they can't all four correspond to $D = S$. Therefore we come to the conclusion that it can only be n-butylbenzene which approximately corresponds to $D = S$, whereas for the higher homologues we have $D > S$.

Our expectation (see Introduction) that for the first additions of an alkylbenzene D will preponderate at last over S when the carbon side chain is sufficiently lenghtened, is therefore confirmed by the experiment.

If we take into consideration that in fig. 1 the G curve of n-butylbenzene goes into a downward direction at further additions, we must conclude (in accordance with point 1 of the Introduction) that the approximate compensation of the two component actions ($D = S$) gives place to a preponderance of the KCl-demanding component action (thus $D = S \rightarrow D > S$).

This relative enforcement of the D component is also present with ethylbenzene and n-propylbenzene, and is indicated by the downward direction of the G curves after the passing of a maximum (in part XI the course of the G curves for these two terms could be followed much further than here; distinct maxima were obtained for both terms, as well as for toluene). We therefore find an illustration of point 1 of the Introduction, and we must conclude that at 0.6 N KCl and at a sufficient hydrocarbon concentration all terms of the n-alkylbenzenes will show a total KCl-demanding action (thus $D > S$).

The next survey summarizes the above discussions:

KCL conc.	hydrocarbon conc.	toluene	ethyl-benzene	n-propyl-benzene	n-butyl-benzene	n-amyl-benzene	n-hexyl-benzene	n-heptyl-benzene
	small	$D < S$	$D < S$	$D < S$	**D = S**	$D > S$	$D > S$	$D > S$
0.6 N	│	$D < S$	$D < S$	**D = S**	$D > S$	$D > S$	$D > S$	$D > S$
	│	$D < S$	**D = S**	$D > S$	$D > S$	$D > S$	$D > S$	$D > S$
	↓	**D = S**	$D > S$	$D > S$	$D > S$	$D > S$	$D > S$	$D > S$
	increasingly larger	$D > S$	$D > S$	$D > S$	$D > S$	$D > S$	$D > S$	$D > S$

We now turn to the results obtained at 1.08 N KCl (see fig. 2). At the first addition of all the investigated alkylbenzenes the G curves go into

an upward direction and the n curves into a downward direction. As 1.08 N KCl represents the KCl concentration of minimum damping of the blank, this means that for all terms of the homologous series is valid: $D < S$.

Apart from ethylbenzene (and possibly n-propylbenzene) the G curves attain a maximum after which they go into a downward direction. This means (compare the diagrams of fig. 2 in part XI) that with increasing hydrocarbon concentration a transition in the type of action ($A \rightarrow B \rightarrow C$) or in the new manner of expression, a transition from $D < S$ to $D > S$ occurs. In connection with the relative position of the G curves in fig. 2 we conclude that the transition ($D = S$) takes place at lower hydrocarbon concentrations when the length of the aliphatic carbon chain is increased. A discussion of the relative position of the n curves in fig. 2 leads to the same conclusion. Whereas at low hydrocarbon concentrations n always decreases, we find at higher concentrations a tendency of the n curves to take a horizontal or even an upward course (the latter for instance with n-amylbenzene).

This change of direction is accompanied by a reversal in the succession of the n curves.

At small hydrocarbon concentrations the curve for ethylbenzene lies at the top, followed by n-propylbenzene and further downwards by the curves of the following hydrocarbons (crowded together). But after the change in direction the curve for n-hexylbenzene is the top one and the curve for ethylbenzene (not showing a change in direction) is the undermost. It is obvious that the concentrations of the hydrocarbons at which the change in direction takes place will be lower when the carbon side chain is longer. These regularities can be explained by a transition of $D < S$ to $D > S$ (a succession of the diagrams 2, 3, 6 and 9 in fig. 2 of part XI). We must expect here that the n curve after attaining a minimum goes upwards to a maximum and thereafter definitively goes into a downward direction. This whole succession is realised with heptylbenzene, though here the maximum is only indicated by the S shape of the curve. For the other terms there are not enough experimentally determined points to show the maximum and the following downward course of the n curves.

The next survey (in which we also incorporate toluene, which is investigated in part XI) summarizes the above discussions:

KCl conc.	hydrocarbon conc.	toluene	ethyl-benzene	n-propyl-benzene	n-butyl-benzene	n-amyl-benzene	n-hexyl-benzene	n-heptyl-benzene
	small	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$
1.08 N	↓	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$\mathbf{D = S}$
	↓	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$\mathbf{D = S}$	$D > S$
	↓	$D < S$	$D < S$	$D < S$	$D < S$	$\mathbf{D = S}$	$D > S$	$D > S$
	increasingly larger	$D < S$	$D < S$	$D < S$	$\mathbf{D = S}$	$D > S$	$D > S$	$D > S$

If we compare this survey with the preceding one, valid for 0.6 N KCl, we perceive the same general character. Only the oblique bar formed by the symbols $D = S$ is displaced considerably to the right, which is in accordance with point 3 mentioned in the Introduction.

The results obtained at 1.5 N KCl (compare fig. 3) show a great simil-

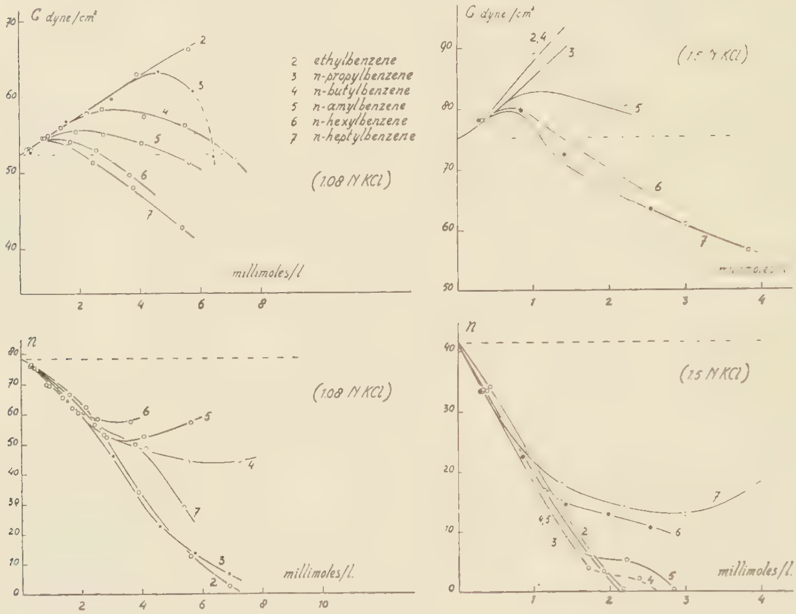

Fig. 2—3

arity [5]) with those obtained at 1.08 N (see fig. 2) and therefore we will not repeat the analogous reasonings (in this case we have to consider the diagrams 3, 6 and 9 in fig. 2 of part XI).

We conclude that at 1.5 N KCl all alkylbenzenes which are investigated exert a KCl-sparing total action at the first additions (thus $D < S$), and that at increase of the hydrocarbon concentration a transition into a KCl-demanding total action (thus $D > S$) occurs for n-amylbenzene, n-hexylbenzene and n-heptylbenzene. From the deflecting to the right of the n curve for n-butylbenzene one may not conclude to this transition because of reasons still to be mentioned in section 5.

We have the impression that at 1.5 N KCl the transition from $D < S$ into $D > S$ is somewhat more difficult than at 1.08 N KCl (deflection of the n curves to the right takes place at proportionally smaller values of n than at 1.08 N KCl). The results at 1.5 N KCl may therefore be represented schematically by the following survey:

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[5]) A difference is that the G curve for n-butylbenzene is situated at the left of the G curve for n-propylbenzene, but this may be the result of an experimental error. The sequence of the corresponding n curves (before butylbenzene deflects to the right) is quite normal.

KCl conc.	hydrocarbon conc.	toluene	ethyl-benzene	*n*-propyl-benzene	*n*-butyl-benzene	*n*-amyl-benzene	*n*-hexyl-benzene	*n*-heptyl-benzene
	small	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$
	↓	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$
1.5 N	↓	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	**D = S**
	↓	$D < S$	$D < S$	$D < S$	$D < S$	$D < S$	**D = S**	$D > S$
	increasingly larger	$D < S$	$D < S$	$D < S$	$D < S$	**D = S**	$D > S$	$D > S$

Comparing the three above surveys (at 0.6 N, 1.08 N and 1.5 N KCl), we may state that the results laid down in this section strongly support the hypothesis that the total action is the resultant of two component actions (D and S), and that they confirm the three factors mentioned in the Introduction which relatively enforce D with regard to S.

3. *The displacement of the maximum of the n-C_{KCl} curve by added alkylbenzenes.*

The n graphs of the figs 1, 2 and 3 allow to read for the blank or for a given concentration of added hydrocarbons the values of n at 0.6 N, 1.08 N and 1.5 N KCl. Therefore three points of the corresponding n-C_{KCl} curve are known. Though this is not enough to draw this n-C_{KCl} curve with certainty, it nevertheless suffices to sketch its position approximately. This has been done in fig. 4 for a number of hydrocarbon concentrations (the numbers added to the individual curves give the hydrocarbon concentration in millimoles/l) [6]. Through the maxima of the n-C_{KCl} curves dotted curves have been drawn which therefore represent the displacement of the maxima by the added hydrocarbons. The latter curves have been united in fig. 5A, from which their relative position can be discerned.

In this figure the curve representing the displacement of the maximum by benzene (calculated from the data represented in fig. 2 of part X), is also drawn [7].

[6]) The blank curves in fig. 4 have been arbitrary sketched in the following way (using the mean values of n at 0.6, 1.08 and 1.5 N KCl): a straight line was drawn through $n = 20.3$ at 0.6 N KCl and $n = 0$ at 0.47 N KCl; similarly one through $n = 41.5$ at 1.5 N KCl and $n = 0$ at 1.77 N KCl. The two straight lines were united at the top by a curve through $n = 78.4$ at 1.08 N KCl. The remaining curves were constructed in a similar way by drawing straight lines through the values of n (read from the graphs of the fig. 1, 2 and 3), parallel to the corresponding ones of the blank, and uniting them at the top by a curve.

[7]) From fig. 2 in part X we read for the positions of the maxima of the n curves expressed in N KCl the values 1.10; 1.04; 0.82; 0.70 and 0.55, whereas the values of n at these maxima amounted to 61, 58, 54, 47,5 and 34. The maximum of the blank n curve and the value of n at this maximum were 1.08 N KCl and 78.4 respectively, in the present experiments. For the plotting of the curve for benzene in fig. 5 A, we have therefore multiplicated the above mentioned values of the KCl concentrations by 1.08/1.10 and those of the n values by 78.4/61.

As the concentrations of the added hydrocarbons are no independent variables in this figure, the relative positions of the curves represent specific differences in action between the individual hydrocarbons. In

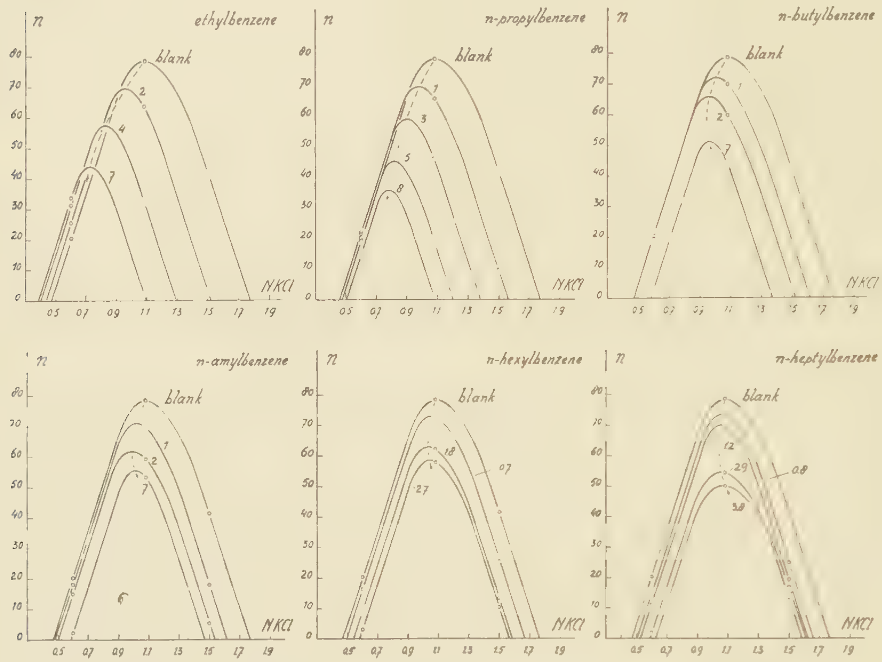

Fig. 4

the sequence from left to right we easily recognize point 2 of the Introduction: increase of the length of the carbon side chain relatively enforces the KCl-demanding component action (D) with regard to the KCl-sparing component action (S).

Point 1 of the Introduction finds its expression in the bent character of each of the curves (D is relatively enforced with regard to S by increase of the hydrocarbon concentration), though point 3 of the Introduction also steps in here. This factor (lowering of the KCl concentration relatively enforces D with regard to S), will hardly play a part in the shape of the curves for n-amylbenzene, n-hexylbenzene and n-heptylbenzene, for these substances do not bring about considerable shifts either to smaller or to larger KCl concentrations.

When the carbon side chain is shortened, this factor becomes more and more of importance. This may explain why the curvature of the curves is greatest for benzene (no side chain) and diminishes with increasing length of the side chain.

In fig. $5B$ we have plotted the positions of the maxima of the n curves (expressed in N KCl) as functions of the concentration of the alkylbenzenes. The sequence of the terms of the homologous series is quite the same as in fig. $5A$. This is to be expected because in the KCl-water

medium the considered alkylbenzenes are practically insoluble. For in this case the specific differences in action of the successive terms must also express themselves in the succession of the curves (compare part

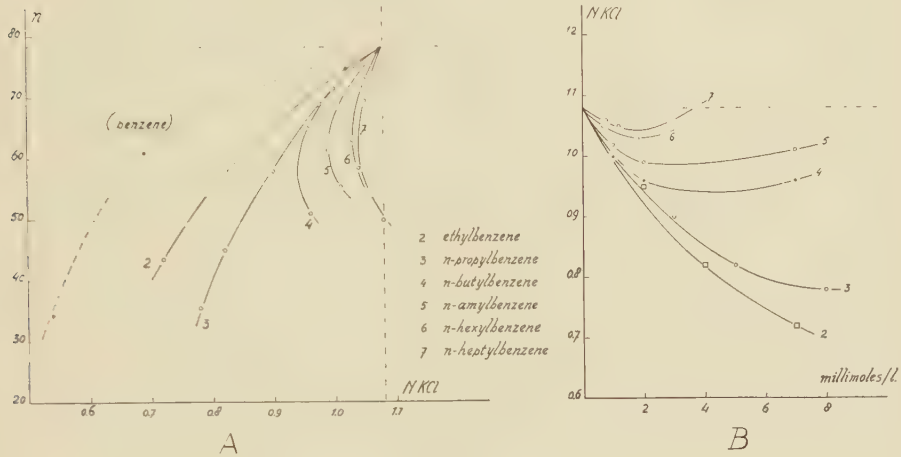

Fig. 5

VIII, section 3). As benzene, however, is already slightly soluble, this may no longer hold for this substance. Indeed if we had designed, in fig. 5*B*, the curve for benzene, it would not lie entirely to the left of the other curves, but would cross them.

4. *The results obtained with the homologous series of the n-alkylbenzenes and the n-alkanes represented in G — n diagrams.*

In the parts VIII and IX of this series we have introduced G—$1/\Lambda$, G—λ and G—n diagrams for the discussion of the results obtained (at constant KCl concentration) with the homologous series of the n-primary alcohols and of the fatty acid anions. Supposing that the premisses in which this method is based are realised (or approximately realised) one can draw conclusions as to specific differences in action exerted by the molecules bound to the oleate micelles of the successive terms of the homologous series from the sequence of the curves in such diagrams. This method only has its right of existence when the substances are markedly soluble in the KCl-water medium in which the oleate micelles are suspended. For in this case (e.g. the n-primary alcohols C_1 — C_6; the fatty acid anions C_7 — C_{11}) one cannot draw conclusions as to the above-mentioned specific differences from the sequence of the curves representing G or n as functions of the concentrations of the added substances [8]).

In the case of the hydrocarbons we may generally neglect their solu-

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[8]) This sequence is mainly the expression here of the distribution equilibrium between the molecules being present free in the medium and the molecules taken up by the oleate micelles, which changes considerable from term to term. Compare parts VIII and IX.

bility in the KCl-water medium [9]) and therefore there is no real need to discuss the results with the aid of $G - n$ diagrams. The latter can in principle give no other information here than the graphs representing G or n as functions of the hydrocarbon concentrations already give. Nevertheless we have represented in this way in fig. 6 the results obtained with the alkylbenzenes at 1.08 N KCl (minimum damping of the blank). Starting from the point representing the blank ($G = 52.4$; $n = 78.4$), all curves first proceed upwards to the left, that is to say they exert a KCl-sparing total action ("Type A", compare diagram 2 given in part IX fig. 1). Apart from ethylbenzene, they are bending downwards, that is to say, at increase of the carbon side chain $D \langle S$ makes place for $D \rangle S$, and this happens so much the sooner when carbon chain is longer.

The shape and the relative position of the curves in the $G - n$ diagram for the n-alkylbenzenes is therefore in the general as we would expect. We will return to this at the end of this section.

In sharp contrast to this stands the $G - n$ diagram of fig. 7, in which

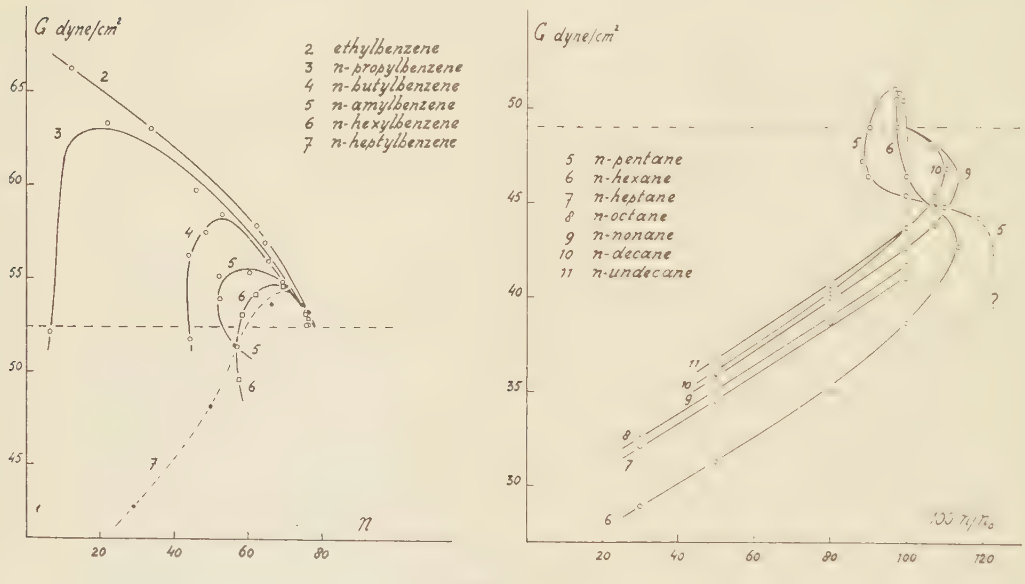

Fig. 6—7

we have plotted the results obtained in part XI (data of tables I and II) with the n-alkanes [10]). We have no objection to the shape of the curves in fig. 7 for each of the alkanes apart. As these experiments have been performed at 1.2 N KCl (KCl $\rangle$ KCl min. damping) we have to consider

[9]) Except the slight solubility of benzene; compare H. L. Booij, these Proceedings 53, 882 (1950).

[10]) The points on the curves of fig. 7 are not the experimentally determined points themselves, but they represent G values corresponding to a number of arbitrary chosen values of 100 n/n_0 (using the G and n graphs represented in fig. 1 of part XI).

the diagrams 3, 6 and 9 in fig. 2 of part XI. The first additions of pentane and hexane show $D < S$, which at higher concentrations changes into $D > S$. The S shape of the $G - n$ curves for pentane and hexane in fig. 7 is in accordance with this (the shape of the curves can be considered as a succession of the initial courses of the $G - n$ curves which are represented in the diagrams 3, 6 and 9 of the above-mentioned figure in part XI).

The shape of the $G - n$ curves for the higher alkanes in fig. 7 (in order not to overcrowd the figure, we have only drawn the complete course for nonane and decane) is in accordance with diagram 9 of the above-mentioned fig. 2 in part XI, because the higher alkanes at once show $D > S$.

Therefore when the shape of the $G - n$ curves for each of the terms of the n-alkanes is considered apart, they are quite as one would expect. But, what is most remarkable, the succession of the curves in the bundle proceeding downwards to the left is in sharp contradiction with our expectations. In this bundle the curve for undecane is the top one and that for hexane (or probably pentane) is the undermost. One would therefore conclude from this succession that the preponderance of D over S is greatest with hexane (pentane) and least with undecane (thus undecane would stand nearest to a substance showing $D = S$). But this conclusion is quite erroneous. The total KCl-demanding action is just greatest with undecane and decreases with the shortening of the carbon chain (see part XI).

The method of the $G - n$ diagrams, which was useful in the case of the n-primary alcohols (part VIII) and of the fatty acid anions (part IX) and does not lead to conclusions which are in principle erroneous in the case of the n-alkylbenzenes, utterly fails in the case of the n-alkanes.

To shed some light on the question why this method fails here, we have investigated the influence of increasing concentrations of n-undecane on the curves which represent G and n as functions of the KCl concentration (using just as in part X the simplified method of half filled 110 ml vessels). The results have been represented in fig. 8, which confirms that n-undecane exerts a powerful KCl-demanding action (the G and n curves are shifted into the direction of higher KCl concentrations). The compound nature of the total action is still visible in the displacement of the right footpoint of the n curve. The first addition (1.13 millimoles/1) shifts this footpoint to the left (thus $D < S$) and at higher concentrations (2.26 millimoles/l and higher) brings about shifts to the right (thus $D < S \rightarrow D = S \rightarrow D > S$).

The maximum and the left footpoint of the n curve, however, is already shifted to the right at the first addition (thus $D > S$), in accordance with point 3 mentioned in the Introduction. We further perceive that the course which is followed by the maximum of the n curve, when this curve is shifted to the right at increasing concentrations of n-undecane, is a curve with a minimum.

Thus the ratio of the vertical and horizontal components of the displacement of the maximum of the n curve does not remain approximately constant at all but even changes its sign at increase of the n-undecane

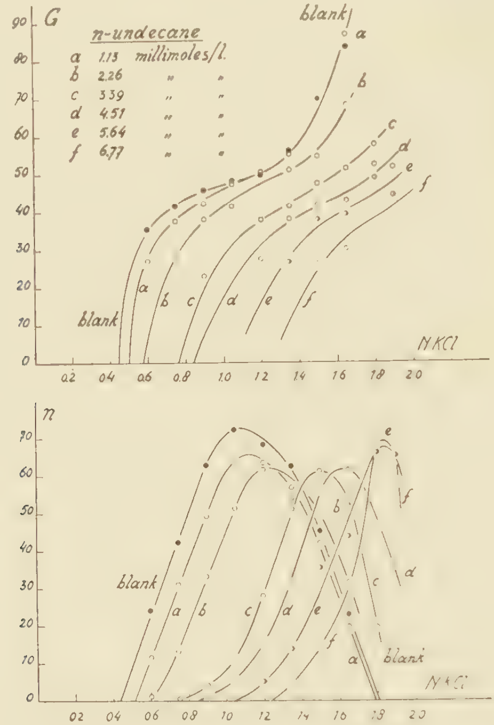

Fig. 8

concentration. Compare the next survey which gives the KCl concentrations corresponding to the maxima of the n curves, the values of n and of G at these maxima.

Conc. n-undecane millimoles/l	blank	1.13	2.26	3.39	4.51	5.64	6.77
KCl conc. moles/l corresp. to max. of the n curves	1.08	1.14	1.23	1.50	1.62	1.83?	1.83?
n at the max. of the n curves .	72	65.5	61.5	61	61.5	67	68,5
G at the max. of the n curves .	48.5	48.5	48	47	44	43	39?

We further observe that the values of G, corresponding to the maxima of the n curves do not remain constant, but decrease at increase of the n-undecane concentration.

However, the method of using $G - n$ diagrams for the discussion of the results with the successive terms of a homologous series, resides on the assumptions that the above-mentioned ratio remains constant and that the value of G on the shifted G curves which correspond to the maxima of the shifted n curves retains its original value.

It therefore seems likely that the failure of the method of $G-n$ diagrams when applied to the homologous series of the n-alkanes [11]) resides in the large deviations from the above assumptions.

We now return to the $G-n$ diagram for the alkylbenzenes which is given in fig. 6. One may wonder why in this diagram the succession of the curves is as one would expect. For, the above-mentioned ratio is not constant in the case of the n-alkylbenzenes too (compare the bent character of the curves in fig. 5A.

To this we must remark that the erroneous succession of the $G-n$ curves of the n-alkanes only appears there in fig. 7 where these curves definitely proceed downwards to the left. The relative positions of those parts of the curves (here only visible for two terms, namely n-pentane and n-hexane) which lie above the dotted horizontal line are quite normal.

Now the experimentally accessible points represented in the $G-n$ diagrams of the n-alkylbenzenes lie in general also above the dotted horizontal line in fig. 6.

This at least allows for a partial reconciliation of the at first sight quite opposite cases of the n-alkanes ($G-n$ diagram leads to erroneous conclusions) and of the n-alkylbenzenes ($G-n$ diagram allows right conclusions). So far the $G-n$ curves proceed upwards to the left, i.e. so long as $D < S$, including the eventual bends downwards (transition via $D=S$ to $D > S$), no erroneous conclusions are obtained in either case. When the curves, however, have definitively taken their course in a direction downwards to the left, i.e. when the transition to $D > S$ has been completely accomplished, the succession of the curves leads to erroneous conclusions in the case of the n-alkanes. As the further course of the $G-n$ curves, but for n-heptylbenzene, is not known in the case of the n-alkylbenzenes we cannot make the same statement here. But if we consider, in fig. 6, the position of the curve for n-heptylbenzene relative to the loops of the curves for n-hexylbenzene and n-amylbenzene, it seems likely that if the further course of the $G-n$ curves would be accessible, here too we would obtain quite the same situation as in the $G-n$ diagrams of the n-alkanes, i.e. a succession of the curves also leading to erroneous conclusions.

5. *Metastable elastic systems at the coacervation limit.*

At just transgressing the coacervation limit, two phase systems (very large coacervate layer + small layer of equilibrium liquid) are obtained,

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[11]) In part X indications may be found that also for other alkanes similar deviations from the simple assumptions on which the use of the $G-n$ diagrams is based do occur. Compare fig. 4 in part X, in which n-hexane causes the maximum of the n curve to shift in a similar way (curve with a minimum!) as is the case with n-undecane in the present fig. 8. See also the survey in section 5 of part X where there are indications that the values of G corresponding to the maxima of the n curves also decrease in the case of n-hexane and n-heptane.

which are transformed to clear elastic systems by agitation or shaking. The latter, however, are metastable and when they are at rest sooner or later turbidity develops, i.e. coacervate drops are formed which gradually coalesce to form the coacervate layer anew.

Such metastable elastic systems were formed in a very pronounced way with the system lying on the dotted part of the n curve for n-butylbenzene in fig. 3 by wheeling round the vessels before applying the electrolytic H_2-mark (see part X, section 3). The metastable character of these systems forbids to draw conclusions from the shape of this n curve analogous to the conclusion drawn from the similar shaped curves for the higher homologues (where the elastic system are stable).

Summary.

1. The influence of added n-alkylbenzenes (from ethylbenzene up to and including n-heptylbenzene) has been investigated on the elastic behaviour of the oleate system at three KCl concentrations.

2. The results support the hypothesis that the total action of hydrocarbon is the resultant of a KCl-demanding (D) and a KCl-sparing (S) component action. They also confirm that D is relatively enforced with regard to S by increase of the hydrocarbon concentration, by increase of the length of the carbon (side) chain and by lowering of the KCl concentration.

3. The expectation that at sufficient length of the carbon side chain D might at once preponderate over S has been confirmed at 0.6 N KCl. At small concentrations ethylbenzene and n-propylbenzene show $D < S$; n-amylbenzene, n-hexylbenzene and n-heptylbenzene show $D > S$, whereas n-butylbenzene here just stands at the transition $D = S$.

4. The method to discuss the experimental results obtained with a homologous series with the aid of $G - n$ diagrams and which proved useful in the case of the n-primary alcohols (part VIII) and the fatty acid anions (part IX) does not lead to erroneous conclusions in the case of the n-alkylbenzenes for the experimentally accessible G and n values. This method, however, utterly fails in the case of the n-alkanes.

5. An investigation has been made about the influence of increasing concentrations of n-undecane on the curves representing G and n as functions of the KCl concentration. The results show that the simple assumptions on which the method of the $G - n$ diagrams is based are not fulfilled here. The failure of this method in the procuring of correct conclusions in the case of the n-alkanes is ascribed to this.

7. At just transgressing the coacervation limit, two phase systems (very large coacervate layer + small layer of equilibrium liquid) are obtained, which by agitation (or shaking) are transformed into clear elastic systems. The latter, however, are metastable and when they are at rest sooner or later coacervation sets in anew.

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CHEMISTRY

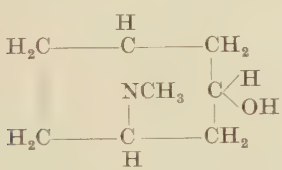
SYNTHESES OF N-ETHYL-NOR-TROPINONE, N-ETHYL-NOR-Ψ-TROPINE AND N-ETHYL-NOR-Ψ-TROPINE BENZOATE *)

BY

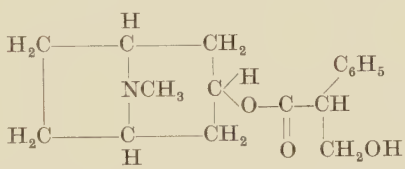
J. P. WIBAUT, A. L. VAN HULSENBEEK AND C. M. SIEGMANN

(Communicated at the meeting of June 24, 1950)

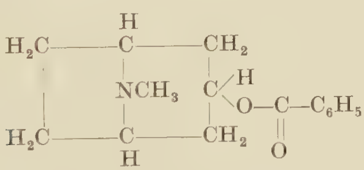
§ 1. Several alkaloids of the tropane group are derivatives of a cyclic amino alcohol (I). This amino alcohol occurs in the two stereoisomeric forms tropine and pseudo-tropine, which are *cis-trans*-isomers and not optical opposites (1). The molecule of tropine is symmetrical; the same holds for the molecule of Ψ-tropine. Hyoscyamine (II) is the ester of tropine with l-tropic acid; atropine (II), the racemic form of hyoscyamine, is the ester of tropine with d, l-tropic acid. Tropacocaine (III), which occurs in coca leaves from Java and from Peru, is the benzoyl ester of Ψ-tropine.



I. tropine and Ψ-tropine

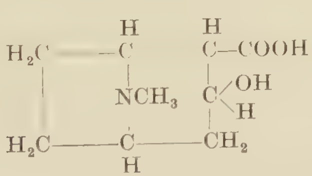


II. hyoscyamine and atropine



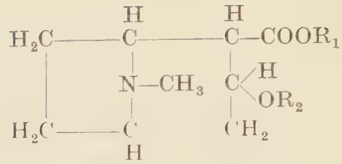
III. tropa-cocaine

Cocaine is a derivative of ecgonine (IV), which is tropine carboxylic acid



IV. ecgonine

*) Complete experimental data will be published in the Recueil des travaux chimiques.



cocaine $R_1 = CH_3$; $R_2 = C_6H_5CO$ (V)

cinnamoylcocaine $R_1 = CH_3$; $R_2 = C_6H_5CH = CHCO$ (VI)

Cocaine (V) is the benzoyl derivative of ecgonine methyl ester; cinnamyl cocaine (VI) is the cinnamoyl derivative of ecgonine methylester.

Many attempts have been made to correlate pharmacological action with chemical constitution by introducing various changes in the cocaine molecule (2). Ecgonine, ecgonine methyl ester or benzoyl ecgonine have no anaesthetic properties.

Therefore it seems to be necessary to block up both free hydroxyl and free carboxyl in ecgonine in order to produce anaesthetic properties. The nature of the alkyl group R_1 plays a minor part only. Benzoyl ecgonine ethyl ester ($R_1 = C_2H_5$) and the analogous propyl and butyl esters are powerful anaesthetics; the benzyl and phenyl-ethyl esters of benzoyl ecgonine are also anaesthetics.

The effect of the acid radical R_2, which esterifies the alcoholic hydroxyl group in the ecgonine molecule, is more important; in order to produce anaesthetic properties the group must be benzoyl or a substituted aromatic residue.

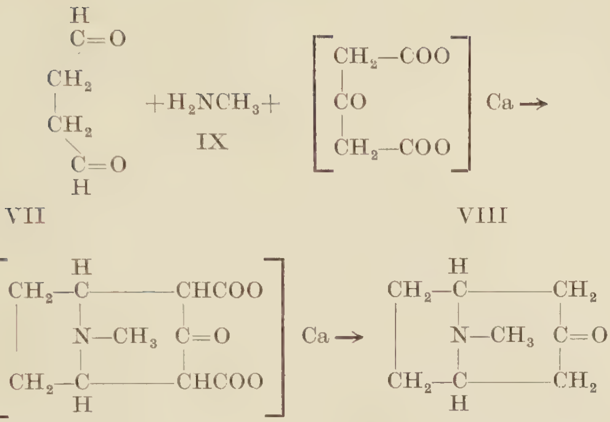
Atropine is a powerful mydriatic. Many other tropeines, i.e. esters of tropine, have been prepared. The esters of tropine with aliphatic acids have no mydriatic effect; the mandelic ester of tropine is a useful mydriatic. Benzoyl-Ψ-tropine (tropacocaine) (III) is a powerful local anaesthetics, but shows no mydriatic activity.

Another possibility of making alterations in the cocaine or in the atropine molecule would be the replacement of the group $>$ N-CH_3 by the group $>$ N-C_2H_5 or $>$ N-R (R = alkyl).

Several investigators have observed that there seems to be a marked difference between the ethyl and the methyl groups as regards pharmacological action. We have therefore started preparing N-homologues of tropine derivatives, wherein an ethyl group or a larger alkyl group is attached to the nitrogen atom.

§ 2. In this paper we describe the preparation of N-ethyl-nor-tropinone (XIV), of N-ethyl-nor-Ψ-tropine (XV) and of its benzoate (XVI).

A simple and elegant synthesis of tropinone (XI) has been discovered by R. Robinson (3). It depends on the condensation of succinic dialdehyde (VII) with calcium acetone dicarboxylate (VIII) and methyl amine (IX)



VII VIII X XI tropinone

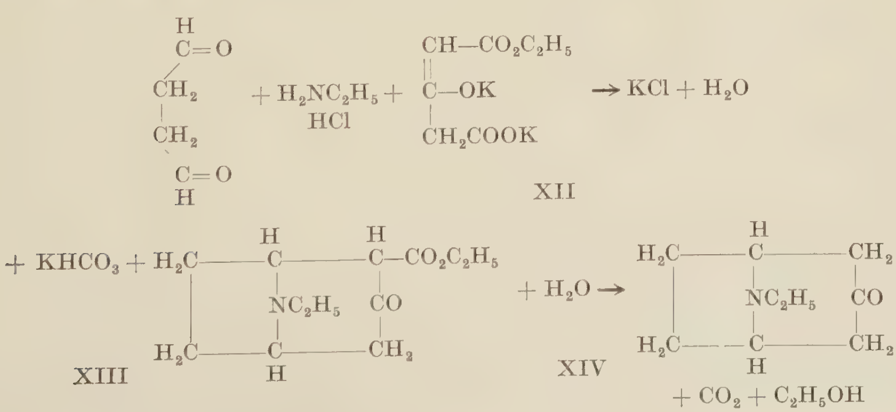
The initial product of this synthesis, calcium tropinone-dicarboxylate (X), has not been isolated. The reaction product was heated with dilute acid, as a result of which the tropinone dicarboxylic acid was decomposed and tropinone was formed.

Schöpf and Lehmann (4) have shown that the conditions under which Robinson's synthesis is carried out affect the yield considerably: in an appropriately buffered solution at P_H 3—11 and at a temperature of 20°—25°, good yields of tropinone can be obtained.

Acetone dicarboxylic acid is an unstable substance, which cannot be kept, but must be used as soon as possible. Therefore the dimethyl or diethyl esters of this acid are more convenient intermediates for synthetic work.

A synthesis of methyl tropinone carboxylate has been carried out by Willstätter, Wolfes and Mäder (5). This synthesis is based on Robinson's synthesis of tropinone and depends on the condensation of succinic dialdehyde, methyl amine and the dipotassium salt of acetone dicarboxylic acid semi ester (XII).

We have extended this synthesis by substituting ethyl amine or other primary amines for methyl amine.

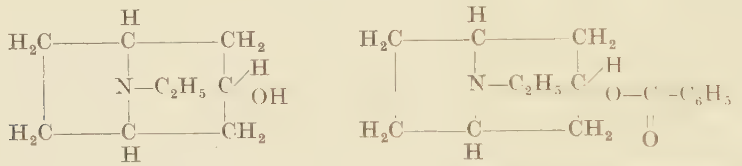


XII XIII XIV

We have not isolated the initial reaction product, i.e. the ester of N-ethyl-nor-tropinone carboxylic acid (XIII): this compound seemed to be decomposed when we attempted to isolate it by distillation at reduced pressure.

Therefore the reaction mixture was acidified by dilute sulphuric acid and heated in order to decompose the carboxylic ester. In this manner N-ethyl-nor-tropinone (XIV) was formed, which was identified in the form of the picrate, the picrolonate and the dipiperonylidene derivative (6). By treating the tropinone homologue (XIV) with sodium amalgam in a weak acid solution we obtained N-ethyl-nor-tropine (XV) in the form of white crystal leaflets (melting point 85°.5).

Willstätter (1) has shown that the reduction of tropinone (XI) by this method leads to Ψ-tropine, which is a stereo-isomeride of the tropine obtained by hydrolysis of cocaine or of hyoscyamine. We therefore assume that the N-ethyl-nor-tropine synthesized by us is N-ethyl-nor-Ψ-tropine. By benzoylation of this compound we obtained the benzoate XVI. As the natural alkaloid tropacocaine is the benzoate of Ψ-tropine, the compound XVI may be designated as *N-ethyl-nor-tropacocaine*. The hydrochloride of this compound has been obtained in the form of a white micro-crystalline powder, melting point 256°.5. This salt has a slightly bitter taste. Its pharmacological action will be investigated.



XV. N-ethyl-nor-tropine XVI. N-ethyl-nor-tropacocaine

Independently of and simultaneously with our work Keagle and Hartung (7) have carried out an investigation on the synthesis of N-homologues of tropinone. These authors have applied Schöpf's modification of Robinson's synthesis of tropinone. They have carried out the condensation of succinic dialdehyde (prepared *via* the dioxime), acetone dicarboxylic acid and ethylamine. They obtained N-ethyl-nor-tropinone, which has been isolated as the dipiperonylidene derivative. They give for the melting point 184—185°, which is in accordance with our observations *); a complete analysis of this compound has not been published by the American authors, only the nitrogen content is given. These authors prepared also N-benzyl-nor-tropanone and N-hydroxymethyl-

*) Thesis of A. L. Hulsenbeek, Amsterdam, appeared in print on June the 26th 1946 (J. B. Wolters' Publishing Co. Groningen).

In this thesis the synthesis of N-ethyl-nor-tropinone is described; complete analytical data are given for the picrate, the picrolonate and the dipiperonylidene derivate of this compound. A short preliminary report of these experiments has been published in Chem. Weekblad, 91 (1945).

nor-tropanone; these compounds have been isolated as their dipiperonylidene derivatives (analysis: nitrogen content).

§ 3. *Description of the syntheses.*

a. Succinic dialdehyde.

Of the necessary intermediates succinic dialdehyde is the most difficult to obtain. HARRIES (8) prepared this dialdehyde by treating dioxime with nitrous acid. The dioxime of succinic dialdehyde is prepared by the method of CIAMICIAN and DENNSTEDT (9), i.e. the reaction of hydroxyl amine hydrochloride with pyrrole in the presence of alkali.

Although this method has been improved by WILLSTÄTTER and HEUBNER (10) and has been used by several investigators, it is not an attractive method for preparative work. We have studied several other methods for the preparation of succinic dialdehyde and prefer a synthesis represented by the following formulae (6).

$$HC \equiv CH + 2\,C_2H_5MgBr \rightarrow BrMgC \equiv CMgBr + 2\,C_2H_6$$

$$BrMgC \equiv CMgBr + 2\,HC(OC_2H_5)_3 \rightarrow$$

$$2\,C_2H_5OMgBr + (C_2H_5O)_2\underset{H}{C}\!-\!C \equiv C\!-\!\underset{H}{C}(OC_2H_5)_2 \qquad \text{XVII}$$

$$(C_2H_5O)_2\underset{H}{C}\!-\!C \equiv C\!-\!\underset{H}{C}(OC_2H_5)_2 + 2\,H_2 \xrightarrow[(Ni)]{}$$

$$(C_2H_5O)_2\underset{H}{C}\!-\!CH_2\!-\!CH_2\!-\!\underset{H}{C}(OC_2H_5)_2 \qquad \text{XVIII}$$

By passing acetylene into an ethereal solution of magnesium ethyl bromide a solution of acetylene dimagnesium bromide is obtained. WOHL and MYLO (11) treated this solution with an ethereal solution of ethyl orthoformate $HC(OC_2H_5)_3$ and obtained the tetra ethyl acetal of butynedial (XVII). WEYGAND and HENKEL (12) have improved the experimental conditions under which this synthesis is carried out. Following their prescription we obtained the compound in a yield of 60—70 %, calculated on magnesium.

This butynedial tetra ethyl acetal (XVII) is an unstable compound, which cannot be kept and must be worked up as soon as possible. The catalytic hydrogenation of XVII is carried out at 20° and an initial hydrogen pressure of 100 atm., using Raney nickel as catalyst. Succinic dialdehyde tetra ethyl acetal (XVIII) (boiling point 114—116°/15 mm) is obtained in a yield of 90 %.

The hydrolytic scission of the tetra ethyl acetal is carried out by shaking it with 0.1 N hydrochloric acid (yield 85 % calculated on acetal). In this manner a solution of succinic dialdehyde is obtained, which is used in the following experiments.

b. N-ethyl-nor-tropinone (XIV).

To a buffer solution consisting of 30.5 g of citric acid ($C_6H_8O_7 \cdot H_2O$)

and 55 g of disodium phosphate ($Na_2HPO_4 \cdot 2H_2O$) in 1.5 l of water is added a solution of succinic dialdehyde (prepared by shaking 32 g of succinic dialdehyde tetra acetal with 160 ml of 0.1 N hydrochloric acid), 70 g of the dipotassium salt of the monoethyl ester of acetone dicarboxylic acid and 21 g of ethyl amine hydrochloride; the volume of the solution is made up to 3 l. This solution is stored at room temperature in the dark for 4½ days.

The solution is saturated with potassium carbonate and thoroughly extracted with ether. By distilling off the ether a brown-coloured oily liquid is obtained, which is refluxed with 150 ml of dilute sulphuric acid. The acid solution is distilled with steam in order to remove the non-basic constituents; to the distillation residue is added an excess of sodium hydroxide and the liquid is extracted with ether. After working up the ethereal solution an oily liquid is obtained, which consists mainly of *N-ethyl-nor-tropinone* (fraction *a*).

The N-ethyl-nor-tropinone has been identified in the form of several derivatives: *the picrate* melts at 189°. (Found: 47.09 % C, 4.68 % H, 14.75 % N; $C_9H_{15}ON \cdot C_6H_3O_7N_3$ requires: 47.12 % C, 4.71 % H, 14.66 %N.) The picrolonate melts at 209°. (Found: 54.53 % C, 5.75 % H, 16.95 % N; $C_9H_{15}ON \cdot C_{10}H_8O_5N_4$ requires: 54.67 % C, 5.56 % H, 16.78 % N.) The dipiperonylidene derivative melts at 183.5°. (Found: 71.97 % C, 5.46 % H, 3.65 % N; $C_{25}H_{23}O_5N$ requires 71.94 % C, 5.56 % H, 3.35 % N.)

c. N-ethyl-nor-*Ψ*-tropine (XV).

The required quantity of sodium amalgam (prepared from 5 g of sodium and 200 g of mercury) is added gradually to a solution of fraction *a.* in dilute hydrochloric acid, in such a manner that a slight excess of acid is present during the reaction.

The liquid is then basified and extracted with ether. By working up the ethereal solution N-ethyl-nor-*Ψ*-tropine is obtained in the form of white leaflets, which are recrystallized from ether: melting point 85°.5. (Found: 69.40 % C, 11.03 % H, 8.66 % N; $C_9H_{17}ON$ requires: 69.63 % C, 11.04 % H, 9.02 % N.) The yield is about 5—6 % based on the initial quantity of succinic dialdehyde. The hydrochloride of N-ethyl-nor-*Ψ*-tropine melts at 219°.

d. Benzoate of N-ethyl-nor-*Ψ*-tropine (XVI).

A mixture of 0.640 g of benzoyl chloride and 0.490 g of XV is heated at 150° for 1½ hours. The mixture is basified with a solution of sodium carbonate and extracted with ether.

The benzoate of N-ethyl-nor-*Ψ*-tropine has been identified in the form of the picrate or in the form of the hydrochloride. The picrate melts at 162°. (Found: 54.19 % C, 4.96 % H, 11.35 % N; $C_{16}H_{21}O_2N \cdot C_6H_3O_7N_3$ requires: 54.09 % C, 4.95 % H, 11.47 % N.)

The hydrochloride of N-ethyl-nor-tropa-cocaine melts at 256°.5. (Found:

64.78 % C, 7.57 % H, 5.08 % N, 12.02 % Cl; $C_{16}H_{21}O_2N \cdot HCl$ requires: 64.96 % C, 7.50 % H, 4.74 % N, 11.99 % Cl.)

Laboratory for Organic Chemistry of the University of Amsterdam.

Amsterdam, June 1950.

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SOME RHIZOCEPHALAN PARASITES OF XANTHID CRABS

BY

H. BOSCHMA

(Communicated at the meeting of May 20, 1950)

The present paper deals with variation in two species of the genus *Loxothylacus*, each occurring as a parasite on two different crabs of the family Xanthidae. As in other species of the genus the variation manifests itself chiefly in the male organs: in the two species the distinctive characters in the first place are those of the excrescences of the external cuticle of the mantle.

Loxothylacus murex nov. spec.

Benkulen, Sumatra, 1 specimen on *Xanthias lamarcki* (H. M. E.), von Martens leg. (collection Zoological Museum Berlin, no. 2788); holotype, $6 \times 5 \times 3$ mm.

Kupang, Timor, 1 specimen on *Cymo melanodactylus* de Haan, Snellius Expedition; paratype, $7\frac{1}{2} \times 5\frac{1}{2} \times 3\frac{1}{2}$ mm.

The two specimens (fig. 1) are of a similar shape; both are roundish to slightly oval. In the two specimens the mantle opening is found in a corresponding part of the mantle, on the left side, at some distance from the posterior margin. In the specimen on *Cymo* the posterior margin of the mantle has a few shallow grooves, the rest of the mantle to the naked eye has a smooth surface. In the two specimens there is a broad groove in the posterior part of the right surface, where the median ridge of the abdomen of the host pressed against the parasite.

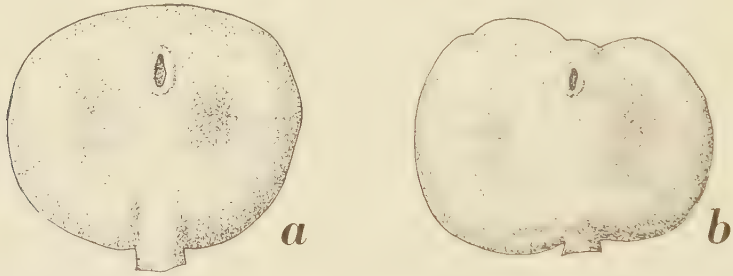

Fig. 1. *Loxothylacus murex* nov. spec., left side. *a*, specimen on *Xanthias lamarcki*; *b*, specimen on *Cymo melanodactylus*.

From each of the two specimens a series of longitudinal sections was made.

A section through the stalk of the specimen on *Xanthias lamarcki*

(fig. 2 *a*) distinctly shows the most striking generic character of *Loxothylacus*, as the visceral mass posteriorly is attached to the mantle at a considerable distance from the stalk. Towards a more dorsal region soon the vasa deferentia appear in the sections; at first both of these are rather narrow canals (fig. 2 *b*), but gradually the right vas deferens distinctly

Fig. 2. *Loxothylacus murex* nov. spec., specimen on *Xanthias lamarcki*. *a—g*, longitudinal sections of the posterior part of the visceral mass; *a*, through the stalk, each following section from a more dorsal region. *h*, *i*, longitudinal sections of one of the colleteric glands. *lt*, left testis; *mc*, mantle cavity; *rt*, right testis; *st*, stalk; *vm*, visceral mass. *a—g*, × 11; *h*, *i*, × 53.

increases in size, whilst the left remains narrow (fig. 2 *c — e*). The cavity of the larger vas deferens is rather irregular as a result of the development of a system of ridges on its inner wall. In its dorsal part the right vas deferens passes into its testis (fig. 2 *f*), which is distinctly curved (the dorsal part of the curvature is shown in fig. 2 *g*); its extremity consequently is pointing in a ventral direction (lower part of fig. 2 *e*). The left testis is much smaller than the right, it does not become appreciably wider than its vas deferens, whilst here the curvature is less pronounced than that of the right, as the extremity of the left testis is extending more or less in an anterior direction (fig. 2 *f*).

The colleteric glands contain a rather compact mass of branched canals, in the longitudinal sections of one of these glands represented here the one (fig. 2 *h*) contains 46 canals, the other (fig. 2 *i*) 42. In this specimen the canals contain a well developed layer of chitin.

The distinct specific characters are those of the external cuticle of the mantle. This cuticle bears excrescences consisting of a hyaline kind of

chitin, differing from that of the main layers of the cuticle. The excrescences are composed of groups of spines pointing in various directions. In the specimen on *Xanthias lamarcki* the total length of the spines in the greater part of the mantle is from 30 to 60 μ (fig. 3 *a* — *c*). The variation in shape and in size is shown in the figure.

The internal cuticle of the mantle bears numerous retinacula which

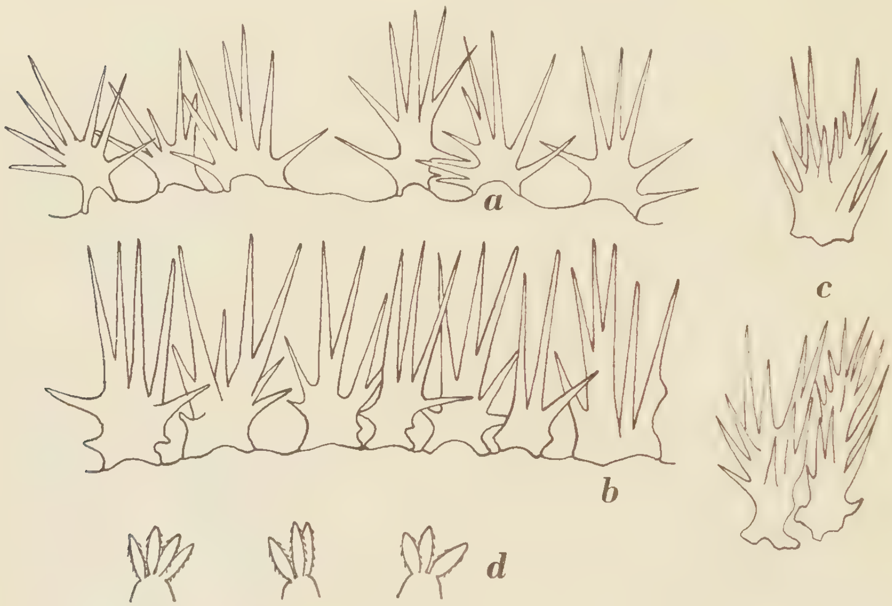

Fig. 3. *Loxothylacus murex* nov. spec., specimen on *Xanthias lamarcki*. *a*, *b*, excrescences on two different parts of the external cuticle; *c*, isolated excrescences of the external cuticle; *d*, retinacula. × 530.

are more or less regularly distributed on its surface. Each retinaculum consists of a basal part and three or four spindles (fig. 3 *d*), the latter have a length of about 12 μ, they possess numerous small barbs.

The specimen on *Cymo melanodactylus* also distinctly shows the generic characters of *Loxothylacus*, as here again the visceral mass is attached to the mantle at a large distance from the stalk (fig. 4 *a*). This section shows the ventral parts of the vasa deferentia, which here appear as narrow canals. Towards the dorsal part of the visceral mass the two vasa deferentia become distinctly larger (fig. 4 *b*), and gradually they pass into the testes (fig. 4 *c*), which possess a distinct curvature towards the anterior region and finally towards the ventral half of the visceral mass (the extremity of the right testis is shown in the lower half of fig. 4 *b*). The inner wall of the vasa deferentia shows some ridges, especially in the region in which these organs pass into the testes, but these ridges are less pronounced than those of the other specimen. Here the two male organs are of about equal size and are of an approximately similar shape.

The structure of the colleteric glands of the specimen on *Cymo melano-*

dactylus is similar to that of these organs in the specimen on *Xanthias lamarcki*. In the specimen on *Cymo* the canal system is slightly more strongly divided, as in a longitudinal section 66 of these canals may be

Fig. 4. *Loxothylacus murex* nov. spec., specimen on *Cymo melanodactylus*. *a—c*, longitudinal sections of the posterior part of the visceral mass; *a*, through the stalk, each following section from a more dorsal region. *d*, longitudinal section of one of the colleteric glands. *lt*, left testis; *mc*, mantle cavity; *rt*, right testis; *st*, stalk; *vm*, visceral mass. *a—c*, × 40; *d*, × 142.

counted (fig. 4 *d*). As in the other specimen the canals have a distinct layer of chitin.

The excrescences of the external cuticle of the mantle in the specimen on *Cymo melanodactylus* in every respect are similar to those of the other specimen. In the larger part of the mantle their size varies from 22 to 60 μ, they consist of groups of spines united on common basal parts, the whole consisting of a kind of chitin differing from the main layers of the cuticle by its more hyaline structure (fig. 5 *a* — *e*). In each excrescence the

spines are pointing in various directions. In some parts of the mantle the basal parts of the excrescences consist of a thin layer of chitin only (fig. 5 *a*), in other parts of the mantle these basal parts are of a more solid structure (fig. 5 *d*).

On the internal cuticle of the mantle there are numerous retinacula, consisting of a basal part and four or five spindles, the latter have a length

Fig. 5. *Loxothylacus murex* nov. spec., specimen on *Cymo malanodactylus*. *a*—*e*, excrescences from various parts of the external cuticle; *f*, retinacula. × 530.

of 15 μ approximately (fig. 5 *f*). As the spindles are covered by fragments of the thin chitinous matter enveloping the egg-masses in the mantle cavity it was not possible to ascertain whether the spindles are barbed or not.

The two specimens of *Loxothylacus murex* closely correspond as far as concerns the structure of the excrescences of the external cuticle. The retinacula are of a similar structure, the differences found in the two specimens may be the result of individual variation. In the two specimens the colleteric glands have a similar structure. The male organs, however, are rather different. In the specimen on *Xanthias lamarcki* one of the male organs (the right) is fully developed, the other remains more or less rudimentary. On the other hand in the specimen on *Cymo melanodactylus* the two male organs have a corresponding shape and a similar size. As, however, variation of this kind is of common occurrence in many species of *Loxothylacus*, the two specimens may be safely regarded as conspecific. The peculiar excrescences separate the new species at once from other representatives of the genus.

Loxothylacus corculum (Kossm.)

Beo, Karakelong, Talaud Islands (Siboga Expedition, Sta. 131), 1 specimen on *Atergatis floridus* (L.), 12 × 10 × 5 mm.

Mozambique, 1 specimen on *Xantho sanguineus* (H. M. E.), C. Cooke leg., May

1863 (collection Museum of Comparative Zoölogy, Cambridge, Mass., no. 1265), $9\frac{1}{2} \times 6 \times 3\frac{1}{2}$ mm.

Zanzibar, 2 specimens on *Xantho sanguineus* (H. M. E.), C. COOKE leg. (collection Museum of Comparative Zoölogy, Cambridge, Mass., no. 1269), one specimen $10 \times 7 \times 4$ mm, the other slightly smaller.

The specimen from the Siboga Expedition is one of the two that were attached to the abdomen of one specimen of *Atergatis floridus*, which explains their irregular asymmetrical shape (VAN KAMPEN and BOSCHMA, 1925). The type specimen of *Loxothylacus corculum*, a parasite of *Atergatis floridus*, was more or less heart-shaped (KOSSMANN, 1872). This shape is also that of the specimens on *Xantho sanguineus*. In the three specimens from this crab the mantle opening is surrounded by a comparatively thick wall; this opening is found on the left side, near the posterior margin of the mantle.

Longitudinal sections have been made of the specimen on *Atergatis floridus*, of the specimen on *Xantho sanguineus* from Mozambique, and of one of the specimens on the same crab from Zanzibar. The sections show that in the three specimens especially the male genital organs present rather striking differences.

In a previous paper (VAN KAMPEN and BOSCHMA, 1925, pl. II fig. 3) there is a figure of a longitudinal section of the visceral mass of the Siboga specimen of *Loxothylacus corculum*, showing that the visceral mass is attached to the mantle at a considerable distance from the stalk. A figure of a section from the same region is fig. 6 *a* in the present paper. It shows that the male organs have a wide curvature, so that the testes are far distant from the vasa deferentia. A section from a more dorsal region is represented in fig. 6 *b*. It shows the dorsal curved part of the male organs, the region in which the vasa deferentia are passing into the testes. Here the inner walls over the whole of their surface are beset with rather thick papillae or ridges, so that the cavities of the male organs here are comparatively narrow.

The colleteric glands of the specimen on *Atergatis floridus* have a rather large number of branched canals; in a longitudinal section of the most strongly branched region there are 80 of these canals (fig. 6 *c*). In this specimen the inner walls of the canals have a distinct layer of chitin.

The large conical excrescences of the external cuticle have been described in a previous paper (l.c., p. 35); they have a length of 115 to 520 μ, and in their basal parts a thickness of 78 to 173 μ. The retinacula have 8 to 12 barbed spindles, the latter have a length of 18 μ.

In the specimen on *Xantho sanguineus* from Zanzibar the two male genital organs are of a rather different size, the right being much larger than the left and being much more strongly curved (fig. 7 *a*—*c*). The left vas deferens too is much smaller than the right (fig. 7 *a*). The left testis is rather narrow, it runs chiefly in an anterior direction, whilst its cavity is narrow on account of numerous papillae and ridges on its inner

wall (fig. 7 *b*, *c*). The right testis has the shape of an enlarged pouch chiefly extending in an anterior direction (fig. 7 *b*, *c*) but the closed end is

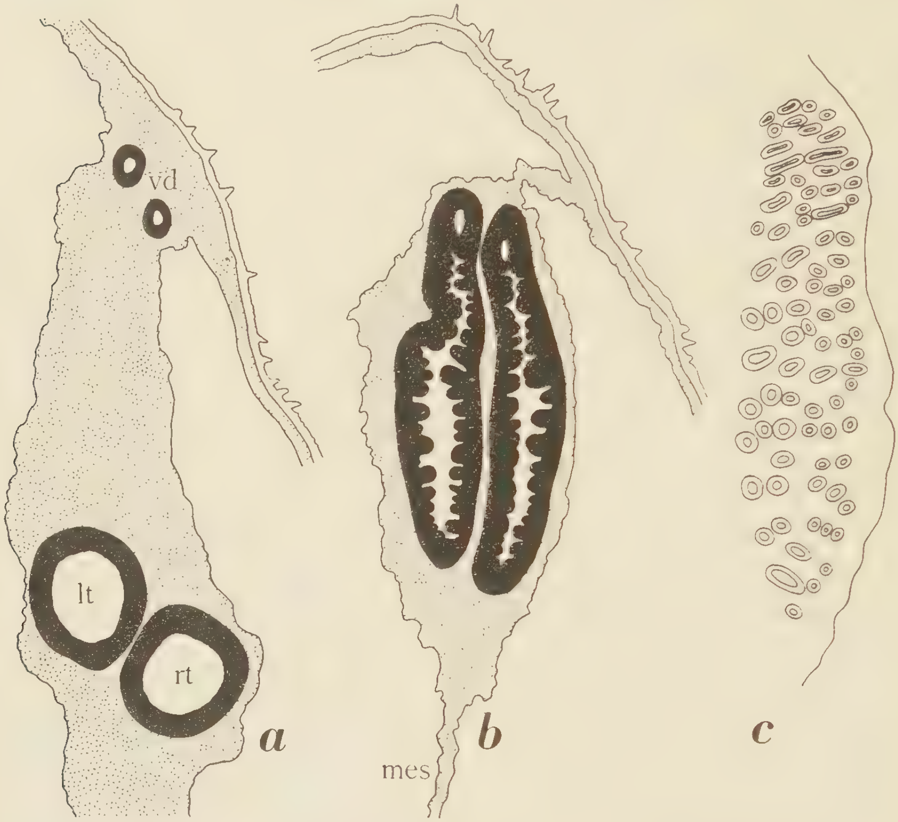

Fig. 6. *Loxothylacus corculum* (Kossm.), specimen on *Atergatis floridus*. *a*, longitudinal section from the neighbourhood of the stalk; *b*, longitudinal section from the dorsal region; *c*, longitudinal section of the right colleteric gland in its most strongly branched part. *lt*, left testis; *mes*, mesentery; *rt*, right testis; *vd*, vasa deferentia. *a*, *b*, × 23; *c*, × 80.

pointing towards the ventral half of the visceral mass (fig. 7 *a*). In the right male organ there are some ridges on the inner wall of the vas deferens, the inner wall of the testis is smooth.

The colleteric glands of the specimen on *Xantho sanguineus* from Zanzibar are very similar to those of the specimen on *Atergatis floridus*. The number of canals is slightly smaller, as in a longitudinal section of the most strongly branched region there are 65 of these canals (fig. 7 *d*). The canals possess a well developed layer of chitin.

The excrescences of the external cuticle of the specimens on *Xantho sanguineus* from Zanzibar are considerably smaller than those of the specimen on *Atergatis floridus*, though being of the same general appearance. They are comparatively slender (fig. 8 *b*, *c*), straight or slightly

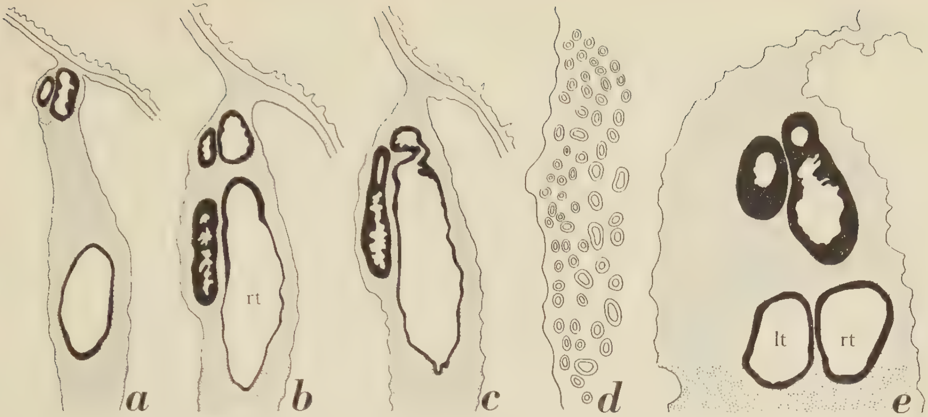

Fig. 7. *Loxothylacus corculum* (Kossm.). *a*—*d*, larger specimen on *Xantho sanguineus* from Zanzibar; *e*, specimen on *Xantho sanguineus* from Mozambique. *a*—*c*, longitudinal sections of the greater part of the visceral mass; *a*, from a more ventral region than *b*; *c*, from a more dorsal region than *b*. *d*, longitudinal section of the left colleteric gland in its most strongly branched part. *e*, posterior part of a longitudinal section of the visceral mass, showing the dorsal parts of the testes. *lt*, left testis; *rt*, right testis. *a*—*c*, × 27; *d*, × 86; *e*, × 27.

Fig. 8. *Loxothylacus corculum* (Kossm.), specimens on *Xantho sanguincus*. *a*—*c*, excrescences of the external cuticle; *a*, from the specimen from Mozambique, *b*, from the smaller specimen from Zanzibar, *c*, from the larger specimen from Zanzibar. *d*, retinacula from the specimen from Mozambique; *e*, retinacula from the smaller specimen from Zanzibar. × 530.

curved; their length varies from 64 to 160 μ, their basal thickness from 18 to 35 μ.

The retinacula of these specimens again are slightly smaller than those of the specimen on *Atergatis floridus*; moreover, they have a smaller number of spindles (up to four). The distinctly barbed spindles here may reach a length of 14 μ (fig. 8 *e*).

Of the specimen on *Xantho sanguineus* from Mozambique one section is shown here (fig. 7 *e*). In this specimen the male genital organs are of approximately equal size; their curvature, however, is much narrower than that in the specimens dealt with above. The vasa deferentia of this specimen have a number of ridges on their inner walls, especially in the region where they are passing into the testes.

The colleteric glands of this specimen do not differ in any important detail from those of the former specimen.

The excrescences of the external cuticle of the specimen on *Xantho sanguineus* from Mozambique may be broadly conical or of a more slender shape, straight or slightly curved (fig. 8 *a*). They vary in length from 150 to 200 μ; their basal thickness is from 32 to 68 μ.

The retinacula of this specimen as a rule bear four spindles, the latter have a length of up to 16 μ (fig. 8 *d*).

In all their salient characters the specimens on *Xantho sanguineus* closely correspond with the specimen on *Atergatis floridus*, though the measurements of the various organs are somewhat smaller. Undoubtedly all the specimens dealt with here belong to the species *Loxothylacus corculum*. It is interesting that in every specimen examined the shape of the male organs is slightly different from that of the others.

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THE INFLUENCE OF HYPERTONICITY ON THE EGGS OF LIMNAEA STAGNALIS

BY

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DE GROOT (1948) studied the effects of LiCl solutions of various concentrations on the eggs of *Limnaea stagnalis* immediately after oviposition. In hypertonic LiCl solutions, development is inhibited. The stage, at which development stops, is dependent on concentration, stage of treatment, temperature and susceptibility of the eggs. In general, the stronger the LiCl solution used and the earlier the eggs are transferred to the solution, the earlier development comes to a standstill.

In 1 % LiCl, the first maturation spindle looses its contact with the animal pole and sinks into the interior of the egg. It shows a degenerated appearance and in its centre a nucleus-like body is situated. In the degenerating spindle and especially at its periphery a great number of vacuoles appear, each containing a heavily-stained rod-shaped body, which, sometimes, seems to be paired. In eggs fixed later, these bodies prove to be scattered over a wider area, and their number, shape and dimensions may vary greatly. The author leaves in doubt whether these bodies originate from chromosomes, or are to be considered as coagulation products of the degenerating spindle.

In 0.5 % LiCl, the first polar body is formed without any visible abnormality. However, the second maturation spindle sinks into the interior of the egg and degenerates. In its centre a nucleus-like body may appear. The degeneration of the second maturation spindle, like that of the first, is accompanied with the appearance of vacuoles in and around the spindle. The heavily-staining bodies in these vacuoles are often lying in pairs; sometimes they exceed many times the dimensions of chromosomes.

In 0.4 % LiCl, several interesting abnormalities occur. Sometimes the chromosomes swell into karyomeres shortly after the extrusion of the first polar body, while the sperm nucleus migrates towards the animal pole and swells into a male pronucleus, which copulates with the female pronucleus formed by the fusion of the egg karyomeres. In other cases a second maturation spindle is formed, but this may sink into the interior and place itself perpendicular to the egg axis. Still other eggs extrude both polar bodies, but the egg karyomeres may sink into the interior of the egg.

Even if copulation of the pronuclei occurs at the animal pole, any further development is inhibited.

Finally, in 0.2 % LiCl, the nuclear phenomena of maturation and fertilization are normal; a first segmentation occurs in most of the eggs.

Besides deviations of the mitotic cycle, DE GROOT observed interesting abnormalities in the distribution of cytoplasmic substances. The animal pole plasm was suppressed in all concentrations studied, whereas an abnormal distribution of the subcortical plasm was most pronounced in an isotonic (0.2 %) solution.

The interpretation of these results met with some difficulties, as they could either be due to a specific influence of LiCl or to the hypertonicity of most of the solutions employed. By a comparison with other investigations, DE GROOT came to the conclusion that the disturbances of the nuclear cycle were chiefly due to hypertonicity, whereas the abnormal distribution of cytoplasmic substances was caused by the specific action of LiCl. Further experiments were needed, however, to test this hypothesis.

Therefore, RAVEN and ROBORGH (1949) studied the effects of isotonic and hypotonic LiCl solutions on the *Limnaea* egg. They showed that, among the phenomena described by DE GROOT, at least the premature swelling of the egg chromosomes to karyomeres immediately after the extrusion of the first polar body, with concomitant swelling and migration of the sperm nucleus, could not be ascribed to the action of hypertonicity, as it occurred also in isotonic and hypotonic LiCl solutions. This phase of nuclear swelling seems to be reversible, in this case; it is followed by deswelling of the chromosomes and completion of the second maturation division. Further phenomena observed in these experiments pointed to the assumption that the primary action of lithium consists in a change of the state of hydration of the protoplasmic colloids.

The present experiments represent a further step in the analysis of the results obtained by DE GROOT, by a study of the effects of hypertonicity as such on the structure of the *Limnaea* egg.

According to RAVEN and KLOMP (1946), freshly-laid eggs of *Limnaea* are isotonic to about 0.1 m solutions of non-electrocytes. We studied, therefore, the effects of a treatment of the eggs with 0.1 to 0.7 m solutions of glucose and urea.

Material and methods.

The egg-masses were obtained in the usual way by stimulation of the snails with *Hydrocharis* (RAVEN and BRETSCHNEIDER 1942). Immediately after oviposition the eggs were decapsulated and washed in distilled water to remove the capsule fluid. Then the eggs of an egg-mass were divided into groups of 10, and transferred to the solutions, in which they remained for 1-4 hours. Controls developed for the same time in distilled water, in which medium development is entirely normal during the first hours. In total, 58 groups of eggs, from 28 egg-masses, have been studied in this

way. The eggs were fixed in Bouin's fluid and sectioned at 5 or 7.5 μ; the sections were stained with iron haematoxylin-saffranin or with azan.

Results

1. *The development in glucose and urea solutions of different concentrations.*

In general, the results of the experiments are only dependent on the molar concentration of the solutions, not on their nature. No regular differences between the actions of glucose and urea during the period studied have been observed. Therefore, the results of both groups are described together.

a. 0.1 *m solutions.* In these solutions, which are about isotonic to the eggs, as a rule development proceeds in a normal way. Both polar bodies are extruded, the pronuclei are formed and about 4 hours after oviposition cleavage begins. However, in some cases development is somewhat delayed as compared with the controls. This delay is only slight in most of these cases, but it is considerable e.g. in batch AAG_4, which exhibited second maturation spindles after 3 h. 5 min. in glucose, whilst the controls possessed pronuclei in copulation at the animal pole. In other batches, however, no delay even after 3 hours in the solutions could be observed.

b. 0.15 *m solutions.* In most cases, both polar bodies are extruded; the egg chromosomes may swell into karyomeres after the extrusion of the second polar body. However, copulation of the pronuclei and cleavage have not been observed, within 5 hours after oviposition. This may be due to the delay of development, which is much more pronounced than in 0.1 m solutions. Only in one batch, fixed after 1 h. 10 min. in the solutions, no delay has been observed. All the other batches show a delay, which is greater according as the eggs have remained longer in the solutions. So in batches KAG_2 and KAU_2, both in glucose and urea after 4 hours the eggs possess second maturation spindles, whereas the controls are cleaving. Moreover, 2 batches exhibit a beginning degeneration of the second maturation spindles.

c. 0.2 *m solutions.* In these solutions, the first polar body is extruded, and a second maturation spindle is formed, as a rule, but development stops at anaphase of this division and the spindle and asters degenerate. Only in one batch the second maturation division had been completed in some eggs. On the other hand, in another batch no second maturation spindle had been formed, and the inner aster of the first maturation division degenerated after the extrusion of the first polar body.

d. 0.3 *m solutions.* Development at this concentration varies with the moment, at which the eggs were put into the solutions. When this occurred early (45-35 min. before the extrusion of the first polar body in the controls), the first maturation spindle degenerated and no polar body was formed. In batches transferred to the solution somewhat later, a first polar body was extruded, then either the inner aster of this spindle

degenerated or a second maturation spindle was formed, which then degenerated. Finally, in one batch, which was put into the solution at the moment of formation of the first polar body, second polar bodies were extruded and development stopped at the karyomere stage.

e. 0.5 *m solutions*. Degeneration either of the first maturation spindle or, after extrusion of the first polar body, of the inner aster and central body remaining in the egg.

f. 0.7 *m solutions*. Degeneration of the first maturation spindle. Only in one batch, put into the solution 10 minutes before the formation of the first polar body, the latter was extruded and the remaining aster and central body degenerated.

It is evident, therefore, that, the stronger the solution employed, the earlier development stops and the degeneration of asters and spindles begins.

2. *The degeneration of spindles and asters.*

Three cases have to be distinguished:

a. *Degeneration of the second maturation spindle.*

The beginning of this degeneration process can be observed in batch JAU_2 (3 h. in 0.15 m urea). The second maturation spindle is in metaphase. It is surrounded by an equatorial girdle of vacuoles. The latter have developed in connexion with the chromosomes: in fact, each chromosome lies at the margin of a vacuole (fig. 1a). The latter are rather irregular in shape and are not surrounded by a distinct membrane; therefore, they are quite different from karyomeres and cannot be considered as such. Somewhat later, the vacuoles also invade the spindle area; this may be seen in fig. 1b, a transverse section at the spindle equator, which shows clearly the relation of the chromosomes to the vacuoles.

Simultaneously with the vacuolar transformation of the chromosomes, peculiar deeply-stained rod-shaped bodies become visible in the periphery of both asters of the spindle. Each of them is surrounded by a narrow vacuolar space in the sections; however, this may be due to shrinking of the rods during fixation. Evidently, these bodies correspond to those described by DE GROOT (1948) in degenerating maturation spindles. It is evident from their manner of formation, however, that they have nothing to do with the chromosomes; they must be coagulation products of the cytoplasm, arising, presumably, by a coalescence of astral rays.

A further stage of degeneration is shown by fig. 1c, from a batch which had been treated for 3 hours with 0.2 m glucose. A row of irregular vacuoles in the spindle equator indicate the place where the chromosomes have been lying. No chromosomes are visible any more, however; presumably, their substance has been dissolved away in the formation of the vacuoles. The above-mentioned rod-shaped bodies are visible in the peripheral part of both asters and in the spindle; partly, they have reached a considerable size.

Fig. 1d gives a final stage of the degeneration of the second maturation spindle. The area formerly occupied by the spindle and asters now consists of a dense heavily-staining and more or less granular mass. It is surrounded

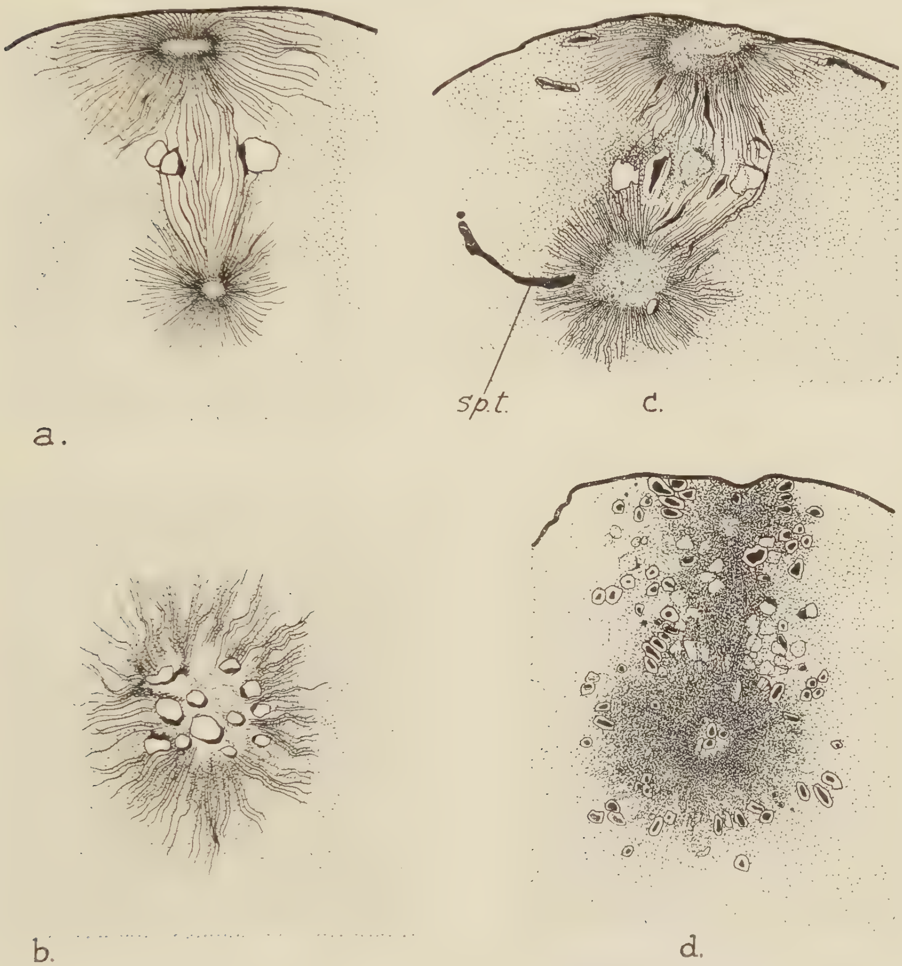

Fig. 1. *Limnaea stagnalis.* Degeneration of second maturation spindle.
a. JAU_2, 3 h. in 0.15 m urea.
b. KAU_2, 4 h. in 0.15 m urea.
c. JAG_1, 3 h. in 0.2 m glucose. Sp.t. = sperm tail.
d. JU, 1 h. 30 min. in 0.3 m urea.

by a halo of vacuoles each containing a rod-shaped body; moreover, in and around the former spindle area numerous small empty vacuoles are present, which may have originated from the degeneration of the chromosomes.

b. *Degeneration of the inner aster and central body of the first maturation spindle.*

In normal development, after the extrusion of the first polar body the inner group of chromosomes have reached the margin of the central body.

The latter then transforms into the second maturation spindle and the chromosomes are arranged into its equatorial plate (RAVEN 1949).

In part of the eggs treated with hypertonic solutions, the first polar body is extruded in a normal way. Immediately afterwards, degenerative changes occur in the inner aster remained in the egg. Its central area becomes dense and deeply-staining. It is surrounded bij a circle of vacuoles, part of which contain such heavily-stained rod-shaped bodies as described above (fig. 2a). In azan-stained sections, these bodies are deep-red: they are surrounded by a vacuolar cavity, which appears empty. Between them, a few vacuoles of a different kind may be seen: their contents stain blue and they contain a vaguely-delimited violet body. Presumably, these are the chromosomes in dissolution.

Fig. 2b shows a further stage of the degeneration process. The central aster area is now quite dense and very deeply stained; with iron haema-

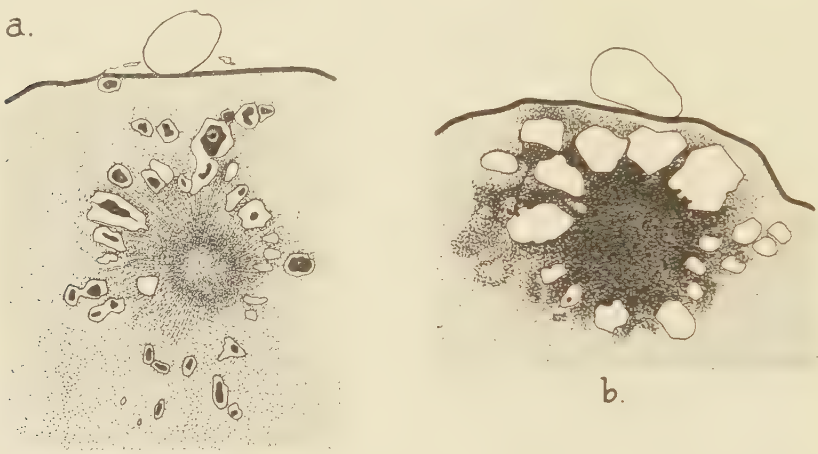

Fig. 2. *Limnaea stagnalis*. Degeneration of inner aster of first maturation spindle.
a. MU, 2 h. 15 min. in 0.3 m urea.
b. BAG_2, 2 h. 5 min. in 0.5 m glucose.

toxylin, it becomes nearly black; in azan-stained sections, it is deep-red. It is surrounded by a circle of big vacuoles, which are empty or contain small vaguely-delimited bodies. No remains of the chromosomes are visible any more.

Still later, the vacuoles disappear and the dense central part falls apart into some fragments, which are gradually resorbed in the cytoplasm.

c. *Degeneration of the first maturation spindle.*

The beginning of the degeneration of the first maturation spindle is shown in fig. 3a (batch WG_2, 1 h. 20 min. in 0.3 m glucose). The chromosomes are still visible within the spindle, but they have lost a great deal of their stainability. Deeply-stained rod-shaped bodies have appeared in the outer margin of the asters; each of them is lying in a vacuole.

In fig. 3b the chromosomes have disappeared. The whole area of spindle

and asters has become quite dense and deeply-stained. Each of the asters is surrounded by a circle of vacuoles; they do no longer contain rod-shaped

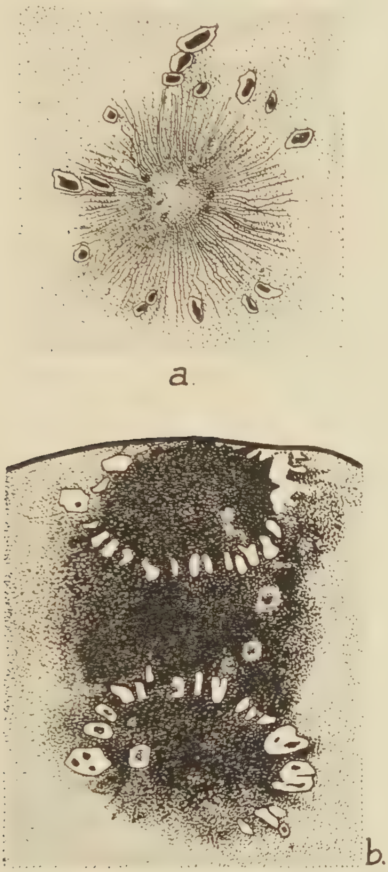

Fig. 3. *Limnaea stagnalis.* Degeneration of first maturation spindle.
a. WG_2, 1 h. 20 min. in 0.3 m glucose.
b. WG_1, 1 h. 20 min. in 0.5 m glucose.
c. AAG_1, 3 h. 5 min. in 0.7 m glucose.

bodies, but in some of the vacuoles several smaller granules are lying, which may have arisen by fragmentation of the rods.

A final stage of degeneration is shown by fig. 3c. The vacuoles present in the previous stage have coalesced to big irregular vacuolar spaces, still surrounding the former asters. The area of spindle and asters consists of a dense mass, staining black with iron haematoxylin and deep-red with azan.

We may conclude from these observations that the degeneration in all 3 cases proceeds along the same lines, but with little variations according to stage of the eggs and rapidity of the degenerative changes. The chromosomes disappear at an early stage of the process by the formation of vacuoles in which they are dissolved. Rod-shaped bodies are formed in

the circumference of the asters, presumably by a fusion of astral rays; they are surrounded from the beginning by a vacuolar space. Later these vacuoles increase in size; their number may diminish by coalescence. Meanwhile the central part of the asters and the spindle area become very dense and their staining capacity increases greatly. The final stage is a complete disorganization of the mitotic apparatus. After disappearance of the vacuoles, the central part may fall apart into fragments and be resorbed in the cytoplasm.

3. *Depolarization phenomena.*

Raven and Mighorst (1946) observed a displacement of the second maturation spindle from the animal pole, followed by a rotation through 90°, by which the spindle places itself perpendicularly to the egg axis, in *Limnaea* eggs treated with 0.5 % $CaCl_2$. This abnormality belongs to the group of phenomena described as 'depolarization' by Dalcq (1925).

Depolarization phenomena were also observed in a number of cases by de Groot (1948) in eggs treated with 0.4 % LiCl. They consisted either in a rotation of a 'submerged' second maturation spindle, or in the displacement of the egg karyomeres from the animal pole towards the centre of the egg, combined with an inhibition of the migration of the sperm nucleus towards the animal pole. Moreover, giant polar bodies were observed by him in a small number in 0.2 – 0.4 % LiCl.

As the solutions, in which these depolarization phenomena occurred, all were weakly hypertonic to the eggs, one might suppose this hypertonicity to be the causal factor. It might be expected, therefore, that depolarization phenomena would be met with in the eggs treated with hypertonic solutions of non-electrolytes.

This does, however, hardly occur. As a matter of fact, in only two batches some eggs have been found, in which the second maturation spindle had lost its contact with the egg surface (JU, 1 h. 50 min. in 0.1 m urea; HAG_2, 3 h. 20 min. in 0.15 m glucose). Moreover, in the latter one egg possessed a 3-polar maturation spindle.

According to de Groot, in greatly hypertonic LiCl solutions both the degenerating first and second maturation spindle loose their contact with the animal pole and sink into the interior of the egg. On the contrary, in hypertonic urea and glucose solutions the degenerating spindles retain their connexion with the egg surface, even till the final stages of degeneration (fig. 1d, 3c). We must conclude, therefore, that, whereas the degeneration of the spindles is due to hypertonicity, their displacement in the experiments of de Groot must be ascribed to the action of LiCl; presumably, it is caused by a weakening of the attractive factors at the animal pole, and belongs, therefore, to the group of depolarization phenomena.

It is evident, therefore, that hypertonicity is not the direct cause of depolarization in *Limnaea*.

4. *Cytoplasmic differentiations.*

According to DE GROOT (1948), LiCl solutions caused disturbances in the distribution of cytoplasmic materials. The animal pole plasm, occurring normally one hour before the first cleavage, was not formed in any of the concentrations studied. The subcortical plasm showed an abnormal distribution especially in isotonic LiCl solutions, to a less degree also in hypertonic solutions. DE GROOT ascribes these abnormalities to the specific properties of LiCl.

In the eggs treated with isotonic and hypertonic urea and glucose solutions, no disturbances in the distribution of the subcortical plasm have been observed. Though in many cases the azan staining did not give quite satisfactory results, in all eggs where the subcortical plasm was distinct it showed a normal localization.

No animal pole plasm was formed in all those eggs, where degeneration of the maturation spindles had occurred. The same holds true as regards those cases, where development was delayed and the eggs did not reach the stage, at which copulation of the pronuclei takes place, within the time limits of the experiment. Apparently, the animal pole plasm cannot be formed before this stage. Hence, only in the lots kept in 0.1 m the formation of the animal pole plasm has been observed, in eggs which remained in the solution for more than 3 hours.

5. *The sperm.*

The evolution of the sperm nucleus coincides with that of the maturation spindles in a normal manner, even when development is delayed in hypertonic solutions. In eggs which possess a 2nd maturation spindle in meta- or anaphase, the sperm heads are lying as compact dark bodies under the egg cortex. In some batches which had been fixed during the second maturation division, only in those eggs where the second polar body had been extruded and the egg chromosomes had swollen into karyomeres, the sperm nucleus had also begun to swell and was found on its way towards the animal pole. Evidently, the beginning of the transformation and migration of the sperm nucleus also in these retarded egg strictly coincides with the swelling of the egg karyomeres after the extrusion of the 2nd polar body. Finally, in eggs which had developed still further the male pronucleus was found, together with the female one, near the animal pole.

DE GROOT (1948) and RAVEN and ROBORGH (1949) observed a premature swelling of the egg chromosomes, accompanied with a swelling and migration of the sperm nucleus, immediately after the extrusion of the first polar body in eggs treated with LiCl solutions. Nothing of the kind has been observed in hypertonic solutions of urea and glucose.

Some observations have been made which indicate that in those eggs where degeneration of the 2nd maturation spindle takes place the sperm nucleus simultaneously begins to degenerate. It is surrounded by a circle

of vacuoles like the degenerating spindle and egg chromosomes. If this be true, it would represent another peculiar example of the 'mise à l'unisson' (BRACHET) which characterizes the behaviour of the sperm nucleus also in normal development.

The evolution of the spermaster, like that of the sperm nucleus, coincides with that of the maturation spindles, even when development is greatly retarded. A striking example of this is given by batch FAG_1 (2 h. 5 min. in 0.2 m glucose). The controls have reached the telophase of the second maturation division, but the experimental eggs are in pro-metaphase of this division. In agreement with this, they possess early spermasters of small size, which have not yet fused with the maturation spindle. In other batches, where the eggs are in meta- or anaphase of the 2nd maturation division, the spermaster has fused with the spindle, forming its deep aster, like in normal development (RAVEN 1949). After the extrusion of the second polar body, the spermasters become free again and soon disappear, leaving behind for some time a spherical clear area in the cytoplasm near the centre of the egg. When the second maturation spindle degenerates, however, the aster is involved in this degeneration process, as has been described above.

A third structure, the evolution of which is strictly synchronous with that of the egg as a whole, is the sperm tail. In normal development, this may be found as a sinuous thread in the cytoplasm till the end of anaphase of the second maturation division; in no case it has been observed after this stage. Often, it is especially conspicuous at the moment when the spermaster approaches the deep end of the maturation spindle prior to its fusion with the latter; in these cases, it is clear that the spermaster is still connected with the end of the tail (cf. RAVEN 1945, fig. 1d). This connexion seems to be broken soon after the spermaster has fused with the spindle.

The same phenomena have been observed in the eggs treated with hypertonic solutions. The sperm tail disappears at a stage which is the same as in normal eggs. In numerous eggs, where the 2nd maturation spindle had not yet reached the telophase stage, sperm tails have been found; after this stage, no one has been observed. This holds true of eggs with greatly delayed development, like batch AAG_4 (3 h. 5 min. in 0.1 m glucose), where the controls possessed copulating pronuclei, whereas many experimental eggs, in anaphase of the second maturation division, showed sperm tails in the cytoplasm. When degeneration of the second or first maturation spindle occurs, the dissolution of the sperm tails seems to be inhibited: In batch JAG_1 (3 h. in 0.2 m glucose), the controls have extruded both polar bodies; the experimental eggs show a degeneration of the 2nd maturation spindle, in many of them a sperm tail is visible (fig. 1c). In batch QU_2 (4 h. 45 min. in 0.5 m urea), where the controls have passed the first cleavage, in the experimental eggs the first maturation spindle has degenerated; in the cytoplasm, sperm tails can still be observed. We

may conclude, therefore, that the dissolution of the sperm tails is dependent on a certain state of the cytoplasm, which is normally reached at the end of the second maturation division; when the evolution of the maturation spindles comes to an untimely end, this condition does not occur.

6. *The cytoplasmic vacuoles.*

Normal eggs, developing in the egg capsules, swell during the uncleaved stage, increasing in volume with about 50 % (Raven 1945). This swelling is an osmotic phenomenon (Raven and Klomp 1946). The absorbed water for the greatest part is used in the swelling of a special kind of inclusions, the γ -granules, forming in this way numerous big vacuoles in the cytoplasm. In sections of eggs fixed during the later part of the uncleaved period, the cytoplasm is crowded with these vacuoles, each containing a γ-granule in its centre.

In eggs kept in isotonic solutions, the swelling is prevented; in hypertonic solutions even a shrinking of the eggs, by loss of water, occurs. It is to be expected that in these cases the formation of vacuoles will be inhibited.

As a matter of fact, in eggs kept for 2 hours in 0.1 m solutions of urea or glucose the cytoplasm is only weakly vacuolar. Each γ-granule in the sections is still surrounded by a narrow empty space, which may have arisen by a shrinking of the granule at fixation or dehydration of the fixed egg. In hypertonic solutions, with increasing concentration, the number and size of these 'vacuolar' spaces is reduced. After 2 hours in 0.2 m solutions, the cytoplasm has a dense structure; only some of the bigger γ-granules are still surrounded by a very narrow 'vacuolar' space. In stronger solutions, hardly any further change in this respect is to be noted; even after 2 hours in 0.7 m, still some γ-granules show the same picture. It must be concluded, then, that indeed the vacuolization of the cytoplasm decreases with increasing hypertonicity of the medium, but that even in the strongest solutions employed the γ-granules are still slightly swollen.

The behaviour of these vacuoles is very different from those arising by degeneration of the maturation spindles; even in the strongest solutions the latter increase in size and may reach great dimensions, forming big empty holes in the sections (cf. fig. 3). Evidently, the nature of these degeneration vacuoles and the manner of their formation differs from that of the normal cytoplasmic vacuoles.

7. *Variation of effects with time of exposure and susceptibility.*

The effect of the treatment with hypertonic solutions depends on the moment at which the eggs are transferred to the solution. In general, the disturbances of development are the more serious and begin at an earlier stage according as the eggs are put earlier in the solutions (with respect to the time of first polar body formation in the controls). However,

differences in susceptibility between the egg-masses apparently play an important part.

For example, the eggs of egg-masses LA and KA were both transferred to the solutions 10 min. before the formation of first polar bodies. However, in 0.15 m and 0.2 m solutions of glucose and urea the eggs of LA only showed a slight delay of development as compared with the controls and completed the second maturation division, whereas those of KA exhibited a degeneration of the second maturation spindle. Moreover, in the same solutions the eggs of egg-mass GA (transferred to the solutions 45 min. before first polar body formation) showed hardly any delay, whereas those of egg-mass FA (put into the solutions 40 min. before first polar body formation) showed an inhibition of the second maturation division or even a degeneration of the inner aster of the first maturation spindle after the extrusion of the first polar body.

Evidently, the egg-masses differ greatly in their susceptibility against the action of hypertonicity. Similar differences have been found to exist with respect to other external agents (LiCl: RAVEN 1942; $CaCl_2$: RAVEN and MIGHORST 1946; thiourea: SOBELS 1948).

Discussion.

1. *The effects of hypertonicity.*

DE GROOT (1948) from his experiments has drawn the conclusion that the nuclear cycle of maturation and fertilization is disturbed especially by hypertonicity, whereas the abnormalities in the distribution of cytoplasmic substances observed by him were especially due to the specific action of LiCl. Our results, in conjunction with those of RAVEN and ROBORGH (1949), now permit a better evaluation of the respective actions of LiCl and hypertonicity in DE GROOT's experiments.

It is evident that the degeneration of the first and second maturation spindles, observed by DE GROOT, is indeed due to hypertonicity. However, the fact that these degenerating spindles, in hypertonic LiCl solutions, loose their contact with the animal pole and sink into the interior, must be ascribed to a specific action of the salt. Probably, this belongs to the phenomena of depolarization, which cannot be considered as specific hypertonicity effects.

Moreover, a detail of DE GROOT's description of the degenerating spindles must be mentioned here. Both in the degenerating first and second maturation spindles he observed a nucleus-like body lying in its centre. No such structure has been observed in our experiments. Apparently, here a difference exists which must be explained by a special action of LiCl in DE GROOT's cases. In this connexion, it must be remembered that LiCl causes a swelling of the chromosomes and nuclei (DE GROOT; RAVEN and ROBORGH 1949; RAVEN and DUDOK DE WIT 1949). It is possible that the nucleus-like body in the centre of the degenerating spindle arises by

a swelling and confluence of the chromosomes. In this case, the course of the degeneration process in hypertonic LiCl and non-electrolyte solutions would differ as regards the fate of the chromosomes.

It is clear that the premature swelling of the egg chromosomes to karyomeres in 0.4 % LiCl solutions, with concomitant swelling and migration of the sperm nucleus, is a specific lithium effect. The same holds true as regards the disturbances in the distribution of cytoplasmic substances. Hence, among the phenomena described by DE GROOT only the spindle degeneration as such can be considered as a pure hypertonicity effect.

2. *The mutual connexions of the processes of maturation and fertilization.*

In the eggs of *Limnaea*, from oviposition till first cleavage a regular series of events takes place, in which various components of the egg take part. Our results permit to draw some conclusions on the interrelations existing between these processes.

The first maturation spindle, present in the egg just after laying, moves to the animal pole, where it takes a position perpendicular to the surface. Then the first maturation division takes place. After the extrusion of the first polar body, the inner aster of the first maturation spindle remaining in the egg transforms into the second maturation spindle (RAVEN 1949). Simultaneously, the spermaster makes its appearance, which is (at least in many cases) still connected with the sperm tail. At first only small, it grows rapidly and fuses with the deep end of the second maturation spindle at the moment the latter rotates to a position at right angles to the surface. In this way, the spermaster, which now breaks away from the sperm tail, forms the deep aster of the second maturation spindle. Now the second maturation division occurs. Meanwhile, the sperm tail becomes invisible. When the second polar body is extruded, the spermaster becomes free again and moves toward the centre of the egg, where its rays become indistinct and the whole aster transforms into a clear and somewhat vacuolated area in the cytoplasm, which gradually disappears. Meanwhile, the egg chromosomes swell into karyomeres at the animal pole. Simultaneously, the sperm nucleus, lying up till now as a compact dark body beneath the egg cortex, swells and begins to migrate towards the animal pole, whereby it seems to halt for some time in the outer border of the vanishing spermaster. Arrived near the animal pole, it meets the karyomeres, which coalesce to the female pronucleus. At this time, the animal pole plasm appears beneath the egg cortex surrounding the animal pole. Finally, the first cleavage spindle makes its appearance.

When development of the eggs is retarded by the action of weakly hypertonic solutions, this series of events is not disturbed. All components of the egg keep pace with each other. The evolution of the spermaster takes place strictly synchronously with that of the maturation spindle; the sperm tail disappears at the right stage; the swelling and migration of the sperm nucleus begins at the right moment, simultaneously with

the swelling of the egg chromosomes into karyomeres; the animal pole plasm does not appear before the phase of copulation of the pronuclei.

When, on the other hand, development is arrested at a certain moment in more hypertonic solutions by the degeneration of the second maturation spindle, all subsequent processes are suppressed. The sperm tail is not dissolved, the swelling and migration of the sperm nucleus does not occur, no animal pole plasm is formed. Similarly, when the first maturation spindle degenerates, no further development of the other egg components takes place; notably, no spermaster is formed.

It appears from these observations that the evolution of the different egg components taking part in the processes of maturation and fertilization are rigidly bound together by a common mechanism, which determines the rhythm and sequence of the developmental steps, and in which every subsequent phase depends on the completion of the preceding one. This mechanism might be conceived in two somewhat different ways: firstly, as a more or less autonomous evolution of a superintending structure, directing the developmental processes of all other egg components at their proper place and time; or, secondly, as a complicated interrelationship of the various egg components, in which each influences the evolution of the other ones.

The following observations are of some interest in this connexion. By the action of hypertonic (DE GROOT 1948), isotonic or hypotonic (RAVEN and ROBORGH 1949) LiCl solutions a premature swelling of the egg chromosomes into karyomeres immediately following the extrusion of the first polar body can be induced. Simultaneously, the sperm nucleus swells and migrates towards the animal pole. It may be concluded that 1° the swelling of the egg chromosomes determines that of the sperm nucleus, or inversely, or both are dependent on a common factor e.g. the state of the cytoplasm; the latter explanation seems the most probable (cf. below); 2° the migration of the sperm nucleus is dependent on its swelling; only in a swollen state it responds to the attractive factors determining its migration. The observation of DE GROOT that in depolarized eggs in which the egg karyomeres are sunken into the interior the sperm nucleus does not ascend towards the animal pole, but remains in the central part of the egg, speaks in favour of the view that it is rather attracted towards the egg karyomeres than to the pole; however, in such eggs the directing factors of the sperm nucleus might be weakened.

The further fate of the eggs with premature swelling of the karyomeres is different according to the LiCl solutions employed. In 0.4 % LiCl (DE GROOT 1948), the karyomeres coalesce to a female pronucleus which copulates with the male pronucleus; then, a cleavage spindle with abnormal arrangement of the chromosomes is formed, but no animal pole plasm could be observed. Hence, the normal sequence of events in such eggs is entirely disturbed, the second maturation division is skipped. In isotonic and hypotonic LiCl solutions (RAVEN and ROBORGH 1949), however, the

swelling of the karyomeres is reversible, a deswelling takes place and development enters again into its normal path.

Presumably, simultaneously with the deswelling of the egg karyomeres in this case the swollen sperm nucleus also undergoes a process of condensation. This speaks again for the supposition that egg chromosomes and sperm nucleus respond alike to a common cytoplasmic factor. Moreover, our observation that the degeneration of the second maturation spindle and chromosomes in hypertonic solutions is, probably, accompanied with a similar degeneration of the sperm nucleus forms a further argument in favour of this opinion.

Finally, the observations in centrifuged eggs in which the polar bodies and animal pole plasm are formed in their proper place independently of the direction of stratification (RAVEN 1945) may be recalled here. They prove that factors directing the topographic relationships of the egg components are located in the egg cortex. Depolarization phenomena, as they occur under the influence of $CaCl_2$ (RAVEN and MIGHORST 1946), LiCl (DE GROOT 1948) and, sporadically, hypertonicity, may be explained, therefore, by a weakening of these cortical factors.

Though these observations clearly demonstrate the multiple interrelations between the components of the egg, still no clear picture of the governing mechanism of these processes can be obtained from them. Further data on the reactions of the eggs to outward stimuli have to be awaited.

Summary.

1. Freshly-laid eggs of *Limnaea stagnalis L.* have been treated with 0.1 — 0.7 m solutions of glucose and urea.

2. In isotonic and weakly hypertonic solutions, development is retarded. In stronger solutions, degeneration of the maturation spindles and asters occurs. With increasing concentration, first the second maturation spindle degenerates, then the inner aster of the first maturation spindle immediately after the extrusion of the first polar body, whereas in the strongest solutions the first maturation spindle degenerates.

3. In this degeneration, the chromosomes disappear at an early stage by the formation of vacuoles in which they are dissolved. Rod-shaped bodies are formed in the circumference of the asters, which are also surrounded by vacuoles; later, these vacuoles increase in size, whilst the central part of the asters and the spindle area become very dense and deeply-staining. In this way, a complete disorganization of the mitotic apparatus is produced.

4. Hypertonicity is not the direct cause of depolarization phenomena.

5. Hypertonicity as such causes no disturbances in the distribution of cytoplasmic materials.

6. The evolution of the sperm nucleus and the spermaster and the disappearance of the sperm tail occurs synchronously with the development of the egg as a whole, even when development is retarded.

7. The vacuolization of the cytoplasm decreases with increasing hypertonicity of the medium.

8. The effects of the treatment with hypertonic solutions vary with the moment of exposure and the susceptibility of the egg-masses.

9. Among the phenomena described by DE GROOT (1948) in *Limnaea* eggs treated with hypertonic LiCl solutions only the spindle and aster degenerations can be considered as a pure hypertonicity effect.

10. The evolution of the different egg components taking part in the processes of maturation and fertilization are rigidly bound together by a common mechanism, which determines the rhythm and sequence of the developmental steps. Interrelations between these components play a part in this mechanism.

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ARM EN BEEN VAN MENSCH EN ANTHROPOMORPHEN

DOOR

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In hetzelfde jaar 1859, waarin ik geboren werd, bracht de verschijning van CHARLES DARWIN's Origin of Species de geheele beschaafde wereld in beroering. Dat: 'de mensch van de apen afstamt' werd een dooddoener, waarmee bekrompen bestrijders der nieuwe inzichten in de levensleer de goegemeente opschrikten. Toen ik tot de jaren des onderscheids kwam, was in de kringen der biologen de strijd reeds vrijwel beslist, maar in die der onkundige menigte bleef hij voortsmeulen en is zelfs thans, bijna een eeuw daarna, nog geenszins uitgedoofd. Van de wetenschappelijke argumenten ter bestrijding of wat men daarvoor wilde laten doorgaan is mij uit de periode 1870 tot '80 bijzonder bijgebleven dat Professor JAN VAN DER HOEVEN in Leiden de helft van een overlangs middendoorgezaagd menschelijk skelet in kruipende houding op een schot had laten bevestigen, ten einde daarmede aan te toonen, dat in dien stand de knie zou stuiten tegen den elleboog, en dus bij iederen stap op handen en voeten een hinderlijke botsing van voorste en achterste ledematen zou plaats vinden. Dat bij de viervoetige voorouders der hedendaagsche tweebeenige menschen de proporties van voorste en achterste ledematen geheel gelijk zouden zijn aan de thans bestaande, scheen voor de tegenstanders der ontwikkelingsleer een uitgemaakte zaak, ofschoon zij toch wel niet onkundig zullen geweest zijn van het feit, dat de onderlinge verhouding in lengte tusschen arm en been niet bij alle menschenrassen dezelfde is, terwijl bovendien de kruipende zuigeling het levend bewijs was van de mogelijkheid dier wijze van voortbeweging, ten minste gedurende een bepaalde levensperiode. Maar ofschoon sedert lang wel geen enkele tot oordeelen bevoegde er aan twijfelt, dat de mensch zich in den loop zijner phylogenetische ontwikkeling uit de viervoetige houding heeft opgericht, toch blijft de vraag hoe die oprichting in haar werk is gegaan, en welken invloed zij op den bouw van het menschelijk lichaam heeft gehad, nog steeds niet bevredigend beantwoord. Van de vier wegen van onderzoek: de ontologische, de vergelijkend anatomische, de teratologische en de palaeontologische, schaft de laatste wel het minste licht, daar bij de weinig talrijke overblijfselen van menschachtige wezens uit vroegere aardperioden juist de ledematen het allergeringst bewaard zijn gebleven. Van Pithecantropus b.v. is wel een dijbeen, maar zijn geen armbeenderen gevonden. Vergelijkt men in het algemeen de hedendaagsche mensch met andere tweebeenige zoogdieren, waarmede

hij niet nader verwant is, b.v. met de kangeroe, dan zou men tot het besluit komen dat de oprichting op de achterste ledematen (die daardoor tot onderste worden), ten gevolge heeft gehad, dat deze langer en krachtiger zijn geworden, terwijl de voorste zwakker en korter werden. Tot een zelfde besluit komt men als men de reptielen in zijn beschouwingen betrekt. Vooral onder de Dinosauriërs zijn vormen met sterk ontwikkelde achterpooten en bijna rudimentair geworden voorste ledematen talrijk. Bij de vogels wordt de verhouding tusschen voorste en achterste ledematen in de eerste plaats beheerscht door de vliegfunctie der eerste. Verliest deze haar beteekenis, dan zien wij bij de loopvogels de gangpooten in lengte, omvang en kracht sterk toenemen, de vleugels daarentegen tot rudimenten inkrimpen. Hoe de reptielachtige voorouders der vogels, die nog niet vermochten te vliegen, maar ongetwijfeld reeds groote zweefsprongen maakten, gebouwd waren, kunnen wij ons slechts door hypothetische onderstellingen voor den geest roepen, maar zwak zullen hun voorpooten wel in geen enkel tijdperk hunner aanpassing aan de vliegfunctie geweest zijn. Bij de vergelijking van den mensch met de overige leden der Primatenorde staan wij voor een groote tegenstelling tusschen de naaste verwanten van den mensch: de Anthropomorphen, en de overige apen. Terwijl de armen der laatstgenoemde toch wel goed ontwikkeld zijn maar in lengte en kracht de achterste ledematen niet in eenigszins beteekenende mate overtreffen of er zelfs bij achterstaan, is dit bij de eerstgenoemde wel het geval, het allermeeste bij de gibbons wier bovenmatig verlengde armen hun korte beentjes meer dan tweemaal in lengte te boven gaan. Dat wij hierbij met secundaire verlenging in verband met het voltigeer-vermogen dezer absolute boombewoners te doen hebben, valt moeilijk te betwijfelen, en geldt schoon in mindere mate ook voor de overige mensch-apen. Maar desniettegenstaande bestaat er een groote tegenstelling tusschen de gibbons ter eene zijde, de orangoetans, chimpansé's en gorilla's ter andere. Terwijl deze laatste zich op hun handen kunnen neerlaten, en dit ook bij het loopen op den vlakken grond bijna altijd doen, gebruiken de eersten hun armen nimmer als steunpilaren, wanneer zij op hun korte beentjes over dwarsuitstekende takken dribbelen, maar steken hen zijdelings omhoog bij wijze van balanceerstokken, Of de gibbons in de vrije natuur wel ooit op den bodem komen, tenzij om te drinken, is mij onbekend. Bij hun verkeer met den mensch, dus in den toestand van geliefden huisgenoot, zitten zij graag op den grond en laten hun lange onderarmen daarop rusten zonder op de bovenarmen te steunen. Of de voorouders der gibbons in het begin der periode van progressieve verlenging hunner armen onder den invloed der oprichting op de achterste ledematen, wel ooit op hun handen gesteund hebben, lijkt mij twijfelachtig, maar in allen gevalle moeten zij dit zeer spoedig hebben nagelaten. Daarentegen moet men aannemen, dat de andere Anthropomorphen bij het terugvallen uit den opgerichten stand telkens weer op hun handen zijn neergekomen, zij het ook dat zij daarbij niet met hun handpalmen maar met hun knokkels den grond raakten. Bij den

primitieven mensch moet het verloop der vervorming gedurende den overgang van het vier- tot het tweebeenige stadium een eenigszins ander karakter gedragen hebben. Of de verlenging der voorste ledematen aanvankelijk die van de hedendaagsche gestaarte apen heeft overtroffen valt niet meer uit te maken, maar van veel beteekenis zal zij wel in geen geval geweest zijn, en op zijn knokkels zal de primitieve mensch wel nooit gesteund hebben. Daarentegen zijn de beenen gaandeweg in lengte en gespierdheid toegenomen, terwijl tegelijkertijd of reeds van te voren, de hielen tot op den grond werden neergedrukt. [1]) In dit laatste opzicht en wat de ontwikkeling in lengte en vermogen van de beenen aangaat, heeft de mensch de hoogste differentiatie bereikt, maar wat de lengte der armen betreft, is hij minder gedifferentieerd dan de Anthropomorphen. Men kan dit ook zoo uitdrukken: De mensch heeft in den loop zijner phylogenetische vervorming nimmer een chimpansé — of gibbon — stadium doorgemaakt. Hij is in de meeste opzichten primitiever gebouwd dan de Anthropomorphen. Zijn differentiatie is in hoofdzaak te danken aan het leven op den vlakken grond; het boomleven heeft slechts geringen en kortstondigen invloed gehad, zoo het al ooit heeft bestaan. Het oorsprongscentrum van den mensch lag niet in een tropische woudstreek, maar in een steppenland.

Summary.

Arm and Leg of Man and Anthropoids.

The relative proportions of their limbs show that Man has remained in a more original condition than the Anthropoids, the arms of the latter still continuing to grow in length after their ancestors had already assumed the erect attitude on their legs, instead of growing shorter, as is the case with most bipedal mammals and reptiles. Therefore this process of elongation may not be considered as a necessary consequence of this change in carriage. Neither can it be explained by the necessity of giving support to the trunk during forward movement, as is proved by the Gibbons, whose hands never touch the soil though their arms attain the greatest relative length of all Apes. Therefore it is probable that Man never passed through a secondary arboricole stage after changing from the quadrupedal to the bipedal attitude.

Résumé.

Bras et Jambes chez l'homme et les Anthropoïdes.

Les relations entre les proportions des membres supérieurs et inférieurs nous prouvent qu'à cet égard l'Homme occupe une position plus primitive

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[1]) Daarbij moet niet uit het oog verloren worden, dat bij iederen stap de hiel weer van den grond opgeheven wordt, en bij het rennen zelfs den bodem in het geheel niet meer raakt. Schuifelende gang is een gevolg van verzwakking. De gewoonte om de schoenen van hakken te voorzien, berust op de behoefte, het optillen der hielen te vergemakkelijken.

que les Anthropoïdes, puisque les bras de ces derniers continuaient à s'allonger encore après qu'ils avaient adopté la position bipède, de sorte que leurs mains touchaient de nouveau la terre. Il me semble improbable que l'Homme aurait jamais passé par un état secondaire de vie arboricole, puisque les Gibbons, qui se sont adoptés au plus haut degré à ce genre de vie ont les bras les plus longs et les jambes les plus courts de tous les Anthropoïdes.

Zusammenfassung.

Arm und Bein beim Menschen und den Anthropomorphen.

Die Beziehungen zwischen den Proportionen der menschlichen Gliedmassen und jenen der Menschen-Affen beweisen uns dass in dieser Hinsicht die Menschen einen primitiveren Zustand vergegenwärtigen als die übrigen Anthropoïden, weil bei den letzteren die Verlängerung der Arme sich noch fortgesetzt hat, nachdem sie den aufrechten Stand schon angenommen hatten. Dies hatte zur Folge dasz beim Orang, Chimpanse und Gorilla die Hände den Boden aufs Neue wieder berührten, während dagegen die Gibbons, deren Arme die grösste Länge erreichten und deren Beine am kürzesten wurden, niemals mit ihren Händen den Boden berühren.

Ob der Mensch jemals ein secündäres Baumleben-stadium durchmachte nach der Annahme der aufrechten Haltung, scheint mir zweifelhaft.

MATHEMATICS

QUELQUES TEXTES MATHÉMATIQUES DE LA MISSION DE SUSE

PAR

E. M. BRUINS

(Communicated by Prof. L. E. J. BROUWER at the meeting of June 24, 1950)

Les tablettes mathématiques faisant l'objet de la présente étude ont été trouvées par Mr R. DE MECQUENEM à Suse au chantier no I de la ville Royale en 1936 [1])

Les fouilles amorcées en 1922 arasèrent le terrain à un niveau inférieur de 3 à 5 m au sol naturel.

Sous des caveaux funéraires se trouvait un grand dallage. C'est seulement en 1933 que fut atteint le substratum du dallage entouré de caveaux funéraires Elamites. Il s'agit donc d'une butte funéraire élevée en bordure de la ville: elle comportait un sanctuaire, qui dut être reconstruit au fur et à mesure de la hauteur atteinte par la superposition des tombes.

Dès 1933 Mr R. DE MECQUENEM a trouvé entre les deux dallages successifs signalés plus haut des tablettes en terre crue de l'époque des "soukkals" de Suse généralement fragmentées et en mauvais état. Les textes scolaires publiés par P. E. VAN DER MEER [2]) appartiennent à ce lot; les textes de cet ouvrage nos. 291 à 296 se rapportent à des excercices de calcul généralement assez simples. Les tablettes de 1936 relatives à de nouveaux problèmes appartiennent à la même série. M. le Docteur G. CONTENAU, Directeur Général des fouilles avait confié ces tablettes à Mlle M. RUTTEN, attachée à la Mission de Suse pour la Publication. Après les avoir copiées Mlle M. RUTTEN a fait appel à ma collaboration pour en donner le commentaire mathématique suivant.

Les tablettes et les problèmes ont été numerotés 1, 2, 3, ... A, B, ... Quelques unes ne contiennent que des choses bien connues. Par exemple: la tablette K donne une table de multiplication pour 25: la tablette F contient deux problèmes: équations linéaires à une variable en formulant les problèmes en largeur et longueur d'un rectangle, donnant la largeur: la tablette O traite problèmes du type $lb + (l \pm b)^2 = A$, $l \pm b = B$, la solution écrite presque complètement en phonétique. Celles, dont les parties abimées peuvent être reconstruites d'une manière univoque (à l'exception peut-être de S) et qui, à l'exception de A. B. donnent des

[1]) Cf. Mém. de la Miss. Archéol. en Iran, tome XXIX; pp. 44–62, Presses Universitaires de France–Paris 1943.

[2]) Mém. de la Miss. Archéol. de Perse, tome XXVII. E. LEROUX, Paris, 1935.

problèmes nouveaux seront discutées ici du point de vue mathématique. Elles sont:

4. Calcul du rayon du cercle circonscrit à un triangle isocèle,
3. Polygones réguliers,
I. Table de constantes mathématiques,
A, *B*. Livre d'instruction,
C. Calcul d'un rectangle par transformation semblable,
D. Solution d'une équation de huitième degré,
H. Changement de variables, *KI—GUB*,
Q. Calcul d'intérêt composé et annuité,
S. Problème de division d'un triangle.

La tablette 4.

Cette tablette (comp. fig. 1) contient évidemment le calcul du rayon d'un cercle circonscrit à un triangle isocèle. La base étant 60 et les deux

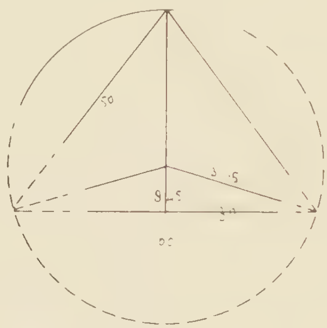

Fig. 1

autres cotés 50, la hauteur sera 40, d'après le théorème de Pythagore. Alors on a l'équation

$$(40 - x)^2 + 30^2 = x^2 \quad \text{ou bien} \quad 4.10\, x = 1.20 \quad x = 31.45 \quad 40 - x = 8.15$$

La tablette 3.

3*a* contient un hexagone régulier (fig. 2). Le coté et le rayon portent

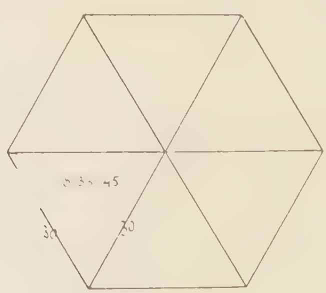

Fig. 2

les valeurs 30 et un des triangles centraux porte le nombre 6.33.45. Nous

avons déduit [3]) de *VAT* 6598, que les formules pour le calcul de la racine carrée d de

$$d^2 = a^2 + b^2$$

étaient

$$a + \frac{b^2}{2a+b} < d < a + \frac{b^2}{2a}.$$

De ces deux formules on tire immédiatement en les résolvant pour a

$$d - \frac{b^2}{2d-b} < a < d - \frac{b^2}{2d}.$$

Cette méthode nous donne pour $x = \sqrt{3}$; $4 = x^2 - 1$, $x = 2 - \frac{1}{4} = 1.45$.

On trouvera donc pour la hauteur d'un triangle régulier: 52.30.

La méthode la plus directe pour obtenir l'aire du triangle indiqué sur 3*a* nous donne, à cause de $2 \times 30 = 1$, $30 - 15^2 = 30 - 3.45 = 26.15$ et par conséquent l'aire est $15 \times 26.15 = 6.33.45$.

On peut donc interpréter le contenu de 3*a* en posant $\sqrt{3} = 7/4$. D'autre part il est évident que le calcul ne peut pas se faire par $x^2 = 30^2 - 15^2$ parce que le radicand est $26^2 - 1$, ce qui donne en tout cas une valeur plus petite que 26. La choix du coté 30 doit donc avoir une autre raison qu'une augmentation de précision. Vraisemblablement, c'est parce que la formule pour l'aire totale du polygone à n côtés prend la forme simple

$$O = nx$$

x représentant la hauteur du triangle central.

Ce point de vue est supporté par 3*b*. L'inscription est presque complètement détruite, mais la figure montre un heptagone, le rayon étant 35, et les facteurs 7 et 4 et le mot "l'aire" sont à peu près les seules choses qui n'ont pas été détruites. En effet, mesurant le coté de l'heptagone on obtient pour la longueur relative 30.

De cette tablette on peut donc déduire que le quotient du rayon du cercle et du coté de l'hexagone et de l'heptagone a été posé 1 et 1.10 et en extrapolant on pourrait poser pour le pentagone 50.

Alors l'aire du pentagone serait $5 \times \frac{1}{2} \times 40 = 1.40$ (comp. Heroon, Metrica) et celle de l'hexagone $4 \times 6 \times 6.33.45 = 2.37.30$.

Pour l'heptagone on a pour la hauteur du triangle central x

$$x^2 = (1.10')^2 - (30')^2 = 1.6'.40'' = 1 + (20')^2$$

donc

$$1.3'.20'' > x > 1 + \frac{20'^2}{2.20} = 1 + \frac{20'}{7}$$

et multipliant par $\frac{1}{2} \times 7 = 3\frac{1}{2}$

$$3.41'.40'' > O_7 > 3.40'.$$

Cette interprétation est tout à fait confirmée par la tablette *I* des constantes mathématiques, qui donne en ligne 26, 27, 28, 29, 31 *)

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[3]) Proc. Kon. Ned. Akad. v. Wetensch. Amsterdam, **51**, 332 seq. (1948).

*) Ligne 30 donne: 57.36 igi-gub ša SAR c.à.d. $\pi = 3^1/_8$, parce que $57.36 = {}^{24}/_{25}$.

1.40	la constante du pentagone
2.37.30	la constante de l'hexagone
3.41	la constante de l'heptagone
52.30	la constante du triangle
1.25	la constante de la diagonale du carré.

La tablette I.

Cette tablette contient 70 constantes mathématiques, parmi lesquelles apparait de nouveau: 18, im–li–im. Les premières 36 données sont des constantes pour des figures géométriques. La tablette commence par les valeurs igigub, ri et pi–ir–ku du cercle 5, 20, 10 pour l'aire, le diamètre et le rayon du cercle exprimé par le périmètre c

$$\frac{1}{4\pi}c^2,\quad \frac{1}{\pi}c,\quad \frac{1}{2\pi}c,\quad \pi = 3.$$

Plus loin on donne 15, 40, 20 pour l'aire, diamètre et rayon d'un uš–qa–ri ce qui dénote le demicercle dont l'aire est calculée comme un quart du produit de l'arc et du diamètre (comp *BM* 85210) et qui met en évidence le procédé suivi en *BM* 85210 pour obtenir l'arc, en prenant la somme du diamètre et du rayon! En ces deux cas par lesquels la table commence l'arc du cercle est pris comme unité. Toute une série de constantes, groupées en triades analogues, qui suit peut être interprétée comme des constantes analogues pour différents segments du cercle, ayant l'arc comme unité de longueur.

Les problèmes A, B.

Ces problèmes ont une importance du point de vue de didactique babylonienne. Il s'agit de deux problèmes d'équations linéaires resp. à deux et à trois inconnues. Le problème A consiste des deux équations, (les inconnues sont nommées uš $= l$ et sag $= b$)

$$\tfrac{1}{4}b + l = 7 \text{ šu} \quad ; \quad l + b = 10\, \check{s}u.$$

Il est remarquable, que partout dans la solution les termes uš et sag signifient "coefficient de uš" et "coefficient de sag". D'abord on donne la solution heuristique. On transforme les šu en doigts: $5 \times 7 = 35$. Décomposition $35 = 30 + 20$; $5 \times 10 = 50$; décomposition $30 + 20$. Alors on multiplie $30 \times 4 = 2.0$ et $4 \times 5 = 20$. On continue: $20 - 20$ et $2.0 - 30$, tu vois 1.30. On évite évidemment le "0 ta–mar" en continuant simplement par "et". Prenant un tiers de 1.30 on obtient 30 pour la longueur et $50 - 30 = 20$ pour la largeur.

Cette démonstration prend 11 lignes sur un total de 16 du problème A. Dans les 5 dernières lignes la solution directe est donnée:

,,De nouveau:

$7 \times 4 = 28$, $28 - 10 = 18$, $18 \times 20 = 6$, $10 - 6 = 4$, $5 \times 6 = 30$, $5 \times 4 = 20$''.

Le problème B contient les trois données pour les inconnues uš, sag et si–a

$$\tfrac{1}{4}b + l = 7 \text{ šu}; \; l + b + t = 11 \text{ šu}; \; t = 5$$

Ce problème est d'abord discuté du point de vue heuristique comme en A. On multiplie

$$5 \times 7 = 35 = 30 + 5 \quad 5 \times 11 = 55 = 30 + 20 + 5$$

et on a réduit le problème au problème A, dont la même solution suit

$$4 \times 5 = 20, \; 4 \times 30 = 2.0, \; 20 - 20 \text{ et } 2.0 - 30 \text{ est } 1.30$$

et le scribe ne fait plus les autres calculs mais ajoute "et 5 le si–a".

De nouveau on donne la solution directe:

$7 \times 4 = 28$, $28 - 11 = 17$ et on diminue le coefficient de uš en $4 - 1 = 3$;

on prend un troisième de $17 = 5.40$ et un troisième de si–a $= 1.40$. On soustrait les 5.40 de 11 en obtenant 5.20 et on ajoute 5 de si–a à $1.40 = 6.40$.

Ensuite on réduit les šu: $5 \times 5.48 = 28.20$ et $5 \times 5.20 = 26.40$. La solution est maintenant trouvée par $28.20 + 1.40 = 30$ et $26.40 - 6.40 = 20$.

Le problème C.

Dans le problème C il s'agit d'un rectangle, l, b, diagonale d. On pose la question:

Largeur sur (a–na) longueur, un quart l'excès. 40 la diagonale.

On pose $l = 1$ et 15 le prolongement (*daḫ*) donc $45 = b$ et la racine carrée de $1 + (45')^2 = 1.33'.45''$, soit $1.15'$, est la diagonale. Parce que "j'ai dit 40, la diagonale, dénoue l'inverse de $1.15'$: c'est 48"

$$48 \times 40 = 32 \quad 32 \times 1 = 32 = l \quad 32 \times 45 = 28 = b.$$

Le problème D.

On donne: 20 l'aire. Longueur multiplié par son carré et par la diagonale 14.48.53.20. Combien longueur, largeur et diagonale?

En symboles modernes on a

$$lb = 1200 = O \quad l^3 d = 3200000 = B \text{ donc}$$
$$B^2 = l^6 d^2 = l^8 + l^6 b^2 = l^8 + l^4 O^2$$

et en résolvant l'équation quadratique en l^4

$$l^4 = -\tfrac{1}{2} O^2 + \sqrt{B^2 + (\tfrac{1}{2} O^2)^2} \;, \; l^2 = \sqrt{l^4}.$$

En effet le scribe obtient la valeur de l^2 par le calcul de ces expressions et en utilisant des nombres à huit sexagésimales il obtient le résultat $b = 30$ par $b^2 = O^2/l^2$ et prenant la racine carrée de cette dernière valeur:

Il carre l'aire $20^2 = 6.40$ et il carre B: $(14.48.53.20)^2 = 3.39.$ [28.43. 27]24.26.40. Il prend la moitié de $6.40 = 3.20$ et la carre 11.6.40 et il

ajoute 3.39.28.43.27.[24.26.40] à 11.6.40 et il écrit pour la somme 3.50.36.43.34.26.40. au lieu de la somme correcte 3.50.35.23.27.24.26.40 ce qui n'est qu'une faute faite en copiant le "brouillon" parce qu'il continue: Combien la racine carrée? 15.11.6.40 est la racine carrée et $(15.11.6.40)^2$ est bien 3.50.35.23.27.24.26.40. Il soustrait ensuite 3.20 de 15.11.6.40 et obtient 11.51.6.40 dont il prend la racine carrée 26.40. La valeur réciproque de 26.40 est 2.15, par laquelle il multiplie 6.40 le carré de l'aire obtenant [15] dont la racine carrée est 30, la largeur.

La tablette H.

La tablette *H* contient trois problèmes concernant le changement de variables. Le premier problème (ligne 1 — 9): la somme de l'aire et la longueur = 40 (20 la largeur, que j'ai dite) et la solution est donnée en ajoutant une unité — *KI GUB-GUB* à la largeur ce qui transforme le problème en: l'aire = 40, largeur 1.20, ki–a–am ta–ša–al, que tu vas investiger, rechercher.

Le deuxième problème (ligne 10 — 18) est le même que le premier, seulement la solution est donnée "comme l'accadien la fait". Il ajoute 1 à la largeur et une unité à la longueur et en ajoutant à l'aire. 40, la largeur nouvelle on obtient le problème: l'aire = 2, largeur = 1.20. C'est à dire que "l'accadien" utilise l'identité

$$(l + 1)(b + 1) = (lb + l + b + 1)$$

pour obtenir des données numériques plus simples.

Le troisième problème (ligne 19 — 51) contient la solution de

$$lb + l + b = 1$$
$$b + \tfrac{1}{17}(3l + 4b) = 30$$

En lignes 19 — 25 on calcule en multipliant par 17 les coefficients de

$$3l + 21b = 8.30$$

En lignes 26 — 30 on ajoute une unité à la largeur et à la longueur et on obtient l'équation

$$LB = 2;\ L = l + 1;\ B = b + 1.$$

En lignes 31 — 34 on transforme par 21 + 3 + 8.30 = 32.30 la deuxième équation en l et b en

$$3L + 21B = 32.30.$$

Ensuite on multiplie 3 × 21 × 2 = 2.6 et on calcule les racines de l'équation quadratique

$$X^2 - 32.30\,X + 2.6 = 0$$

dont les racines sont $3L$ et $21B$, desquelles on obtient $l = 30$ et $b = 20$.

Les signes *KI-GUB* désignent donc dans cette tablette un changement de variable pour simplifier le calcul.

Remarque. *KI–GUB* se trouve aussi en *BM* 85210 et l'interprétation de ces signes donnée par la tablette *H* peut éliminer aussi toutes les difficultés pour *BM* 85210. Le problème serait alors (fig. 3): Volume

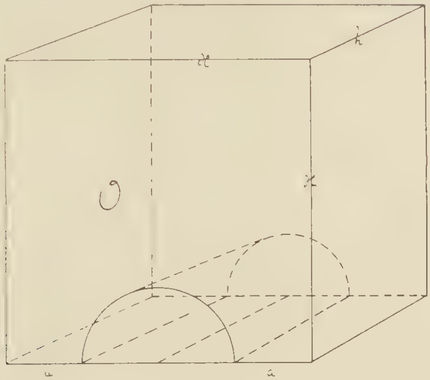

Fig. 3

1.12.30, $h = 3$, $a = 10$. Cela nous donne $\frac{1}{3} \times 1.12.30 = 24.10 = O$. Alors

$$x^2 - \frac{\pi}{8}(x - 20)^2 = 24.10$$

ou

$$\left[1 - \frac{\pi}{8}\right] x^2 + 5\pi x - \frac{\pi}{8}(20)^2 = 24.10.$$

Le coefficient $\pi/8 \times (20)^2$ est calculé par $KI–GUB = 10$ comme $\frac{1}{6}$ $(30)^2$ c.à.d. au lieu de $20^2 = 6.40$; $3 \times 6.40 = 20$; $\frac{1}{8} = 7.30$, $7.30 \times 20 = 2.30$ le scribe calcule $30^2 = 15$, $10 \times 15 = 2.30$ parce que $\frac{1}{6} = 10$ est fondamental pour le système sexagésimale. Ensuite on trouve $2.30 + 24.10 = 26.40$.

Tout de même on obtient $[1 - \pi/8]$ par $1 - \frac{1}{6}(\pi/2)^2 = 1 - \frac{1}{6}(1.30)^2 = 22.30$ et $\frac{1}{2} \times 5 \times \pi = \frac{1}{6} \times 45 = 7.30$. Ainsi on a toutes les constantes nécessaires pour la solution; $x = 40$.

La tablette Q.

La tablette *Q* nous donne le problème, partant d'une quantité initiale de 1; 31.58.48 še, l'intérêt de 3.30 še étant 1.30 še par an et en prenant chaque année 1 še; à calculer le total après trois années.

L'intérêt est calculé par $\frac{1}{2}(A - \frac{1}{7}A)$ donc

1.31.58.48. — 13.8.24 = 1.18.50.24	1.31.58.48 + 39.25.12 = 2.11.24
1.11.24 — 10.12 = 1.1.12	1.11.24 + 30.36 = 1.42
42 — 6 = 36	42 + 18 = 1.0

La tablette S.

Le contenu de la tablette *S* nous donne

> . . .[4]) longueur inférieure multiplié par longueur supérieure est . . .
> aire supérieure multiplié par aire inférieure: 36

[4]) Le signe est la figure 4, dont l'hypotenuse a été détruite.

La somme du carré de ... rieure et du carré de la ligne de séparation: 20.2x (ou $x = 4$, 5, ou 6)
Toi, le produit de l'aire par l'aire, 36, multiplie par 4: 2.24. Dénoue l'inverse de 10, le produit du prolongement et prolongement, 6: $6 \times 2.24 = 14.24$. Carre 14.24: 3.27.21. (36) 14.24 par 2 ....

Sauf quelques traces le reste a été détruit. Mais un 30 ta-mar et un 20 ta-mar ont été conservés vers la fin de la solution.

Par un hasard la reconstruction de cette tablette est de nouveau univoque. Parce qu'il s'agit de trois inconnues, dont deux sont des segments contigues d'une même ligne (ce qui est prouvé par le terme daḫ, comp. tablette C) et d'une ligne de séparation, même dans le cas d'un triangle, il faut que l'on ait consideré un triangle spécial. Le produit des deux segments de la longueur k.a doit être 10, selon la valeur utilisée au commencement de la solution. De plus les nombres 20.24; 20.25; 20.26; ne peuvent être décomposés en somme de deux carrés que par

$$
\begin{aligned}
20.24 &= (30)^2 + (18)^2 \\
20.25 &= (21)^2 + (28)^2 \\
20.26 &= (35)^2 + 1
\end{aligned}
$$

et par hasard, sauf dans le premier cas on a des nombres irréguliers: 21, 28, 35 qui ne sauraient donner un produit régulier en les multipliant par un nombre entier, casu quo, une fraction sexagésimale finie. La combinaison 1, 35 donnerait de plus une division vraiment bizarre. Les seules nombres que l'on puisse avoir sont donc longueur supérieure $a = 30$, ligne de séparation $r = 18$ et par conséquent ligne inférieure $k = 20$. En supposant que le triangle soit rectangulaire comme dans la plupart des problèmes de division (comp. *M L C* 1950) on obtient pour l'aire supérieure $A = 30 \times 9 = 4.30$ et pour l'aire inférieure $K = 10(30 + 18) = 8$ et le produit $AK = 36.0.0.$

Le problème posé était alors (fig. 4)

$$
\begin{gathered}
ak = 10 \\
AK = 36.0.0. \\
r^2 + a^2 = 20.24
\end{gathered}
$$

Puisque l'on a $A = \frac{1}{2}\, ar$; $K = \frac{1}{2}\, kr(2a + k)/a$ on obtient

$$AK = \tfrac{1}{4}\, r^2\, (2a + k)k$$

ou bien

$$AK\, a^2 = \tfrac{1}{4}\, r^2 (2a^2 + ak)ak$$

et la réduction du problème à une équation de quatrième degré, soluble par une équation quadratique en a^2 devra commencer par le calcul de $4AK$ et par la division du résultat obtenu par ak - - - et celui est exactement ce que le scribe fait. Il multiplie 36.0.0. par 4 et multiplie le résultat

par 6, l'inverse de 10. Alors un calcul des coefficients de l'équation pour a^2 ou r^2 doit commencer (et celle pour a est

$$14.24\, a^2 = (20.24 - a^2)\,(2a^2 + 10.0).)$$

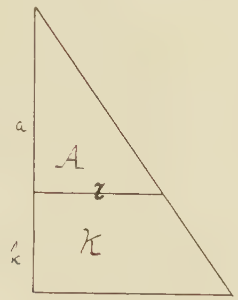

Fig. 4

Conclusion.

Les tablettes mathématiques de la Mission de Suse jettent une lumière tout à fait nouvelle sur "les mathématiques des Babyloniens". Sauf la table des igi–gub, des constantes mathématiques, elles nous donnent la démonstration de l'existence d'une théorie des polygones réguliers à 5, 6 et 7 cotés, qui est du point de vue numérique identique à la théorie de Héroon. Soit que l'application pratique des résultats obtenu pour les polygones réguliers n'est guère évidente, l'opinion commune, que les mathématiques babylonniennes ne considéraient que des problèmes pratiques est nettement contredite par les tablettes D et S. Le cube de la longueur multiplié par la diagonale d'un champ rectangulaire n'a pas une signification pratique, économique. La division d'un champ triangulaire de telle façon, que le produit des aires des parties ait une valeur prescrite n'a pas une signification pour "la vie de tous les jours". Ces problèmes ont vraiment été posés et résolus du point de vue purement scientifique; la science pour la science.

MATHEMATICS

THE MEDIAN AND INTERQUARTILE RANGE TEST APPLIED TO FREQUENCY DISTRIBUTIONS PLOTTED ON A CIRCULAR AXIS

BY

J. WESTENBERG

(Communicated by Prof. M. W. WOERDEMAN at the meeting of May 20, 1950)

In statistical science, univariate populations and samples are specified by the distribution of the frequencies for the different values of the variate under consideration. These frequency distributions may be represented graphically by plotting the frequencies against the corresponding values of the variate, measured along a straight axis.

For the purpose of a concise description or a comparison of populations or samples, some general characteristics are introduced viz.

1. a *measure of location* (some mean value, median or mode) i.e. some central value giving the whereabouts of the distribution along the axis.
2. a *measure of dispersion*, giving the degree of scatter about this value.
3. a *measure of skewness*, giving the departure of the distribution from symmetry.

In some less common cases, as for instance the number of events during a given period of time coinciding with different directions of wind or water current, or occurring in different seasons, or at different times of the day &c., a suitable graphical representation should rather be given along a circular axis instead of a straight one (fig. 1). This

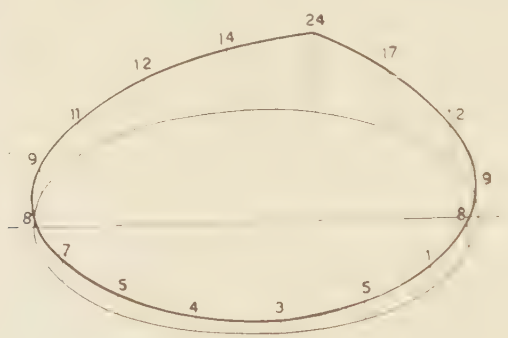

Fig. 1

is especially the case when, in grouped or in discontinuous distributions, all intervals on the axis are occupied.

It is worth noting that in graphical representation on a circular base line with all intervals occupied, the most common measures of location, viz. the means and the median, do not find application. On the contrary,

the mode, the least appreciated among the measures of location, is still valid as usual for plottings on a circular axis. This does not permit, however, the application of a significance test in the comparison of two samples.

Although the concept of the median does not apply to this kind of frequency distributions, the principle of the median and the interquartile range test, needs only a slight extension to yield a significance test for the kind of distributions under discussion.

The median and the quartiles together divide a population into four equal portions. Now a division into four equal portions could as well be started at any other point of the straight axis; the two remaining outer parts together will then constitute one portion.

The same procedure could be started at every point of a circular axis. To achieve as close a resemblance as possible to the use of the median in the case of a straight axis, we could start the division in the region of minimum frequencies, between two frequency classes. This starting point will make the zero point of an ogive curve that can be imagined along the cirular axis (fig. 2). The point at which this curve

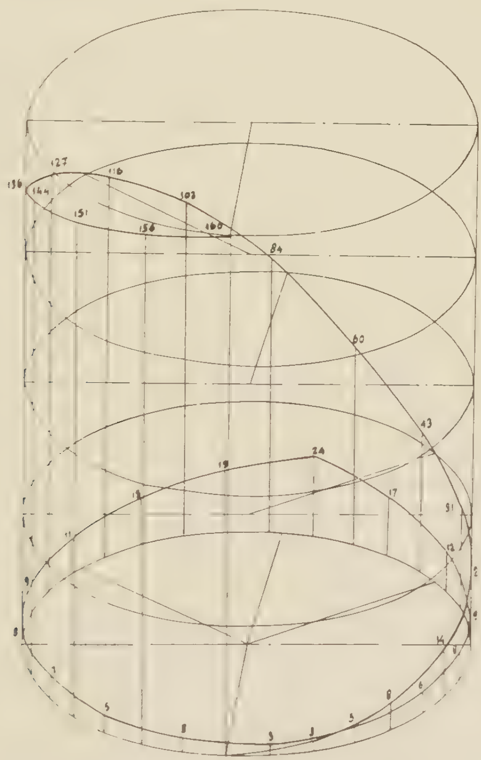

Fig. 2

cuts the horizontal plane, corresponding to a frequency equal to one half of the sample, gives the median, or rather a value measured on the circular axis that can be used as median. Similarly the points of the ogive

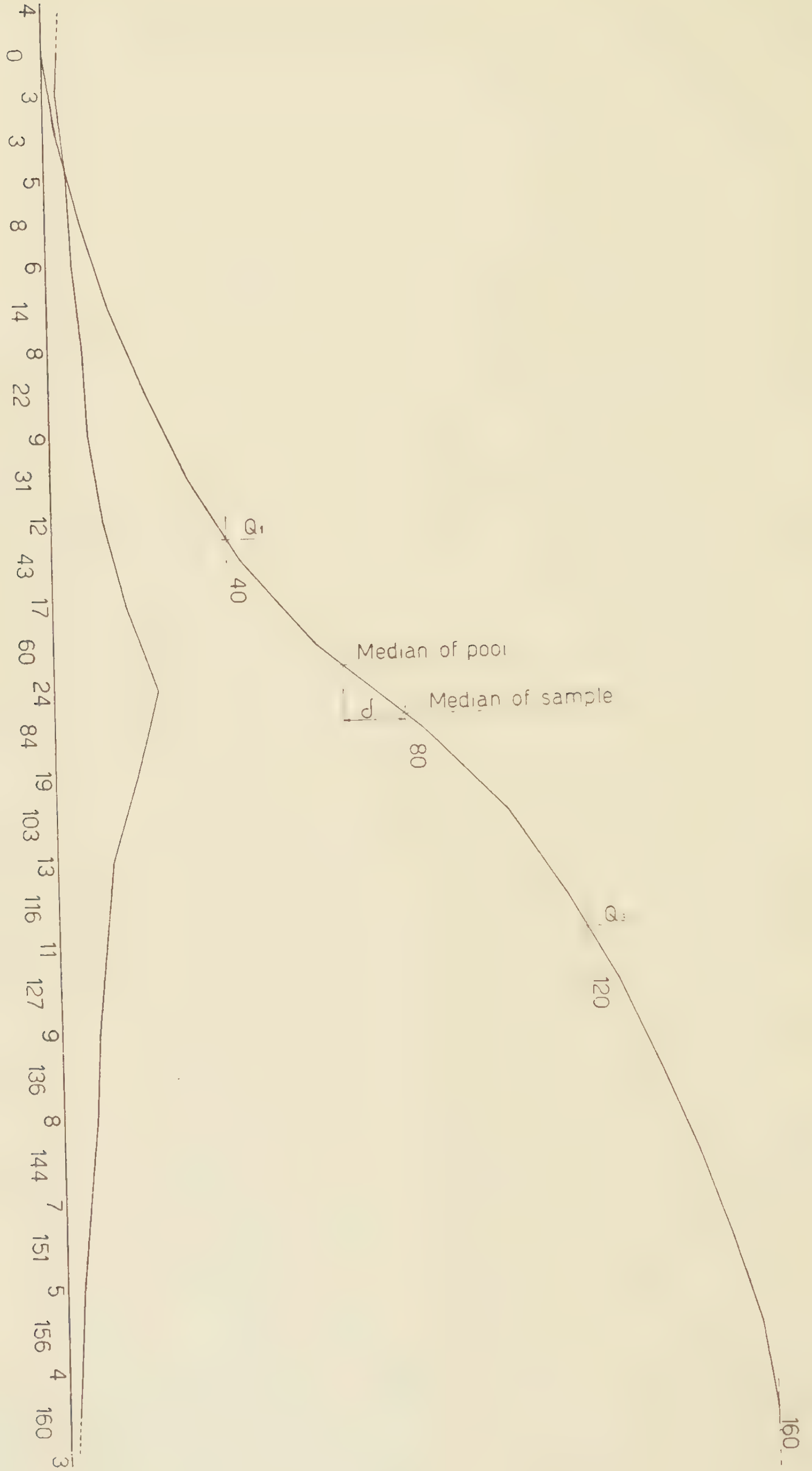

Fig. 3

corresponding to frequencies of $\frac{1}{4}$ and $\frac{3}{4}$ of the sample respectively, will yield values that can be regarded as the quartiles. This construction can suitably be carried out on ordinary graph paper (fig. 3). The resulting division of the sample in four equal parts can again be demonstrated on the circular axis (fig. 4).

In the comparison between two samples, the same procedure can

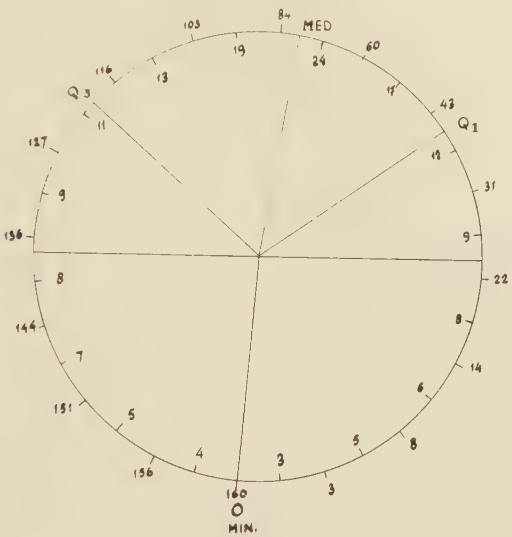

Fig. 4

be carried out with the pool of the samples. The median of the pool is then inserted in the graph representing one of the samples. The number of observations in this sample, between the median of the pool and the median of the sample proper will correspond to $|\delta|$ in the ordinary median test, which can hence be applied without any difficulty.

Similarly, the number of specimens between two quartiles (e.g. on the side of greater density) can be used for testing the significance of differences between two samples. To this end, the quartiles of the pool are again inserted in the graph representing one of the samples. The difference in number of observations between the quartiles of the pool, and the quartiles of the sample proper corresponds to $|\delta|$, so that the significance of differences can be tested with the same $|\delta|$ table.

In the event of no satisfactory significance being revealed, other starting points for the ogive curve may be tried.

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MATHEMATICS

UNIFORM DISTRIBUTION (MOD 1) IN SEQUENCES OF INTERVALS

BY

L. KUIPERS AND B. MEULENBELD

(Communicated by Prof. J. G. van der Corput at the meeting of May 20, 1950)

§ 1. *Introduction. New definitions.*

In our previous papers [1], [2] and [3] we studied the C-distribution (mod 1) of the values of measurable functions $f(t)$ for $t \geqq 0$. In the definition of C^{I}-uniform distribution (see [1], p. 1152), Definition I, we introduced the expression

$$\| \mathfrak{S}_T(\alpha,\beta) \| \equiv \int_0^T \theta(\alpha,\beta; f(t))\, dt \quad (T>0),$$

which is the measure of the set of points t in the interval $0 \leqq t \leqq T$ with $0 \leqq \alpha \leqq f(t) < \beta < 1$ (mod 1).
If now

$$\lim_{T\to\infty} (1/T) \| \mathfrak{S}_T(\alpha,\beta) \| = \beta - \alpha, \tag{1}$$

for every pair of fixed numbers α, β $(0 \leqq \alpha < \beta < 1)$, then $f(t)$ is C^{I}-uniformly distributed (mod 1) (shortly C-u-d (mod 1)). It is clear that (1) must hold independent of the manner in which T tends to infinity.

We remark that there exist functions $f(t)$ for which (1) holds if T runs through a special fixed sequence $T_1, T_2, \ldots \to \infty$, and for which (1) is not true if T runs through another special fixed sequence $T_1^*, T_2^*, \ldots \to \infty$. In § 2 we give an example. According to Definition I such functions are not C-u-d (mod 1).

It is reasonable to introduce the following

Definition:

The function $f(t)$, defined for $t \geqq 0$, is said to be C-u-d (mod 1) *in the (fixed) sequence F of intervals*

$$Q:\ 0 \leqq t \leqq T_k \ (k=1,2,\ldots) \text{ with } T_k \to \infty \ (k\to\infty), \tag{2}$$

if, for every pair of fixed numbers α, β $(0 \leqq \alpha < \beta < 1)$,

$$(1/T_k) \| \mathfrak{S}_{T_k}(\alpha,\beta) \| \to \beta - \alpha,$$

if Q runs through F. Otherwise we call $f(t)$ not C-u-d (mod 1) *in the sequence F of intervals* (2).

By means of this definition we are able to extend our former definitions of C-distribution in the following way.

Definitions (*a*), (*b*), (*b* 1), (*b* 2).

(*a*) *The function* $f(t)$ *is said to be C-u-d* (mod 1) *if* $f(t)$ *is C-u-d* (mod 1) *in all sequences* F *of intervals* (2).

(*b*) *The function* $f(t)$ *is said to be not C-u-d* (mod 1) *if there exists at least one sequence* F *of intervals* (2) *in which* $f(t)$ *is not C-u-d* (mod 1).

Case (*b*) *includes two subcases:*

(*b* 1) *The function* $f(t)$ *is said to be relatively C-u-d* (mod 1), *if there exists at least one sequence* F_1 *in which* $f(t)$ *is C-u-d* (mod 1), *and at least one sequence* F_2 *in which* $f(t)$ *is not C-u-d* (mod 1).

(*b* 2) *The function* $f(t)$ *is said to be totally not C-u-d* (mod 1) *if there is no sequence* F *of intervals* (2) *in which* $f(t)$ *is C-u-d* (mod 1).

The C-test for the case of *C-u-d* (mod 1) in a sequence of intervals takes the form:

Necessary and sufficient for the C-u-d (mod 1) *in the sequence* F *of* (2) *is, that for every* h, *integer*, $\neq 0$, *the expression*

$$(1/T_k) \int_0^{T_k} e^{2\pi i h f(t)}\, dt$$

tends to zero if $k \to \infty$.

The proof is quite analogous to that given in [4] by the first of us.

Another problem studied in the present note deals with the distribution of the rests (mod 1) of the values of $f(t)$ in a (fixed) sequence F of intervals

(3) $\quad Q\colon\ 0 < S_k \leqq t \leqq T_k\ (k = 1, 2, \ldots)$ with $T_k - S_k \to \infty\ (k \to \infty)$.

Here we formulate the following

Definition:

The function $f(t)$ *is said to be C-u-d* (mod 1) *in the sequence* F *of intervals* (3) *if*

$$\frac{1}{T_k - S_k} \int_{S_k}^{T_k} \theta\,(\alpha, \beta\,; f(t))\, dt \to \beta - \alpha,$$

for every pair of fixed numbers α, β $(0 \leqq \alpha < \beta < 1)$, *if* Q *runs through* F.

This definition is a special case of the more general one (for n dimensions and m functions) given by the second of us in [5].

The C-test here to be used takes the following form: *Necessary and sufficient for the C-u-d* (mod 1) *of* $f(t)$ *in the sequence* F *of intervals* (3) *is that, for every* h, *integer*, $\neq 0$,

$$\frac{1}{T_k - S_k} \int_{S_k}^{T_k} e^{2\pi i h f(t)}\, dt \to 0,$$

if Q *runs through* F.

In our investigations concerning the distribution in sequences (S_k, T_k) we assume that $S_k \to \infty$ if $k \to \infty$. Otherwise (S_k is bounded) it is obvious that we have the same situation as $S_k = 0$ $(k = 1, 2, \ldots)$.

In the present note we consider the connections between the two kinds of distribution, say the case $(0, T)$ and the case (S, T) of a function $f(t)$.

In § 3 we show that, if a function $f(t)$ is totally not C-u-d (mod 1) (case b 2), this does not imply, that there would be no sequence F of intervals (3) in which $f(t)$ is C-u-d (mod 1), and in § 4 we prove that, if a function $f(t)$ is C-u-d (mod 1) (case a), it is not excluded there exist sequences F of intervals (3) in which $f(t)$ is not C-u-d (mod 1).

In §§ 5 and 6 we give sufficient conditions to make either on $f(t)$ or on the intervals (3) such that from the behaviour of $f(t)$ in the case $(0, T)$ conclusions can be drawn about the properties of $f(t)$ in the case (S, T).

In § 7 we deduce some theorems concerning the distribution merely in the case (S, T).

§ 2. *Relative C-uniform distribution* (mod 1).

In the present § we give an example of a measurable function $f(t)$ which is C-u-d (mod 1) in a special fixed sequence $(0, T_k)$, and which is not C-u-d (mod 1) in another special fixed sequence $(0, T_k^*)$ $(T_k, T_k^* \to \infty)$.

Firstly we construct both sequences.

Let α be an arbitrary fixed number with

$$0 < \alpha < \frac{1}{\sqrt{1 + 4\pi^2}}.$$

Since, for $T_1 > 0$ and for every h, integer, and $\neq 0$,

$$I_1 = |(1/T_1) \int_0^{T_1} e^{2\pi iht}\, dt| \leq (1/T_1 \pi |h| \leq (1/T_1 \pi),$$

it follows that there exists a fixed $T_1 > 0$ such that $I_1 < \alpha$.
From

$$I_1^* = |(1/T_1^*) \int_0^{T_1} e^{2\pi it}\, dt + (1/T_1^*) \int_{T_1}^{T_1^*} e^{2\pi i \log t}\, dt| \geq$$

$$\left| \frac{T_1^* e^{2\pi i \log T_1^*} - T_1 e^{2\pi i \log T_1}}{T_1^* (1 + 2\pi i)} \right| - (T_1/T_1^*) =$$

$$(1 - (T_1/T_1^*)) \frac{1}{\sqrt{1 + 4\pi^2}} - (T_1/T_1^*),$$

it follows that there exists a fixed $T_1^* > 2\, T_1$ such that $I_1^* > \alpha$.

Then we determine a fixed T_2 ($> T_1^*$) such that

$$I_2 = |(1/T_2) \int_0^{T_1} e^{2\pi iht}\, dt + (1/T_2) \int_{T_1}^{T_1^*} e^{2\pi ih \log t}\, dt + (1/T_2) \int_{T_1^*}^{T_2} e^{2\pi iht}\, dt| \leqq$$

$$(T_1^*/T_2) + \frac{1}{T_2 \pi |h|} \leqq (T_1^*/T_2) + \frac{1}{T_2 \pi} < a/2,$$

and a fixed $T_2^* > 2\, T_2$ such that

$$I_2^* = |(1/T_2^*) \int_0^{T_1} e^{2\pi it}\, dt + (1/T_2^*) \int_{T_1}^{T_1^*} e^{2\pi i \log t}\, dt +$$

$$(1/T_2^*) \int_{T_1^*}^{T_2} e^{2\pi it}\, dt + (1/T_2^*) \int_{T_2}^{T_2^*} e^{2\pi i \log t}\, dt | > a.$$

We proceed in this way, and find a sequence

$0 = T_0^* < T_1 < T_1^* < T_2 < T_2^* < \ldots$ with $T_k, T_k^* \to \infty$ $(k \to \infty)$

such that the corresponding expressions I_k and I_k^* satisfy the inequalities:

(4) $$I_k < a/k \;,\; I_k^* > a.$$

Now we define $f(t)$ in the following way:

$$f(t) = t \text{ for } T_k^* \leqq t < T_{k+1} \; (k = 0, 1, \ldots)$$

$$f(t) = \log t \text{ for } T_k \leqq t < T_k^* \; (k = 1, 2, \ldots).$$

We assert

(*) $f(t)$ is C-u-d (mod 1) in the intervals $(0, T_k)$,

(**) $f(t)$ is not C-u-d (mod 1) in the intervals $(0, T_k^*)$.

Proof of (*).

Let ε be an arbitrarily small positive number, then for a suitably chosen integer N we have $a/N < \varepsilon$. Hence, from the first inequality of (4), we have for $k \geqq N$, and every h, integer, $\neq 0$, the relation

$$I_k = |(1/T_k) \int_0^{T_k} e^{2\pi ihf(t)}\, dt| < (a/k) \leqq (a/N) < \varepsilon.$$

This means that I_k tends to zero if $k \to \infty$, and so (*) is proved. The exactness of assertion (**) follows immediately from the second inequality of (4).

§ 3. *Example of a totally not C-u-d function* (mod 1) *which is C-u-d* (mod 1) *in a special sequence F of intervals* (3).

Let F be a sequence of intervals (S_k, T_k) with

(5) $$S_k = k^3 - k,\; T_k = k^3 \; (k = 1, 2, \ldots).$$

Let $f(t)$ be a function defined in the following way:

$$f(t) = \log t \text{ for } T_k \leqq t < S_{k+1},$$
$$f(t) = t \text{ for } S_k \leqq t < T_k \ (k = 1, 2, \ldots).$$

Assertions:

(*) $f(t)$ is totally not C-u-d (mod 1),

(**) $f(t)$ is C-u-d (mod 1) in the sequence F of intervals (5).

Proof of (*).

For arbitrary $T > 0$ we have two possibilities:

(6) $\qquad (a) \quad T_k \leqq T < S_{k+1} \quad \text{or} \quad (b) \quad S_k \leqq T < T_k.$

Case (a). We have

$$I_k = (1/T) \int_0^T e^{2\pi i f(t)}\, dt = (1/T) \int_1^T e^{2\pi i \log t} -$$
$$(1/T) \sum_{i=2}^{k} \int_{S_i}^{T_i} e^{2\pi i \log t}\, dt + (1/T) \sum_{i=1}^{k} \int_{S_i}^{T_i} e^{2\pi i t}\, dt,$$

so that

$$|I_k| \geqq (1-(1/T)) \frac{1}{\sqrt{1+4\pi^2}} - (2/T) \sum_{i=1}^{k} (T_i - S_i).$$

Case (b).

$$I_k = (1/T) \int_0^T e^{2\pi i f(t)}\, dt = (1/T) \int_1^T e^{2\pi i \log t}\, dt - (1/T) \sum_{i=2}^{k} \int_{S_i}^{T_i} e^{2\pi i \log t}\, dt +$$
$$(1/T) \sum_{i=1}^{k} \int_{S_i}^{T_i} e^{2\pi i t}\, dt - (1/T) \int_{S_k}^{T} (e^{2\pi i \log t} - e^{2\pi i t})\, dt,$$

so that

$$|I_k| \geqq (1-(1/T)) \frac{1}{\sqrt{1+4\pi^2}} - (2/T) \sum_{i=1}^{k-1} (T_i - S_i) - (2/T)(T - S_k) \geqq$$
$$(1-(1/T)) \frac{1}{\sqrt{1+4\pi^2}} - (2/T) \sum_{i=1}^{k} (T_i - S_i).$$

Now we have

$$(2/T) \sum_{i=1}^{k} (T_i - S_i) = (2/T) \sum_{i=1}^{k} i = \frac{k(k+1)}{T} < \frac{k(k+1)}{(k-1)^3}.$$

If ε is an arbitrarily chosen positive small number, then a T^* exists such that for $T > T^*$, in both cases,

$$|I_k| > \frac{1}{\sqrt{1+4\pi^2}} - \varepsilon,$$

so that I_k does not tend to zero if T (in an arbitrary way) $\to \infty$. This completes the proof of (*).

Proof of (**).

In the intervals (S_k, T_k) $(k = 1, 2, \ldots)$ we have $f(t) = t$, hence

$$I_k = \frac{1}{T_k - S_k} \int_{S_k}^{T_k} e^{2\pi i h t}\, dt = \frac{e^{2\pi i h T_k} - e^{2\pi i h S_k}}{2\pi i h (T_k - S_k)},$$

consequently

$$|I_k| \leqq \frac{1}{\pi |h| (T_k - S_k)}, \text{ and } I_k \to 0 \text{ if } k \to \infty,$$

which proves (**).

§ 4. *Example of a C-u-d* (mod 1) *function which is not C-u-d* (mod 1) *in a special sequence F of intervals* (3).

Let F be the sequence (S_k, T_k) with (5).

Let $f(t)$ be the function defined in the following way:

$$f(t) = \log t \text{ for } S_k \leqq t < T_k,$$

$$f(t) = t \qquad \text{for } T_k \leqq t < S_{k+1}\,, k = 1, 2, \ldots$$

Assertions:

(*) $f(t)$ is C-u-d (mod 1),

(**) $f(t)$ is not C-u-d (mod 1) in the sequence F.

Proof of (*).

We distinguish the cases (6).

Case (a).

$$I_k = (1/T) \int_0^T e^{2\pi i h f(t)}\, dt = (1/T) \int_0^T e^{2\pi i h t}\, dt -$$

$$(1/T) \sum_{i=1}^{k} \int_{S_i}^{T_i} (e^{2\pi i h t} - e^{2\pi i h \log t})\, dt,$$

So

$$|I_k| \leqq (1/T\pi|h|) + (2/T) \sum_{i=1}^{k} (T_i - S_i) < (1/T\pi) + \frac{k(k+1)}{(k-1)^3}.$$

Case (b).

$$|I_k| \leqq (1/T\pi|h|) + (2/T) \sum_{i=1}^{k} (T_i - S_i) + (2(T - S_k)/T) < (1/T\pi) + \frac{k(k+1)}{(k-1)^3}.$$

In both cases we have $I_k \to 0$ if $k \to \infty$.

So (*) is proved.

Proof of (**).

Even, if F is an arbitrary sequence of intervals (3), we have

$$I_k = \frac{1}{T_k - S_k} \int_{S_k}^{T_k} e^{2\pi i \log t}\, dt = \frac{T_k\, e^{2\pi i \log T_k} - S_k\, e^{2\pi i \log S_k}}{(T_k - S_k)(1 + 2\pi i)},$$

so

$$|I_k| = \frac{\{T_k^2 + S_k^2 - 2\,T_k S_k \cos 2\pi (\log T_k - \log S_k)\}^{1/2}}{(T_k - S_k)\sqrt{1+4\pi^2}}$$

$$= \frac{\{(T_k - S_k)^2 + 4\,T_k S_k \sin^2 \pi (\log T_k - \log S_k)\}^{1/2}}{(T_k - S_k)\sqrt{1+4\pi^2}}$$

$$\geqq \frac{1}{\sqrt{1+4\pi^2}},$$

hence I_k does not tend to zero if $k \to \infty$, and (**) is proved.

§ 5. *Connection between totally not C-u-d* (mod 1) *and C-distribution* (mod 1) *in sequences F of intervals* (3).

Theorem I.

If $f(t)$ is totally not C-u-d (mod 1), *then $f(t)$ is not C-u-d* (mod 1) *in the sequences F of intervals*

$$S_k \leqq t \leqq T_k \; (k = 1, 2, \ldots) \text{ with } (S_k/T_k) \to 0 \; (k \to \infty).$$

Proof. Put

$$\varphi(T) = \int_0^T e^{2\pi i h f(t)}\, dt, \tag{7}$$

then

$$\left\{ \begin{aligned} I_k &= \frac{1}{T_k - S_k} \int_{S_k}^{T_k} e^{2\pi i h f(t)}\, dt = \frac{\varphi(T_k) - \varphi(S_k)}{T_k - S_k} = \\ &\frac{T_k}{T_k - S_k} \cdot \frac{\varphi(T_k)}{T_k} - \frac{S_k}{T_k - S_k} \cdot \frac{\varphi(S_k)}{S_k}. \end{aligned} \right. \tag{8}$$

From our assumptions it follows:

$$\frac{T_k}{T_k - S_k} \to 1 \text{ and } \frac{S_k}{T_k - S_k} \to 0 \text{ if } k \to \infty,$$

$(\varphi(T_k)/T_k)$ does not tend to zero if $k \to \infty$,
$|(\varphi(S_k)/S_k)|$ is bounded,
so that I_k does not tend to zero if $k \to \infty$, as we see by (8).

Theorem 2.

If in no sequence F of intervals (3) *$f(t)$ is C-u-d* (mod 1), *then $f(t)$ is totally not C-u-d* (mod 1).

Proof. Put (7). From our assumption it follows that for all sequence (S_k, T_k) the right side of (8) does not tend to zero if $k \to \infty$. If we choose (S_k, T_k) in such way that $(S_k/T_k) \to 0$, then it is clear that $(\varphi(T_k)/T_k)$ does not tend to zero if $T_k \to \infty$. Since, in connection with $(S_k/T_k) \to 0$, the T_k may be arbitrarily chosen, if follows that $f(t)$ is totally not *C-u-d* (mod 1).

§ 6. *Connection between C-u-d* (mod 1) *and C-distribution in sequences F of intervals* (3).

Theorem 3.

If $f(t)$ *is C-u-d* (mod 1) *then* $f(t)$ *is C-u-d* (mod 1) *in those sequences* (S_k, T_k) *for which*

$$(S_k/T_k) \leqq \lambda \quad \text{(constant)} \quad < 1 \; (k = 1, 2, \ldots).$$

Proof. Again putting (7), we have

$$I_k = \frac{T_k}{T_k - S_k} \cdot \frac{\varphi(T_k)}{T_k} - \frac{S_k}{T_k - S_k} \cdot \frac{\varphi(S_k)}{S_k}.$$

From our assumptions it follows:

$\frac{T_k}{T_k - S_k}, \frac{S_k}{T_k - S_k}$ are bounded,

$(\varphi(T_k)/T_k)$, $(\varphi(S_k)/S_k) \to 0$ if $k \to \infty$,

and we find that $I_k \to 0$ if $k \to \infty$.

Theorem 4.

If $f(t)$ *is C-u-d* (mod 1) *in all sequences F of intervals* (3), *then* $f(t)$ *is C-u-d* (mod 1).

Proof. We write (8). If we choose (S_k, T_k) in such way that $(S_k/T_k) \to 0$, then $(\varphi(T_k)/T_k) \to 0$ $(k \to \infty)$.

Since the T_k may be arbitrarily chosen, we have proved all.

§ 7. *C-distribution in sequences F of intervals* (3).

Theorem 5.

Let $f(t)$ *be a differentiable function.*

Let F be an arbitrary sequence of intervals (S_k, T_k) $(k = 1, 2, \ldots)$.

Let there exist a fixed $T^* \geqq 0$, *such that for* $T > T^*$ *the function* $f(t)$ *satisfies*

$$0 \leqq |t f'(t) - A| < B, \tag{9}$$

where A and B are fixed numbers with

$$2 \pi B < 1. \tag{10}$$

Then $f(t)$ *is not C-u-d* (mod 1) *in the intervals of F.*

Proof. For $S_k > T^*$ we have

$$I_k = \frac{1}{T_k - S_k} \int\limits_{S_k}^{T_k} e^{2\pi i f(t)}\, dt = \frac{T_k\, e^{2\pi i f(T_k)} - S_k\, e^{2\pi i f(S_k)}}{T_k - S_k} -$$

$$\frac{2\pi i}{T_k - S_k} \int\limits_{S_k}^{T_k} t f'(t)\, e^{2\pi i f(t)}\, dt,$$

hence, if Q_k denotes the first term on the right,

$$(11) \qquad (1+2\pi i A)\, I_k = Q_k - \frac{2\pi i}{T_k - S_k} \int_{S_k}^{T_k} \{t f'(t) - A\}\, e^{2\pi i f(t)}\, dt.$$

Now

$$Q_k = \frac{1}{T_k - S_k} \{(T_k - S_k)^2 + 4\, T_k\, S_k \sin^2 \pi\, (f(T_k) - f(S_k))\}^{1/2} \geqq 1.$$

From the last inequality, (10) and (11) it follows that

$$|I_k| \sqrt{1 + 4\pi^2 A^2} > 1 - 2\pi B > 0,$$

so that I_k does not tend to zero if $k \to \infty$. The application of the C-test with $h = 1$ completes the proof.

A special case of Theorem 5 is

Theorem 6.

Let $f(t)$ be a differentiable function with

$$t f'(t) \to A \ \ (\textit{constant}) \ \ \textit{if} \ \ t \to \infty.$$

Then in no sequence F of intervals (3) *the function $f(t)$ is C-u-d* (mod 1).

Remarks.

1. From Theorem 6 we find again (see the proof of (**) in § 4) that in no sequence F the function $\log t$ is C-u-d (mod 1).
2. For $A = 0$ Theorem 6 is proved in [5].
3. We remember that, if $|t f'(t)|$ is bounded, $f(t)$ is totally not C-u-d (mod 1), see [6]. The condition, made on $f(t)$ here, is weaker than in the case (S, T).

Theorem 7.

Let F be a sequence of intervals (3), *and let $f(t)$ be a differentiable function, such that*

$$(12) \qquad (T_k - S_k)\, M_k \to 0 \quad \text{if} \quad k \to \infty,$$

where M_k is the maximum of $|f'(t)|$ in $S_k \leqq t \leqq T_k$.

Then $f(t)$ is not C-u-d (mod 1) *in the sequence F.*

Proof. We have successively:

$$\frac{1}{T_k - S_k} \int_{S_k}^{T_k} \cos 2\pi h f(t)\, dt = \cos 2\pi h f(S_k) - (T_k - S_k)\, \pi h f'(\xi_1) \sin 2\pi h f(\xi_1),$$

$$\frac{1}{T_k - S_k} \int_{S_k}^{T_k} \sin 2\pi h f(t)\, dt = \sin 2\pi h f(S_k) + (T_k - S_k)\, \pi h f'(\xi_2) \cos 2\pi h f(\xi_2),$$

$$(S_k < \xi_1, \xi_2 < T_k),$$

hence

$$I_k = \frac{1}{T_k - S_k} \int_{S_k}^{T_k} e^{2\pi i h f(t)}\, dt = e^{2\pi i h f(S_k)} + A_k + B_k,$$

with A_k, $B_k \to 0$ if $k \to \infty$, as follows from (12), so that $|I_k| \to 1$ if $k \to \infty$. This completes the proof.

Theorem 8.

Let F be an arbitrary sequence of intervals (3).
Let F be a differentiable function with

(13) $$f'(t) \to c \neq 0$$

where c is a fixed number. Then $f(t)$ is C-u-d (mod 1) *in the sequence F.*

Proof. Without loss of generality we assume $c > 0$.
From (13) we have for $t > T^* = T^*(\varepsilon)$

$$|(1/f'(t)) - (1/c)| < \varepsilon.$$

Now for $S_k > T^*$:

$$I_k = \frac{1}{T_k - S_k} \int_{S_k}^{T_k} e^{2\pi i h f(t)}\, dt = \frac{1}{T_k - S_k} \int_{f(S_k)}^{f(T_k)} \frac{e^{2\pi i h u}\, du}{f'(t)} =$$

$$\frac{1}{T_k - S_k} \int_{f(S_k)}^{f(T_k)} \{(1/f'(t)) - (1/c)\}\, e^{2\pi i h u}\, du + (1/c(T_k - S_k)) \int_{f(S_k)}^{f(T_k)} e^{2\pi i h u}\, du,$$

hence

$$|I_k| \leqq \frac{\varepsilon \{f(T_k) - f(S_k)\}}{T_k - S_k} + \frac{1}{c\pi |h| (T_k - S_k)} = \varepsilon f'(\xi) + \frac{1}{c\pi |h| (T_k - S_k)} \quad (S_k < \xi < T_k),$$

so that, from (13), $(T_k - S_k) \to \infty$ and the arbitrary choice of ε, we see that

$$I_k \to 0 \text{ if } k \to \infty.$$

For the sake of completeness we mention the following two theorems which are special cases of theorems proved in [5].

Theorem 9.

Let $f(t)$ be a differentiable function with
$f'(t)$ is monotonically non-decreasing,

$$f'(t) \geqq \lambda > 0 \text{ for fixed } \lambda.$$

Then $f(t)$ is C-u-d (mod 1) *in every sequence F of intervals* (3).

Theorem 10.

Let F be a sequence of intervals (3), *and let $f(t)$ be a differentiable function with*

$f'(t) > 0$, monotonically non-increasing, and continuous,

$$(T_k - S_k)\, f'(T_k) \to \infty \text{ if } k \to \infty.$$

Then $f(t)$ is C-u-d (mod 1) *in the sequence F.*

Bandung, April 26, 1950. *University of Indonesia.*

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A NOTE ON THE THEORY OF STELLAR DYNAMICS, I

BY

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Abstract. — The components of average velocity and the values of $\overline{U^2}$, $\overline{V^2}$, $\overline{UV}$ have been derived for a flat and rotationally symmetrical stellar system with the mass concentrated in the centre. The values found satisfy, of course, the hydrodynamical equations of motion.

1. *Introduction.* — Theories of the dynamics of stellar systems are generally based on various assumptions concerning the individual velocities of the stars in the system. Here with individual velocity is meant the deviation of the velocity of a star from the average velocity of all stars in its neighbourhood. CHANDRASEKHAR (1), for instance, assumes that the density of the stars in phase space depends, as far as the velocity is concerned, only on quadratic terms of the components of the individual velocities. In LINDBLAD's (2) studies of the formation of spiral arms it has been assumed that $\overline{UV}$, $\overline{UW}$, $\overline{VW}$ (U, V, W are the components of the individual velocity) vanish identically and that $\overline{U^2} = \overline{V^2} = \overline{W^2}$. In some of his papers he assumes that $\overline{UV}$, $\overline{UW}$, $\overline{VW}$, if they do not vanish identically, nevertheless remain unchanged when the distribution of the density in the system changes. The purpose of the present and the following note is to help in obtaining an idea of the significance of the assumptions made by LINDBLAD.

I have considered the very simple system in which the whole mass is concentrated in the centre *). In this case the orbits of the stars can simply be described by KEPLER's laws. Further it has been assumed that the inclinations vanish, so that a two-dimensional system is obtained. How the orbits are distributed over different excentricities, and how they are orientated in space, will be explained in the sequel. In any case we assume that all excentricities are small. In the present note we moreover assume that the system is rotationally symmetrical, while in the next note we shall consider the most "oval" system possible for a given distribution of the excentricities of the orbits.

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*) In a stellar system collisions as well as mutual gravitational interactions between neighbouring stars are generally believed to be negligible.

At the end of the second note we shall discuss what consequences the results of our calculations may have for the theory of the formation of spiral arms in stellar systems.

2. *The Motion of Individual Stars.* — The orbit of a star in the system can be conveniently described by means of the following elements, which have been used by EDDINGTON [3] is his investigation of a globular stellar system:

r_0 = the apcentric distance,
ϑ_0 = the apcentric longitude,
T_0 = transverse linear velocity at the apcentron,
t_0 = time of passage through the apcentron.

Let r be the distance of the star from the centre, ϑ its longitude, T its transverse linear velocity component, R the radial component of the velocity, t the time, μ a constant and Φ the gravitational potential defined by

(2, 1) $$\Phi = \mu / r.$$

The equations which describe the motion of an individual star are then 1. the equation of areas,

(2, 2) $$rT = r_0 T_0,$$

2. the equation of energy (R_0 vanishes),

(2, 3) $$\tfrac{1}{2}(R^2 + T^2) - \Phi = \tfrac{1}{2} T_0^2 - \Phi_0.$$

3. the relation between $\vartheta - \vartheta_0$ and $t - t_0$. The latter relation (which is an integral of the two first ones), however, cannot be expressed in closed form. Since we limit our investigation to small values of the excentricity, e, the following series may be used [4]

(2, 4) $$\vartheta - \vartheta_0 = M' - 2e \sin M' + \tfrac{5}{4} e^2 \sin 2M' + \dots$$

where M' is the mean anomaly calculated from the apcentron,

(2, 4)′ $$M' = \frac{\sqrt{\mu}}{[r_0/(1+e)]^{3/2}} (t - t_0).$$

Reversing the series (2, 4) we obtain

$$M' = (\vartheta - \vartheta_0) + 2e \sin(\vartheta - \vartheta_0) + \tfrac{3}{4} e^2 \sin 2(\vartheta - \vartheta_0) + \cdots$$

or, from (2, 4)′,

(2, 5) $$t - t_0 = \frac{1}{\sqrt{\mu}} \left(\frac{r_0}{1+e}\right)^{3/2} [(\vartheta - \vartheta_0) + 2e \sin(\vartheta - \vartheta_0) + \tfrac{3}{4} e^2 \sin 2(\vartheta - \vartheta_0) + \dots].$$

Inserting (2, 1) and then solving equations (2, 2) and (2, 3) we obtain:

(2, 6) $$T = \frac{r_0}{r} T_0; \quad R = \sqrt{\left(\frac{r_0}{r} - 1\right)\left[\frac{2\mu}{r_0} - T_0^2 \left(\frac{r_0}{r} + 1\right)\right]}.$$

In a circular orbit $T_0 = \sqrt{\mu/r_0}$. Instead of T_0, r_0 it will be convenient to introduce new parameters s and x which for a given value of r are defined by the relations

(2, 7) $$T_0 = \sqrt{\mu/r_0}\cdot s;\quad r_0/r = 1 + \frac{2\,(1-s^2)}{s^2}\,x.$$

These relations give further

(2, 7)′ $$T_0 = \sqrt{\frac{\mu}{r\left(1+\frac{2\,(1-s^2)}{s^2}\,x\right)}}\cdot s.$$

Circular orbits correspond to $s=1$ while for other orbits $s<1$. Hence s determines the excentricity. $x=0$ corresponds to any orbit having its apcentric distance r_0 equal to r, while $x=1$ corresponds to any orbit having its pericentric distance equal to r. Then

(2, 8) $$T = \sqrt{\frac{\mu}{r}}\cdot s\sqrt{1+\frac{2\,(1-s^2)}{s^2}\,x};\qquad R = \sqrt{\frac{\mu}{r}}\,\frac{2\,(1-s^2)}{s}\sqrt{\frac{x\,(1-x)}{1+\frac{2\,(1-s^2)}{s^2}\,x}}.$$

Now x will be expressed by means of the quantity $\vartheta-\vartheta_0$ and e by means of s, so that the final equations will contain $\vartheta-\vartheta_0$ and s, and not x and e. These dependences are obtained from the equation of an ellipse in polar coordinates,

$$r = \frac{r_0\,(1-e)}{1-e\cos(\vartheta-\vartheta_0)},$$

(remembering that $\vartheta-\vartheta_0$ is calculated from the apcentron) and the relation (4)

$$r_0 T_0 = \sqrt{\mu r_0\,(1-e)}.$$

The latter equation combined with the first equation (2, 7) gives

(2, 9) $$1-e = s^2;\qquad \text{i.e.}\qquad e = 1-s^2.$$

Hence

$$\cos(\vartheta-\vartheta_0) = \frac{1-(r_0/r)\,s^2}{1-s^2},$$

or, according to the second equation (2, 7),

(2, 10) $$\cos(\vartheta-\vartheta_0) = 1-2x,\quad \text{i.e.}\quad x=\sin^2\tfrac{1}{2}(\vartheta-\vartheta_0).$$

When the relations (2. 9) and (2, 10) are inserted in equations (2,7) — (2, 8) and (2, 5), the equations given below are found:

(2, 11) $$\left\{\begin{aligned} r_0 &= \frac{r}{s^2}\,[1-(1-s^2)\cos(\vartheta-\vartheta_0)] \\ T_0 &= \sqrt{\frac{\mu}{r\,[1-(1-s^2)\cos(\vartheta-\vartheta_0)]}}\cdot s^2 \\ t-t_0 &= \frac{1}{\sqrt{\mu}}\left[\frac{r\,[1-(1-s^2)\cos(\vartheta-\vartheta_0)]}{1-(1-s^2)^2}\right]^{3/2}[(\vartheta-\vartheta_0)+ \\ &\quad + 2\,(1-s^2)\sin(\vartheta-\vartheta_0) + \tfrac{3}{4}\,(1-s^2)^2\sin 2\,(\vartheta-\vartheta_0)+\ldots] \end{aligned}\right.$$

$$(2,12)\quad \begin{cases} T = \sqrt{\dfrac{\mu}{r}} \cdot \sqrt{1-(1-s^2)\cos(\vartheta-\vartheta_0)} = \sqrt{\dfrac{\mu}{r}}\, s \sqrt{1+\dfrac{2\,(1-s^2)}{s^2}\sin^2\tfrac{1}{2}(\vartheta-\vartheta_0)} \\ R = -\sqrt{\dfrac{\mu}{r}} \cdot \dfrac{(1-s^2)\sin(\vartheta-\vartheta_0)}{\sqrt{1-(1-s^2)\cos(\vartheta-\vartheta_0)}} = -\sqrt{\dfrac{\mu}{r}}\,\dfrac{1-s^2}{s}\,\dfrac{\sin(\vartheta-\vartheta_0)}{\sqrt{1+\dfrac{2\,(1-s^2)}{s^2}\sin^2\tfrac{1}{2}(\vartheta-\vartheta_0)}} \end{cases}$$

These equations express r_0, T_0, t_0; T, R as functions of r, $\vartheta-\vartheta_0$ and s.

Let us further write

$$(2,13)\qquad D = \frac{\partial\,(r_0, T_0, t-t_0)}{\partial\,(r, \vartheta, s)}.$$

After somewhat lengthy calculations it will be found that

$$(2,14)\quad \begin{cases} D = -\dfrac{r}{s}\,\dfrac{[1-(1-s^2)\cos(\vartheta-\vartheta_0)]^2}{[1-(1-s^2)^2]^{3/2}}\,[1+2\,(1-s^2)\cos(\vartheta-\vartheta_0)+ \\ +\tfrac{3}{2}\,(1-s^2)^2\cos 2\,(\vartheta-\vartheta_0)+\ldots]. \end{cases}$$

This can also be written in the form

$$(2,15)\qquad D = -\frac{r}{s}\,\frac{1-\tfrac{3}{2}\,(1-s^2)^2+\ldots}{[1-(1-s^2)^2]^{3/2}}$$

so that, neglecting third order terms of $1-s^2$,

$$(2,16)\qquad D = -r/s.$$

3. *Density and Components of Mean Velocity.* — Let

$$(3,1)\qquad \frac{1}{2\pi}\,f\,(r_0, T_0, t_0)\,dr_0\,dT_0\,dt_0\,d\vartheta_0$$

denote the number of stars with apcentric distances between r_0 and r_0+dr_0, apcentric velocities between T_0 and T_0+dT_0, the times of passage through the apcentron between t_0 and t_0+dt_0, and the apcentric longitudes between ϑ_0 and $\vartheta_0+d\vartheta_0$. The integral of the expression (3, 1) over all values of r_0, T_0, t_0 and ϑ_0 gives the total number of stars in the system. f is supposed to be independent of ϑ_0 because we are considering a rotationally symmetrical system.

The density has to be defined as the density per unit area. In order to obtain it, we change the variables r_0, T_0, t_0 in (3, 1) into r, ϑ, s, then divide the expression by $dr \cdot r\,d\vartheta$, and finally integrate over s and ϑ_0. First we have

$$dr_0 dT_0 dt_0 d\vartheta_0 = -D \cdot dr\,d\vartheta\,ds\,d\vartheta_0.$$

It will be seen that the assumption

$$(3,2)\quad \begin{cases} f(r_0, T_0, t_0) = N, & \text{from } s=1 \text{ to } s=s_0, \\ f(r_0, T_0, t_0) = 0, & \text{for } s < s_0, \end{cases}$$

where $N=$ const. and $1-s_0 \ll 1$, leads to a constant density, in the limits of accuracy. Since, according to the first equation (2, 7), s is a

function of r_0 and T_0 only, $f(r_0, T_0, t_0)$ will then be a function of r_0 and T_0 only, and independent of t_0. Then

$$(3, 3) \qquad \varrho = \frac{N}{2\pi} \int_{s_0}^{1} \int_{0}^{2\pi} (1/s)\, ds\, d\vartheta_0$$

i.e.

$$\varrho = N \ln (1/s_0).$$

Introducing

$$(3, 4) \qquad 1 - s = \sigma; \qquad 1 - s_0 = \sigma_0$$

which are small quantities, we also obtain:

$$(3, 5) \qquad \varrho = N \sigma_0 (1 + \tfrac{1}{2} \sigma_0 + \tfrac{1}{3} \sigma_0^2 + \ldots).$$

The terms written down in this equation are not influenced by the terms neglected in equation (2, 16).

Without calculation it can be seen that, for any given r, the mean radial component $u = \overline{R} = 0$. Further

$$(3, 6) \qquad \varrho \overline{T} = \sqrt{\frac{\mu}{r}} \frac{N}{2\pi} \int_{s_0}^{1} \int_{0}^{2\pi} \sqrt{1 + \frac{2(1-s^2)}{s^2} \sin^2 \tfrac{1}{2} (\vartheta - \vartheta_0)}\, ds\, d\vartheta_0.$$

We have:

$$\sqrt{1 + \frac{2(1-s^2)}{s^2} \sin^2 \tfrac{1}{2} (\vartheta - \vartheta_0)} = 1 + \\ + \frac{1-s^2}{s^2} \sin^2 \tfrac{1}{2} (\vartheta - \vartheta_0) - \tfrac{1}{2} \frac{(1-s^2)^2}{s^4} \sin^4 \tfrac{1}{2} (\vartheta - \vartheta_0) + \ldots$$

Excluding terms of higher order than the second in $1 - s = \sigma$, we have:

$$\sqrt{1 + \frac{2(1-s^2)}{s^2} \sin^2 \tfrac{1}{2} (\vartheta - \vartheta_0)} = 1 + \\ + \frac{1-s^2}{s^2} \sin^2 \tfrac{1}{2} (\vartheta - \vartheta_0) - 2 (1-s)^2 \sin^4 \tfrac{1}{2} (\vartheta - \vartheta_0).$$

Integrating (3, 6) we then find:

$$\varrho \overline{T} = \sqrt{\frac{\mu}{r}} N \left[(1 - s_0) + \tfrac{1}{2} \frac{(1-s_0)^2}{s_0} - \tfrac{1}{4} (1 - s_0)^3 \right]$$

which expression is correct to the third power of $\sigma_0 = 1 - s_0$. Accordingly:

$$(3, 7) \qquad \varrho \overline{T} = \sqrt{\mu / r}\, N \sigma_0 (1 + \tfrac{1}{2} \sigma_0 + \tfrac{1}{4} \sigma_0^2 + \ldots)$$

and from (3, 5),

$$(3, 8) \qquad v = \overline{T} = \sqrt{\mu / r}\, (1 - \tfrac{1}{12} \sigma_0^2 + \ldots).$$

4. *The Quantities* $\overline{U^2}$, $\overline{V^2}$, $\overline{UV}$. — Including the lowest powers of $1 - s$ only, equations (2, 12) and (3, 8) give the following values for the deviations

of the velocities of the stars from the average velocity at the point considered:

$$(4,\,1)\qquad \begin{cases} U = R = -2\sqrt{\mu/r}\,(1-s)\sin(\vartheta-\vartheta_0); \\ V = T-\overline{T} = -\sqrt{\mu/r}\,(1-s)\cos(\vartheta-\vartheta_0). \end{cases}$$

When the squares of these expressions are inserted into the integral of equation (3, 3) the following values are found:

$$(4,\,2)\qquad \varrho\overline{U^2} = \tfrac{2}{3}N\sigma_0^3\frac{\mu}{r}; \qquad \varrho\overline{V^2} = \tfrac{1}{6}N\sigma_0^3\frac{\mu}{r}.$$

Hence

$$(4,\,3)\qquad \overline{U^2} = \tfrac{2}{3}\sigma_0^2\frac{\mu}{r}; \qquad \overline{V^2} = \tfrac{1}{6}\sigma_0^2\frac{\mu}{r}.$$

When, as is assumed in the present note, the system is rotationally symmetrical, without calculation we see that, for any given r, $\overline{UV}$ vanishes, because positive and negative values of UV are equally possible.

The following list summarizes the values found in sections 3 and 4, to the accuracy used in the present note:

$$(4,\,4)\qquad \begin{cases} u = 0; \qquad v = \sqrt{\mu/r}\,(1-\tfrac{1}{12}\sigma_0^2); \\ \overline{U^2} = \tfrac{2}{3}\frac{\mu}{r}\sigma_0^2; \qquad \overline{V^2} = \tfrac{1}{6}\frac{\mu}{r}\sigma_0^2; \qquad \overline{UV} = 0. \end{cases}$$

5. *The Hydrodynamical Equations of Motion.* — When the vertical component disappears, the hydrodynamical equations of motion will be the following:

$$(5,\,1)\qquad \begin{cases} \varrho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial r} + \dfrac{v}{r}\dfrac{\partial u}{\partial\vartheta} - \dfrac{v^2}{r}\right) + \dfrac{\partial(\varrho\overline{U^2})}{\partial r} + \dfrac{1}{r}\dfrac{\partial(\varrho\overline{UV})}{\partial\vartheta} + \\ \qquad\qquad + \dfrac{\varrho\overline{U^2}}{r} - \dfrac{\varrho\overline{V^2}}{r} = \varrho\dfrac{\partial\Phi}{\partial r} \\ \varrho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial r} + \dfrac{v}{r}\dfrac{\partial v}{\partial\vartheta} + \dfrac{uv}{r}\right) + \dfrac{\partial(\varrho\overline{UV})}{\partial r} + \dfrac{1}{r}\dfrac{\partial(\varrho\overline{V^2})}{\partial\vartheta} + 2\dfrac{\varrho\overline{UV}}{r} = O. \end{cases}$$

The gravitational potential is given by the expression

$$(5,\,2)\qquad \Phi = \mu/r$$

while the equation of continuity will be:

$$(5,\,3)\qquad \frac{\partial\varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial r} + \frac{1}{r}\frac{\partial(\varrho v)}{\partial\vartheta} + \frac{\varrho u}{r} = 0.$$

In the present case these equations are further simplified because ϱ is constant and the derivatives with regard to t vanish. We see immediately that the equations are satisfied by the values (4, 4), up to the second order terms of σ_0, as, of course, they should be.

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3. See e.g. Smart: Stellar Dynamics. Cambridge, 333 (1938).
4. See e.g. Moulton: An Introduction to Celestial Mechanics (Second Edition), New York, N.Y. (1947).

PALAEONTOLOGY

AGATHICERAS SUNDAICUM HAN., A LOWER PERMIAN FOSSIL FROM TIMOR

BY

P. KRUIZINGA

(Communicated by Prof. J. H. F. UMBGROVE at the meeting of June 24, 1950)

Up to now no paleozoic fossils were described from the island Billiton. The first specimen, discussed in the following pages was found on the so-called "kong" of the "kollong" [1]) Berajung (D.S.B., 10 — 1), mine Kelatung in the district Lenggang of this island. Therefore, very probably, it derives from the "kaksa" that was originally on it.

It was made a present by the Billiton Company to the collections of the Rijksmuseum voor Geologie en Mineralogie (Museum for Geology and Mineralogy) at Leyden and was kindly offered to the present author for identification. I wish to thank both, the BILITON COMPANY at The Hague and Professor VAN DER VLERK at Leyden for their courtesy.

The fossil consists of a little lump of cassiterite (the weight is about 11.25 gr) that proved to be nearly one half of a broken ammonoid. The cassiterite is dark coloured and rather coarse-grained and it has been formed at a high temperature as is shown by the pleochroism.

Although the fossil is very incomplete there are sufficient characteristics to allow its identification as *Agathiceras sundaicum* Han., a species which occurs abundantly in the Lower Permian of Timor, chiefly near Bitauni. Evidently it derives from the old complex of sedimentary rocks that existed in Billiton at the time of the post-Triassic (Jurassic) intrusion of the tin ore bearing granite. The rocks became locally intensely metamorfosed by this intrusion but up to this moment the exact age of the invaded complex was unknown, as hitherto only *Radiolaria* had been found in cherts of disputed age. The ammonoid therefore is of great scientific importance as it shows that Lower Permian sediments are represented in the intruded complex of rocks.

Permo-carboniferous deposits are already known from the western part of the East Indies viz., from the islands Sumatra, Banka and Borneo. Limestones containing *Fusulinidae* have been described from these three islands.

In Atcheen (northern Sumatra) volcanic tuffs of Permian age have been found in the neighbourhood of Aluë Kaluë in the district Tamiang together with corals, brachiopods and trilobites. In the eastern part of

[1]) A kollong (chinese word) is a pit in the secondary tin ore deposits of Banka and Billiton, whereas kaksa, also a chinese word, is the name of the secondary tin ore deposits on the kong (bedrock) of the floor of the valleys on those islands.

the East Indies the same fauna is represented on the islands Timor and Letti. Especially from Timor numerous Permian ammonoids have been described.

The diameter of the ammonoid from Billiton must have been at least 23.3 mm, but we cannot establish how much of the shell is missing, as the body-chamber is not preserved. The form of the shell is robust the venter of the extern whorl being rounded, the flanks being somewhat flattened. The flanks pass with a rather abrupt "shoulder" into a narrow and deep umbilicus (diam. 4.3 mm). The specimen is an involute type as the inner whorls are practically embraced by the extern ones. Rather near the umbilicus this extern whorl has its greatest thickness (13.7mm), its height is about 11.5 mm. The last whorl is 3.8 mm wide.

Originally the surface was covered by a great number of fine but distinct spiral lines, equally firm and with equal distances between them (about twice as large as the lines themselves). The spiral lines are characteristic and important, for they are known only from two related genera of the Permian (*Agathiceras* and *Adrianites*) and from *Cladiscites* of the Upper-Trias. *Cladiscites*, however, has a very different shape. A smooth fossil with the same form might belong either to the Permian or to the Trias.

Eleven lines per 5 mm can be counted. A transverse section through a rather big specimen from Timor shows that the distance between the lines may increase with the growth of the shell to about three times the thickness of a spiral line, because as HANIEL already mentioned, rarely new lines are intercalated. In the same section we can see also that the ornamentation is limited to the extern surface of the shell, the intern surface being smooth (fig. 4). When well preserved the spiral lines are blunt or rounded, not sharp. It is impossible, however, to obtain a good impression of the form of the lines from the Billiton specimen. Some seem to be rounded, sometimes they seem rather sharp but it cannot be decided wether this actually is the original form. An ornamentation by radial ribs or lines is not represented and no constrictions on the shell are to be seen. Therefore the presence of the spiral lines proves that the fossil from Billiton is not an intern mould of cassiterite.

The shell as well as its original contents, whatever this may have been, became metasomatically changed into cassiterite; all the $CaCO_3$ has disappeared.

The coarse-grained cassiterite obliterated the internal structure; in some parts, however, septa can be seen. Without much loss of material a polished section through the shell has been made. It shows some of the inner whorls as lines running regularly through the cassiterite (fig. 2 and 3).

One umbilicus is filled up with brownish coloured material, that also proved to be cassiterite. In this umbilicus the embryonic stage of another ammonoid was found (fig. 2 and 3). Its shell is rather thick in proportion to its diameter (the ratio of thickness and diameter is about 2 : 3). This little ammonoid is, in its present embryonic stage at least more evolute

than *Agathiceras sundaicum*. Its surface shows some radially arranged and rounded ribs obliquely directed forwards. They begin near the umbilicus and become less distinct in an outward direction. To what species this ammonoid may belong cannot be established because the shells of many ammonoids may change considerably during the development of the animal.

HANIEL[2]) described 8 species of the genus *Agathiceras* and several varieties from the Permian of Timor.

PERRIN SMITH[3]) described a ninth species from Timor, but the thought that 6 of the species described so far from the East Indies belong to the genus *Adrianites* and only 3 to *Agathiceras*, viz. *Agathiceras brouweri* Smith, *Ag. martini* Han., and *Ag. sundaicum* Han.

These six species of *Adrianites* are: *Adr. beyrichi* Han., *Adr. cancellatus* Han.. *Adr. oyensi* Han., *Adr. rothpletzi* Han., *Adr. timorensis* Han. and *Adr. wichmanni* Han. All these species have much thicker shells than the specimen from Billiton and some of them have a wide umbilicus. The ornamentation of their shells is also different. Not only spiral lines occur — and these are on some species rather coarse — but also radial lines are represented at about equal distances and resembling the spiral lines. As a result the surface shows a pattern of diamond-shaped figures.

Therefore the Billiton specimen has to be compared only with the three species retained by PERRIN SMITH in the genus *Agathiceras*. *Ag. martini* has no spiral lines on the surface of its shell, but fine radial growth-lines. The ammonoid from Billiton therefore cannot be considered as belonging to that species.

Agathiceras brouweri and *Ag. sundaicum* agree in many respects. They both have spiral lines but they are rather weakly developed on the holotype of the first named species. On the shells of both species these are however at equal distances. Here we can count also 11 spiral lines per 5 mm.

Besides this slight difference of the spiral lines *Ag. brouweri* is more flattened and the flanks are converging to the narrower venter of the whorls. Fig. 16 on plate 9 of the paper of PERRIN SMITH as well as the holotype itself prove that the height of the chamber of the outer whorl is larger than the average height of *Ag. sundaicum*.

So the identity with the last named species is striking, as well with the description and the figures of both authors as with the specimens available. Therefore the specimen of Billiton must undoubtedly belong to *Agathiceras sundaicum* Han.

As mentioned by HANIEL the variability of this species is considerable and both the extreme forms of the series might be considered as separate species if no connecting links had been found. Their nearly imperceptable

[2]) HANIEL, C. A., Die Cephalopoden der Dyas von Timor. Palaeontologie von Timor, 3e Lief., 66—72 (1915).

[3]) SMITH, J. PERRIN, Permian Ammonoids of Timor. Jaarb. v. h. Mijnwezen in Ned.-Indië. 45ste jaarg. (1926). Verhandelingen 1ste part. 38—40 (1927).

intergradation induced HANIEL to join all specimens into one species. Only two specimens from Portuguese Timor, that are characterized by alternating firm and weak spiral lines have been described as a variety viz. var. *atsabensis*. In all other respects they agree with the species proper. We can, however, retain this variety because the difference is characteristic and striking. Among the specimens from the rest of Timor not a single one of this variety has been found. So when reading the description of *Ag. brouweri* one is inclined to consider it as an extreme form of variability of *Ag. sundaicum*.

In order to settle this problem, I measured several specimens of both species and determined the variation of different proportions. HANIEL has already given some data concerning specimens of *Ag. sundaicum* on page 66 and 67 of his paper. Moreover he published an illustration of a cross-section showing the inner whorls; finally a number of specimens have been figured by him on the plates of his treatise. Some of the figured specimens are in the Geological Museum at Delft, while the others are in Bonn. The specimens of the same species figured by PERRIN SMITH as well as the holotype of *Ag. brouweri* are in Delft.

In a paper on the ammonoids of the Sula islands [4]) in the Moluccas I advocated the use of the following ratios thickness: entire diameter, thickness: height of the whorl, height of the chamber: entire diameter, height of the chamber: height of the whorl, *etcetera*.

For the successive whorls of one specimen these ratios can be calculated from a cross-section in order to establish whether the form of the whorls has changed during the life of the animal and if so how much they have changed. These quotients multiplied by 100 furnish us index numbers or indices.

Application of this method to *Ag. sundaicum* immediately confirmed its variability (table I — V), but it also shows that the indices of *Ag. brouweri* fall within the variability range of *Ag. sundaicum* (table I).

According to PERRIN SMITH *Ag. brouweri* has been exceptionally found in the Basleo beds near Basleo. In the collections of Delft, however, it is represented only by (1) the holotype, (2) part of a whorl of an intern mould from the same locality, and (3) two specimens from Bitauni. I believe, however, that the specimens from Bitauni have not been determined by PERRIN SMITH himself, although their slender form and high whorls agree very well with his description and with his figures. The smaller one is an intern mould but on the other one parts of the shell are preserved. Similarly, rather slender forms with high whorls and spacious chambers are represented among the specimens of *Ag. sundaicum* from Bitauni, identified by HANIEL, as one can easily see for example in his figure 13 of plate IV (coll. Delft no. 12378). An intern mould of this species from Sufa

[4]) KRUIZINGA, P., Ammonieten en eenige andere fossielen uit de Jurassische afzettingen der Soela-eilanden. Jaarb. van het Mijnwezen in Ned.-Indië (1925). Verhandelingen I, 11—85 (1926).

TABLE I. Indices of specimens arranged according to increasing diameter.

	Indices of the ratio thickness: diameter of shell	Indices of the ratio thic- ness: height of the wh
HANIEL, pl. IV, fig. 9	$100 \times \frac{7.4}{9.9} = 74$	$100 \times \frac{7.4}{5.9} = 125$
HANIEL, table pp. 66 and 67	$100 \times \frac{7}{10} = 70$	$100 \times \frac{7}{6.5} = 107$
HANIEL, table pp. 66 and 67	$100 \times \frac{9.5}{12.5} = 76$	$100 \times \frac{9.5}{7} = 134$
HANIEL, table pp. 66 and 67	$100 \times \frac{10.25}{15.25} = 67$	$100 \times \frac{10.25}{9} = 114$
Ag. brouweri from Bitauni	$100 \times \frac{8.5}{17.8} = 47$	$100 \times \frac{8.5}{10.3} = 82$
Ag. sundaicum from Sufa	$100 \times \frac{8.7}{18.4} = 47$	$100 \times \frac{8.7}{11} = 79$
HANIEL, table pp. 66 and 67	$100 \times \frac{11.5}{19} = 60$	$100 \times \frac{11.5}{11} = 104$
HANIEL, table pp. 66 and 67	$100 \times \frac{9}{19} = 47$	$100 \times \frac{9}{12} = 75$
HANIEL, pl. IV, fig. 13	$100 \times \frac{8.8}{19.5} = 45$	$100 \times \frac{8.8}{12.4} = 71$
PERRIN SMITH, pl. 9, fig. 12 and 13	$100 \times \frac{13.7}{20.1} = 68$	$100 \times \frac{13.7}{12.5} = 112$
HANIEL, table pp. 66 and 67	$100 \times \frac{14}{20.5} = 68$	$100 \times \frac{14}{12} = 116$
Ag. brouweri from Bitauni	$100 \times \frac{11.5}{20.8} = 55$	$100 \times \frac{11.5}{13.8} = 83$
Specimen from Billiton	$100 \times \frac{13.7}{23.3} = 58$	$100 \times \frac{13.7}{11.5} = 119$
PERRIN SMITH, pl. 9, fig. 10 and 11	$100 \times \frac{13.5}{24.2} = 58$	$100 \times \frac{13.5}{12.3} = 109$
HANIEL, pl. IV, fig. 12	$100 \times \frac{12.8}{24.3} = 52$	$100 \times \frac{12.8}{15.8} = 8$[illegible]
HANIEL, pl. IV, fig. 16 (*Ag. sund.* var. *atsabensis*)	$100 \times \frac{14.5}{24.6} = 58$	$100 \times \frac{14.5}{14.3} = 101$
HANIEL, table pp. 66 and 67	$100 \times \frac{13}{25} = 52$	$100 \times \frac{13}{14} = 9$[illegible]
HANIEL, table pp. 66 and 67	$100 \times \frac{14}{28} = 50$	$100 \times \frac{14}{15} = 9$[illegible]
HANIEL, p. 67, textfig.	$100 \times \frac{17.2}{28.5} = 60$	$100 \times \frac{17.2}{15.6} = 108$
HANIEL, table pp. 66 and 67	$100 \times \frac{18.25}{29} = 52$	$100 \times \frac{18.25}{16} = 114$
HANIEL, pl. IV, fig. 7	$100 \times \frac{17}{29.1} = 58$	$100 \times \frac{7}{15.7} = 108$
PERRIN SMITH, pl. 9, fig. 9	$100 \times \frac{18}{29.1} = 61$	$100 \times \frac{18}{17} = 106$
HANIEL, pl. IV, fig. 8		
HANIEL, pl. IV, fig. 11	$100 \times \frac{17}{30.4} = 55$	$100 \times \frac{17}{18.3} = 9$[illegible]
PERRIN SMITH, pl. 9, fig. 15 and 16	$100 \times \frac{20.9}{34.8} = 60$	$100 \times \frac{20.9}{20.6} = 103$
PERRIN SMITH, pl. 9, fig. 7 and 8	$100 \times \frac{20.2}{35.5} = 56$	$100 \times \frac{20.2}{19.6} = 103$
HANIEL, pl. IV, fig. 15	$100 \times \frac{19.1}{37} = 51$	$100 \times \frac{19.1}{20.3} = 94$
HANIEL, table pp. 66 and 67	$100 \times \frac{20}{38} = 52$	$100 \times \frac{20}{20} = 100$
HANIEL, table pp. 66 and 67	$100 \times \frac{24.25}{45} = 53$	$100 \times \frac{24.25}{24} = 101$
HANIEL, pl. IV, fig. 14	$100 \times \frac{24.1}{45.2} = 53$	$100 \times \frac{24.1}{23.3} = 103$
The indices are variing from:	45 – 76	71 – 125

When not mentioned otherwise the specimens have been found near Bitauni, Timor.

ndices of the ratio eight of the whorl: diameter of shell	Indices of the ratio umbilicus: diameter of entire shell	Indices of the ratio width of chamber: diameter of shell	Indices of the ratio width of the chamber: height of whorl
$100 \times \frac{5.9}{10} = 59$	$100 \times \frac{1}{10} = 10$	$100 \times \frac{3.7}{9.9} = 37$	$100 \times \frac{3.7}{5.9} = 62$
$100 \times \frac{6.5}{10} = 65$	$100 \times \frac{1}{10} = 10$	—	—
$100 \times \frac{7}{12.5} = 56$	$100 \times \frac{1.5}{12.5} = 12$	—	—
$100 \times \frac{9}{15.25} = 59$	$100 \times \frac{2}{15.25} = 13$	—	—
$100 \times \frac{10.3}{17.8} = 58$	—	—	—
$100 \times \frac{11}{18.4} = 60$	$100 \times \frac{1.6}{17.8} = 9$	$100 \times \frac{6.3}{17.8} = 35$	$100 \times \frac{6.3}{10.3} = 61$
$100 \times \frac{11}{19} = 58$	$100 \times \frac{1}{17.8} = 5$	$100 \times \frac{5.4}{18.4} = 29$	$100 \times \frac{5.4}{11} = 49$
$100 \times \frac{12}{19} = 63$	$100 \times \frac{1}{19} = 5$	—	—
$100 \times \frac{12.4}{19.5} = 63$	$100 \times \frac{1.5}{19.5} = 8$	$100 \times \frac{7.8}{19.5} = 40$	$100 \times \frac{7.8}{12.4} = 62$
$100 \times \frac{12.2}{20.1} = 60$	$100 \times \frac{1.5}{20.1} = 7$	$100 \times \frac{6.4}{20.1} = 32$	$100 \times \frac{6.4}{12.2} = 52$
$100 \times \frac{12}{20.5} = 58$	$100 \times \frac{2}{20.5} = 9$	—	—
$100 \times \frac{13.8}{20.8} = 65$	$100 \times \frac{1.8}{20.8} = 8$	$100 \times \frac{6}{20.8} = 28$	$100 \times \frac{6}{13.8} = 43$
$100 \times \frac{11.5}{23.3} = 49$	$100 \times \frac{4.3}{23.3} = 18$	$100 \times \frac{3.8}{23.3} = 16$	$100 \times \frac{3.8}{11.5} = 33$
$100 \times \frac{12.5}{24.2} = 51$	$100 \times \frac{3.3}{24.2} = 13$	$100 \times \frac{6.5}{24.2} = 26$	$100 \times \frac{6.5}{12.3} = 52$
$100 \times \frac{15.8}{24.3} = 65$	—	$100 \times \frac{8}{24.3} = 33$	$100 \times \frac{8}{15.8} = 50$
$100 \times \frac{14.3}{24.6} = 58$	$100 \times \frac{1}{24.6} = 4$	$100 \times \frac{8}{24.6} = 32$	$100 \times \frac{8}{14.3} = 56$
$100 \times \frac{14}{25} = 56$	$100 \times \frac{2}{25} = 8$	—	—
$100 \times \frac{15}{28} = 53$	$100 \times \frac{3.5}{28} = 12.5$	—	—
$100 \times \frac{15.6}{28.5} = 54$	$100 \times \frac{1.5}{28.3} = 5$	$100 \times \frac{7.3}{28.5} = 25$	$100 \times \frac{7.3}{15.6} = 46$
$100 \times \frac{16}{29} = 55$	$100 \times \frac{3.6}{29} = 12$	—	—
$100 \times \frac{15.7}{29.1} = 54$	$100 \times \frac{3}{29.1} = 10$	$100 \times \frac{6.5}{29.1} = 22$	$100 \times \frac{6.5}{15.7} = 41$
—	$100 \times \frac{2}{29.1} = 7$	$100 \times \frac{8.1}{29.1} = 27$	$100 \times \frac{8.1}{17} = 47$
$100 \times \frac{18.7}{29.1} = 54$	—	—	—
$100 \times \frac{18.3}{30.4} = 60$	—	$100 \times \frac{7.8}{30.4} = 25$	$100 \times \frac{7.8}{18.3} = 42$
$100 \times \frac{20.6}{34.8} = 60$	$100 \times \frac{2.4}{34.8} = 7$	$100 \times \frac{10.5}{34.8} = 30$	$100 \times \frac{10.5}{20.6} = 51$
$100 \times \frac{19.6}{35.5} = 55$	$100 \times \frac{4.1}{35.5} = 11$	—	—
$100 \times \frac{20.3}{37} = 54$	$100 \times \frac{2.2}{37} = 8$	—	—
$100 \times \frac{20}{38} = 52$	$100 \times \frac{5}{28} = 13$	—	—
$100 \times \frac{24}{45} = 53$	$100 \times \frac{6.25}{45} = 14$	—	—
$100 \times \frac{23.3}{45.2} = 51$	$100 \times \frac{5.8}{45.2} = 12$	$100 \times \frac{11.8}{45.2} = 26$	$100 \times \frac{11.8}{23.3} = 50$
49—65	4—18	16—40	33—62

TABLE II

Indices of the intern whorls of *Agathiceras sundaicum* from Billiton

Indices of the ratio thickness: height of whorl	Indices of the ratio width of chamber: height of the whorl
$100 \times \frac{13.7}{11.5} = 119$	$100 \times \frac{\pm 4}{11.5} = \pm 34$
$100 \times \frac{10}{8} = 125$	$100 \times \frac{4}{8} = 50$
$100 \times \frac{5.11}{3.5} = 151$	$100 \times \frac{2}{3.5} = 57$

TABLE III

Indices of ratio of thickness and width of chamber to other proportions of the successive whorls of a cross-section of one specimen of *Agathiceras sundaicum* viz. HANIEL p. 67, textfig. 19 [1])

Indices of the ratio thickness: diameter of shell	Indices of the ratio thickness: height of whorl	Indices of ratio width of chamber: height of whorl
$100 \times \frac{17.2}{28.5} = 60$	$100 \times \frac{17.2}{15.8} = 108$	$100 \times \frac{6.8}{15.8} = 43$
$100 \times \frac{14.8}{22} = 67$	$100 \times \frac{14.8}{13} = 114$	$100 \times \frac{5.6}{13} = 43$
$100 \times \frac{11.3}{16} = 70$	$100 \times \frac{11.3}{9} = 125$	$100 \times \frac{4}{9} = 44$
$100 \times \frac{10.2}{12.2} = 83$	$100 \times \frac{10.2}{7.2} = 141$	$100 \times \frac{3}{7.2} = 41$
$100 \times \frac{6.8}{9} = 73$	$100 \times \frac{6.8}{4.5} = 151$	$100 \times \frac{2}{4.5} = 44$
$100 \times \frac{5.6}{6.8} = 82$	$100 \times \frac{5.8}{4} = 145$	$100 \times \frac{1.8}{4} = 42$
$100 \times \frac{4}{5} = 80$	$100 \times \frac{4}{2.9} = 153$	$100 \times \frac{1.2}{2.6} = 46$
$100 \times \frac{3.2}{3.6} = 88$	$100 \times \frac{3.5}{2.3} = 152$	$100 \times \frac{1.1}{2.3} = 48$
In younger stages the indices of the ratio thickness: diameter and the ratio thickness; height of whorl, are greater than those of older stages		In younger stages the indices of the ratio width of chamber: height of chamber, are greater than those of older stages.

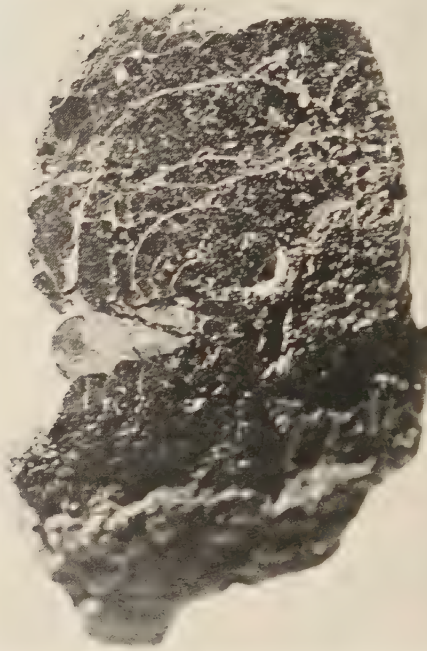
[1]) The change of the indices is not regular. Probably this is caused by inaccuracies of the measuring.

TABLE IV

Indices of ratio width of the chamber: height of the whorl of successive whorls in a cross-section measured in 3 specimens from Bitauni

$100 \times \frac{6.3}{16} = 39$	$100 \times \frac{13.1}{26.4} = 49$	$100 \times \frac{9.5}{21} = 45$
$100 \times \frac{4.9}{12} = 40$	$100 \times \frac{9}{18.2} = 49$	$100 \times \frac{7.8}{17.5} = 44$
$100 \times \frac{4.5}{10.3} = 43$	$100 \times \frac{7}{13.5} = 51$	$100 \times \frac{6.4}{13} = 46$
$100 \times \frac{3.5}{7.3} = 48$	$100 \times \frac{4.8}{9.6} = 50$	$100 \times \frac{5}{10} = 50$
$100 \times \frac{2.7}{5} = 54$	$100 \times \frac{2.9}{6.4} = 45$	
$100 \times \frac{2.1}{3.8} = 55$	$100 \times \frac{2.1}{4.8} = 48$	
	$100 \times \frac{1.7}{3.7} = 46$	

1



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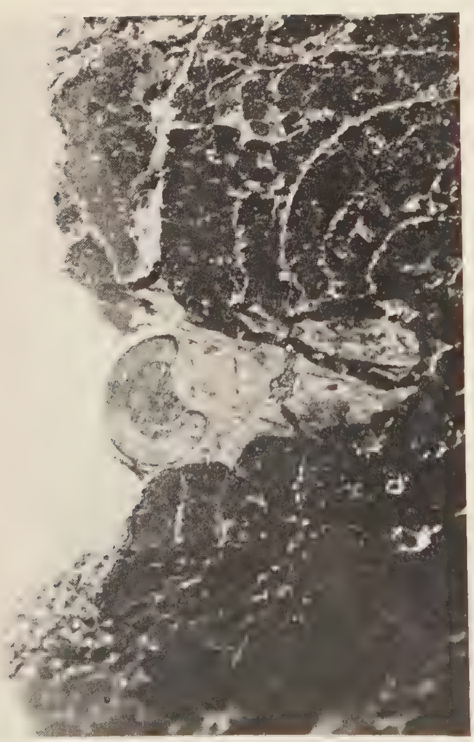

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TABLE V

Indices of ratio thickness: height of the whorl of successive whorls in a cross-section measured in three specimens, from Bitauni

$100 \times \frac{16}{16} = 100$	$100 \times \frac{23.2}{26.4} = 84$	$100 \times \frac{22}{22} = 100$
$100 \times \frac{13}{12} = 108$	$100 \times \frac{18}{18.2} = 99$	$100 \times \frac{18}{17.5} = 102$
$100 \times \frac{11.5}{10} = 115$	$100 \times \frac{14.3}{13.5} = 105$	$100 \times \frac{16.5}{13} = 127$
$100 \times \frac{8.7}{7.3} = 119$	$100 \times \frac{10.2}{9.6} = 106$	$100 \times \frac{14}{10} = 140$
$100 \times \frac{6.4}{5} = 128$	$100 \times \frac{7.6}{6.4} = 118$	
$100 \times \frac{4.6}{3.8} = 121$	$100 \times \frac{5.7}{4.8} = 118$	
	$100 \times \frac{4.6}{3.7} = 124$	
	$100 \times \frac{3.4}{2.6} = 125$	

(identification by HANIEL) resembles very well this figure and also *Ag. brouweri*. The chamber shown in HANIEL's figure is even more spacious. When only slender forms of *Ag. sundaicum* are compared the characteristics of them agree very well with *Ag. brouweri*. However, as already mentioned the spiral lines of *Ag. brouweri* are weaker than those of *Ag. sundaicum*. I do not believe that this slight difference is of specific value, as there exists a continuous series from this type to the more robust *Ag. sundaicum* proper.

In the Bitauni beds of the Lower Permian of Timor specimens of *Ag. sundaicum* are numerous. These beds are separated only from the Basleo beds, which are considered to form the transition beds to the Upper Permian, by the Tea Wei beds [5]).

It seems therefore very probable to me that the specimen from Billiton belongs to the Lower Permian.

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[5]) BEMMELEN, R. W. VAN, The geology of Indonesia. pp. 237 and 315—319 (The Hague 1949).

Fig. 1. *Agathiceras sundaicum* Han. from Billiton, showing the spiral lines on its surface. Enlarged ± 3.5 ×

Fig. 2. Polished cross-section of the same specimen, showing the inner structure and the small ammonoid in the umbilicus. Enlarged ± 3.5 ×

Fig. 3. Part of fig. 2 showing an oblique section through the small ammonoid and the inner structure of *Agathiceras*. Enlarged ± 8 ×

Fig. 4. Part of a polished cross-section through a specimen of *Ag. sundaicum* Han. from the Lower Permian of Bitauni, Timor, showing a section of the spiral lines on the extern side of the shell. Enlarged ± 3.5 ×

PHYSIOLOGY

ON THE STUDY OF THE EFFECTS OF LIGHT OF VARIOUS SPECTRAL REGIONS ON PLANT GROWTH AND DEVELOPMENT

BY

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(Communicated by Prof. A. J. KLUYVER at the meeting of June 24, 1950)

Introduction.

Various studies on the influence of spectral light on aspects of plant life have been made. We will for the moment exclude those in which micro-organisms were the object. They require special methods and in many cases they are advantageous over higher plants, from the view point of the plant physiologist. But many phenomena, induced by light, are restricted to or at least very specially manifest in higher plants. As those may be mentioned phototropism (in part), various phenomena induced by the duration of the light period, generally called 'photoperiodic' effects, and other, especially formative influences of light. We may mention the formation of flowers, of tubers, the regulation of sprouting, and so on. In the course of time, many of these effects have been shown to answer differently to light of various spectral regions (see, *e. g.*, 1, 2, 3, 4, 5).

In a few consecutive papers we describe some observations on the influence of light of various spectral regions on the sprouting of potatoes and on some effects related with the photoperiod, or, at least, with additional coloured illumination.

In this paper we describe a few principles of our working method and of our equipment.

Previous investigations on the effect of spectral light on higher plants can be divided into two types: 1) Studies using continuous spectra obtained by a spectrograph (*e. g.*, 1, 5); 2) Studies using coloured filters (2, 3, 4).

The first type offers the possibility to use well-defined and limited spectral regions of high purity, but, even in its most advanced form (5) it allows illumination of only small fractions of a higher plant. It is preferably adapted to illuminations of brief duration.

In order to achieve illuminations of appreciable duration and intensity over large areas it seems indicated to use the second method. Drawbacks of this method in many of the previous studies were: The use of filters of rather low specificity in transmission (*e.g.*, 3, 4), often combined with inconstant illumination (daylight), a very limited knowledge especially

of the infrared transmission, and its effect upon the plants (an estimation of the latter, *e.g.*, was attempted in (2)). In most cases the expositions were combined with unfiltered daylight as 'short day'.

In the studies being carried out in our laboratory we attempt at achieving the following conditions:

1) Constant illumination over long duration and, if required, large area and appreciable intensity, with limited spectral regions.

2) Control of other factors — humidity, temperature and aeration — during the experiment, as carefully as possible.

3) If additional illumination with white light is required, artificial light of controlled intensity, duration, and spectral composition is used (if not too high intensities are required).

Description.

Our present equipment consists of a chief part, used for various purposes, which is supplemented by smaller buildings if in an experiment special spectral regions are required, for which large areas and high intensities are not essential. The central part is built to fulfill the requirements listed under 1), above. Its major — spectral — part consists of 6 compartments of 210 cm length, 45 cm width and 120 cm height. They are built side by side, separated by rigid, hard board walls. The bottom consists of wood. The top consists partly of hard board, partly of a glass filter of 30×110 cm, mounted at one end of the compartment. Along the sides of the filter are light proof ventilation holes. Above the filters, at a distance of ~ 5 cm, the lamps are mounted.

Six spectral regions are produced, *viz.*, violet, blue, green, yellow, red, and infrared. The large areas required prevented the use of special, so called optical colored glasses, the price of which is about *f*. 30.— per dm², so that one filter plate would cost about *f*. 1000.— (about £ 100). And even then in many cases special additional filtering would be required to remove, *e.g.*, the near infrared. It seemed, however, that the newly developed, fluorescent tubes, would offer an opportunity to reach our aims in an easier way. These tubes are, essentially, mercury vapour tubes, with a highly fluorescent internal coating. In the normal trade specimens a few phosphores are mixed, yielding white, additional emission (the hue of the white being regulated by the proportion of the used phosphores). By introducing, however, only one special phosphor, tubes are produced with additional emission in one spectral region. Through the courtesy of the Physical Research Laboratory of the Philips Factories, Eindhoven, we obtained a limited supply of fluorescent tubes with, respectively, preferably blue, green, and pink emissions. [1]) These tubes have only a limited emission in the near infrared, according to determinations by the

[1]) We are especially indebted to Prof. Dr H. B. G. Casimir, director, and to Dr J. Voogd and Dr R. van der Veen for their kindness in helping us to obtain these tubes.

Philips Laboratory. It now seemed feasible to restrict the spectral extension of their emission still further by colored glasses. To this purpose a large number of samples of colored glass from the trade were examined spectroscopically, and a number of them were selected as filters in the equipment. The following combinations were made (*cf.* fig. 1):

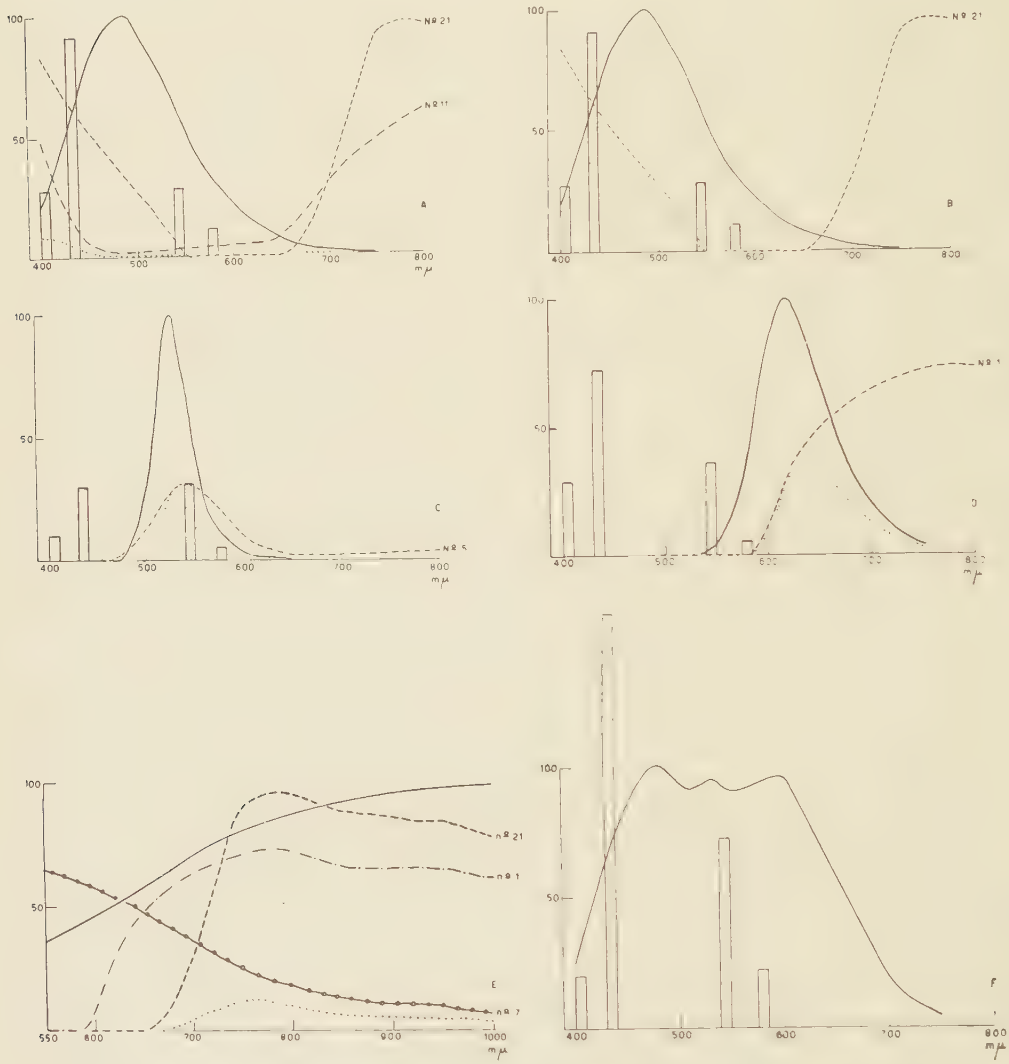

Fig. 1. Spectral emission of lamps and transmission of filters (in %) used. Lamps: ————. Filters: numbered. Spectral energy distribution of radiation: .... A: violet, B: blue, C: green, D: red, E: infrared compartment; F: daylight fluorescent tube. Yellow compartment: sodium line (not given here).

1) 7 blue fluorescent tubes (40 W, 120 cm long) with a blue filter (No. 14)[1]) and a purple filter (No. 11), yielding *violet* light. Owing to limitations in the supply of blue tubes, in various experiments white 'daylight'

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1) Very similar to No. 21.

tubes had to be used, which yielded much less pure light, with, especially, more near infrared.

2) 4 blue fluorescent tubes (40 W.) combined with a blue glass (No. 21), yielding *blue* light.

3) 4 green fluorescent tubes (40 W.) with a dark green glass (No. 5), yielding *green* light.

4) 2 sodium vapour lamps (100 W. each) with a pale green, infrared absorbing glass (No. 7), yielding *yellow* light.

5) 4 pink fluorescent tubes (40 W) with a red glass (No. 1), yielding *red* light.

6) 6 incandescent lamps (60 W) with the filters mentioned under 2), 4), and 5), yielding *infrared* light, with relatively much near infrared.

The spectral composition of the light in the compartments up to 1 μ is given in fig. 2. It was computed as follows. The emission curves of the

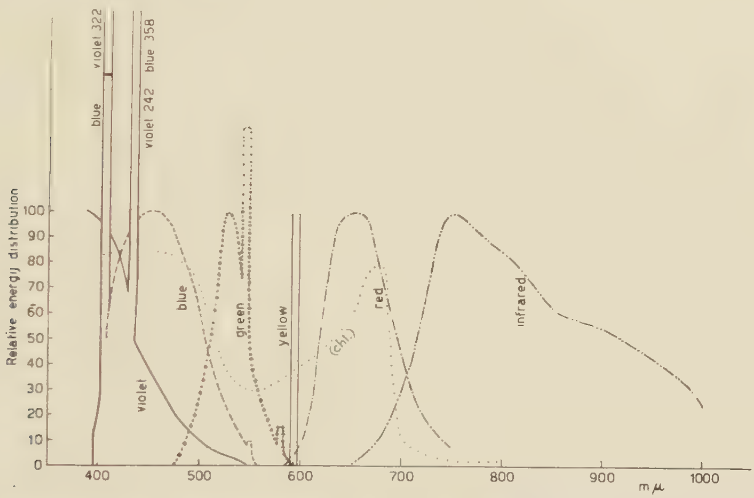

Fig. 2. Relative energy distribution in the various compartments up to 1μ. Maximum of fluorescent emission = 100; (chl): absorption of chloroplast suspension of spinach after RABIDEAU, FRENCH and HOLT, Am. J. Bot. Dec. 1946.

fluorescent tubes were taken as indicated by the Philips laboratory. The transmission of the glasses was estimated with a monochromator-amplifier set-up, between 0.4 and 1 μ.[1]) Notwithstanding the fact that the fluorescent tubes are much more restricted in their emission of near and further infrared than, *e.g.* incandescent lamps, this emission,even after having passed the glass could not be altogether neglected. So far, we are not in a position to measure directly the emission beyond 1 μ. With the aid of the properties of various SCHOTT-glasses, as mentioned in the catalogue of SCHOTT and Gen, and in the lists supplied by each specimen of their glasses, we have tried to obtain an idea of the transmitted energy in the regions

[1]) These measurements were made in the Physical Institute of the University of Utrecht. We are indebted to Professor J. M. W. MILATZ who kindly allowed us to use the apparatus.

specially viewed upon by the filtering, and of the spurious light, especially in the regions of $\sim 0.7-1\ \mu$, and $1-3\ \mu$. The results of the measurements made for this purpose are collected in Table I and fig. 3. In Table I also

TABLE I

Data regarding the partition of the energy in each compartment over various spectral regions (measured in the middle of filter).

Colour	Filter	Region (μ)	Emission (relative units)	In %	% without far infrared
Violet	WG 3	0.33—3	3.53	100	(70 = 100)
	WG 3—GG 11	0.33—0.5	1.68	48 *	68 *
	GG 11—RG 5	0.5 —0.67	0.07	2	3
	RG 5—RG 7	0.67—$\sim$ 0.96	0.73	20 ≈	29
	RG 7	$\sim$ 0.96—3	1.05	30 †	—
Blue	WG 3	0.33—3	7.32	100	(92 = 100)
	WG 3—GG 3	0.33—0.43	4.52	62 *	67 *
	GG 3—OG 2	0.43—0.56	1.50	20 *	22 *
	OG 2—RG 5	0.56—0.67	0.08	1	1
	RG 5—RG 7	0.67—$\sim$ 0.96	0.65	9 ≈	10
	RG 7	$\sim$ 0.96—3	0.57	8 †	
Green	WG 3	0.33—3	8.60	100	(84 = 100)
	WG 3—GG 7	0.33—0.47	0.50	6	7
	GG 7—GG 11	0.47—0.50	0.3	3	3.5
	GG 11—RG 1	0.50—0.60	6.45	75 *	89 *
	RG 1—RG 5	0.60—0.67	0.13	0 ≈	0
	RG 5—RG 7	0.67—$\sim$ 0.96	— 0.13		
	RG 7	$\sim$ 0.96—3	1.35	16 †	—
Yellow	WG 3	0.33—3	23.2	100	(50 = 100)
	WG 3—OG 2	0.33—0.56	0.2	1	2
	OG 2—RG 2	0.56—0.63	10.5	45 *	90 *
	RG 2—RG 5	0.63—0.67	— 0.3	0	0
	RG 5—RG 7	0.67—$\sim$ 0.96	1.0	4 ≈	8
	RG 7	$\sim$ 0.96—3	11.8	50 †	—
Red	WG 3	0.33—3	11.00	100	(85 = 100)
	WG 3—RG 1	0.33—0.60	0.6	5	6
	RG 1, RG 8, RG 7 and graph, see text	0.60—0.70	7.13	65 *	76 *
		0.70—$\sim$0.96	1.63	15 ≈	18
	RG 7	$\sim$ 0.96—3	1.64	15 †	—
Infrared	WG 3	0.33—3	14.7	100	(28 = 100)
	WG 3—RG 5	0.33—0.67	0.2	1	4
	RG 5, RG 8, RG 7 and graph, see text	0.67—0.70	0.2	1	4
		0.70—$\sim$0.96	3.8	26 *	92 *
	RG 7	$\sim$ 0.96—3	10.5	72 †	—

Legend: * region(s) required ≈ near infrared † far infrared

the fractions of the total emission for the various regions are indicated as computed from these measurements.

Further, special measurements and comparisons indicated that the emission between 1 and 3 μ of our filter no. 7 answered closely that of Schott NG 3, that of no. 21 answered closely to that of Schott BG 3, which knowledge was used in the computation of the curves for additional infrared expositions as given in (6).

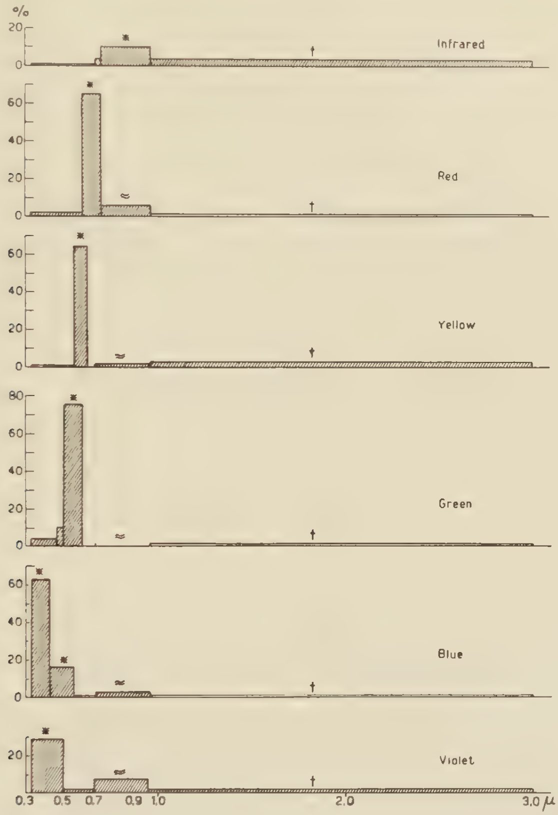

Fig. 3. Energy distribution (%) in the various compartments differentiated according to spectral regions up to 3 μ. * region(s) required, $\approx$ near infrared, † far infrared.

By combining the filter no. 7 with Schott RG 5 and BG 23, it was found that the transmission of no. 7 was about $^1/_3$ in the region between 1 and 3 μ. A similar test, substituting BG 12 for BG 23 led to the conclusion that, between 2 and 3 μ, the transmission of no. 7 was about $^1/_3$ as well.

It appeared worth while, furthermore, to evaluate the amount of near infrared (as generated in the infrared compartment) present in the red compartment. This was approached in two ways.

The total radiation, passing filter WG 3 of Schott and Gen. was measured. Then, the radiation, passing RG 7, *i.e.* above ~ 0.96 μ (50 % transmission of RG 7) was determined, and the value found was subtracted from the previous one. This yielded the radiation with wave lengths < 0.96 μ. In the same way, the radiation passing RG 7 was subtracted from that passing RG 1, RG 5, RG 8, and no. 21. In the latter case, however, a much smaller amount had to be subtracted, since no. 21, like BG 3, only transmits part of the radiation > 1 μ. Also the amount of

energy transmitted by BG 17, absorbing very strongly the far infrared, was determined.

On the other hand, the spectrum as given in fig. 1, combined from the Philips' data on lamp emission and the transmission of filter no. 1, was taken, and the percentages of the light, which were transmitted by RG 5, RG 8, 21 and BG 17, were estimated graphically, up to 0.8 μ. The good agreement, obtained in both types of estimation (by energy measurements and graphically, see Table II) indicates

TABLE II

Estimation of energy distribution in the red compartment (see text).

Filter	Region (μ)	Transmission in relative units from thermopile measurements		Transmission in % without far infrared	
		— 3 μ	— 1 μ	Estimated from measurement with thermopile	Estimated graphically
WG 3	0.3 —3	11.0	9.36 [1])	100	100
RG 1	0.6 —3	10.40	8.76 [1])	92.5	
RG 5	0.66—3	5.37	3.73 [1])	40	48.5
RG 8	0.69—3	3.75	2.11 [1])	22.7	21.6
BG 17	0.3 —1	7.15	6.38	68	75.8
No. 21	0.72—3	1.90	1.43	15.3	14.6
No. 21 [2]) + RG 7		0.47			
RG 7	∼ 0.96—3	1.64			

[1]) Transmission of RG 7 subtracted.

[2]) This filter has a strong absorption > 1 μ.

that the lamp emits little radiation between 0.8 and 0.96 μ. As a good average for the short wave length side of filter 21 (giving the short wave length boundary in the infrared compartment) under the type of lamp in the red compartment, 0.7 μ was chosen. In this way the figures of table 1 were obtained.

In the yellow compartment, the glass no. 7 was substituted by the dark green glass, no. 5, with a lower transmission of the sodium line, to bring the intensity in this compartment more in a line with that in the other compartments. The infrared transmission of this glass, however, was higher than that of no. 7.

The maximum energy available in the various compartments was not fully equal, it was from about 1×10^3 ergs / cm^2 sec. in the violet to 10×10^3 ergs / cm^2 sec. in yellow, red and infrared. By variation of the distance from the filter, both horizontally and vertically, equal intensities could be adjusted in the various colours, or a range of expositions, mutually overlapping, could be employed in the various compartments. The intensities are sufficient for inducing various photoperiodic and photoformative effects in higher plants. They are, however, not high enough to raise plants fully under colored light.

To this purpose other constructions are necessary which are now being built up. If full growth and development of plants was required, in our

present experiments we had to supply white light (derived from fluorescent tubes) of sufficient intensity during part of the daily cycle. Cultures of *Chlorella*, however, could be grown in the compartments with sufficient strength.

The energy measurements in the compartments were made in absolute units with the aid of a standardized thermopile and galvanometer; afterwards a barrier layer photocell combined with a μ-amp. meter, calibrated for each colour against the thermopile was often used for convenience.

A special problem is the regulation of temperature in each compartment and how to secure a reasonable equality between the various compartments. Various devices have been tried in the course of time and the development of these matters is still in flow. At present, the front side of the compartments is closed by a 'door' of board, the back side by a dark blanket. Above the 'door' in front, a ventilator is mounted in a solid part of the 'wall', light is shielded off by an overhanging dark blanket. The ventilator sucks in air from outside which enters the space between blanket and compartment from below into the compartment where it is blown between the filter and a transparent sheet of glass, mounted about 5 cm below the filter. The filter is thus cooled, and the air is used for some additional warming of the darker side of the compartment. Additional ventilators, removing warm air from the surroundings of the light sources are placed on the top of the construction, outside.

In the way described it was possible to maintain the temperature of the set of 6 compartments constant within $1 - 2°$ C. Placing plants into a compartment yielded slight disturbances of the regularity of the temperature. A sufficient humidity is obtained by placing a layer of moist *Sphagnum* in a zinc box in each compartment.

The whole set-up is placed in a sub-floor room with three windows at the south side, with central heating but no special temperature regulations. A high reed mat is placed in front of the south wall of the room at a distance of about 1 m, to prevent direct solar radiation against the windows and the wall. During the winter season, from about Sept. — April, the room could be kept fairly closely at 18° C by sucking in cold air from outside by a ventilator, operated via a mercury thermo-regulator. During the summer season air was sucked in from an artificially cooled room. So far the capacity of this cooling was not sufficient to prevent increase of the temperature of the room. On excessively hot days the room temperature was about 24° C, that of the compartments rose to about 23° C, these being rather near to the entrance of the cooled air.

Alongside with the equipment containing the compartments for irradiation with limited wavelength regions, cases for dark controls, for additional illumination with 'daylight' from fluorescent tubes (see fig. 3F) and for growing the plants under strong 'daylight' illumination have been built up in the same room.

Summary.

An equipment is described which allows illumination of higher plants during long times with light of limited spectral regions: violet, blue, green, yellow, red, infrared, with intensities up to 10^4 ergs / cm^2 sec. under controlled conditions. The equipment is suitable for photoperiodic and photoformative studies, and for the growth of algae. The intensities are not high enough for the cultivation of most higher plants in coloured light only.

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THEORY ON CENTRAL RECTILINEAR RECESSION OF SLOPES. III

BY

J. P. BAKKER AND J. W. N. LE HEUX

(Communicated by Prof. F. A. VENING MEINESZ at the meeting of June 24, 1950)

Central rectilinear recession with decreasing h-values (crest-recession)

1. *Introduction.*

In the following lines, we shall subject the central rectilinear recession of crests with a straight-lined triangular profile FT_1H (fig. 12) [1]) to a mathematical treatment.

From the steep rockwall, in a very small unit of time, a part T_1FT_2, T_2PRT_3 etc. is removed, while in the same period a screes volume $I'PF$, $II'I'PR$ etc. is deposited on an almost horizontal form FII' at the foot. Supposing further the same conditions as in the first part of our theory (16, I, p. 961 — 962) [2]) the coordinates of the points P and R (fig. 12) are (x, y) and $(x + dx, y + dy)$. The constant base FD being k, the decreasing height

$$T_2D = k\,\frac{y}{x} \text{ and } T_3D = k\,\frac{y+dy}{x+dx}.$$

So

$$T_2T_3 = k\,\frac{y\,dx - x\,dy}{x\,(x+dx)}\,;\quad FT_3T_2 = \tfrac{1}{2}\,k^2\,\frac{y\,dx - x\,dy}{x\,(x+dx)}\,;$$

$$FSP = \frac{x^2}{k^2}\cdot FT_3T_2\,;\quad PST_3T_2 = \left(1 - \frac{x^2}{k^2}\right) FT_3T_2 = \frac{k^2 - x^2}{2\,x}\cdot\frac{y\,dx - x\,dy}{x+dx}\,.$$

Now, considering, that for

$$R \to P,\ \text{Lim.}\,\frac{II'\,I'\,PR}{II'\,I'PK} = 1 \text{ and Lim.}\,\frac{PRT_3T_2}{PST_3T_2} = 1$$

and putting

$$(1-c)\,II'\,I'\,PK = PST_3T_2$$

we have

$$y\,(1-c)\,(a\,dy - dx) = \frac{k^2 - x^2}{2\,x^2}\,(y\,dx - x\,dy). \tag{17}$$

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[1]) See PHILIPPSON, A., literature (5, II, 2, p. 63) in the second part (p. 1162) and the introduction of the first part of our theory (p. 959—961).

[2]) The numbers in parentheses refer to the list of literature at the end of part IV of this article. For the literaturenumbers 1—15, see p. 1162 of part I—II of our theory.

It may be remembered, that in the case of central rectilinear recession of a slope, bordered at the top by a horizontal plateau with constant height h, we have found:

$$y\,(1-c)\,(a\,dy-dx) = \frac{h^2-y^2}{2\,y^2}\,(y\,dx-x\,dy). \tag{10}$$

(See 16, I, p. 962).

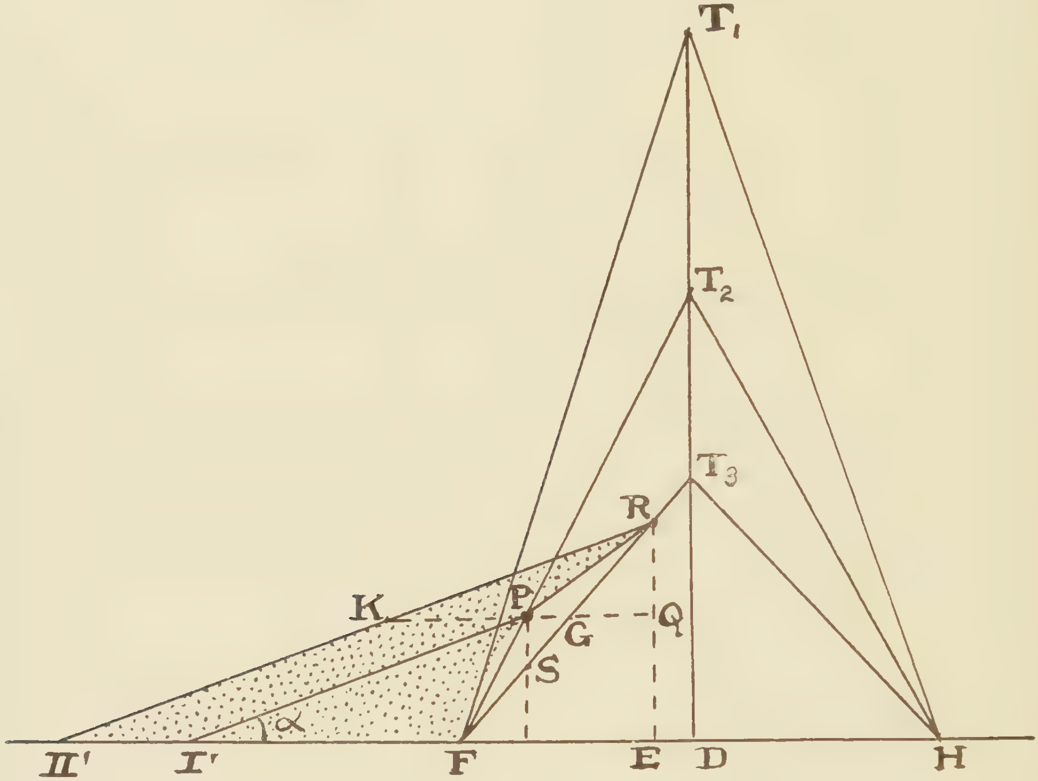

Fig. 12

2. *Projective-geometric treatment of our formulae* (10) *and* (17).

In our projective-geometric treatment of LEHMANN's theory on parallel recession of mountain slopes, we observed, that the form in which LEHMANN had cast his theory did not lend itself to a rapid construction of the curves required and a ready comparison with the phenomena observable in Nature (10, p. 541).

In our own theory on central rectilinear recession with stable position of the basis point of the slopes, it is mostly also fairly difficult and laborious to find and to draw the integral curves of our differential equations (10) and (17).

The geomorphologist wants a quick survey of possible rock-profiles underneath the screes, drawn if desired, with sufficient exactness for given values of β (initial slope-angle of the wall of a plateau or crest), α (slope-

angle of the screes) and c (ratio between rock-volume and screes-volume; 16, I, p. 961 — 962). Therefore, we will join the well-known graphical solution of a differential equation of the first degree by means of isoclines (17, p. 124 — 126) to a nomographical construction of these isoclines. In this way we find a method for discussing the possible forms of the rock-profile underneath the screes for given values of a ($= \cot \alpha$), b ($= \cot \beta$) and c, *without drawing any curve* (cf. part 6: The ratio between the flat part ...).

Although several integral curves of the equation

$$y\,(1-c)\,(a\,dy-dx) = \frac{h^2-y^2}{2\,y^2}\,(y\,dx-x\,dy) \tag{10}$$

have already been discussed in the second part of our theory (16, II, p. 1154 — 1161) we shall begin to treat it again with the new method, in order to have a control. The more so as the general form of both equations (10) and (17) is the same.

By introducing $\frac{dy}{dx} = p$, the equation (10) may be written

$$y\,(1-c)\,(ap-1) = \frac{h^2-y^2}{2\,y^2}\,(y-px)$$

or $$y^3\{2(1-c)\ (ap-1)+1\} - h^2y = -px\ (h^2-y^2).$$

Putting $2(1-c)\ (ap-1)+1 = m$, we have

$$px = y\,\frac{h^2-my^2}{h^2-y^2} \tag{18}$$

and in the same way for central linear crest-weathering

$$px = y\,\frac{k^2-mx^2}{k^2-x^2}. \tag{19}$$

For the constant p-value, the curve of formula (18) is the locus of points with line-elements of the integral-curves of the differential-equation

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{h^2-my^2}{h^2-y^2},$$

parallel to p (isocline).

The construction of the desired rock-profile may be obtained by drawing a family of such isoclines (formula 18) for p-values from $\tan \beta$ to $\tan \alpha$, decreasing by $d = 0{,}1$ or a smaller interval and joining the successive elements, $p = \tan \beta$ being in the footpoint of the slope we started from (= zero point of our coordination system). Each new element $p = 0{,}1\,n$ has to be drawn in the point of intersection of the element $p = 0{,}1\,(n+1)$ with the isocline $p = 0{,}1\,n$ (cf.: sub 4, construction of the integral-curve).

The same may be said for the construction of the integral-curve of

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{k^2-mx^2}{k^2-x^2}$$

(considering $\frac{dy}{dx} = \tan \beta$ for $x = 0$ and $y = 0$) by means of the family of isoclines

$$px = y\,\frac{k^2-mx^2}{k^2-x^2}.$$

3. *Construction of the isoclines (curve of formula 18 for p = constant).*

The curve of formula (18), written in the form

$$y^2(my - px) + h^2(px - y) = 0$$

has a tangent $y = px$ in the zeropoint of our coordination-system and the asymptotes $y = h$ and $y = \frac{p}{m}x$.

Putting $s = \frac{h^2 - my^2}{h^2 - y^2}$ we get (from form. 18)

$$\frac{p}{s} = \frac{y}{x}$$

p being given and m depending upon given values of a, c and p, s may be obtained for a chosen value y_0 of y by projective transformation of a quadratic scale of y. The direction of $\frac{y_0}{x_0}$ thus being given, x_0 is known.

Changing y into x, the same may be said for the drawing of the curve $px = y\ \frac{k^2 - mx^2}{k^2 - x^2}$.

In this case, however, the equation

$$x^2(my - px) + k^2(px - y) = 0$$

shows the asymptotes $x = \frac{k}{\sqrt{m}}$ and $y = \frac{p}{m}x$.

Now, if a plateau DC with a slope OD is given (fig. 13), the X-axis is laid along OB and the Y-axis is drawn at right angles with OB in O. DA is drawn, parallel to OY. Following the nomographical rules for constructing the projective scale $s = \frac{h^2 - my^2}{h^2 - y^2}$ on OB (S-axis), we get

$$\text{for}\begin{cases} y = 0\ , s = 1 \\ y = \infty, s = m \end{cases} \text{and for}\begin{cases} y = \frac{h}{\sqrt{m}}, \ s = 0 \\ y = h \qquad , s = \infty. \end{cases}$$

Choosing OA as unit, the point A is numbered o ($s = 1$). If $OB = m$, the point B is numbered ∞. The origin ($s = o$) is numbered $\frac{h}{\sqrt{m}}$. These three points are sufficient to construct the projective scale. Let the zero-point of the quadratic y-scale coincide with the zero-point A. This quadratic scale may be drawn in any direction and with any modulus (unit). In our case, the scale is laid along AD with a modulus of 1 mm (if $AD = h = 100$ mm) and numbered (the distance $A - 1 = 1$ mm; $A - 2 = 4$ mm; $A - 3 = 9$ mm etc.). Now, these points of division 1, 2, 3 etc. of the quadratic scale should be projected on the S-axis OB from a centre C to find the scale $s = \frac{h^2 - my^2}{h^2 - y^2}$. The centre C is the point of intersection of two lines, joining similarly numbered points of both scales OB and AD. The point at infinity of AD corresponds to B (∞), so we draw BC parallel

to AD ($OB = s - m$). The origin, numbered $\frac{h}{\sqrt{m}}$ should be joined to the point numbered $\frac{h}{\sqrt{m}}$ of the quadratic scale, but it is easier to join the point D of the quadratic scale ($y = h$) to the point at infinity of OB, for we have

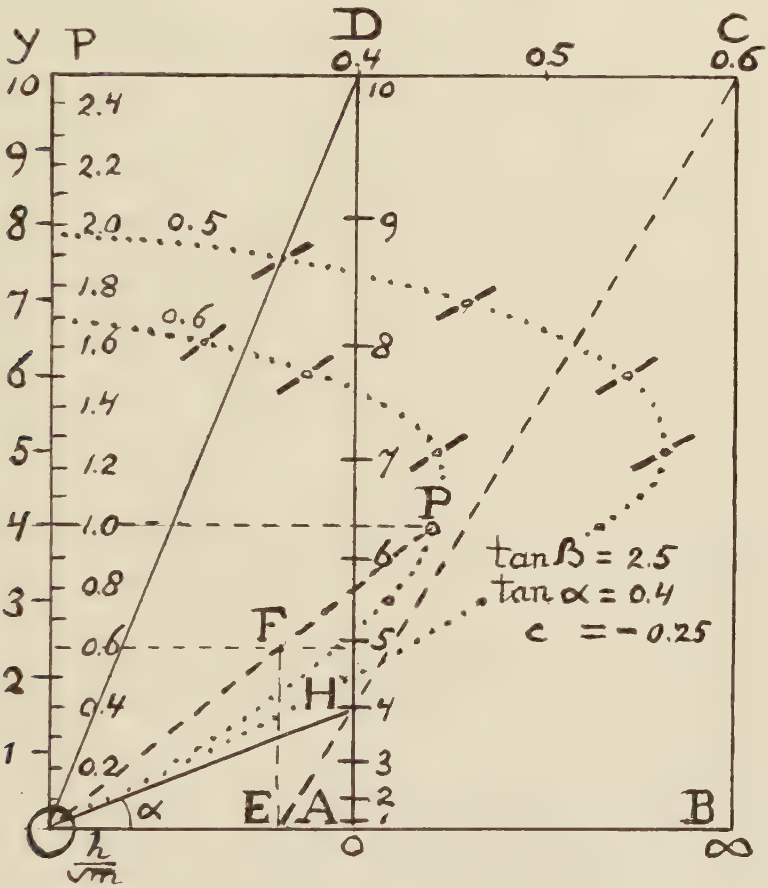

Fig. 13

found before, that for $y = h$, $s = \infty$. The place of the centre C depends upon the value of m and therefore of p. *Thus, each isocline has its own centre of projection* C, numbered as the isocline ($p = 0{,}6$ in fig. 13).

As we have proposed sub 2, the isoclines will be drawn for p-values from $\tan\beta$ to $\tan\alpha$, decreasing by 0,1. The required p-scale has been constructed on the right side of OY by dividing $OY = AD = \tan\beta$ ($OA = 1$) in 25 equal parts (for a value of $\tan\beta = 2{,}5$).

To construct a point P of isocline 0,6 we proceed as follows (fig. 13). The central projection of any point H of the quadratic scale, numbered y_0 from C on OB is E. Now $OE = \frac{h^2 - m y_0^{\,2}}{h^2 - y_0^{\,2}} = s$. The perpendicular in E

meets the line $p = 0{,}6$ parallel to OB in F, so, the direction of $OF = \frac{0{,}6}{s}$. The line $y = y_0 = 4$ meets OF in the desired point $P\,(x_0, y_0)$ for $\frac{0{,}6}{s} = \frac{y_0}{x_0}$. Repeating the same construction for $y_0 = 1, 2, 3$ etc. we get the dotted curve 0,6, locus of line elements of the integral-curves with direction $\frac{dy}{dx} = p = 0{,}6$. In like manner the isocline 0,5 is constructed by means of the centre 0,5.

As the modulus of the quadratic scale on AD has no influence upon the projective scale on OA, the same construction may be applied to find $s = \frac{k^2 - mx^2}{k^2 - x^2}$. But now we must take the point of intersection of OF with $x = x_0$. The modulus of the X-scale is $0{,}1\,k$.

In fig. 13, $m = 2\,(1 - c)\,(ap - 1) + 1 = 2{,}25$; $a = 2{,}5$; $c = -0{,}25$ for $p = 0{,}6$.

4. *Construction of the integral-curve, determined by*

$$x = 0,\ y = 0,\ \frac{dy}{dx} = \tan\beta.$$

a. Case of a plateau.

In fig. 14 an example is given of the construction of the integral-curve of the equation

$$y\,(1 - c)\,(a\,dy - dx) = \frac{h^2 - y^2}{2\,y^2}\,(y\,dx - x\,dy)$$

determined by $x = o$, $y = o$, $\frac{dy}{dx} = 1$, by means of a family of isoclines $px = y\,\frac{h^2 - my^2}{h^2 - y^2}$.

The data are $\beta = 45°$; $a \sim 22°$; $\tan a = 0{,}4$; $a = \cot a = 2{,}5$; $c = 0$; m (for $p = 0{,}6$) $= YC = 2$. The centra 0,6; 0,55; 0,5 etc. on the line YC refer to the isoclines 0,6; 0,55; 0,5 etc. (distances from point O: 2; 1,75; 1,5 etc.). To construct point P (3) of isocline 0,6, the centre C (0,6) is joined to point 3 of the quadratic scale. The perpendicular in A meets the line $p = 0{,}6$ parallel to OA in B. The line OB meets the line $y = 3$ in the desired point P (see sub 3). In this way, isocline 0,6 and the other isoclines are drawn. To construct the integral-curve we proceed as follows. The line-element in O with the direction $\frac{dy}{dx} = p = 1$ meets the first isocline 0,9 in N. The line-element in N with the direction $\frac{dy}{dx} = 0{,}9$ meets the following isocline 0,8 in L. The line-element in L with the direction 0,8 meets the isocline 0,7 in K etc. We obtain a curved part, hidden underneath the screes and a flat part (approximately beginning with isocline 0,45), where the screes no longer protects the rocky nucleus, so that a further softening may be assumed (16, p. 1155 — 1156 and fig. 7).

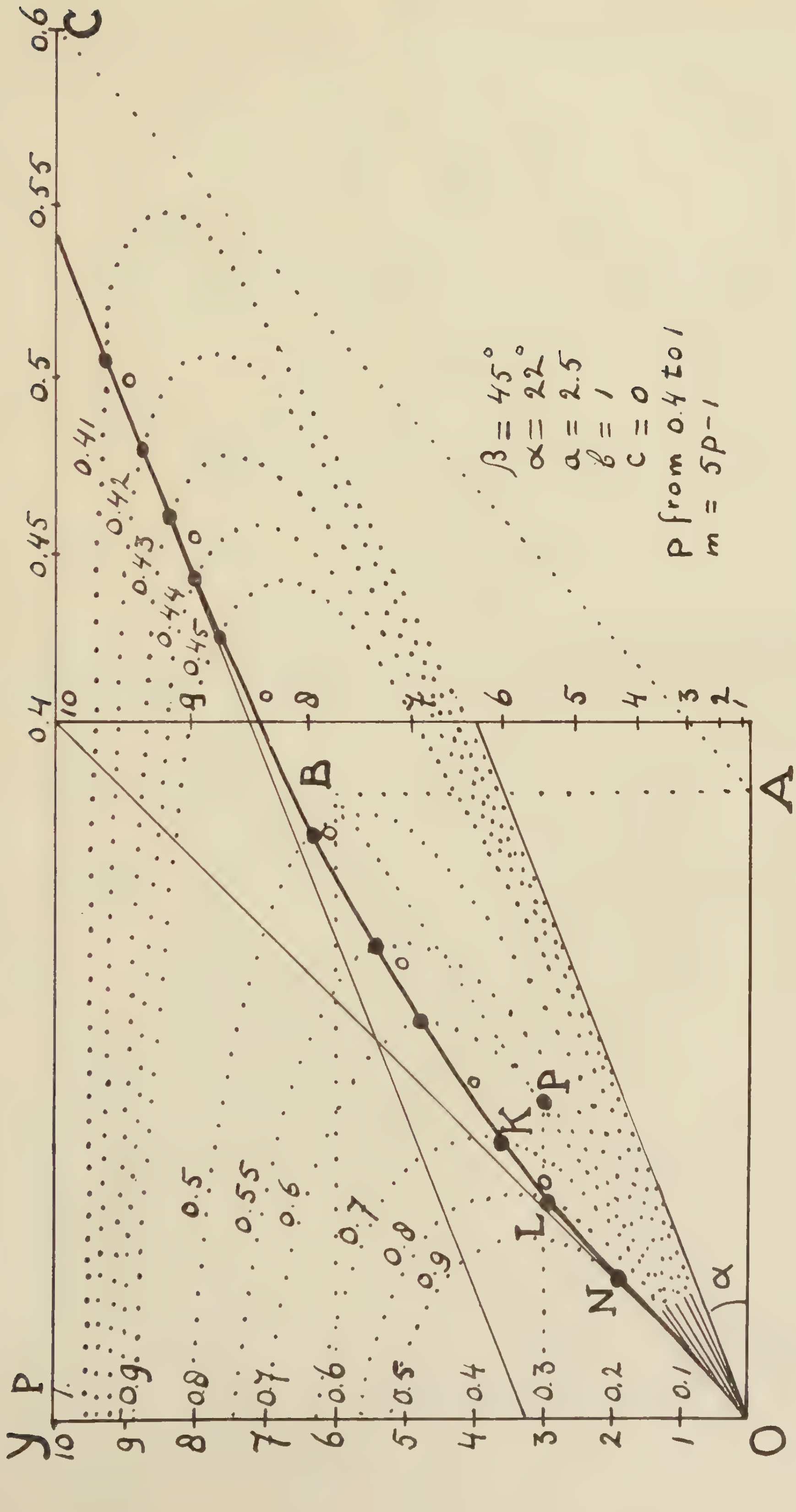

Fig. 14

b. Control by the results, found before.

With the same data, the equation of the integral curve, found before (16, p. 963)

$$x = ay - (a-b)\, y \left\{ \frac{h^2 + (1-2c)\, y^2}{h^2} \right\}^{\frac{c-1}{1-2c}} \tag{14}$$

becomes $x = 2{,}5\, y - 1{,}5\, \frac{y}{1+y^2}$.

From this equation, we find by calculation the encircled points of fig. 14. Our curve runs nearer to these points, when the interval between the isoclines is smaller.

c. Case of a crest.

In fig. 15 an example is given of the construction of the integral-curve of the equation

$$y\,(1-c)\,(a\,dy - dx) = \frac{k^2 - x^2}{2\,x^2}\,(y\,dx - x\,dy)$$

determined by $x = o$, $y = 0$, $\frac{dy}{dx} = 1$ by means of a family of isoclines

$$px = \frac{k^2 - mx^2}{k^2 - x^2}\,.$$

The data are the same as for fig. 14.

To find point $P\,(x_0 = 7)$ of isocline 0,45, the centre of projection C $(m = 5p - 1 = 1{,}25;\; p = 0{,}45)$ is joined to point 7 of the quadratic

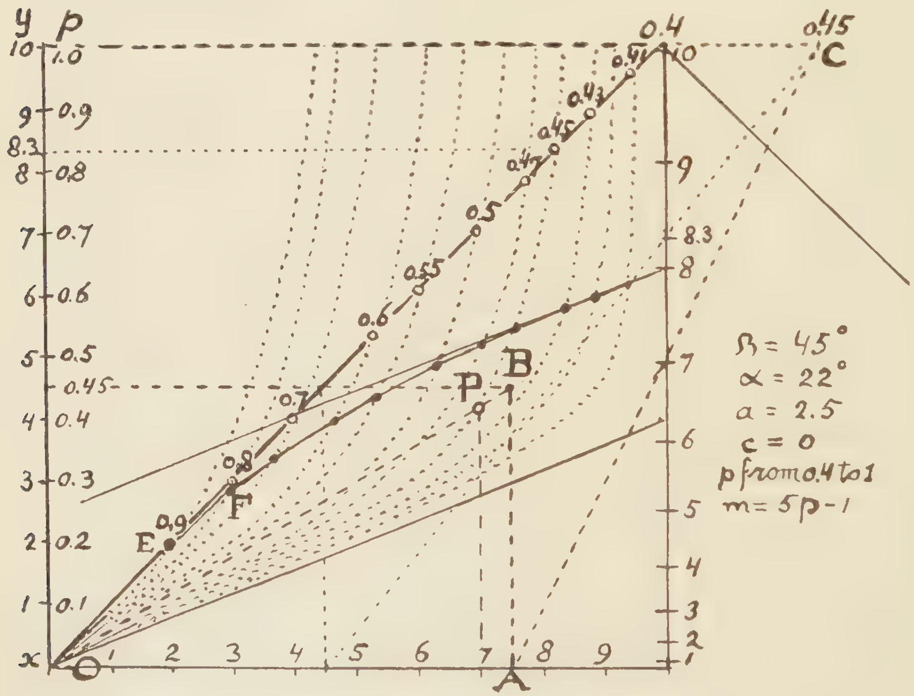

Fig. 15

scale. This line meets the S-scale in A. The perpendicular in A meets the line $p = 0{,}45$ parallel to OA in B. The line $x = x_0 = 7$ meets the line OB in the desired point P. Repeating this construction for several points of the quadratic scale, we get isocline 0,45. Repeating it again for centra on the line YC, we get a family of isoclines.

The integral-curve is constructed in the same way as sub a. The line-element in O with the direction $\frac{dy}{dx} = p = 1$ meets the first isocline 0,9 in E. The line-element in E with direction 0,9 meets isocline 0,8 in F etc. In the case of a crest, the rock-profile has a somewhat greater curvature, due to the vertical direction of the isoclines.

5. *Simplification of the construction.*

Although it is now possible to construct as many points of the isoclines as we desire, practical use requires a more rapid construction of the integral-curve and a conception of the ratio between the flat part and the curved part.

In the first place, we only want some two or three points of each isocline near the probable course of the integral-curve.

In the second place, we can sketch the general course of an isocline, if we bear in mind, that each curve has a tangent $y = px$ in the origin O and if we know some special points.

We have already remarked (sub 3), that the centre C must lie upon the line, joining O (numbered $\frac{h}{\sqrt{m}}$ in the case of a plateau and $\frac{k}{\sqrt{m}}$ in the case of a crest) to the similarly numbered point of the quadratic scale.

Reversely: joining O to the centra of all isoclines, the values $\frac{h}{\sqrt{m}}$ or $\frac{k}{\sqrt{m}}$ are read off upon the quadratic scale. But $\frac{h}{\sqrt{m}}$ is the ordinate of the point where isocline $px = y\,\frac{h^2-my^2}{h^2-y^2}$ ... (18) meets the Y-axis (putting $x = o$) and $\frac{k}{\sqrt{m}}$ is the abscissa of the point where the asymptote of $px = y\,\frac{k^2-mx^2}{k^2-x^2}$ (19) parallel to OY meets the X-axis. Thus, these values are immediately found.

In fig. 16 isocline 0,5 (plateau) will meet the Y-axis in A (7,5) and isocline 0,5 (crest) has an asymptote $x = 7{,}5$ (meets YH in D). The tangent in O is OG for both isoclines.

The meeting-points of the isoclines with the initial slope are also easily to be found in both cases. From the construction of point 6,8 of isocline 0,5 (fig. 16) it follows, that the perpendicular in E meets the line $p = 0{,}5$ parallel to OP just upon the initial slope OH. Consequently, the desired point B must also lie on the slope and therefore, it is the meeting point of isocline 0,5 with the slope. Reversely: to find the ordinates of the meeting points of all isoclines

with the initial slope, we must proceed as follows. Draw the lines 0,5; 0,6; 0,7 etc. parallel to OP till they meet the slope. From these points of intersection, let the perpendiculars fall upon OP and join the feet E etc. to the centra 0,5; 0,6; 0,7 etc. The numbers of the meeting-points of these

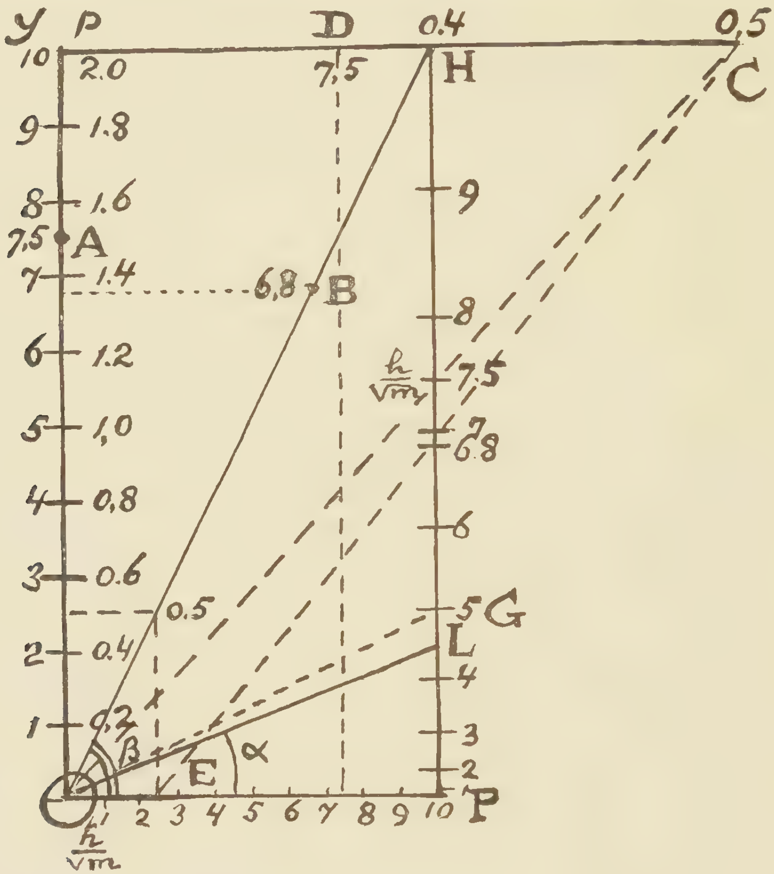

Fig. 16

lines with the quadratic scale PH will give the ordinates of the points of intersection of the isoclines with the initial slope. It must be noted, that, with the same data, we find the *same* meeting-points, whether the profile is that of a plateau or that of a crest. For, the equation of the slope being $y : h = x : k$, the equations

$$px = y\,\frac{h^2 - my^2}{h^2 - y^2} \quad \text{and} \quad px = y\,\frac{k^2 - mx^2}{k^2 - x^2}$$

will give the same values of x and y.

Examples:

The slope in fig. 14 has the same scale of intersection-points as the slope in fig. 15. In fig. 17a, O has been joined to the centra of the isoclines

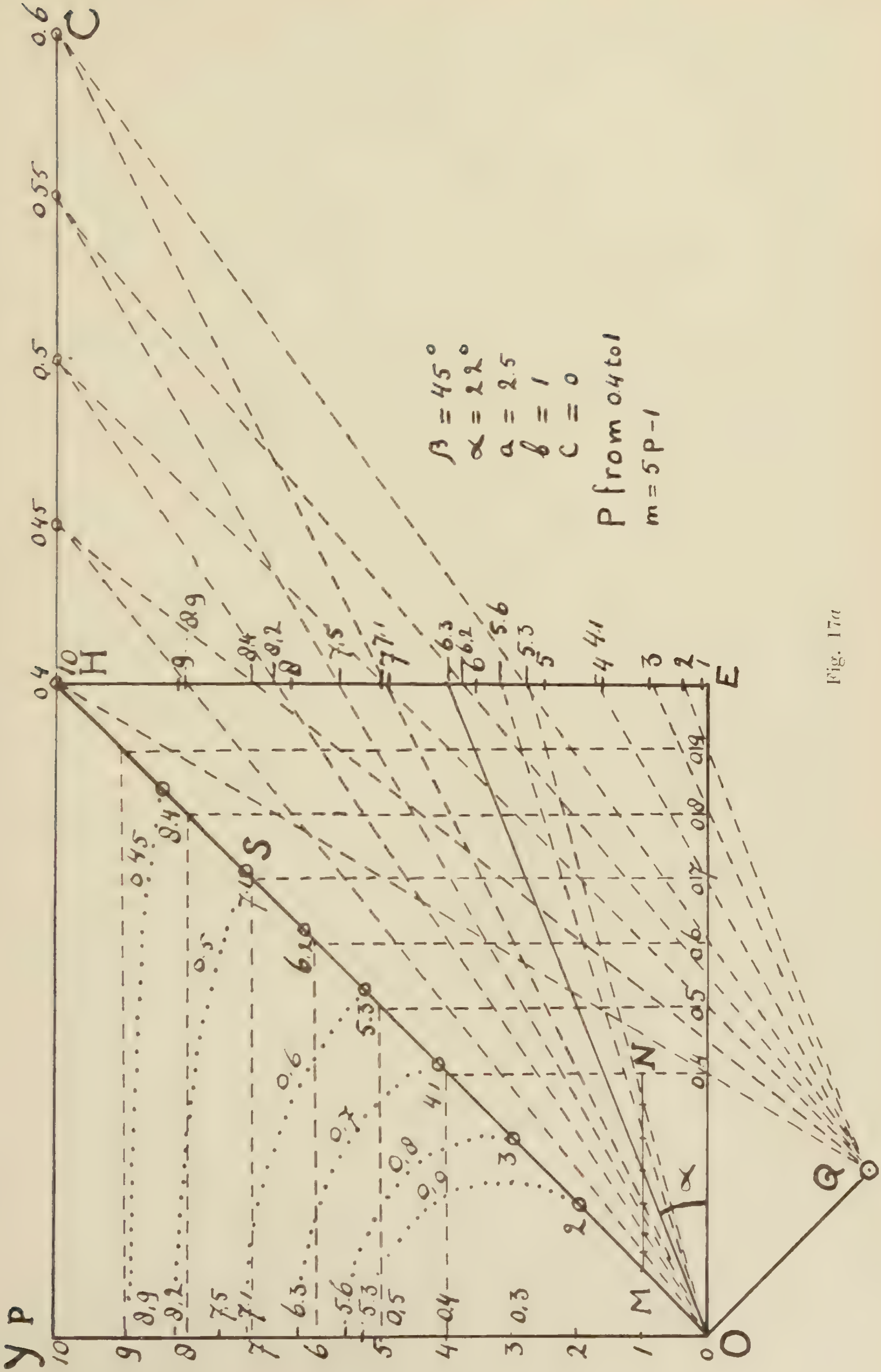

Fig. 17a

0,45; 0,5; 0,55; 0,6; 0,7; 0,8 and 0,9 (auxiliary-scale MN 1 : 10). The quadratic scale HE shows, that the isoclines will meet the Y-axis in points with ordinates 8,9; 8,2; 7,5; 7,1; 6,3; 5,6 and 5,3.

To find the points where the isoclines meet the slope, the feet of the perpendiculars should be joined to the corresponding centres. But these lines, joining up similarly numbered points of two parallel, regular scales, will pass through a fixed point Q. Making use of this point, the regular scale of the centra may be omitted and the figure for constructing the points of intersection on the slope remains limited between the parallels OY and EH. By joining Q to the points of division upon OE, we find the ordinates 8,4; 7,1 etc. of the meeting-points of the isoclines with the slope.

By making use of the new guise of our theory, we shall present the problem of the ratio between the flat part and the curved one of the rocky nucleus underneath the screes in the fourth part of our treatment.

TRANSGRESSIEVE VARIABILITEIT EN TRANSGRESSIE-SPLITSING. I

DOOR

G. P. FRETS

(Communicated by Prof. J. BOEKE at the meeting of June 24, 1950)

De naam transgressieve variabiliteit is door HUGO DE VRIES gegeven (gecit. JOHANNSEN, 1926, blz. 177). HUGO DE VRIES beschrijft ze uitvoerig in zijn "Die Mutationstheorie" (1902 en 1903) bij kleine soorten, rassen en bastaarden. Als voorbeeld noemt hij de grootte der bloembladeren van de verwante soorten Oenothera Lamarckiana en Oen. biennis. Als men over vele en velerlei bloemen van beide planten beschikt en de bloembladeren in rijen legt, blijkt, dat de kleinste bloembladeren van Oen. Lamarckiana kleiner zijn dan de grootste van Oen. biennis: er is transgressie (I, blz. 305).

NILLSON–EHLE stelde bij zijn "Kreuzungsuntersuchungen an Hafer und Weizen" (I, 1909; II, 1911) transgressiesplitsingen vast. Het zijn overschrijdingen in beide richtingen van de eigenschappen van de oudervormen (1909, p. 103). En als hij het resultaat van velerlei onderzoekingen samenvat, zegt hij, "dass bei Kreuzungen zweier Abstufungen quantitativer Merkmale neue erbliche (konstantbleibende) Abstufungen, seien es mehr oder weniger stark *transgressive*, seien es intermediäre, so gut wie regelmässig zustande kommen, was zu zeigen scheint, dass die vielen Abstufungen (auch hier) das Resultat verschiedener Kombination einer relativ geringen Anzahl von Einheiten ist". (Evenzo 1911, II, p. 5; II, p. 17 en 54.) Hij vindt de transgressieve variaties in F_2- en bewijst, door F_3- te kweken, dat ze erfelijk zijn. Transgressiesplitsingen zijn erfelijke transgressieve variaties.

Transgressiesplitsingen komen vooral voor bij kruisingen, waar de onderzochte eigenschappen bij de oudervormen niet veel verschillen. Bij de twee zuivere lijnen van de zaden van Phaseolus vulgaris van ons materiaal is dit, van de afmetingen, voor de lengte en de dikte niet het geval. We zullen dus transgressieve variabiliteit vooral bij de breedte mogen verwachten. Dit volgt uit de polymerietheorie.

In 1935 was het verschil van de gemiddelde lengten van het totale aantal gemeten en gewogen bonen van de I- en de II-lijn 4.1 mm, van de breedten 0.7, van de dikten — 1.2 mm en van de gemiddelde gewichten 10.1 cg. In 1936 waren deze cijfers 3.1, 0.4 en — 1.2 mm en 6 cg. Voor het lengteverschil van de bonen van de I- en de II-lijn hebben we een groter aantal genen aangenomen dan voor dat van de breedte. We geven nu een

uitbreiding aan de hypothese en laten de 6, resp. de 3 genenparen van het lengte- en van het breedte-verschil over 7, resp. 4 genen-paren verspreid zijn. In de splitsingsgeneraties ontstaan dan $L_1 \ldots L_7$ en $l_1 \ldots l_7$ als transgressieve variaties van de lengte en $B_1 \ldots B_4$ en $b_1 \ldots b_4$ als transgressieve variaties voor de breedte (blz. 15). Op een zelfde aantal bonen ontstaan aldus meer transgressieve variaties voor de breedte dan voor de lengte. JOHANNSEN (1926, p. 490) geeft een goed schema van deze verhoudingen.

Transgressiesplitsing, erfelijke transgressieve variatie, wil niet zeggen, dat de transgrederende eigenschap in geheel homozygote vorm uitgesplitst is. Door voortkweken en selectie kan de homozygotie in toenemende mate worden bereikt.

Nu volgt een overzicht van het voorkomen er van in de op elkaar volgende splitsingsgeneraties. We onderscheiden transgressieve plus- en minusvariaties en bespreken ze van de 3 afmetingen, het gewicht en de vorm (de indices) ieder afzonderlijk. We gaan daarbij uit van F_5-1936. In vroegere publicaties hebben we enkele malen op het voorkomen van transgressieve variabiliteit in ons materiaal gewezen. (Genetica 1950 en Proc. 1950). De gemiddelde en de distributiekromme van een eigenschap van de bonen ener bonenopbrengst, drukken goed het phaenotype er van uit. We maken er bij ons overzicht van gebruik.

a. Transgressieve plusvariaties van F_5-1936.

Uitgaande van bonenopbrengsten van de I- en de II-lijn van 1936 ter vergelijking, zijn in tab. 1 a—d van de bonenopbrengsten van F_5-1936 de grootste gemiddelde afmetingen, het grootste gemidd. gewicht en de grootste afmetingen en het grootste gewicht van individuele bonen opgenomen.

De *lengte*. De gemidd. lengte (tab. 1a) is het grootst van de bonenopbrengst van *pl. 450*. De uitgangsboon is van pl. 119, F_4-1935 en heeft een zeer grote lengte en breedte (cl 2a); deze boon vertoont geen transgr.-variabiliteit. De bonenopbrengst van pl. 450 heeft een grote gemidd. lengte en breedte en een groot gemidd. gewicht. De gemidd. dikte is niet zeer groot, zodat de indices zeer overeenkomen met die van bonenopbrengsten van de I-lijn.

De gemidd. lengte is groter dan de grootste gemidd. lengte van bonenopbrengsten van de I-lijn van 1936. Van de indiv. bonen van de bonenopbrengst komt de grootste lengte overeen met de grootste lengte van de bonen van de I-lijn van 1936. De distributiekromme in haar geheel ligt meer naar rechts dan distributiekrommen van bonenopbrengsten van de I-lijn van 1936: er is transgressieve variabiliteit (fig. 1a).

Pl. 463. De uitgangsboon is van pl. 121 en heeft geen zeer grote afmetingen en gewicht (cl 2b). De gemidd. lengte van de bonenopbrengst van pl. 463 is groter dan de grootste gemidd. lengte van bonenopbrengsten van de I-lijn van 1936. De grootste lengte van de indiv. bonen is kleiner

dan de grootste lengte van de bonen van de I-lijn van 1936. De distributiekromme ligt duidelijk iets meer naar rechts (fig. 1b).

Pl. 783. De uitgangsboon is van pl. 297 en heeft zeer grote afmetingen en gewicht (cl 1a). De gemidd. lengte van de bonenopbrengst van pl. 783 is groter dan de grootste gemidd. lengte van bonenopbrengsten van de I-lijn van 1936. Van de indiv. bonen is de grootste lengte, l = 18.1 (dan = 17.9) mm (fig. 1c). Van de bonenopbrengst (pl. 81) met de grootste gemidd. lengte van de I-lijn van 1936 is de grootste lengte der indiv. bonen, l = 17.0 mm (fig. 1c). Een bonenopbrengst (pl. 48) met wat kleiner gemidd. lengte (l_m = 15.1 mm) bevat de boon met de grootste lengte, l = 18.1 (dan = 16.5) mm van de bonen van de I-lijn van 1936 (fig. 1g). Als we de kromme van de lengte van de bonenopbrengst van pl. 783 vergelijken met de krommen van de lengte van de pl. 81 en 48 van de I-lijn van 1936, blijkt de transgressieve variabiliteit: ze ligt meer naar rechts, d.i. meer in het gebied van de grote afmetingen (fig. 1c en fig. 1f en 1g).

Pl. 674. De uitg. boon behoort tot cl 2b, form. L_1 l_2 B th en heeft geen zeer grote afmetingen en gewicht. De formule der gemiddelden van pl. 674 is L_1 L_2 B th, cl 2a. De lengte vertoont transgr. variabiliteit (fig. 1d).

Pl. 666. De uitg. boon is van pl. 206, F_4-1935, cl 2a, form. L_1 L_2 B th. De form. van de gemiddelden van pl. 666 is L_2 l_2 B th, cl 2b. De gemidd. lengte is even groot als de grootste gemidd. lengte van bonenopbrengsten van de I-lijn van 1936. De grootste lengte van de indiv. bonen is iets kleiner dan die van bonen van de I-lijn van 1936 (fig. 1e).

Meer bonenopbrengsten met een gemidd. lengte groter dan of even groot als de gemidd. lengte van bonenopbrengsten van de I-lijn van 1936 bevat F_5-1936 niet.

De *breedte* (tab. 1b). *Pl. 783* (z. boven). De gemidd. breedte van de bonenopbrengst van pl. 783 is b_m = 10.0 mm. De grootste gemidd. breedte van bonenopbrengsten van de I-lijn is van pl. 31 (n = 16) met b_m = 9.7 mm. Pl. 31 heeft slechts een kleine bonenopbrengst. Dan volgt pl. 81 met b_m = 9.45 mm. De grootste breedte van de indiv. bonen van pl. 783 is b = 10.9 mm. De grootste breedte van de indiv. bonen van het totale bonenaantal van de I-lijn van 1936 is b = 10.5 mm (van pl. 81). De bonenopbrengst van pl. 783 toont transgr. variabiliteit: ook bij bezichtiging van de krommen (fig. 2a, fig. 2e en 2f).

Pl. 450 (blz. 1086). De gemidd. breedte is groter dan de grootste gemidd. breedte van bonenopbrengsten van de I-lijn van 1936. De grootste breedte van de indiv. bonen is groter dan van indiv. bonen van de I-lijn van 1936 (fig. 2b): er is dus transgr. variabiliteit.

Pl. 463 (blz. 1086). De gemiddelde breedte is groter dan de grootste gemidd. breedte van bonenopbrengsten van de I-lijn van 1936. De grootste breedte van de individuele bonen is groter dan de grootste breedte van de bonen van de I-lijn van 1936 (tab. 1b). Er is transgr. variabiliteit van de breedte.

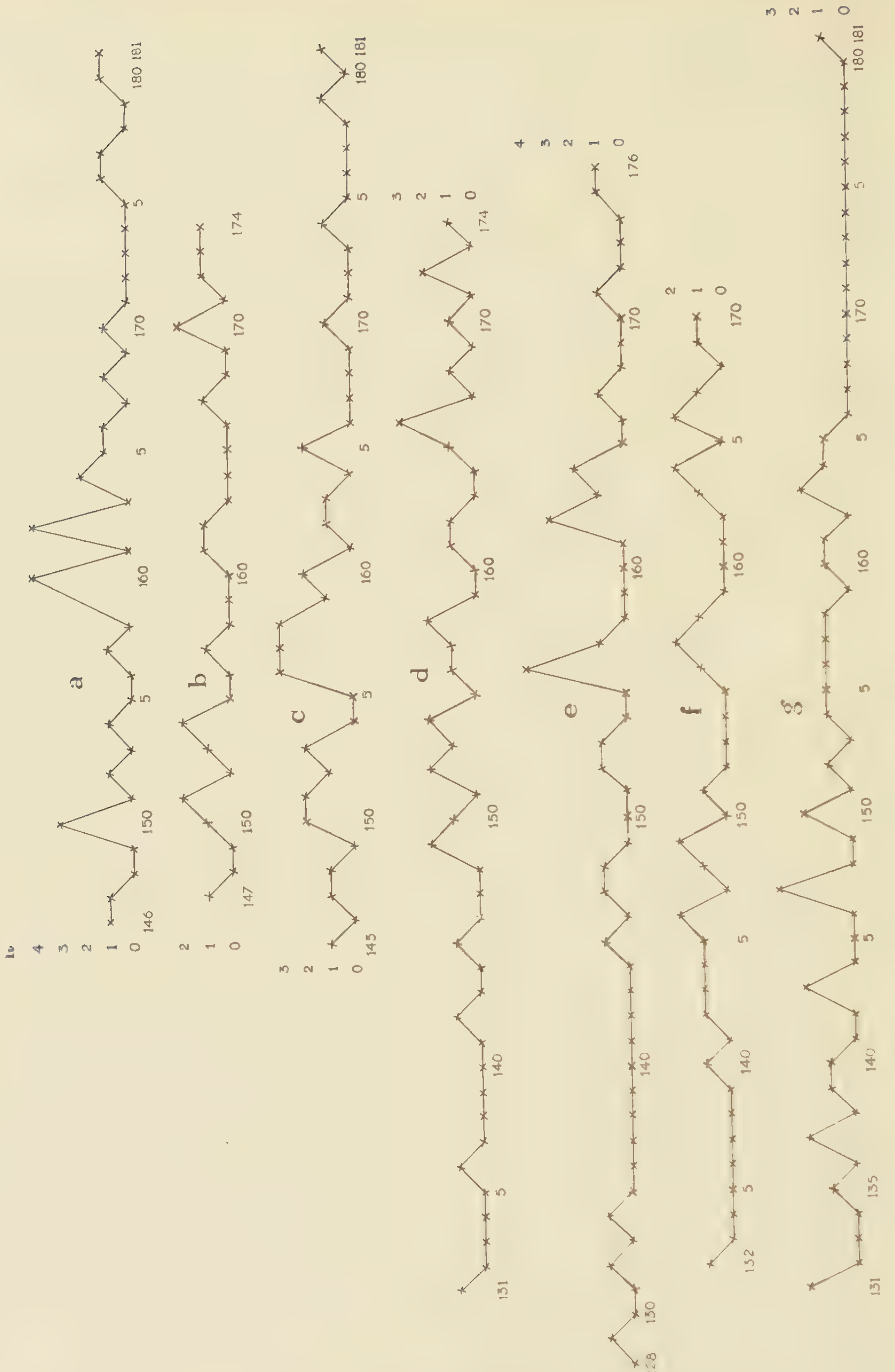

Fig. 1. Plus variations. Fig. 1a—e, F_5-1936. Frequency curves of the length. Plus variations. Fig. 1a, Frequency curve of the length of pl. 450, n = 28, l_m = 16.2 mm. Fig. 1b, Id. of pl. 463, n = 16, l_m = 16.0 mm. Fig. 1c, Id. of pl. 783, n = 30, l_m = 15.8 mm. Fig. 1d, Id. of pl. 674, n = 27, l_m = 15.7 mm. Fig. 1e, Id. of pl. 666, n = 27, l_m = 15.5 mm. The length of the smallest bean is l = 11.4 mm.

Fig. 1f—g, I-line 1936. Frequency curves of the length of comparison beanyields. Fig. 1f, Frequency curve of the length of pl. 81, n = 24, l_m = 15.5 mm. Fig. 1g, Id. of pl. 48, n = 26, l_m = 15.1 mm.

Pl. 457. De gemidd. breedte is zo groot als de grootste gemidd. breedte van bonenopbrengsten van de I-lijn van 1936 (pl. 31 met een slechts kleine bonenopbrengst van de I-lijn van 1936; tab. 1b. fig. 2e en f). De grootste breedte van de individuele bonen is zo groot als de grootste breedte (op één na) van de indiv. bonen van de I-lijn. Er is bij pl. 457 waarschijnlijk transgr. variabiliteit van de breedte.

Pl. 784 (vgl. ook pl. 783 blz. 1087). De gemidd. breedte vertoont transgr. variabiliteit (fig. 2c).

Pl. 451. De grootste gemidd. breedte is groter dan bij bonenopbrengsten van de I-lijn voorkomt. Ook van de indiv. bonen is de grootste breedte groter dan bij de I-lijn.

Pl. 785 (vgl. ook pl. 784 en 783, blz. 1089 en 1087). De gemidd. breedte vertoont transgr. variabiliteit (fig. 2d). *Pl. 776* (zie vorige). Er is waarschijnlijk enige transgressieve variabiliteit. Ook van de pl. *441*, *446*, *719*, *674*, *475* en *568* is de gem. breedte, $b_m = 9.6$ mm en van de indiv. bonen is de breedte van de grootste boon groter dan bij de I-lijn, $b = 10.8$—10.5; van pl. 441, $b = 10.1$ mm. We hebben hier met onzekere gevallen van transgr. variabiliteit te doen.

Nu volgen in ons materiaal 4 bonenopbrengsten van de pl. *447*, *556*, *489* en *546* met de gemidd. breedte, $b_m = 9.5$ mm.

De *dikte* (tab. 1c). *Pl. 783* (blz. 1087). De gemidd. dikte van de bonenopbrengst is $th_m = 7.3$ mm. De grootste gemidd. dikte van bonenopbrengsten van de II-lijn van 1936 is $th_m = 7.4$ mm (pl. 100) en 7.3 mm (pl. 113 en 95). De grootste dikte van de individuele bonen van pl. 783 is $th = 8.4$ mm. De boon met de grootste dikte van de II-lijn is van pl. 105, $th = 8.3$ mm. Van pl. 95 is van de boon met de grootste dikte, $th = 8.1$ mm. De gemidd. dikte is dus iets kleiner dan de grootste gemidd. dikte van de bonenopbrengsten van de II-lijn van 1936: de dikte der indiv. bonen toont enige (onzekere) transgressieve variabiliteit. De distributiekromme ligt iets naar rechts (fig. 3a).

Pl. 526. De gemidd. dikte is kleiner dan de grootste gemidd. dikte van bonenopbrengsten van de II-lijn van 1936. De grootste indiv. dikte is zo groot als de grootste indiv. dikte van bonen van de II-lijn van 1936 (dan volgt $th = 8.1$ mm).

Van de *pl. 485*, *463*, *568*, *1087* en *578* is de gemidd. dikte ook $th_m = 7.0$ mm. Dan volgt *pl. 487* met $th_m = 6.9$ mm. De grootste dikte der indiv. bonen is $th = 7.9$, 7.9, 7.5, 7.3 en 7.3 mm.

Er zijn dus geen zekere gevallen met transgr. variabiliteit van de dikte in het materiaal.

Het *gewicht* (tab. 1d). Het gemidd. gewicht van de bonenopbrengst van pl. *783* (blz. 1087) is 83 cg. d.i. veel groter dan het grootste gemidd. gewicht ($w_m = 65$ cg) van bonenopbrengsten van de I-lijn van 1936 en dan het grootste gemidd. gewicht ($w_m = 55$ cg) van bonenopbrengsten van de II-lijn van 1936. Er is hier een zo grote transgressie van het gemidd. gewicht, omdat de formule van de gemidd. afmetingen van pl. 783,

L_1 L_2 B Th is, d.w.z. de beide grote afmetingen, lengte en breedte van de bonen van de I-lijn zijn er verenigd met de grote dikte van de II-lijn (fig. 4a).

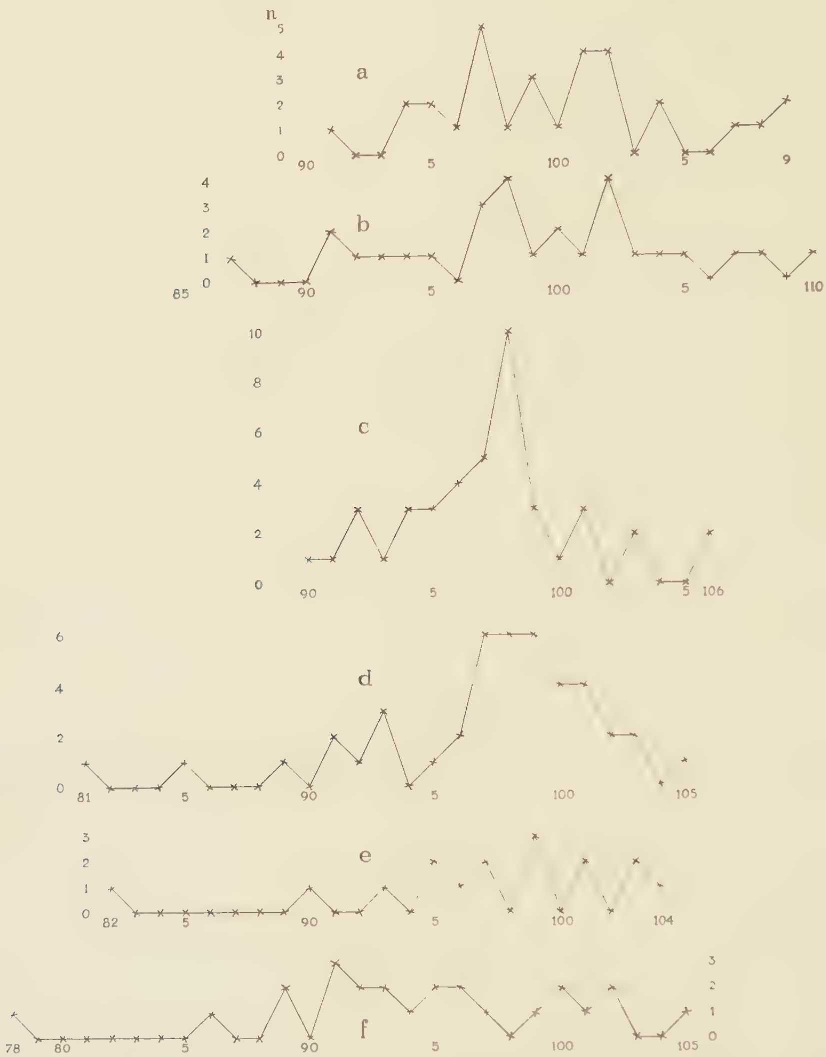

Fig. 2. Fig. a—d, F_5-1936. Frequency curves of the breadth. Plus variations. Fig. 2a, Frequency curve of the breadth of pl. 783, n = 30, b_m = 10.0 mm. Fig. 2b, Id. of pl. 450, n = 28, b_m = 9.9 mm. Fig. 2c, id. of pl. 784, n = 42, b_m = 9.7 mm. Fig. 2d, id. of pl. 785, n = 35, b_m = 9.7 mm. Fig. 2, e—f. Frequency curves of the breadth of comparison beanyields of the I-line of 1936. Fig. 2e, pl. 31, n = 16, b_m = 9.7 mm. Fig. 2f, pl. 81, n = 24, b_m = 9.45 mm.

Pl. 463 en *pl. 450* tonen zowel voor het gemidd. gewicht als voor het gewicht van individuele bonen duidelijk transgr. variabiliteit (tab. 1d en fig. 4b).

Pl. 784 vertoont transgr. variabiliteit (fig. 4c). Niet pl. 785. Ook de

pl. *674*, *556*, *568* en *719* vertonen transgr. variabiliteit van het gewicht.

We zien (tab. 1, a—d), dat van de afmetingen de breedte de meeste gevallen van transgressieve variabiliteit vertoont. De transgressies van het gewicht zijn het grootst. Van pl. 783 vertonen alle drie gemidd. afme-

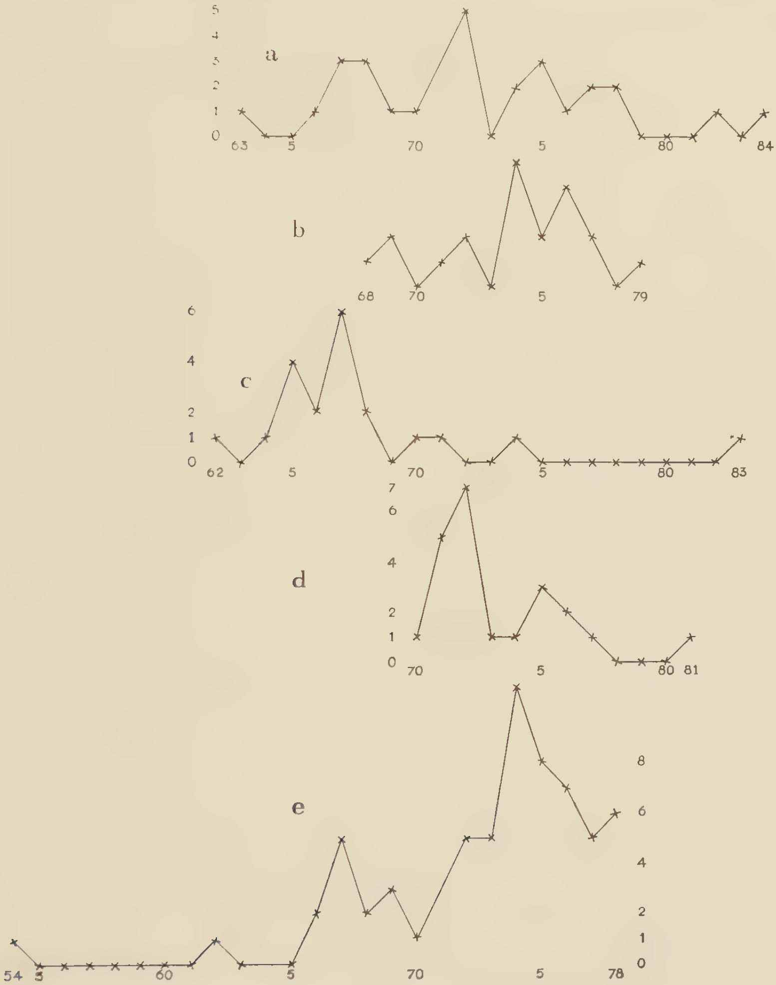

Fig. 3. Fig. 3a, F_5-1936. Plus variation. Frequency curve of the thickness of pl. 783, $n = 30$, $th_m = 7.3$ mm.

Fig. 3, b—e. II-line 1936. Frequency curves of the thickness of comparison beanyields. Fig. 3b, pl. 100, $n = 20$, $th_m = 7.4$ mm. Fig. 3c, pl. 105, $n = 20$, $th_m = 6.8$ mm. Fig. 3d, pl. 95, $n = 22$, $th_m = 7.3$ mm. Fig. 3e, pl. 117, $n = 65$, $th_m = 7.3$ mm.

tingen en het gemidd. gewicht transgr. variabiliteit. Evenzo van pl. 463 en 450, doch met uitzondering van de dikte. Van de pl. 674 en 719 tonen, behalve de lengte, ook het gewicht transgr. variabiliteit.

De *vorm* van de bonen wordt uitgedrukt door de indices. Ook van deze gingen we de transgressieve variabiliteit na.

a. De hoogste plusvariaties van de indices (tab. 2a—c).

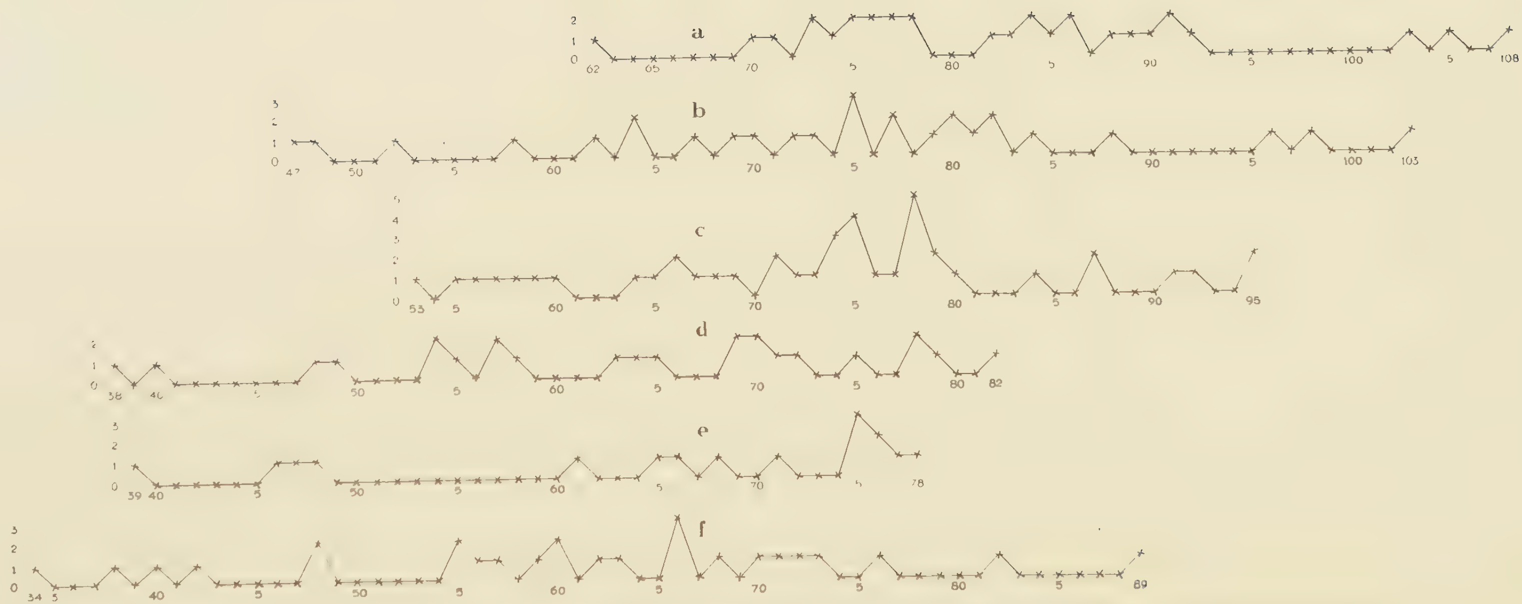

Fig. 4, a—c. F_5-1936. Frequency curves of the weight. Plus variations. Fig. 4a, pl. 783, n = 30, $W_m = 83$ cg. Fig. 4b, pl. 450, n = 28, $W_m = 74$ cg. Fig. 4c, pl. 784, n = 42, $W_m = 73$ cg.

Fig. 4, d—f. I-line 1936. Frequency curves of comparison beanyields. Fig. 4d, pl. 81, n = 24, $W_m = 63$ cg. Fig. 4e, pl. 31, n = 16, $W_m = 65$ cg. Fig. 4f, pl. 48, n = 26, $W_m = 61$ cg.

De *LB-index* (tab. 2a). De hoogste gemidd. LB-index is lager dan de hoogste gemidd. LB-index van bonenopbrengsten van de II-lijn van 1936. De bonenopbrengsten van F_5-1936 met de hoogste gemidd. LB-indices hebben steeds een kleine gemidd. lengte, geen grote gemidd. breedte. De formule van de gemiddelde afmetingen is enkele malen die van cl 5, 6 en 7, in de grote meerderheid der gevallen die van cl 8b en 8c. Er is geen geval met de form. lb th, I, cl 8a, d.i. de formule met kleine afmetingen en indices als van de I-lijn. Van de bonenopbrengsten met de formule van cl 5, 6 en 7 der gemiddelden is de grootste gemidd. breedte van pl. 1087 met $b_m = 9.0$ mm.

De *L Th-index* (tab. 2b). De hoogste gemidd. L Th-index is lager dan de hoogste gemidd. L Th-index van bonenopbrengsten van de II-lijn van 1936. Ook de hoogste L Th-index van indiv. bonen, is lager dan de hoogste L Th-index van bonen van de II-lijn van 1936.

De *B Th-index* (tab. 2c). De hoogste gemidd. B Th-index is van *pl. 492* en is hoger dan de hoogste gemidd. B Th-index van bonenopbrengsten van de II-lijn van 1936. Ook de hoogste indiv. B Th-index is hoog. Er is hier transgressieve variabiliteit (fig. 5).

De bonenopbrengst van pl. 492 met de form. lb Th cl 7 van de uitgangsboon en de form. lb th II, cl 8b van de gemiddelden der bonenopbrengst en de bonenopbrengst van pl. 497 met de form. l BTh, cl 5 van de uitg. boon en de form. lb Th, cl 7 van de gemiddelden van de bonenopbrengst hebben hoogste gemidd. LB-, LTh- en BTh-indices (de gem. LB-indices zijn resp. 71 en 72) en ook hoogste indiv. indices.

De transgressieve variabiliteit van de B Th-index van pl. 492 volgt uit de vrij grote gemidd. dikte en de zeer kleine gemiddelde breedte.

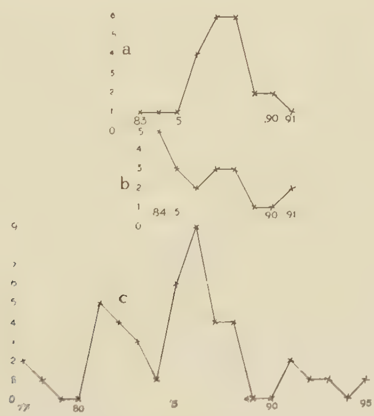

Fig. 5a. F_5-1936. Plus variation. Frequency of the B Th-index. Pl. 492, $n = 24$, $B\ Th_m = 87$.

Fig. 5b—c. II-line 1936. Frequency curves of comparison beanyields. Fig. 5b, pl. 100, $n = 20$, $B\ Th_m = 86$. Fig. 5c, pl. 101, $n = 46$, $B\ Th_m = 85$.

b. Transgressieve minusvariaties. F_5-1936 (tab. 3, a—d).

De lengte. Van geen der bonenopbrengsten is de gemidd. lengte kleiner

dan de kleinste gemidd. lengte van de bonenopbrengsten van de II-lijn van 1936 (fig. 6). De kleinste gemidd. lengte is van pl. 492 met

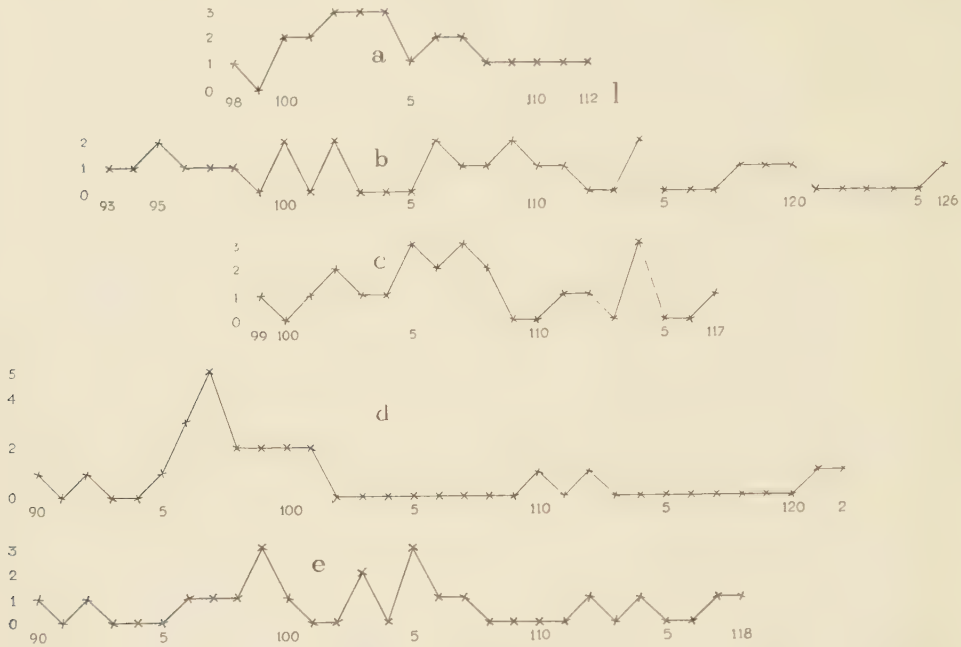

Fig. 6, a—c, F_5-1936. Minus variations. Frequency curves of the length. a, pl. 492, n = 24, l_m = 10.5 mm. b, pl. 656, n = 25, l_m = 10.6 mm. c, pl. 637, n = 22, l_m = 10.7
Fig. 6, d—e. II-line 1936. Frequency curves of comparison beanyields. d, pl. 97, n = 23, l_m = 10.1 mm. e, pl. 119, n = 20, l_m = 10.3 mm.

l_m = 10.5 mm, de kleinste gemidd. lengte van bonenopbrengsten van de II-lijn van 1936 is van pl. 97 met l_m = 10.1 mm (van de I-lijn van pl. 78 met l_m = 13 mm).

De *breedte*. In enige gevallen is de gemidd. breedte kleiner dan de kleinste gemidd. breedte van bonenopbrengsten van de II-lijn van 1936 (tab. 3b en fig. 7).

Van *pl. 664* met een zeer kleine bonenopbrengst (n = 9) is b_m = 7.0 mm; de kleinste gem. breedte van bonenopbrengsten van de II-lijn is van pl. 97 met b_m = 7.5 mm. *Pl. 788.* Van de bonenopbrengst van 40 bonen is de gemidd. breedte duidelijk kleiner dan de kleinste gemidd. breedte van de II-lijn van 1936 (fig. 7a). Pl. 1083 heeft slechts een zeer kleine bonenopbrengst (n = 5). Van *pl. 492* met n = 24, is de gemidd. breedte kleiner dan van de kleinste gemidd. breedte van bonenopbrengsten van de II-lijn. Er volgen nu de *pl. 647*, *565* en *650*, waarvan de gemidd. breedte even groot is als de kleinste gemidd. breedte van bonenopbrengsten van de II-lijn van 1936. Er zijn in het materiaal van F_5-1936, 15 bonenopbrengsten, waarvan de gemidd. breedte kleiner dan 7.8 mm is; bij de bonenopbrengsten van de II-lijn zijn het er twee.

De dikte. Er zijn 2 bonenopbrengsten, waarvan de gemidd. dikte

kleiner is dan de gemidd. dikte van bonenopbrengsten van de II-lijn van 1936. *Pl. 759*. Van deze bonenopbrengst zijn er vele gerimpeld en bruin. Het is waarschijnlijk, dat de transgressieve minusvariabiliteit hier de uiting is van modifikabiliteit. Dit geldt ook voor de zeer kleine bonenopbrengst (n = 9) van *pl. 664*; ze is "slecht". De uitgangsboon is van pl. 205; ze heeft kleine afmetingen. Er zijn nog 3 bonen van pl. 205 uitgezaaid, die eveneens klein zijn. De gemiddelden van hun bonenopbreng-

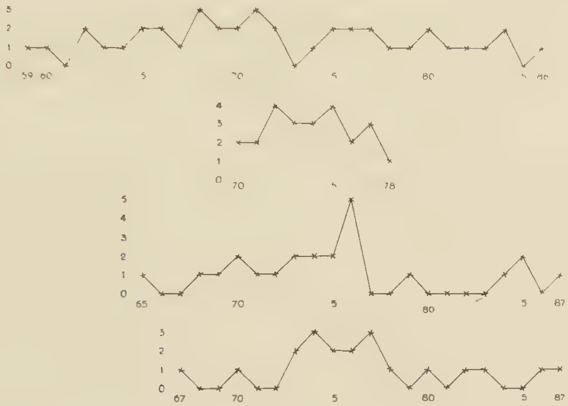

Fig. 7, a—b. F_5-1936. Frequency curves of the breadth. Minus variations. a, pl. 788, n = 40, b_m = 7.2 mm. b, pl. 492, n = 24, b_m = 7.4 mm.
Fig. 7, c—d. II-line 1936. Frequency curves of comparison beanyields. c, pl. 97, n = 23, b_m = 7.5 mm. d, pl. 119, n = 20, b_m = 7.7 mm.

sten de pl. 662, 663 en 665 zijn ook klein. De kleine gemidd. dikte van pl. 664 is dus waarschijnlijk bepaald door erfelijkheid en door het milieu. *Pl. 1066*. De gemidd. dikte is kleiner dan op één na de kleinste gemidd. dikte van de II-lijn van 1936. Van de uitgangsboon van pl. 174 voor pl. 1066 zijn de afmetingen middelmatig groot (tab. 3c, cl 1b). De bonenopbrengst van pl. 174 bestaat uit bonen met een kleine dikte en een klein aantal met een niet zeer grote dikte. De kleine gemidd. dikte van pl. 1066 is een erfelijke variatie aan de grens van transgressieve minusvariabiliteit (fig. 8a).

Het gewicht (tab. 3d). Van de *pl. 664, 759* en *788* is het gemidd. gewicht kleiner dan het kleinste gemidd. gewicht van bonenopbrengsten van de II-lijn van 1936. Van 11 bonenopbrengsten van F_5-1936 is het gemidd. gewicht kleiner dan 37 cg; bij de II-lijn van 1936 zijn dit er slechts 2. Van 20 bonenopbrengsten van F_5-1936 is het gemidd. gewicht kleiner dan 40 cg; van de bonenopbrengsten van de II-lijn is van 3 bonenopbrengsten het gemidd. gewicht kleiner dan 40 cg.

De vorm. De indices.

Er is, in enkele gevallen, een geringe transgressieve variabiliteit van de gemidd. *LB*- en *L Th-indices* en een iets grotere voor de *B Th-index* (tab. 4). De kleine bonenopbrengst (n = 18) van pl. 759 bevat meerdere gerimpelde platte bonen.

We hebben dus gevonden, dat van pl. 664 de breedte, de dikte en het gewicht transgressieve minusvariabiliteit vertonen (blz. 1094 en 1095). De gemidd. lengte is klein, $l_m = 11.5$ mm. Van pl. 788 vertoont, behalve de

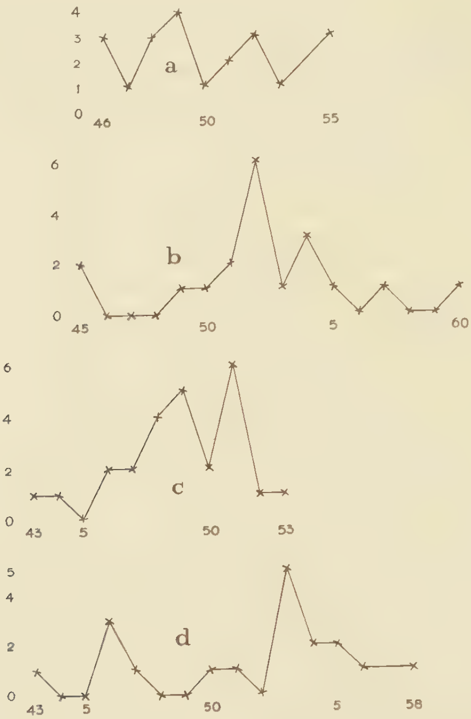

Fig. 8, a—b, F_5-1936. Minus variations. Frequency curves of the thickness. a, pl. 1066, n = 23, $th_m = 5.0$ mm. b, pl. 673, n = 19, $th_m = 5.2$ mm.
Fig. 8, c—d. I-line 1936. Frequency curves of comparison beanyields. c, pl. 78, n = 25, $th_m = 4.9$ mm. d, pl. 53, n = 19, $th_m = 5.2$ mm.

breedte, ook het gewicht transgressieve minusvariabiliteit. De gemidd. lengte is klein, $l_m = 11.2$ mm; de gemidd. dikte is nog al groot, $th_m = 6.5$ mm. De uitgangsboon voor pl. 788 is van pl. 315; de bonenopbrengst van pl. 315 heeft kleine gemiddelden. De transgr. variabiliteit van het gewicht van pl. 788 is terug te voeren op de zeer kleine gemidd. breedte en de kleine gemidd. lengte. Van pl. 492 (blz. 1094) vertoont van de afmetingen alleen de breedte transgr. minusvariabiliteit. De gemidd. lengte (tab. 3a) en het gemidd. gewicht zijn zeer klein. De gemidd. dikte is vrij groot, $th_m = 6.4$ mm. De gemidd. indices zijn hoog: de L Th- en de B Th-indices tonen transgr. minusvariabiliteit (blz. 1095 en tab. 4). De gemidd. LB-index is hoog, $LB_m = 71$. De formule van de gemiddelden is lbth II, cl 8b. De uitgangsboon voor pl. 492 is van pl. 127 met de form lb Th, cl 7. Ook de formule van de gemiddelden van de bonenopbrengst van pl. 127 is lb Th, cl 7. Pl. 492 is één van de beste voorbeelden van bonenopbrengsten als van de II-lijn in het materiaal van F_5-1936 (vgl. 1950). Van pl. 759 vertonen, behalve de gemiddelde dikte, ook het gemidd. gewicht en de gemidd.

L Th- en B Th-indices transgressieve minus-variabiliteit. Pl. 759 behoort, ook door zijn afkomst, tot het gebied van de I-lijn.

TABLE 1. The beanyields of F_5-1936 with the greatest mean dimensions and mean weights and the greatest dimensions and weights of individual beans and some comparison beanyields of the I- and II-line of 1936

Pl [1]	cl [2]	pl	n	m	gr.v.	cl [3]	pl	n	m	gr.v.
a. The length. F_5-1936							The I-line of 1936			
119	2a	450	28	16.2	18.1 [4]	1a	81	24	15.5	17.0
121	2b	463	16	16.0	17.4	1a	93	12	15.2	18.0 [6]
297	1a	783	30	15.8	18.1 [5]	1a	31	16	15.2	16.9
211	2b	674	27	15.7	17.4	2a	36	22	14.3	17.5
206	2a	666	24	15.5	17.6	2a	48	26	15.1	18.1 [7]
225	2a	719	23	15.4	17.6	2a	89	24	14.9	17.4
b. The breadth. F_5-1936							The I-line of 1936			
297	1a	783	30	10.0	10.9	1a	31	16	9.7	10.4
119	2a	450	28	9.9	11.0	2a	81	24	9.45	10.5
121	2b	463	16	10.0	10.7	2b	89	24	9.3	9.9
121	1a	457	18	9.7	10.4	1a	92	26	9.2	10.0
300	1a	784	42	9.7	10.6	1a	25	28	9.2	9.9
119	1a	451	24	9.7	11.0 [8]	1a	93	12	9.1	10.4
300	1a	785	43	9.7	10.5	1a	87	25	9.0	10.3
295	1a	776	40	9.6	10.7 [9]	1a	86	30	8.8	10.4 [10]
c. The thickness F_5-1936							The II-line of 1936			
297	1a	783	30	73	84	1a	100	20	74	79
143	2b	526	26	70	82	2b	105	20	68	83 [11]
d. The weight. F_5-1936							The I-line of 1936			
297	1a	783	30	83	108	1a	31	16	65	78
121	2b	463	16	80	97	2b	92	26	64	72
119	2a	450	28	74	103	2a	81	24	63	82 [12]
300	1a	784	42	73	95	1a	48	26	61	89 [13]
211	2b	674	27	73	92	2b	89	24	61	79
160	1b	556	23	71	86	1b				
250	3	568	23	71	95	3				
225	2a	719	23	68	84	2a				

[1]) Plant of F_4-1935, of which the initial bean for F_5-1936 was taken. [2]) The class of the initial bean. [3]) The class of the beanyield. [4]) then follows 18.1. [5]) then 17.9. [6]) then 16.7. [7]) then 16.5 mm. [8]) then follows 10.5 mm. [9]) then 10.6. [10]) then 9.9 mm. [11]) then follows 7.4 mm. [12]) then follows 7.6. [13]) then 82 cg.

TABLE 2. The beanyields of F_5-1936 with the highest mean indices and the highest indices of individual beans and some comparison beanyields of the I- and the II-line of 1936

l_m	b_m	th_m	pl	n	m	gr.v.	cl_i [1]	cl_m [2]	l_m	pl	n	m	gr.v.
a. The L B-index. F_5-1936									II-line 1936				
10.6	7.8		656	25	74	79	5	8c	11.1	98	21	76	80
12.6	9.0		1087	34	71	77	5	5	11.2	101	46	76	86 [4]
b. The L Th-index. F_5-1936									II-line 1936				
10.5		6.4	492	24	62	66	7	8b	11.1	98	21	65	70
11.6		6.8	497	24	59	67	5	7	9.9	100	20	64	69
c. The B Th-index. F_5-1936									II-line 1936				
	7.4	6.4	492	25	87	91	7	8b	9.9	100	20	86	91
	7.5	6.3	647	17	84	93 [3]	5	8b	11.1	98	21	85	92
	8.4	6.8	497	24	81	86	5	7	11.2	101	46	85	95 [5]
									11.8	95	22	83	95

[1]) class of initial been. [2]) cl. of mean dimensions of beanyield. [3]) then follows 88. [4]) then follows 83. [5]) then 93.

TABLE 3. The beanyields of F_5-1936 with the smallest mean dimensions etc.

Pl	cl	pl	n	m	sm.v.	cl	pl	n	m	sm.v.
a. The length. F_5-1936							The II-line of 1936			
127	7	492	24	105	98	8b	97	23	101	90
201	5	656	25	106	93	8c	119	20	103	92
b. The breadth. F_5-1936							The II-line of 1936			
205	6	664	9	70	64	8a	97	23	75	65
315	2b	788	40	72	59	8b	119	20	77	67
174	4	1083	5	73	64	8c				
127	7	492	24	74	70	8b				
c. The thickness. F_5-1936							The I-line of 1936			
282	2a	759	18	47	40	4	78	25	49	43
205	6	664	9	47	39	8a	33	11	51	48
174	1b	1066	23	50	46	8a	53	19	52	43
d. The weight. F_5-1936							II-line of 1936			
205	6	664	9	28	20	8a	8a	97	23	33
282	2a	759	18	31	18 [1])	4	4	119	20	33
315	2b	788	40	32	13 [2])	8b	8b	102	20	37
168	7	565	28	33		8c				

Dimensions in 0.1 mm. [1]) then 15. [2]) then 21 cg.

TABLE 4. The beanyields of F_5-1936 with the lowest mean indices etc.

l_m	b_m	th_m	pl	n	m	sm.v.	cl_i	cl_m	l_m	pl	n	m	sm.v.
		a. The L B-index. F_5-1936							I-line 1936				
137	80		729	25	58	52	4	4	142	26	5	59	58
141	83		559	25	59	55	2a	4	145	70	25	59	54
		b. The L Th-index. F_5-1936							I-line 1936				
131		47	759	18	36	30	2a	4	133	78	25	37	34
128		50	1066	23	40	36	1b	8a	145	86	30	38	27
		c. The B Th-index. F_5-1936							I-line 1936				
	78	47	759	18	59	49	2a	4	141	41	25	64	59
	89	56	675	30	63	53	2a	2b	155	81	24	64	57
	79	50	1066	23	64	55	1b	8a	152	31	16	65	57

Dimensions in 0.1 mm.

MATHEMATICS

SOME ANALOGUES OF THE SHEFFER STROKE FUNCTION IN n-VALUED LOGIC

BY

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(Communicated by Prof. A. HEYTING at the meeting of June 24, 1950)

The interpretation of the ordinary (two-valued) propositional logic in terms of a truth-table system with values "true" and "false" or, more abstractly, the numbers "1" and "2" has become customary. This has been useful in giving an algorithm for the concept "analytic", in giving an adequacy criterion for the definability of one function by another and it has made possible the proof of adequacy of a list of primitive terms for the definition of all truth-functions (functional completeness). If we generalize the concept of truth-function so as to allow for systems of functions of 3, 4, etc. values (preserving the "extensionality" requirement on functions examined) we obtain systems of functions of more than 2 values analogous to the truth-table interpretation of the usual propositional calculus. The problem of functional completeness (the term is due to TURQUETTE) arises in each of the resulting systems. Strictly speaking, this problem is not closely connected with problems of deducibility but is rather a combinatorial question. It will be the purpose of this paper to examine the problem of functional completeness of functions in n-valued logic where by n-valued logic we mean that system of functions such that each function of the system determines, by substitution of an arbitrary numeral a for the symbol n in the definition of the n-valued function, a function in the truth table system of a values. A function is functionally complete in n-valued logic if for any natural number a, the substitution of a for n in the definition of the function will yield a functionally complete function for the system of truth tables (of the type described above) with a values.

In the two-valued propositional logic, the classical work was done with the use of the operations $\overline{}$, &, V, and $\rightarrow$ of which it was early discovered that $\overline{}$ and any of the other three will suffice to express anything desired. Formal proof of this has been provided by POST [1]). SHEFFER showed that the stroke function can define the above mentioned

[1]) EMIL L. POST, "Introduction to a General Theory of Elementary Propositions", *Am. Jl. of Math.* 43, 167–168 (1921).

functions and hence is functionally complete [2]). In an extension to the n-valued case POST has shown that two functions, $s(p)$, an analogue of negation, and pVq, an analogue of disjunction, will suffice [3]). WEBB has proved the existence of a single function which will suffice [4]). The first part of the proof of theorem 1 is equivalent to POST's proof although it differs in details as to method. In the following, we will use the expression "n-valued SHEFFER function" to mean a two-place function which generate all the truth functions in the Logic of n truth-values [5]).

Theorem 1. *$f_{1.1.1}(p, q)$ is an n-valued* SHEFFER *function.*

The following functions (and families of functions) will be used in the proof of the theorem:

	Function	*Truth-value properties*
1.	$N_{m.x.y}(p, q)$	Takes the value m when p takes the value x and q, y. Otherwise takes the value n.
2.	pVq	Takes the minimum of the value of p and the value of q.
3.	$\sum\limits_{i=a}^{a+m} f_i(p)$	Takes the minimum value of $f_a(p), f_{a+1}(p), \ldots f_{a+m}(p)$.
4.	$D_{m.i}(p)$	Takes the value m if p takes the value i, otherwise takes n.
5.	$p \& q$	Takes the maximum of the value of p and the value of q.
6.	$\overline{p}$	Takes as value $n+1$ minus the value of p.
7.	$T_m(p)$	Takes the value m everywhere [6]).

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[2]) H. M. SHEFFER, "A Set of Five Independent Postulates for Boolean Algebras, with Application to Logical Constants", *Trans. Am. Math. Soc.* **14**, 487—488 (1913), Properly speaking, SHEFFER did not show that the stroke function is functionally complete, but only that it can define negation and disjunction. Since POST subsequently showed the latter to be functionally complete, the result follows from SHEFFER's article.

[3]) POST, *op. cit.*, p. 180—181.

[4]) DONALD L. WEBB, "Generation of any n-valued Logic by One Binary Operator", *Proc. Nat. Acad. Sci*, 21, 252—254 (1935).

[5]) In the remainder of this paper when a number is stated it is to be understood unless otherwise asserted as referring to that number congruent to the stated number modulo n which is greater than zero and less or equal, to n. When truth-tables are refered to the following arrangement is presupposed (for one-place functions): The i^{th} line of the table is that line where the argument takes the value i, (for two-place functions) if the value of the first argument is x and of the second is y then the corresponding line of the truth-table is the ith line when $i = (x-1)n + y$.

[6]) The idea of this function is a generalisation of the T-function of SLUPECKI which together with the negation and implication of LUCASIEWICZ's system form a functionally complete set. (Cf. note 6), "Die Logik und das Grundlagenproblem" in Gonseth, *Les Entretiens de Zurich*, p. 97.

8. $s(p)$ Takes as value the value of p plus one, 1 if p takes n.

9. $s^m(p)$ The m-fold application of $s(p)$ — thus takes as value the value of p plus m.

10. $f_{1,1,1}(p, q)$ Takes as value 1 plus the minimum of the value of p and the value of q, takes 1 if the value of $p =$ the value of $q = n$.

The family $N_{m,x,y}(p, q)$, $m = 1, 2, \ldots n-1$; $x = 1, 2, \ldots n$; $y = 1, 2, \ldots n$ together with the function V is functionally complete since any function (except one) can be represented by a disjunction of members of the family, where $N_{m,x,y}(p, q)$ takes the value m when p takes x and q takes y: the function which takes n everywhere is the only exception and this is equivalent to

$$N_{1,1,2}(N_{1,1,1}(p, q), N_{1,1,1}(p, q))$$

$N_{mx,y}(p, q)$ for $m > 1$ can be defined in terms of $N_{1,x,y}(p, q)$, $^-$, V and $T_m(p)$ $[1 \leqslant m \leqslant n]$ as follows

$$N_{m,x,y}(p, q) = \overline{T_{n-m+1}(p) \; V \; \overline{N_{1,x,y}(p, q)}}$$

$N_{1,x,y}(p, q)$ can in turn be defined in terms of $D_{1,i}(p)$ $[1 \leqslant i \leqslant n]$ and &:

$$N_{1,x,y}(p, q) = D_{1,x}(p) \;\&\; D_{1,y}(q)$$

& is to be defined by $^-$ and V by DE MORGAN's law:

$$p \;\&\; q = \overline{\bar{p} \; V \; \bar{q}}$$

Accordingly, the set $^-$, V, with the families $D_{1,i}(p)$ $[1 \leqslant i \leqslant n]$ and $T_m(p)$ $[1 \leqslant m \leqslant n]$ is functionally complete.

Furthermore if we define the sum of a function $\sum_{i=r}^{a+1} f_i(p)$ as:

$$\begin{cases} \sum_{i=a}^{a} f_i(p) = f_a(p) \\ \sum_{i=a}^{a+k+1} f_i(p) = f_{a+1+k}(p) \; V \sum_{i=a}^{a+k} f_i(p) \end{cases}$$

we can define $^-$ in terms of V and $D_{m,i}$ $[1 \leqslant m \leqslant n,\ 1 \leqslant i \leqslant n]$

$$\bar{p} = \sum_{i=1}^{n} D_{n+1-i,i}(p).$$

With the aid of the multiple application of $s(p)$, $s^r(p)$ (this can also be defined recursively, as follows $s^0(p) = p$, $s^{k+1}(p) = s(s^k(p))$), we can define $D_{m,i}(p)$ $[m > 1]$ in terms of $D_{1,i}(p)$, V, and s:

$$D_{m,i}(p) = s^{m-1}(T_{n-m+1}(p) \; V \; D_{1,i}(p))$$

$D_{1,i}(p)$ and $T_m(p)$ are likewise defineable in terms of V, and s:

$$D_{1,1}(p) = s^{n-1} \sum_{i=1}^{n-1} s^i(p)$$

(for $i \neq 1$) $D_{1,i}(p) = s^{n-1}\left(\sum_{j=1}^{n-i} s^j(p)\ V \sum_{j=n-i+2}^{n} s^j(p)\right)$

$$T_1(p) = \sum_{i=1}^{n} s^i(p)$$

(for $m \neq 1$)

$$T_m(p) = s^{m-1}\, T_1(p).$$

Accordingly, as shown by POST $s(p)$ and V are functionally complete. Both of these can be defined in terms $f_{1,1,1}(p, q)$

$$s(p) = f_{1,1,1}(p, p)$$
$$pVq = s^{n-1}\, f_{1,1,1}(p, q)\ ^{7)}$$

We define $f'_{n,1,1}(p, q)$ as the function which takes 1 if either p or q takes n and otherwise takes the maximum of the value of p and that of q, plus one.

Theorem 2. *$f'_{n,1,1}(p, q)$ is an n-valued* SHEFFER *function.*

This can be shown by the following series of definitions: ($E_{m,1}(p)$ is the function, which takes m when p takes 1, otherwise 1).

$$s(p) = f'_{n,1,1}(p, p)$$

$$s^m(p) = \begin{cases} s^0(p) = p \\ s^{k+1}(p) = s(s^k(p)) \end{cases}$$

$$p\ \&\ q = s^{n-1}\, f'_{n,1,1}(p, q)$$

$$\prod_{i=m}^{m+k} f_i(p) \begin{cases} \prod_{i=m}^{m} f_i(p) = f_m(p) \\ \prod_{i=m}^{m+k+1} f_i(p) = f_{m+k+1}(p) \text{ and } \prod_{i=m}^{m+k} f_i(p) \end{cases}$$

$$T_n(p) = \prod_{i=1}^{n} s^i(p)$$

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7) As in corollary to Theorem 1, the LUCASIEWICZ–SLUPECKI primitives can be easily shown to be functionally complete, by the following definitions:

$s(p) = NCC\,TpNpNCpNp$
$N_{1,1,1}(p, q) = NCCpNqNCpNq$
$N_{2,1,1}(p, q) = s\,(NCTpNN_{1,1,1}(p, q))$
$A'pq = CNpq$
$f_{1,1,1}(p, q) = A'A'A'A'A'\ N_{2,1,1}(p, q)\ N_{2,1,1}(s(p), q)$
$N_{2,1,1}(s^2(p), q)\ N_{2,1,1}(p, s(q))\ N_{2,1,1}(p, s^2(q))$
$N_{1,1,1}(s(p), s(q))$

for $m \neq n$, $T_m(p) = s^m T_n(p)$

$$E_{n,1}(p) = s \prod_{i=1}^{n-1} s^{i-1}(p)$$

for $m \neq 1$, $E_{n,m}(p) = s\left[\prod_{i=1}^{n-m-1} s^i(p) \quad \& \quad \prod_{i=n-m+1}^{n} s^i(p)\right]$

$$E_{k,m}(p) = s^k (T_{n+1-k}(p) \quad \& \quad E_{n,m}(p))$$

$$\bar{p} = \prod_{i=1}^{n} E_{n-i+1,i}(p)$$

$$pVq = \overline{\bar{p} \;\&\; \bar{q}}$$

$$f_{1,1,1}(p, q) = s(pVq).$$

On the basis of the isomorphism of $s(p)$ in n-valued logic to the "successor" function for congruence classes modulo n, we can easily obtain a considerably larger number of SHEFFER functions.

Theorem 3. *If a and n are relatively prime, $f_{1,1,a}(p, q)$ and $f'_{n,1,a}(p, q)$ are n-valued* SHEFFER *functions.*

$f_{1,1,a}(p, q)$ is the function which takes as value the minimum of the values of p and q, plus a ($n + a$ being considered as equal to a). $f'_{n,1,a}(p, q)$ takes the maximum of the values, plus a.

$s_a(p) = f_{1,1,a}(p, p)$. By EULER's theorem $S_a^{a^{\varphi(n)-1}}(p)$ takes as value the value of p plus one. Hence

$$s(p) = S_a^{a^{\varphi(n)-1}}(p). \quad \text{Thus} \quad f_{1,1,1}(p, q) = s^{n-a+1} f_{1,1,a}(p, q).$$

Analogously for $f'_{n,1,a}(p, q)$.

$f_{i,1,a}(p, q)$ is the function which takes $i + a$ if p or q takes i and takes $i + a + b$, if p and q do not take any j $[i \leqslant j < i + b \leqslant i - 1]$ and p or q takes $i + b$ (where $n + c$ is taken as equal to c). $f'_{i,1,a}(p, q)$ is the function which takes $i + a$ if p or q takes i and takes $i + a - b$ if p and q do not take any j $[i + 1 \leqslant i - b < j \leqslant i]$ and p or q take $i - b$ (where $-c$ is taken as equal to $n - c$).

Theorem 4. *If a and n are relatively prime, $f_{i,1,a}(p, q)$ and $f'_{i,1,a}(p,q)$ are n-valued* SHEFFER *functions.*

$s_a(p) = f_{i,1,a}(p, p)$, By EULER's theorem $s_a^{a^{\varphi(n)-1}}(p) = s(p)$. Then $f_{i,1,a}(s^{i-1}(p), s^{i-1}(q)) = f_{1,1,i+a-1}(p, q)$. $s^{n+2-(a+i)} f_{1,1,i+a-1} = f_{1,1,1}(p, q)$.

Analogously for $f'_{i,1,a}(p, q)$.

$f_{1,d,a}(p, q)$ is the function which takes $a + 1$ if p or q takes 1; if not and p or q takes $d + 1$, $f_{1,d,a}(p, q)$ takes $a + d + 1$; if not but either take $2d + 1$, $f_{1,d,a}(p, q)$ takes $a + 2d + 1$, etc. until $md + 1$ where md is the greatest multiple of d less than n; if neither p nor q take any of these but one of them takes 2, $f_{1,d,a}(p, q)$ takes $a + 2$; if not and either takes $d + 2$, $f_{1,d,a}(p, q)$ takes $a + d + 2$ etc. until $Xd + 2$, where $n - d < Xd + 2 \leqslant n$; if neither p nor q take any of these, but one of them takes 3,

$f_{1,d,a}(p, q)$ takes $a + 3$ etc. $f_{i,d,a}(p, q)$ is the result of $n + 1 - i$ applications of $s(p)$ to the arguments of $f_{1,d,a}(p, q)$.

If in the definition of $f_{1,d,a}(p, q)$ in place of starting with 1 and adding units of d with n as a limit, we start with n and subtract units of d with 1 as the limit, we obtain the function $f'_{n,d,a}(p, q)$. $f'_{i,d,a}(p, q)$ is the result of $n - i$ applications of $s(p)$ to the arguments of $f'_{n,d,a}(p, q)$.

Theorem 5. *If d is not a divisor of n and a is relatively prime to n, $f_{i,d,a}(p, q)$ and $f'_{i,d,a}(p, q)$ are n-valued* Sheffer *functions.*

$s(p)$ can be defined as in theorem 4. Then $f_{i,d,a}(s^{i-1}(p), s^{i-1}(q)) = f_{1,d,i+a-1}(p, q)$. $f_{1,d,o}(p, q) = s^{n-(i+a)} f_{1,d,i+a}(p, q)$.

If $\overset{m+k}{\underset{i=m}{\Phi}} g_i(p)$ is defined recursively:

$$\overset{m}{\underset{i=m}{\Phi}} g_i(p) = g_m(p) \text{ and } \overset{m+k+1}{\underset{i=m}{\Phi}} g_i(p) = f_{1,d,o}(g_{m+k+1}(p), \overset{m+k}{\underset{i=m}{\Phi}} g_i(p)),$$

then $G_{1,d}(p) = \overset{n-1}{\underset{i=1}{\Phi}} s^i(p)$ takes $d + 1$ on line 1 and 1 elsewhere, and $K(p) = G_{1,d}(G_{1,d}(p))$ which takes 1 on line 1 and $d + 1$ elsewhere.

(*a*) *Case* 1. There is an x such that $n = xd - 1$.

$G_1(p) = (s^{n-d} G_{1,d}(p))$ takes 1 on line 1 and $n - d + 1$ elsewhere. Then $G_k(p) = s^{k-1} f_{1,d,o}(G_1(p), T_{n+2-(d+k)}(p))$ takes k on line 1 and $n - d + 1$ elsewhere. Define $H_{k.m}(p)$ as follows:

$$H_{k,1}(p) = G_k(p) \text{ and } H_{k,m}(p) = G_k(s^{m-1}(p)) \text{ for } m \neq 1.$$

Thus $H_{k,m}(p)$ takes k if p takes m and takes $n - d + 1$ otherwise. Any one-place function can be defined by the H-functions combined with $f_{1,d,o}(p, q)$ as is obvious, since if p takes $n - d + 1$ and q takes j, $f_{1,d,o}(p, q)$ takes j and $f_{1,d,o}$ is symmetrical. Hence also the function $W_1(p)$ which takes 1 on line 1, $d + 1$ on line 2, etc. and $W_2(p)$ the function such that $W_2 W_1(p) = p$, can be defined.
Then

$$sW_2 f_{1,d,o}(W_1(p), W_1(q)) = f_{1,1,1}(p, q).$$

(*b*) *Case* 2. There are numbers x and y $(1 < y < d)$ such that $n = xd - y$. $s^{n+y-(d+1)} K(p)$, takes $n - d + y$ on line 1 and y elsewhere. $H_{n,1}(p) = s^{n-d} f_{1,d,o}(s^{n+y-(d+1)} K(p), T_d(p))$ takes n on line 1 and $n - d + y$ elsewhere. $G_1(p) = s f_{1,d,o}(H_{n,1}(p), T_{n+y-(d+1)}(p))$ takes 1 on line 1 and $n - d + y$ elsewhere. The remainder of the proof is analogous to case 1 after the definition of G_1.

Similarly, the proof for $f'_{i,d,a}(p, q)$ is analogous to the proof for $f_{i,d,a}(p, q)$.

$b_{a,c}(p, q)$ is the function which takes the value $i + a$ if p and q take the value i and otherwise takes the value c.

Theorem 6. *If a is relatively prime to n, $b_{a,c}(p, q)$ is an n-valued* Sheffer *function.*

$$s(p) = b_{1,n}(p, p).\ T_n(p, q) = b_{1,n}(p, s(p)).$$
$$N_{1,1,1}(p, q) = b_{1,n}(s^{n-1} b_{1,n}(p, q),\ T_n(p, q)).$$

Then $N_{1,x,y}$ can be defined as $N_{1,x,y}(p, q) = N_{1,1,1}(s^{n+1-x}(p), s^{n+1-y}(q))$. $N_{i,x,y} = b_{1,n}(s^{i-2} N_{1,x,y}(p,q), s^{i-1} T_n(p,q))$: We define the following functions:

$$A_0(p, q) = b_{1,n}(p, s^{n-1}(q))$$
$$A_1(p, q) = b_{1,n}(N_{1,n,n}(p, q), T_n(p, q))$$
$$(i \geqslant 2) \quad A_i(p, q) = A_0(L_{i-1}(p, q), s(M_{i-1}(p, q)).$$

where

$$M_0(p, q) = T_n(p, q)$$
$$M_h(p, q) = A_h(M_{h-1}(p, q), A_h(N_{h,n,h+1}(p, q), N_{h,h+1,n}(p, q))$$
$$L_h(p, q) = A_h(M_h(p, q), N_{1,n,n}(p, q).$$

By induction for every i, $[0 < i \leqslant n-1]$ if p and q take n then A_i takes n and if p takes the value j $[0 < j \leqslant i]$ and q, n or if p takes n and q, j then $A_i(p, q)$ takes the value j. Then the remarks made in theorem 1 with respect to the completeness of the family $N_{i,x,y}(p, q)$ together with V, hold when A_{n-1} is substituted for V. Therefore $b_{1,n}(p, q)$ is a SHEFFER function [8]). Then $b_{1,c}(p, q)$ is a SHEFFER function for any c since $s(p) = b_{1,c}(p, p)$ and $b_{1,n}(p, q) = s^{n-c} b_{1,c}((s^c(p), s^c(q))$. If a and n are relatively prime then $s_a(p) = b_{a,c}(p, p)$ and by EULER's theorem there is an x such that $s(p) = s_a^x(p)$.

Then $s^{n+1-a} b_{a,c} = b_{1,c+n+1-a}(p, q)$ which is a member of the family $b_{1,c}(p, q)$ and hence has been shown to be a SHEFFER function.

The present position of the investigation is such that the above theorems certainly do not exhaust all the SHEFFER functions. On the contrary, the present investigator is aware of many special cases which do not fall under the theorems, SWIFT has recently discovered 90 in the 3-valued case while the above theorems give only 18 [9]). In the present state, the writer does not feel in a position even to conjecture as to what may be the necessary conditions (aside from the defining condition) for a function to be a SHEFFER function [10]).

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[8]) The part of the proof of theorem 6 above is due to WEBB, *loc. cit.*

[9]) J. D. SWIFT, "Analogues of the SHEFFER Stroke in the 3-valued Case", *Bull. Am. Math. Soc.* 56, 63 (1950).

[10]) It is to be noted that any function which can define all functions of two-arguments will also define any function of any number of arguments. For $f_{1,1\,1}(p, q)$ this can be done by constructing the appropriate analogues of the N-functions by taking the logical product of the appropriate D-functions for the number of arguments desired (e.g. the function which takes i if p, q and r take x, y respectively, and n otherwise can be defined as:

$$D_{i,x}(p)\ \&\ D_{i,y}(q)\ \&\ D_{i,z}(r).$$

Since all other functions proved to define all two-place functions also define $f_{1,1,1}(p, q)$, the result follows.

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ON THE OZONIZATION OF NAPHTHALENE AND 2,3-DIMETHYL-NAPHTHALENES IN CONNECTION WITH THE STRUCTURE OF THE RING SYSTEM

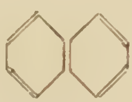
BY

J. P. WIBAUT AND L. W. F. KAMPSCHMIDT

(communicated at the meeting of Sept. 30, 1950)

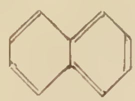
§ 1. *Introduction*

The classical structural theory of the organic chemistry mainly discusses the following valency structures for naphthalene:



I

ERLENMEYER-GRAEBE
"symmetrical structure"

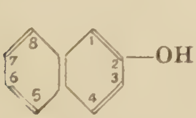


II III

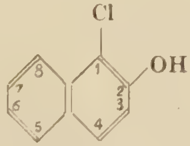
ERDMANN structures
"unsymmetrical structures"

On the strength of the chemical behaviour of naphthalene derivatives MARCKWALD [1] assumed that the naphthalene ring reacts exclusively according to structure I. This view was supported by later investigations of FRIES [2] and of FIESER [3] and their collaborators.

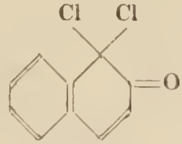
The argumentation is based among other things on the behaviour of β-naphthol (IV) in substitution reactions. Introduction of the chlorine atom into β-naphthol leads to the formation of: 1-chloro-2-hydroxy-naphthalene (V). Introduction of another chlorine atom causes the formation of dichloroketone (VI) and not of 1,3-dichloro-2-hydroxynaphthalene [4].



IV



V



VI

When β-naphthol is coupled with a diazotized amine $[Ar-N\equiv N]Cl$, the $Ar-N=N$-group is linked to carbon atom 1; if position 1 is occupied by a methyl group, no coupling takes place. The hydrogen atom in position 3 is not substituted. To account for these and a number of similar phenomena FIESER assumes that the substitution is preceded by addition to the double bond of the enolic group: $-\underset{H}{C}=\underset{|}{C}-OH$.

As the ERLENMEYER formula contains a double bond between carbon atoms 1 and 2, but not between 2 and 3, FIESER concludes that the naphthalene ring in β-naphthol occurs in structure IV only. Certain regularities in the substitution in β-naphthol and related compounds, which are caused by differences in reaction velocity with respect to the 1 and 3 positions, lead to a conclusion about the stationary state of the molecule. Assuming that the structure valid for the ring system of naphthalene is the same as that for β-naphthol, FIESER [5] concludes that naphthalene has the symmetrical structure suggested by ERLENMEYER and that the double and single bonds are fixed in the 1,2 and 2,3-positions respectively.

On the strength of extensive investigations on tricyclic compounds, which are related to naphthalene, FRIES arrives at the same conclusion, i.e. that naphthalene has little tendency to exist in the unsymmetrical form (II or III), because one of the ring systems would then have to depart from the aromatic condition and acquire the bond structure of the highly reactive o-benzoquinone.

Accurate X-ray studies by ROBERTSON have shown, however, that the distances between two adjacent carbon atoms in the six-membered rings of the naphthalene molecule are nearly all the same and equal to 1.40 Å. A rigid ERLENMEYER–GRAEBE structure would require a difference of about 0.20 Å between the distance of carbon atoms 1—2 and the distance of carbon atoms 2—3.

FIESER's conclusion that naphthalene has a fixed ERLENMEYER-GRAEBE structure is therefore unacceptable. According to the theory of resonance naphthalene is a resonance hybrid like benzene: in its ground state the molecule does not contain single or double carbon bonds: two adjacent C atoms are linked by an "aromatic carbon bond".

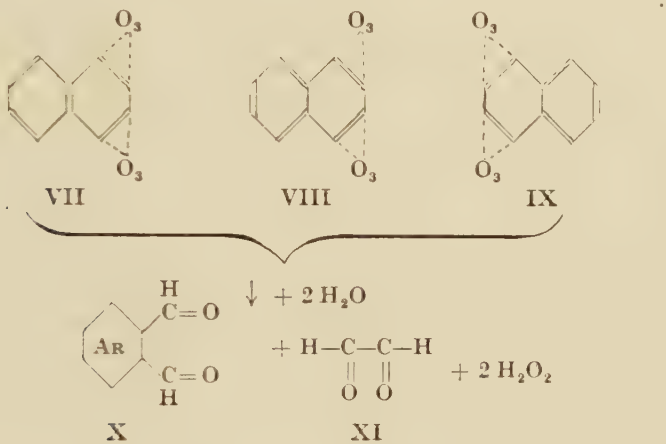
In the present paper we describe the behaviour of naphthalene and 2,3-dimethylnaphthalene towards ozone, the most characteristic reagent to the double carbon bond.

In aliphatic compounds, which contain a double carbon bond, an ozone molecule reacts with the atomic group $> C = C <$ to form an ozonide, whose structure was in one case determined by RIECHE. The way in which the oxygen atoms are bound to form ozonides, derived from aromatic hydrocarbons, is not yet known with certainty. We know that these ozonides are decomposed by water, as a result of which hydrogen peroxide is formed and the six-membered ring is broken; two adjoining carbon atoms each take up one O atom, so that the group $O = \underset{|}{C} - \underset{|}{C} = O$ is formed. During this reaction the ring structures which in the classical structural theory are designated as double bonds, are broken.

A study of the ozonolysis of naphthalene and its homologues may therefore show whether these compounds are capable of reacting according to different valency structures (I, II, III).

§ 2. *Ozonolysis of naphthalene*

The action of ozone on naphthalene was first investigated by HARRIES and WEISS [6], who found that an explosive diozonide $C_{10}H_8O_6$ is formed. By decomposing this diozonide with water these investigators obtained phthaldialdehyde (X) and phthalic acid; glyoxal (XI), which might have been expected, was not found. This scission product has been identified by us. The following formulae give a schematic representation of the naphthalene diozonide; obviously, the above scission products can be formed both from the diozonide of an ERLENMEYER–GRAEBE structure (VII) and from a diozonide of an ERDMANN structure (VIII or IX):



SEEKLES [7], in again investigating the action of ozone on naphthalene, aimed at finding a method to prepare o-phthalaldehyde; he obtained it in a yield of 10 % of the theoretical value and in addition o-phthalaldehydic acid in a yield of 70 %, i.e. 80 % of the quantity of aromatic decomposition products to be expected.

This shows that the action of ozone on naphthalene *chiefly* results in the formation of a diozonide. This is in agreement with measurements on the rate of reaction, which we carried out in collaboration with H. BOER.

The curve in figure 1 represents the consumption of moles of ozone per mole of naphthalene as a function of the time (solvent chloroform, temperature — 40° C).

The rate of reaction drops appreciably after 2.02 moles of ozone have been taken up per mole of naphthalene. The diozonide formed continues to react very slowly with ozone, the second six-membered ring being attacked with formation of a pentozonide. The fact that the bend in the curve occurs at 2.02 moles and not at 2 moles of ozone implies that already during the first stage of the reaction some pentozonide is formed; a rough calculation shows that this quantity is about 0.01 mole.

The question arises whether a triozonide may be formed as an isolable product. This is improbable, however, because a triozonide molecule con-

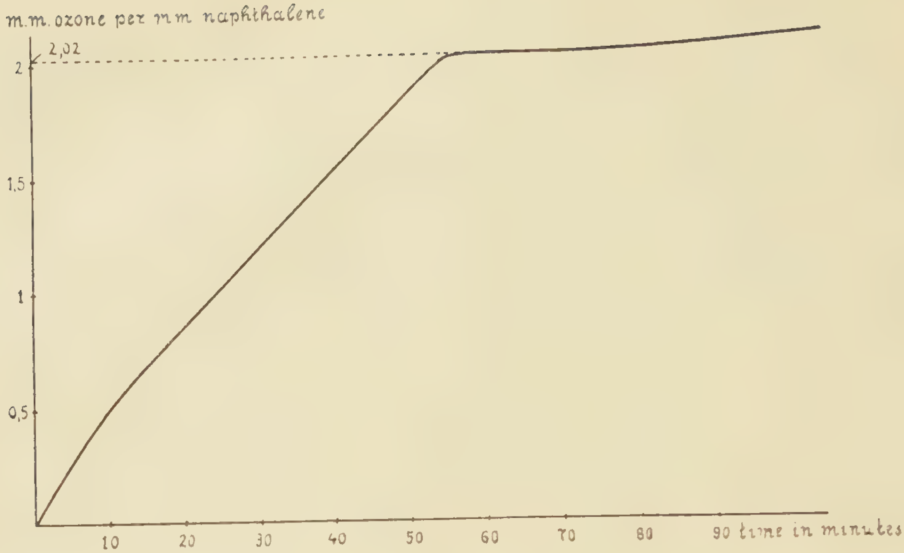
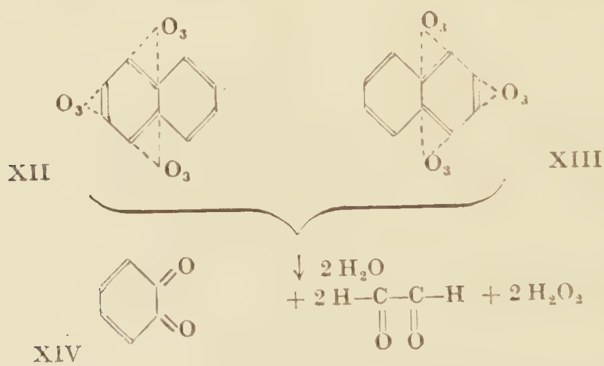


Fig. 1

tains two double carbon bonds, which will rapidly continue their reaction with ozone. Hydrolysis of a triozonide from an ERDMANN structure (XII or XIII) would yield o-quinone (XIV):



However, we found no indication of the formation of o-quinone in the ozonolysis of naphthalene.

Below follows a brief description of the experiments [1]) which were partly carried out in collaboration with H. BOER. For 2—4 hours ozone-containing oxygen is passed at $-25°$ C through a solution of 13—30 millimoles of naphthalene in 30—40 ml of chloroform. The rate of flow of the gas mixture is 3.8 litres (+ 20° C, 760 mm) per hour, the ozone content being 12—15 % by wt, so that each hour 13—17 millimoles of ozone is passed through.

[1]) Thesis of L. W. F. KAMPSCHMIDT; (Amsterdam 1950). The complete experimental data will be published in the Recueil des Travaux Chimiques.

After the ozonization the bulk of the chloroform is distilled off in vacuo at room temperature, the remaining solution (5 ml) being poured into pure ethyl ether cooled to — 20° C. The diozonide of naphthalene precipitates and is filtered off at — 20° C (glass filter) and washed with cold ether. The ethereal filtrate contains non-converted naphthalene, the quantity of which can be determined.

The ozonide, which is very explosive, is worked up in various ways. Decomposition with water yields a solution whose phthalaldehyde content is betrayed first by a deep blue colour and then by a black precipitate after addition of ammonia and acidification with acetic acid. If the ozonide is treated with a solution of hydroxylamine hydrochloride and soda, it is decomposed, the dialdehydes formed reacting with hydroxylamine. From the reaction mixture we isolated the *dioxime of glyoxal* (identified by melting point and mixed melting point) and the *mono-oxime of phthalimide* (melting point 254° C; mixed melting point with an authentic sample showed no depression). The latter compound is formed, as has been known for a long time already, by the action of hydroxylamine on phthalaldehyde [1]). The question whether in the ozonolysis of naphthalene a small quantity of o-quinone is formed cannot be easily solved by experiments, because o-quinone is an unstable compound, which is decomposed by water. WILLSTÄTTER indicates that o-quinone is readily reduced to o-dihydroxybenzene by sulphurous acid. Therefore we treated the chloroform solution, as obtained after the ozonization of naphthalene, with an aqueous solution of sulphurous acid. However, we did not succeed in identifying o-dihydroxybenzene.

§ 3. *The ozonolysis of* 2,3-*dimethylnaphthalene*

This reaction was investigated by one of us (W.) in 1941 and 1942 in collaboration with J. VAN DIJK [8]. In this investigation, which could not be completed owing to the war conditions, only the aliphatic scission products of the ozonide were identified, i.e. dimethylglyoxal and small quantities of methylglyoxal and glyoxal.

In the first place we determined the consumption of ozone per mole of dimethylnaphthalene as a function of the time. The curve obtained has the same shape as that of naphthalene (cf. fig. 1). The reaction of ozone (formation of diozonides) at — 40° C proceeds rapidly until 2.04 millimoles per millimole of dimethylnaphthalene have been taken up; at this point the curve shows a sharp bend and the reaction continues to proceed very slowly (formation of pentozonide).

The investigation of the aromatic scission products shows that two isomeric diozonides (XV and XVI) are formed. In the main reaction (A) the methylated six-membered ring is attacked by ozone, as had already become probable by the investigation of WIBAUT and VAN DIJK; in

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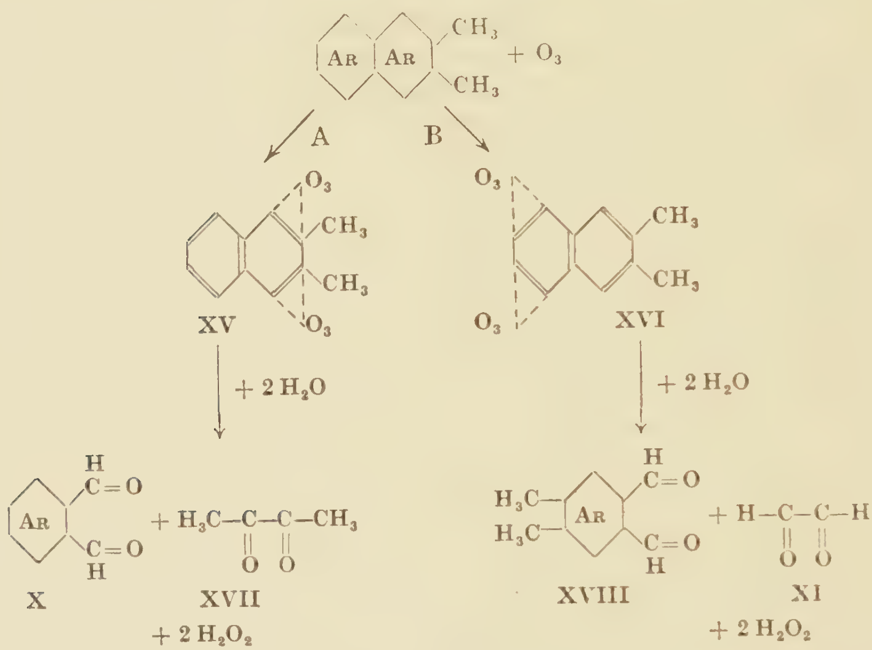
[1]) The chemism of this reaction will be described in a future paper.

addition, a side reaction (B) develops, in which the non-methylated six-membered ring reacts with ozone. The formation of the diozonide XXI was already assumed by KOOYMAN [9], who pointed out that this diozonide may react to form a pentozonide which, on being decomposed, yields methylglyoxal.

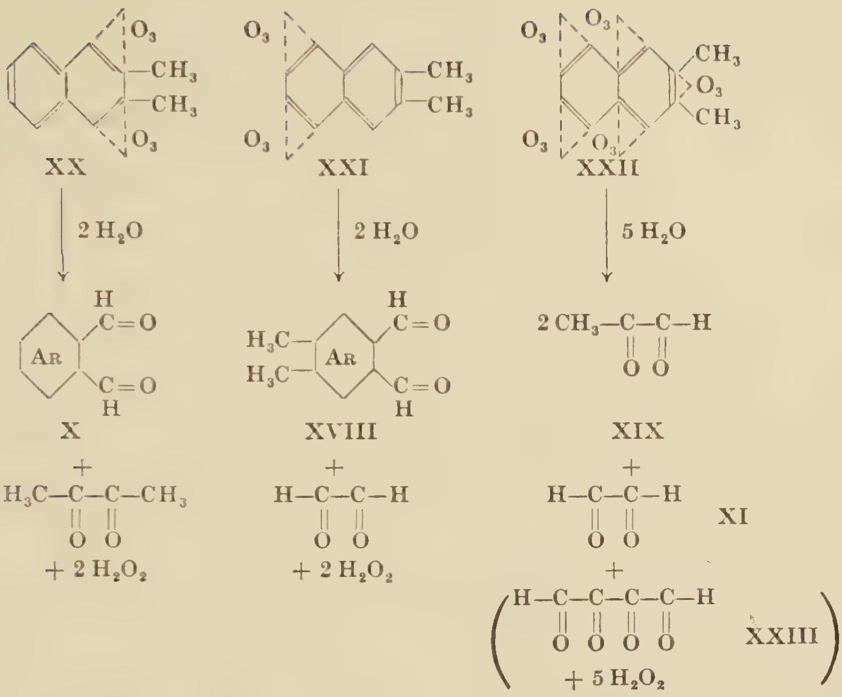
After the hydrolytic splitting of the mixture of isomeric diozonides we found as aromatic decomposition products mainly derivatives of phthalaldehyde (X) and a smaller quantity of derivatives of 4.5-dimethylphthalaldehyde (XVIII). As aliphatic scission products we found — in agreement with the observations of WIBAUT and VAN DIJK — dimethylglyoxal (XVII), glyoxal (XI) and methylglyoxal (XIX).



Phthaldialdehyde and dimethylglyoxal may be formed from the diozonide (XV), derived from the ERLENMEYER-GRAEBE structure, but also from the diozonide (XX), derived from an ERDMANN structure. Dimethylphthalaldehyde and glyoxal may be formed from the diozonide (XVI), derived from the ERLENMEYER-GRAEBE structure, as well as from the diozonide (XXI), derived from one of the two ERDMANN structures. Hydrolysis of the pentozonide (XXII), which may be formed from this ERDMANN structure, yields methylglyoxal (XIX), glyoxal and the hypothetical tetra-oxobutane (XXIII), which is further decomposed:

(See form. following page).

It is therefore to be expected that, as the action of ozone on dimethylnaphthalene is continued, the quantity of methylglyoxal will increase with respect to dimethylglyoxal, for in the first stage of the ozonization



almost exclusively diozonides are formed. The following experimental data are in agreement with this assumption:

TABLE I

Ozonization of 2,3 dimethylnaphthalene in chloroform at — 25° C.

No.	dimethyl-naphthalene in millimoles	ozone passed through in millimoles	dimethyl-glyoxime in millimoles	methyl-glyoxime in millimoles	gloyxime in millimoles
1	50	100	4.55	0.21	0.7
2	20	300	1.9	0.19	0.66

In experiment 1, where two moles of ozone were passed through per mole of dimethylnaphthalene, the molecular ratio dimethylglyoxime: methylglyoxime = 22; in experiment 2, where 15 moles of ozone per mole of hydrocarbon were passed through, this ratio is 10. It should be observed that the quantity of dimethylglyoxime found in these experiments is only 9 % of what might have been expected according to the reaction scheme. This is caused by the fact that the dimethylglyoxal formed in the hydrolytic splitting of the ozonides is oxidized by the hydrogen peroxide, which is also formed, thus causing the formation of acetic acid. We actually found in experiments 1 and 2 acetic acid in yields of 70 and 75 % respectively of the quantities calculated on dimethylnaphthalene. Oxidation of methylglyoxal may yield acetic acid in addition to formic acid. However, only traces of formic acid were found; this acid was possibly further

decomposed in the oxidizing medium. The values found for the ratio dimethylglyoxime : methylglyoxime thus only warrant the conclusion that the experimental result agrees qualitatively with expectations.

The ozonization is carried out in a chloroform solution at — 25 to — 30° C. The solution of the ozonides is treated with a solution of hydroxylamine hydrochloride and soda in water. As primary aromatic scission products of the ozonides phthaldialdehyde and dimethylphthalaldehyde are formed; part of these dialdehydes is oxidized by the hydrogen peroxide, which is also formed, to the corresponding phthalaldehydic and phthalic acids. The remaining dialdehydes and the aldehydic acids react with hydroxylamine and yield various derivatives.

As primary aliphatic scission products of the ozonides are formed: dimethylglyoxal — a major part of which is oxidized to acetic acid, the remaining part being converted by hydroxylamine into dimethylglyoxime, methylglyoxal and glyoxal, which are also converted into dioximes.

The following compounds were isolated:

aromatic scission products	phthalic acid (35%), phthalimide (5 %), mono-oxime of phthalimide (10 %), dioxime of phthalimide (about 1 %), 4,5-dimethylphthalic acid, 4,5-dimethylphthalic acid imide.
aliphatic scission products	dimethylglyoxime (9 %), methylglyoxime (about 0.4 %), glyoxime (1.4 %), acetic acid (70—75 %).

The isolated compounds (except acetic acid) were identified with a synthetic sample by melting point and mixed melting point. The 4.5-dimethylphthalic acid imide was identified by the infra-red spectrum, which was identical with that of a synthetic sample. The yield figures placed between brackets have been calculated on the quantity of original dimethylnaphthalene. For some decomposition products the yield cannot be stated. The above figures represent minimum values, because a quantitative determination of the yields was impossible.

From the available data we conclude that the diozonide formation proceeds to the extent of about 90 % according to reaction A.

From the view point of the classical structural theory the experiments described in § 2 and § 3 warrant the following conclusion: the main ozonolysis products are formed by decomposition of diozonides, which may be formed from the Erlenmeyer Graebe structure as well as from one of the Erdmann structures. A slight quantity of by-product, methylglyoxal, is formed by decomposition of a pentozonide, which may be formed from one of the Erdmann structures. The assumption that the naphthalene ring reacts mainly according to an Erlenmeyer–Graebe structure is therefore not in contradiction with our experiments.

§ 4. *The mechanism of the ozonization according to the localization hypothesis*

Kooyman and Ketelaar [10] are of opinion that the assumption that

the naphthalene molecule is converted into one of its valency structures, which subsequently reacts with ozone, must be rejected on energetic grounds. They justly observe that in the ozonization of benzene or naphthalene some consecutive reactions develop. First of all an ozone molecule will react to form a mono-ozonide. The mono-ozonide of benzene contains a non-aromatic ring system in which two double bonds occur; the mono-ozonide of naphthalene contains one double bond. These mono-ozonides will then rapidly react with ozone to form a triozonide or a diozonide.

This theory has been confirmed by measurements on the rate of the ozonization reaction of aromatic compounds, carried out in our laboratory by SIXMA [11] and some co-workers.

By representing the velocity constants of the three consecutive reactions by k_1, k_2 and k_3

$$\text{(1)} \qquad C_6H_6 + O_3 \xrightarrow{k_1} C_6H_6O_3$$

$$\text{(2)} \qquad C_6H_6O_3 + O_3 \xrightarrow{k_2} C_6H_6O_6$$

$$\text{(3)} \qquad C_6H_6O_6 + O_3 \xrightarrow{k_3} C_6H_6O_9$$

it follows from our measurements that the reaction velocity is determined by the slowest reaction, i.e.

$$\frac{d\,[C_6H_6]}{dt} = k_1\,[C_6H_6]\,[O_3].$$

The value of k_1 at $-31°$ C is 4.9×10^{-5} (m.mol^{-1}, liter, min.$^{-1}$).

Series of experiments, in which either the initial concentration of benzene, or that of ozone was varied, showed that the reaction is of the first order, both with respect to the benzene concentration and the ozone concentration.

We determined the value of k_1 at 6 temperatures ranging between $-39.8°$ C and $-25°$ C and calculated for the energy of activation: 12.4 ± 0.3 kcal/mol.

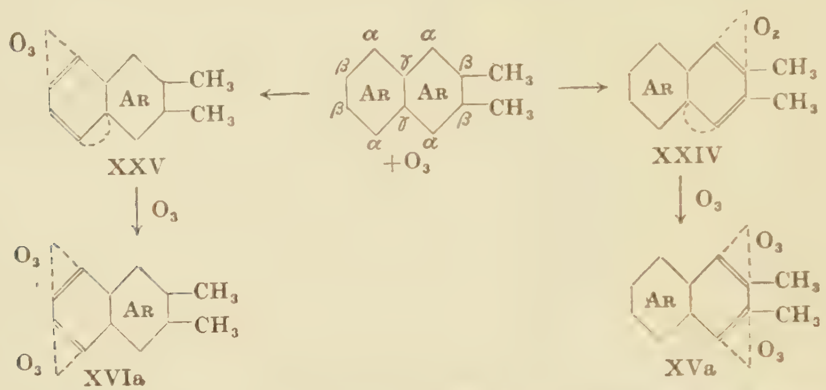
If the reacting benzene molecule is converted into a valency structure the reaction complex will, according to KOOYMAN and KETELAAR, consist of an ozone molecule and a hydrocarbon molecule with single and double bonds. This state cannot be attained until the total resonance energy has been taken up; this resonance energy thus forms part of the activation energy required for the reaction between ozone and benzene. The difference in activation energy of the ozonization of naphthalene and of benzene will be approximately the same as the difference in resonance energy of naphthalene and benzene, i.e. 75—39 = 36 kcal/mol.

It might therefore be expected that naphthalene would react far more slowly with ozone than benzene. However, our measurements showed that at $-30°$ naphthalene reacts about 10^4 times as rapidly as benzene. KOOYMAN and KETELAAR therefore developed a different mechanism,

based on the localization hypothesis [10]. They assume that in the reaction between an ozone molecule and the aromatic molecule the carbon bond, which is attacked by ozone, owing to localization of two π-electrons, is transformed into a polarized double bond, the other π-electrons tending to a state of minimum energy.

By starting from this assumption the authors deduce that in the case of naphthalene, localization of the 1,2-bond requires a smaller amount of energy than localization of the 2,3-bond, so that a reaction in which an α—β-bond is attacked, will show a far smaller energy of activation than a reaction in which a β—β-bond is attacked.

According to KOOYMAN and KETELAAR localization to an α—β-bond in 2,3-dimethylnaphthalene can take place in two ways:



By a further reaction with ozone, XXIV is converted into the diozonide (XVa) which, on being decomposed, yields phthalaldehyde and dimethylglyoxal; XXV is converted into the diozonide (XVIa) from which dimethylphthalaldehyde and glyoxal are formed. XVIa may yield two isomeric pentozonides, one of which (formula XXII in § 3), on being decomposed, yields glyoxal and methylglyoxal.

The view of KOOYMAN and KETELAAR that the α—β-bond in the ring system of naphthalene is more reactive than the β—β-bond leads to the same conclusions as the view that naphthalene and its derivatives react preferentially according to the ERLENMEYER–GRAEBE structure.

Contrary to the classical structural-chemical views of FIESER and FRIES, however, the theory of KETELAAR and KOOYMAN allows for the arguments and results of the theory of resonance.

As appears from our investigation, the methylglyoxal can only be formed from the pentozonide (XXII); as we have shown that the formation of a pentozonide proceeds far more slowly than that of a diozonide, it is clear that only a small quantity of methylglyoxal is formed.

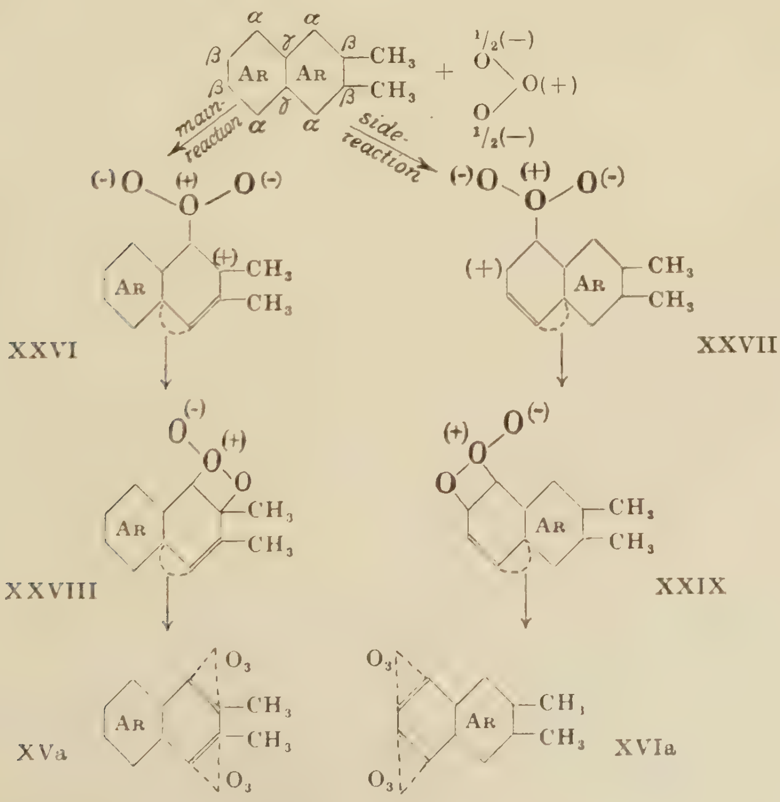
§ 5. *The electrophylic mechanism of the ozonization*

Experiments started in our laboratory with SIXMA, KAMPSCHMIDT and BOER and which are being continued by SIXMA, showed that the influence

of substituents in the nucleus on the rate of ozonization of aromatic compounds is the same as that on halogenation and nitration. The methyl homologues of benzene are more rapidly attacked by ozone than benzene itself, the halogen substituted benzenes on the other hand more slowly. Moreover, the ozonization of chlorobenzene or bromobenzene is catalytically accelerated by BF_3, $AlCl_3$ or $FeCl_3$, which halogenides are typical catalysts for the substitution of halogen in aromatic rings. For these reasons it is probable that the ozonization of aromatic compounds, just like bromination, chlorination and nitration, takes place according to an electrophylic mechanism [12].

We assume that the ozone molecule reacts according to a polar structure, the central O-atom of which carries a positive charge. The aromatic nucleus will be polarized under the influence of the polar ozone molecule, so that in the activated state two π-electrons will be localized to one of the carbon atoms of the aromatic ring. Thus a homopolar bond can be formed between this carbon atom and the ozone molecule, resulting in the formation of a reaction complex by an electrophylic addition.

The hypothesis of the localization of two π-electrons to a carbon atom of the naphthalene ring, which is an extension to the localization hypothesis of KOOYMAN and KETELAAR, has been introduced by WIBAUT and SIXMA [13] to account for the bromination process of naphthalene.



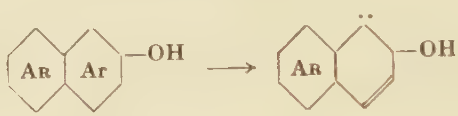
SIXMA [14] demonstrated by a calculation according to the molecular orbital method that localization of two π-electrons to an α—C-atom of the naphthalene ring requires less energy than localization to a β—C-atom. The calculated difference in localization energy for the α and β positions is of the same order as the difference in activation energy calculated from the bromination experiments by SUYVER and WIBAUT.

From this it follows therefore that an ozone molecule will react preferentially with an α—C-atom, the α—β-bond being polarized, for in this case the primary products formed contain an aromatic ring system. Formation of a primary product with polarization of the α—γ-bond is, on energetic grounds, less probable, as may be deduced from the localization theory of KOOYMAN and KETELAAR.

The ozonization of 2,3-dimethylnaphthalene can therefore be represented by the formulae on the preceding page. The primary products XXVI and XXVII will be stabilized to form the mono-ozonides XXVIII and XXIX, from which the diozonides XVa and XVIa are formed by the action of ozone.

Ultimately, the same diozonides are formed as according to the formulations given in § 3 and § 4.

The theory on the development of the monobromination of naphthalene suggested by WIBAUT and SIXMA also applies to the substitution in β-naphthol (see § 1). In this case too it may be assumed that the localization of two π-electrons occurs preferentially to an α—C-atom, so that the activated state of the α-naphthol molecule is represented by:



The reaction of β-naphthol with chlorine or with a diazotised amine allows of the same interpretation as the non-catalytic bromination of naphthalene.

The localization theory thus accounts for the experimental results formed in the ozonolysis of naphthalene and dimethylnaphthalene and for the substitution in β-naphthol and related phenomena. The results obtained by FIESER, FRIES, et. al. are therefore not in contradiction with the results of our investigations.

Summary

The action of ozone on naphthalene at — 25° C to — 40° C results in the formation of a diozonide, which continues to react slowly with ozone to form a pentozonide.

From 2,3-dimethylnaphthalene two isomeric diozonides are formed: the diozonide in which the methylated six-membered ring has taken up two ozone molecules, is formed as main product. The other diozonide, in which

the non-methylated six-membered ring has taken up two ozone molecules, is formed as by-product.

The first stage in the action of ozone on the naphthalene ring system is the addition of one ozone molecule according to an electrophylic mechanism. The ozone reacts with an α—C-atom of one of the six-membered rings, an α—β-bond being polarized. The primary reaction product is stabilized to a mono-ozonide, which is rapidly converted into a diozonide.

The primary decomposition products formed by hydrolysis of the diozonide of naphthalene are phthaldialdehyde and glyoxal. In the case of dimethylnaphthalene the diozonide, the main product, is converted into phthaldialdehyde and dimethylglyoxal. From the diozonide formed as by-product, 4,5-dimethylphthalaldehyde and glyoxal are formed as primary decomposition products. A small part of this diozonide is converted into a pentozonide, from which by hydrolysis methylglyoxal is formed.

The theory on the mechanism of mono-ozonide formation developed by us accounts for the experimental data without being in contradiction with the theory of resonance.

From the viewpoint of the classical structural theory the result can be formulated by stating that naphthalene and dimethylnaphthalene, when acted upon by ozone, chiefly react according to the ERLENMEYER–GRAEBE structure.

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Amsterdam, September 1950.

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CHEMISTRY

ELASTIC VISCOUS OLEATE SYSTEMS CONTAINING KCl. XIII [1])

1. *Influence of the pH and of the temperature on the G—C_{KCl} and n—C_{KCl} curves.*
2. *Substitution of K or Cl in KCl by other cations or anions.*
3. *The G—C_{KCl} and n—C_{KCl} curve for chemically pure K-oleate.*

BY

H. G. BUNGENBERG DE JONG, W. A. LOEVEN *) AND W. W. H. WEIJZEN *) [2])

(Communicated at the meeting of June 24, 1950)

1. *Introduction and methods.*

The influence of the KCl concentration on the elastic behaviour has first been investigated in part VI of this series. In the parts VI — XII we have studied with direct or indirect methods, the effects of added organic substances on the G—C_{KCl}, $1/\Lambda$—C_{KCl}, λ—C_{KCl} and n—C_{KCl} curves. The latter subject is no longer continued in the present communication, in which we will discuss the further investigation of the $G-C_{KCl}$ and $n-C_{KCl}$ curves, as is indicated in the title. We started from Na-oleate, neutral powder from BAKER [3]) (remarks on this preparation in part X, section 4, small print and section 7) and in section 5 from chemically pure oleic acid. The methods are the usual ones: rotational oscillation, contrivance to excite them (see part VII), marking with electrolytic H_2 (see part X). The experiments were performed in completely filled spherical vessels of 110 or 500 ml capacity, except in the case of the experiments with chemically pure oleic acid, where for economical reasons we had to be content with half filled 110 ml vessels (see part X, section 8, for the conflict between the strive for economical use of the preparations and for smaller experimental errors).

2. *Influence of the pH on the $G-C_{KCl}$ and $n-C_{KCl}$ curves.*

In all previous parts of this series the elastic oleate systems contained a small constant concentration of KOH (usually 0.05 N) to ensure a pH

*) Aided by grants from the "Netherlands Organisation for Basic Research (Z.W.O.)".

1) Part I has appeared in these Proceedings 51, 1197 (1948); Parts II—VI in these Proceedings 52, 15, 99, 363, 377, 465 (1949); Parts VII—XII in these Proceedings 53, 7, 109, 233, 743, 759, 975 (1950).

2) Publication no. 8 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

3) A generous gift of Na-oleate from The Rockefeller Foundation provided the means for the experiments described in this paper.

higher than 12, which excludes all hydrolysis of the oleate. From experiments on the behaviour of dilute oleate systems, performed in other fields of research [4]), this precaution did seem to be the indicated one as it could be feared that oleic acid set free by partial hydrolysis would but complicate the elastic behaviour. The aim of the following experiment was to control this expectation. Four 2.4 % stock oleate solutions were made, containing respectively 0.1 N KOH, 0.01 N KOH, no added KOH and 0.00388 N HCl. In ERLEMEYER flasks (with rubber stoppers), a series of mixtures were made with each of these stock solutions according to the receipt: 70 ml stock solution + 70 ml KCl solution (of known and respectively increasing concentrations). After vigorous shaking, a series of 110 ml spherical Pyrex vessels were completely filled (till into the neck) with the above mixtures and the vessels placed overnight in a thermostate of 20°, to become free of enclosed air bubbles. After measuring T and n, the vessels were placed overnight in a thermostate of 30° and the measurements were repeated. Of each series three mixtures with different KCl concentrations were taken every time to determine with a glass electrode (COLEMAN) the pH at room temperature and the average of these results was taken as the "pH" of the series [5]).

The results have been plotted in the figs. 1A and 1B, which show that when the pH is lowered, both the $G - C_{\mathrm{KCl}}$ and the $n - C_{\mathrm{KCl}}$ curves are shifted in the direction of smaller KCl concentrations and that at the same time the maximum of the n curve decreases (i.e. the damping increases). Compare the survey below.

Lowering of the pH has therefore the same effect as addition of organic substances showing the "action type A" (see part VI), or — using the nomenclature proposed in part XI of this series — exerting a KCl-sparing influence on the oleate system.

Now, when the pH is lowered, oleic acid is set free by hydrolysis and a KCl-sparing influence exerted by the oleic acid is just what we would expect. The unionised oleic acid possesses the same physical chemical character (a not too small carbon chain carrying an unionised polar group at one end) as the n-primary alcohols (of not too short a length of the carbon chain), which typically exert a KCl-sparing action (Compare parts VI and VIII).

If we compare fig. 1A and 1B with figure 2 in part VI, showing the KCl sparing influence of n-hexylalcohol, there is still a difference. In the case

[4]) Coacervation of oleate at sufficiently high KCl concentration, see H. G. BUNGENBERG DE JONG, H. L. BOOIJ and G. G. P. SAUBERT, Protoplasma 29, 536 (1938); viscosity of the elastic viscous oleate system at smaller KCl concentrations, see H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, these Proceedings **50**, 849 (1947).

[5]) We are aware that the calculated pH values may be systematically erroneous, because of difficulties inherent with the glass electrode at high pH. But they give at least an impression of the order of magnitude of the pH differences between the four series of experiments.

of n-hexylalcohol the G value corresponding to the maximum of the n curve remains practically the same when this n curve is shifted by n-hexylalcohol in the direction of smaller KCl concentrations.

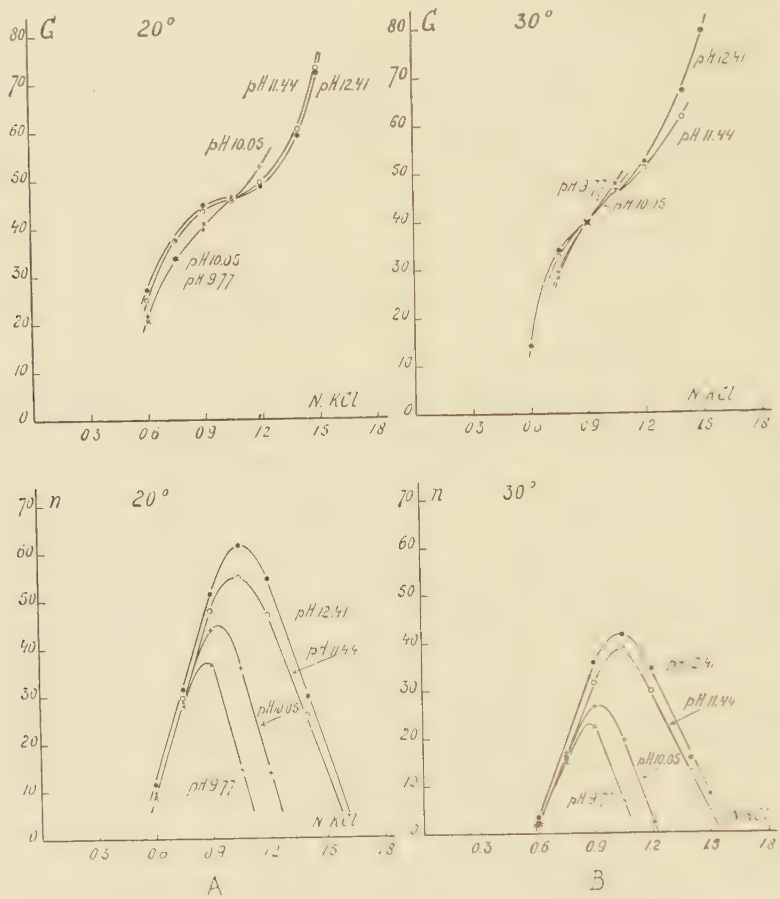

Fig. 1. Influence of the pH on the $G - C_{KCl}$ and $n - C_{KCl}$ curves at 20° and 30°.

In fig. 1, however, the G value corresponding to the maximum of the n curve decreases considerably when the n curve is shifted in the direction of smaller KCl concentrations, as a consequence of the lowering of the pH. See the survey below.

	temp.	pH 12.41	pH 11.44	pH 10.05	pH 9.77
KCl conc. (moles/l) corresp. to max. of the n curve	20 30°	1.06 1.03	1.04 1.02	0.95 0.93	0.87 0.87
n at the max. of the n curve	20° 30°	61.5 41.5	54.5 38.5	45 37	37 23
G at the max. of the n curve	20 30	46.5 44.5	45.5 44	42.5 41	39 38.5

This difference is easily explained if we consider that G depends on the oleate concentration. If n-hexylalcohol is added at a pH $\geq$ 12 the oleate concentration is not altered. When the pH is lowered oleic acid is produced at the cost of oleate. Thus the actual oleate concentration diminishes. As G has been shown to be approximately proportional to the square of the oleate concentration (see part III) a percentually small hydrolysis may therefore have a relative large effect on G.

3. *Influence of the temperature of the $G - C_{KCl}$ and $n - C_{KCl}$ curves and analogous curves with other salts.*

The experiments have been performed with completely filled 500 ml vessels. The endconcentration of Na-oleate (BAKER's) was 1.2 % and 0.05 N KOH was present to prevent partial hydrolysis of the oleate. The elastic systems were measured consecutively at 15°, 20°, 25°, 31.3° and 15° (the vessels remained standing overnight in a thermostate of that temperature whereby the measurements were performed the following day). In fig. 2 we give the results.

Comparing the two series of measurements at 15°, we see that the elastic properties have changed to a small extent during the four days in which we worked at higher temperatures (compare dotted curves in fig. 2). This change (probably due to a chemical alteration of the oleate) is, however, small compared with the difference in position of the curves

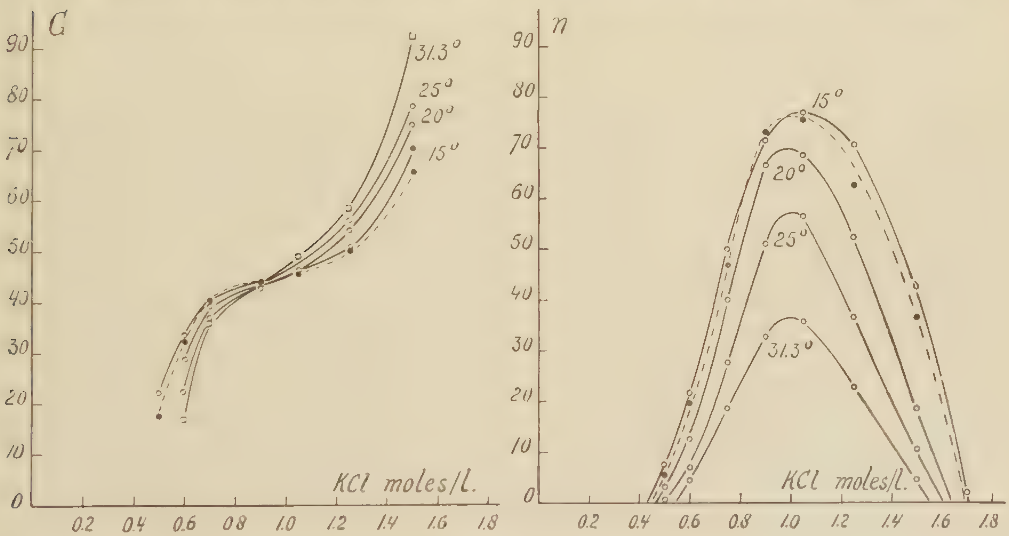

Fig. 2. Influence of the temperature on the G-C_{KCl} and n-C_{KCl} curves.

at 15°, 20°, 25° and 31.3°. Therefore these different positions of the G and n curves give expression to a reversible change of the elastic proporties with the temperature. We may formulate these changes as follows:

1. Dependent on the KCl concentration the change of the shear modulus with temperature has different signs. At increase of the temperature G

decreases at 0.5, 0.6 and 0.75 N KCl and G increases at 1.05, 1.25 and 1.5 N KCl. The G curves of different temperature intersect between 0.9 and 1.0 N KCl.

2. At all KCl concentrations n decreases (consequently the damping increases) at increase of the temperature. The maximum of the n curve becomes lower (compare the survey below) and the footpoints of the n curve draw nearer to one another (the left footpoint shifts to the right and the right footpoint to the left).

3. At increase of the temperature, the KCl concentration corresponding to that of minimum damping (maximum of the n curve) shifts slightly in the direction of smaller KCl concentrations. The shear modulus corresponding to the maxima of the n curves is, however, practically not changed by this shift of the maximum, to the left. Compare the survey below.

temp.	15°	20°	25°	31.3	15
KCl concen. (moles/l) corresp. to max. of the n curve	1.06	1.01	0.99	0.98	1.02?
n at the maxim. of the n curve	77	70	57	36	76
G at the maxim. of the n curve	46.3	45.1	46.4	46.6	45.2

The experiments in the sections 2 and 4 have been performed at two temperatures, so that the effect of the temperature could also be seen. Taking into account the reduced accuracy of these experiments with smaller vessels (110 ml), there is still no reason to doubt that the above three points hold very generally. Compare for instance fig. 4, in which this is evident for the six salts with different anions. If we had made analogous graphs for the remaining experiments in the sections 2 and 4, these three points would also have been plainly visible.

4. *Substitution of Cl or K in KCl by other ions.*

The right footpoint of the curve representing n as a function of the KCl concentration, lies just before the KCl concentration at which coacervation sets in. Supposing this to hold for other K salts too, the determination of their coacervation limits will give us the concentrations below which elastic systems may be expected. We started with a 4 % solution of BAKER's oleate containing 0.167 N KOH and made in test tubes, a series of mixtures according to the receipt: 3 ml stock oleate solution + x cc stock K salt solution + $(7-x)$ ml H_2O, which gives mixtures containing, apart from a variable concentration of the investigated K salt, the usual oleate concentration (1.2 %) and the usual KOH

concentration (0.05 N). After closing the tubes with rubber stoppers, the contents were mixed thoroughly by shaking; the test tubes were put in a thermostate of 25° and after standing overnight the volume of the coacervate layer was read. The results are represented in fig. 3, which gives

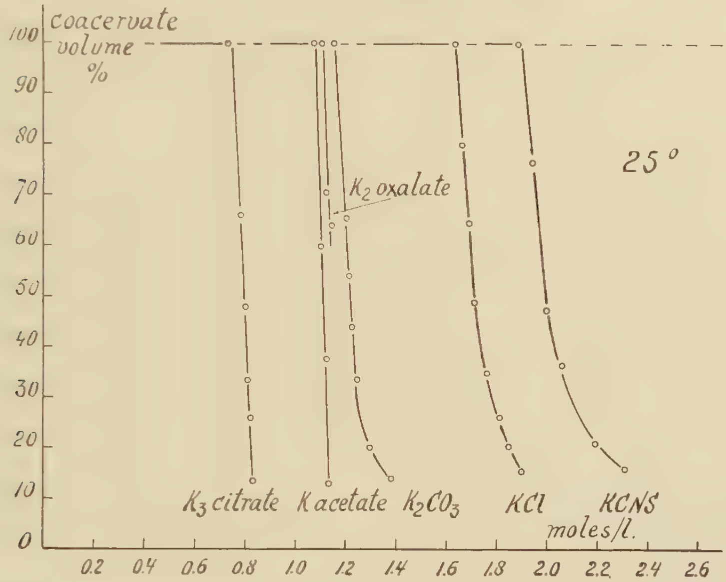

Fig. 3. Determination of the coacervation limit for six potassium salts at 25°.

the volume of the coacervate layer (in percents of the total volume of the mixture) as a function of the salt concentration (in moles/1).

The sequence of the curves in fig. 3 corresponds in the main to the lyotropic series of the anions.

With the same salts we investigated, at 20° and 30°, in completely filled 110 ml vessels, the elastic properties of the 1.2 % oleate systems containing 0.05 N KOH.

For the preparation of the mixtures in the series with potassium oxalate (because of its too small solubility), we could not start from the usual 2.4 % stock oleate solution, but had to use a more concentrated one, viz. a 4 % solution (containing 0.167 N KOH) of which 45 ml were mixed with x ml stock potassium oxalate solution + $(105 - x)$ ml H_2O. The end concentration of oleate and KOH is then the same as in the other salt series.

The results obtained with each of the salts apart, have been represented in the diagrams of fig. 4, which show that the typical elastic systems are not restricted to KCl but can also be realized with other K salts. The shape of the $G - C_{salt}$ curves and $n - C_{salt}$ curves is the usual one and in general the same effects of temperature as are discussed in section 3 hold here too.

Fig. 5 enables us to compare the relative positions of the G and n curves

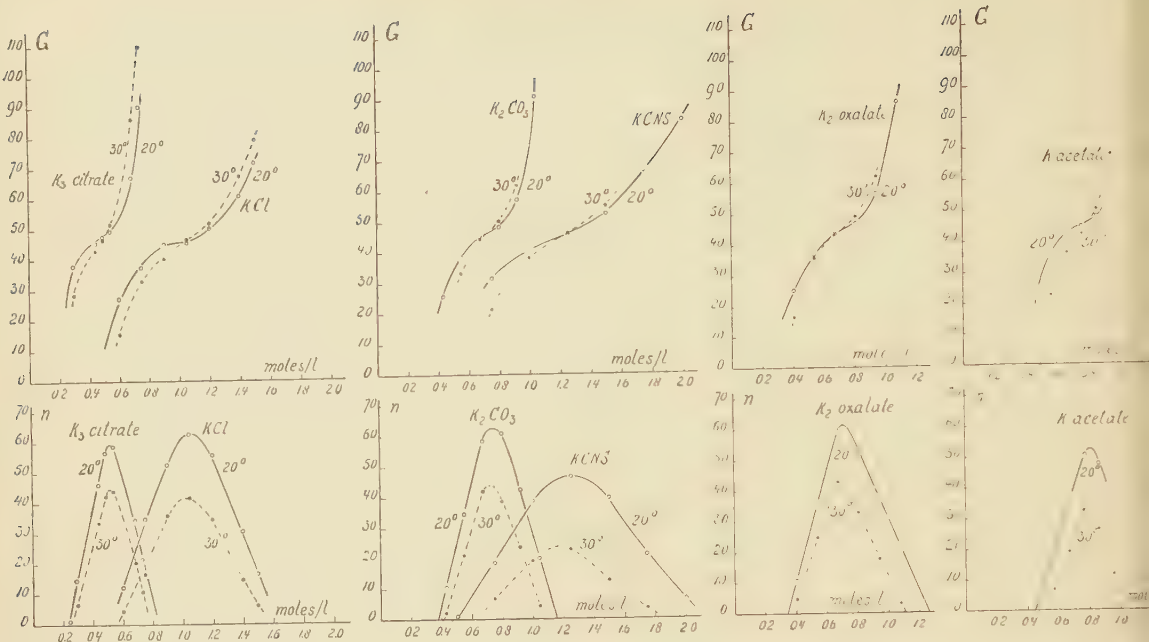

Fig. 4. $G - C_{\text{salt}}$ and $n - C_{\text{salt}}$ curves for six potassium salts at two temperatures.

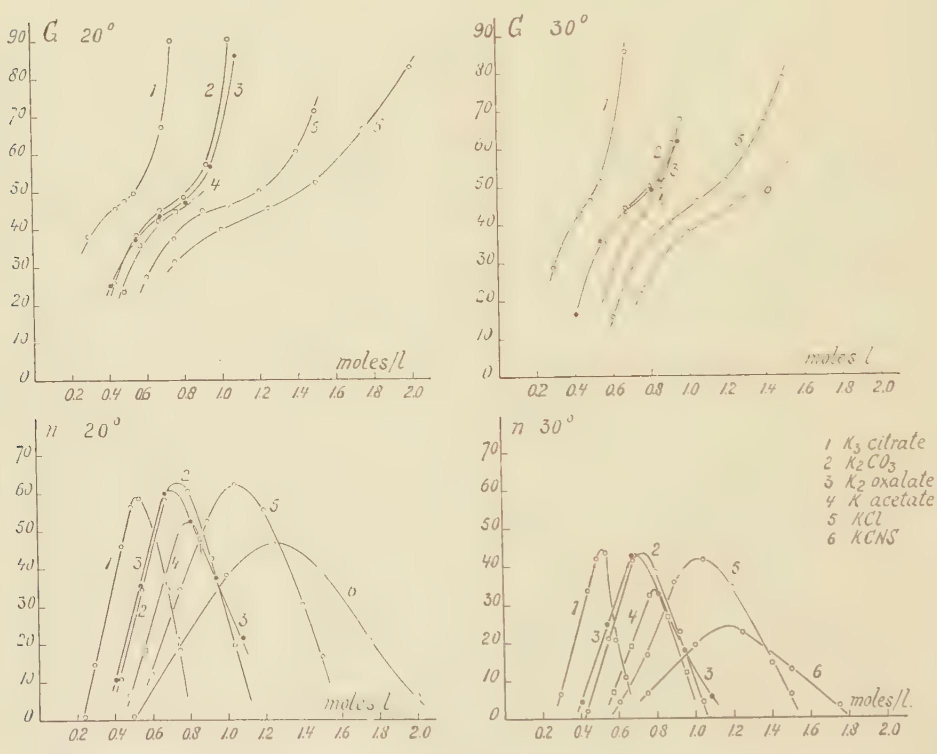

Fig. 5. $G - C_{\text{salt}}$ and $n - C_{\text{salt}}$ curves for six potassium salts at two temperatures.

with different K salts at the same temperatures. These diagrams show that the order in which the following G and n curves rank from left to right viz.

citrate $<$ carbonate $<$ chloride $<$ thiocyanate

is the same as found for the coacervation limit in fig. 3. The curves for acetate, oxalate and carbonate lie close together. Their relative position is not only different from that in fig. 3, but is even different in the upper and lower graphs of fig. 5.

The next survey gives the salt concentration of the maximum of the n curves, the values of n at the maxima and the values of G corresponding to these maxima:

	temper. °C	K_3-citrate	K_2-oxalate	K_2CO_3	K-acetate	KCl	KCNS
salt concentr. moles/l	20°	0.52	0.71	0.76	0.80	1.06	1.24
	30°	0.52	0.70	0.73	0.79	1.03	1.16
n	20°	59	61	63	52	63	47.5
	30°	44	43	43	34	42	24
G	20°	49	44	46.5	45.5	45.5	45
	30°	49	44	46.5	44.5	45	43.5

At the comparison of the numerical values of G and n in the above survey, we must take into consideration that the different salt series have not been investigated simultaneously and that for the series with oxalate another stock oleate solution had to be used (see above).

Therefore we may not consider as being with certainty different those G and n values which differ only a few units in G or in n.

We therefore get as a general impression that at the same temperature the n values at the maxima of the n curve are equal in the case of citrate, oxalate, carbonate and chloride, but that the acetate and the thiocyanate ion exert in addition, a lowering influence on the maximum of the n curve. The G values corresponding to the maxima of the n curves are in general also equal (apart from a too high value for citrate [6]) which may not be real).

The possibility to realize elastic oleate systems with NaCl proved to be restricted to a narrow tract of temperatures (in practice 25° — 35°). At too low temperatures (e.g. 15°, 20°) no stable elastic systems are formed because of transgression of the solubility of the Na-oleate ("curd"-forma-

[6]) We must add the remark that the elastic systems containing citrate were perfectly clear, whereas those with the other salts showed a marked tubidity. It looks as if citrate, by complex formation, removes a contaminating cation present in BAKER's oleate, which is responsible for the formation at room temperature of the above turbidity.

tion); at too high temperatures (e.g. 40° and higher) the elastic phenomena disappear because of the highly increased damping.

The actual experiments served at the same time the aim to investigate the influence of the partial replacement of KCl by NaCl. Four series of experiments (1.2 % oleate systems, completely filled 110 ml vessels) were made, one with only KCl, one with only NaCl and two, whereby for the preparation of the mixtures, stock solutions containing KCl + NaCl were used, in which the molar proportions were 2 : 1 and 1 : 2 respectively. The results have been plotted in fig. 6 in which the abscissa represent the total salt concentrations (moles NaCl/l + moles KCl/l). The curves for

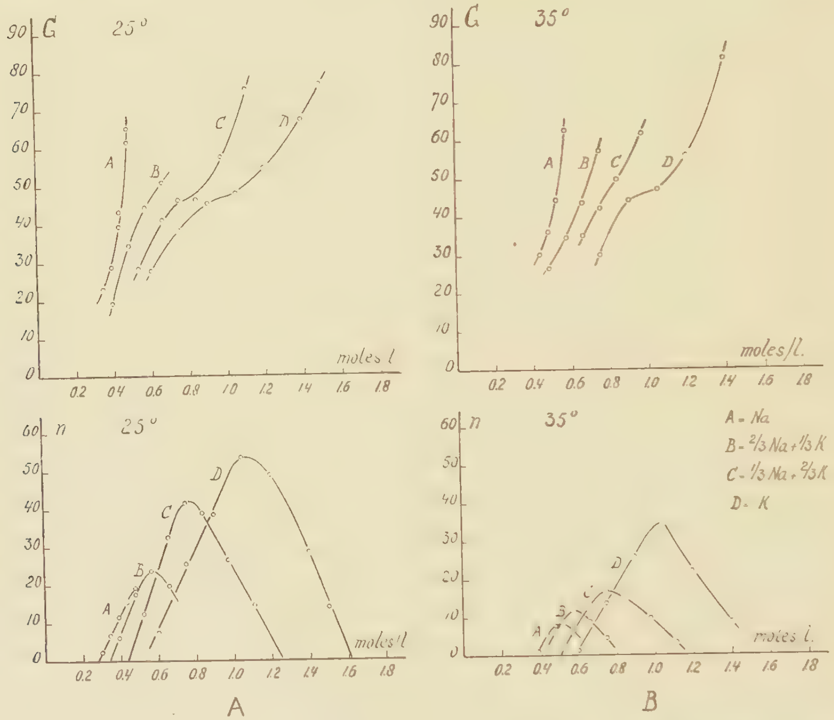

Fig. 6. $G - C_{salt}$ and $n - C_{salt}$ curves for KCl, NaCl and two mixtures of these salts ($K : Na = 2 : 1$ and $1 : 2$) at two temperatures.

NaCl in this figure show the difficulties announced above. At 25° curd formation sets in just at or just before the maximum of the n curve, at 35°, however, though the maximum of the n curve could be passed, the damping is now so high that there are only 8 turning points left. The difficulties are smaller in the next curve (Na : K = 2 : 1), the maximum of the n curve being realizable at 25°, though at higher total salt concentration curd formation still occurred.

The next survey gives the total salt concentrations corresponding to the maxima of the n curves, the values of n at these maxima and the values of G corresponding to the maxima of the n curves.

	temp.	NaCl	2 NaCl + 1 KCl	1 NaCl + 2 KCl	KCl
total salt concentration moles/l	25° 35°	? 0.51?	0.58 0.58	0.77 0.75	1.06 1.04
n	25° 35°	≥ 19 8	24 12	41.5 17	53.5 35
G	25° 35°	? 38?	45 35.5	46 42	48 46.5

We perceive from this survey that in substituting K by Na, both the salt concentration at the maximum of the n curve (= minimum damping) and the value of n at this maximum decrease considerably, whereas G corresponding to the maximum of the n curve also decreases, though much less pronounced.

Using the data of the above survey, the three diagrams of fig. 7 have been constructed. The diagrams A and B give the values of G and of $1/n$ (an approximate measure for Δ) as functions of the ratio KCl/KCl + NaCl.

Diagram C gives the concentration of KCl apart (abscissa) and of NaCl apart (ordinates), this being the KCl and NaCl present in the total salt concentration mentioned in the survey for the position of the maximum of the n curve (for the calculation we used here the averages of the values given for 25° and 35°).

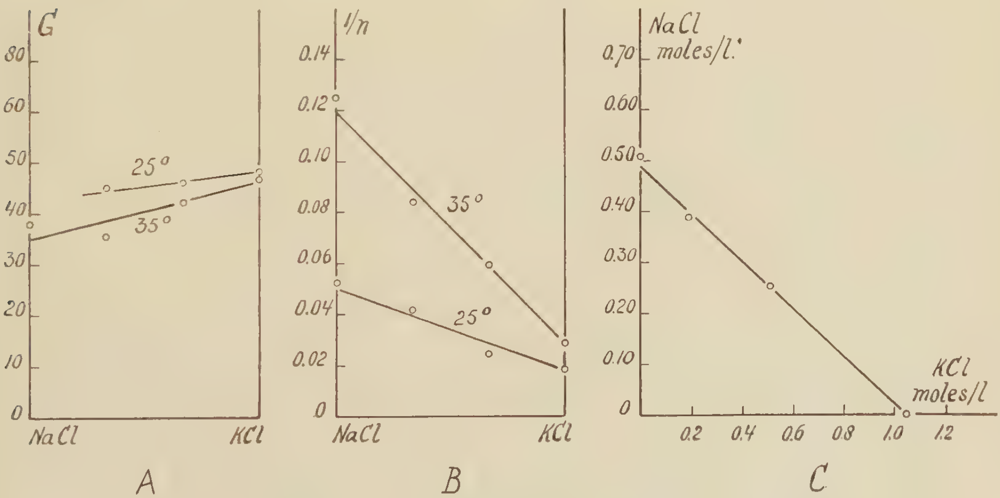

Fig. 7. Additive behaviour of KCl and NaCl in mixtures.

The three diagrams of fig. 7, suggest that in essential points KCl and NaCl behave additively in mixtures.

We may summarize the results obtained hitherto in this section as follows:

1. By substitution of Cl or of K in KCl by other ions the G and n curves are generally shifted in the direction of lower or higher concentrations.

2. In the case of substitution of Cl by other ions (leaving out of consideration here acetate and thiocyanate which show additional effects) the numerical values of G and n at characteristic points of the G curve (inflexion point) and n curve (maximum) are not altered.

3. In the case of substitution of K by Na, however, the above numerical values assume other values.

For a theory of the elastic, KCl containing oleate systems, the above difference between anions and cations seems to be of great importance.

We now turn to the question if elastic oleate systems may be obtained with the chlorides of the other alkalimetals. With LiCl this was not possible; already at small concentrations a fine precipitation was formed (somewhat similar to that with $CaCl_2$). With RbCl and CsCl, however, elastic systems could be obtained. The quantities of the latter salts at our disposal (a few grams) excluded to investigate those elastic systems in the usual way. By a micromethod [7]) we could determine approximately the range of saltconcentrations in which the elastic system is present. For comparison we determined this range for KCl with the same method too.

The following results were obtained at approx. 20°.

KCl	0.4	moles/l	— 1.7	moles/l
RbCl	0.55	,,	— 2.2	,,
CsCl	0.7	,,	— 2.5	,,

The values obtained with this method for KCl do not differ more than 0.05 moles/l from the values one obtains with the macromethod at the same temperature. For NaCl at 25° the elastic properties just begin at about 0.28 moles/l, and at 20° this limit will be still lower (compare footpoints to the left of the n curve for NaCl in fig. 6).

Thus, in the case of the alkalications the concentrations needed to obtain elastic systems increase in the order

$$Na < K < Rb < Cs$$

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[7]) For determination of both limits 100 mgr of the salt was weighted in a small glassvessel. With a drippingpipette we added increasing amounts of a 1.2 % Na-oleate solution (containing 0.05 N KOH). After each addition the contents were thoroughly mixed with a pointed glassrod and it was observed under the microscope if the system was still coacervated. Thus we determined the coacervationlimit, which was taken as the upperlimit of the region of elastic systems. These systems have the pecular property that by pulling out of them the glassneedle threads are formed; compare these Proceedings 50, 1227 (1947). At further successive additions of the oleate solution this property becomes at first pronounced, thereafter it diminished. As the lower limit of the region of elastic systems we took that added amount of oleate solution where we could just no longer observe this property.

From the number of drops added, the dropweight, the weighted quantity of salt, its spec. grav. and its molecular weight we calculated the approximate salt concentration given in the survey above for the two limits of the elastic systems.

We would expect Li at the beginning of this series, but, as already stated, the observation of this is prevented by precipitation.

5. *The $G - C_{KCl}$ and $n - C_{KCl}$ curve for potassium oleate prepared from pure oleic acid.*

In all previous experiments on the elastic behaviour of KCl containing oleate systems we used, because of practical and economical reasons, commercial Na-oleate preparations (see part X, section 7). Most experiments were performed with 1.2 % oleate systems, which is approximately 40 millimoles/1. Thus we never have worked with $Na^{\cdot}$ free systems, but always with such containing a mixture of $K^{\cdot}$ and $Na^{\cdot}$.

At the KCl (+ KOH) concentration corresponding to the minimum damping (MERCK's oleate 1.43 N KCl + 0.05 N KOH; BAKER's oleate 1.05 N KCl + 0.05 N KOH) the $Na^{\cdot}$ concentration is small compared to the $K^{\cdot}$ concentration, viz. the ratio Na/K + Na is 2.6 % and 3.5 % respectively. Taking into account the additive behaviour of KCl and NaCl found in section 4, one is inclined to conclude that this small fraction of Na ions will scarcely alter the elastic behaviour which is to be expected if instead of Na-oleate we had used K-oleate at the preparation of the elastic systems [8]).

This conclusion is based on the assumption, that the Na ions from Na oleate are freely interchangeable with K ions and thus no unforeseen effects are present.

Therefore, it seemed desirable to investigate if starting from oleic acid and using KOH to dissolve it, one obtains with KCl the usual $G - C_{KCl}$ and $n - C_{KCl}$ curves.

The actual experiments to control this, served another aim at the same time. The commercial oleate preparations which were used cannot be considered as being chemically pure, as they contain other fatty acids as well (saturated e.g. palmitate and unsaturated ones).

One may therefore ask if the elastic phenomena shown by these oleate preparations are not possibly due to for instance palmitate (BAKER's oleate contains relatively much palmitate, compare part X, section 7), instead of to the oleate itself.

We used for the following experiment chemically pure oleic acid [9]).

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[8]) In the case of a 4 % substitution of K by Na one calculates a decrease of G from $48 \to 47.8$ dyne/cm² and a shift of the KCl concentration corresponding to the minimum damping from $1.06\ N \to 1.03\ N$, which are differences of the same order as the experimental errors or even smaller. The influence of a 4 % substitution would only be detectable in the change of n (at the maximum of the n curve), viz. at 25° from $53.5 \to 49.9$ and at 35° from $35 \to 30.8$ (calculated from the additivity of $1/n$).

[9]) We are much indebted to Mr. C. DE BOCK for preparing a sample of chemically pure oleic acid.

Commercial oleic acid was freed by mercuri acetate from saturated fatty acids, the resulting "oleic acid" distilled in vacuo, the Li-salts of the resulting "oleic

It was dissolved in KOH and to the stock solution of K-oleate, the usual excess of KOH was added to ensure a sufficiently high pH ($>$ 12).

As only a relatively small quantity of pure oleic acid was available and part of it had to be used for more fundamental investigations which are to be published in a next part of this series, we had to use as economical a method as possible. We thus used the so called simplified technique with half filled 110 ml vessels, described in part X. Fig. 8 gives the result at

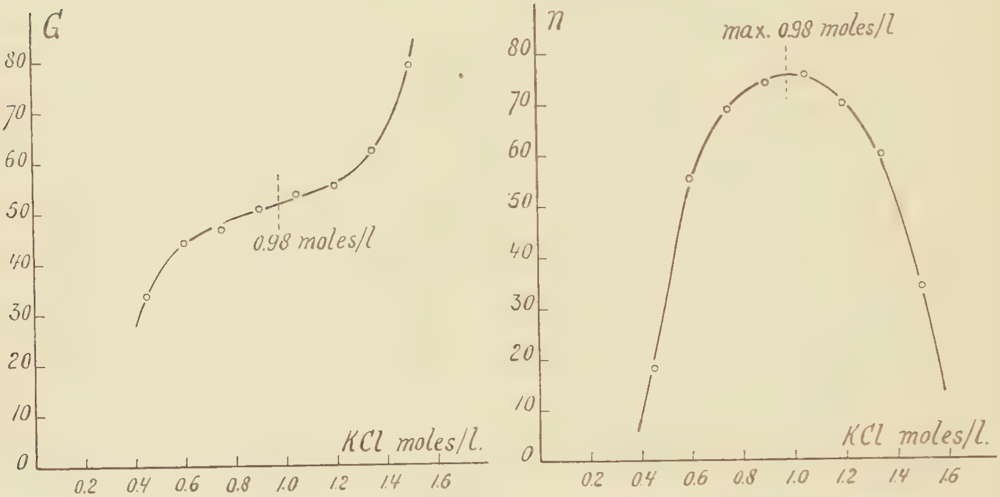

Fig. 8. $G - C_{KCl}$ and $n - C_{KCl}$ curves for potassium oleate prepared from chemically pure oleic acid at 17.8°.

17.8° for an oleate concentration of 37.5 millimoles/l. The $G - C_{KCl}$ and $n - C_{KCl}$ curves which were obtained have the usual form and quantitatively do not differ so very much of the analogous curves with the (palmitate containing) Na-oleate from BAKER (the KCl concentration corresponding to the minimum damping is ± 0.98 N: G at this KCl concentration is 50; n at the maximum of the n curve is 74.3).

We may conclude from this experiment that the elastic phenomena studied hitherto, whereby commercial Na oleate preparations were used, are not due to impurities (e.g. palmitate) in these preparations, nor were they bound to appear only in the presence of an amount of Na ions, equivalent to the oleate ions.

6. *Summary.*

1. The influence of the pH, the temperature and the substitution of each of the ions of KCl by others, on the curves representing G (shear

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acid" recrystallized three times from 50 % ethanol to free it from higher unsaturated fatty acids and the oleic acid which was set free was finally distilled in vacuo. Melting point 13°. (See S. H. BERTRAM, Rec. Trav. Chim. **46**, 397 (1927) and H. W. SCHEFFERS, Rec. Trav. Chim. **46**, 293 (1927).

modulus) and n (approximate measure for $1/\Lambda$; Λ = logarithmic decrement) as functions of the KCl (or in general: of the salt) concentration, has been investigated.

2. The effects, in consequence of the lowering of the pH, can be explained by partial hydrolysis of the oleate, the undissociated oleic acid which is formed exerting a KCl-sparing action on the elastic oleate system.

3. Increase of the temperature has only a slight influence on the KCl concentration corresponding to the maximum of the n curve, decreases G at KCl concentrations lower than the above-mentioned one and increases G at KCl concentrations higher than the above-mentioned one. At all KCl concentrations n decreases considerably at increase of temperature.

4. Other potassium salts (citrate, oxalate, acetate, carbonate, thiocyanate) give analogous $G - C_{\text{salt}}$ and $n - C_{\text{salt}}$ curves, which show practically the same values of G and of n (n is only lower with acetate and thiocyanate, which is attributed to a secondary effect) at the maximum of the n curve. The sequence of the curves in $G - C_{\text{salt}}$ and $n - C_{\text{salt}}$ diagrams is in the main that of the lyotropic series of the anions.

5. By substituting K (in KCl) partially or wholly by Na, the salt concentration corresponding to the minimum damping is displaced towards lower values, the value of G at the maximum of the n curve is sensibly decreased and the value of n at this maximum is strongly decreased. In mixtures KCl and NaCl behave additively.

6. Also with RbCl and CsCl elastic systems may be obtained, not, however, with LiCl. The minimum concentrations needed, as well as the coacervation limits (not realisable with Na) increase in the order

$$\text{Na} < \text{K} < \text{Rb} < \text{Cs}$$

7. Starting from chemically pure oleic acid, and using KOH to dissolve it, we obtained with KCl, typical elastic systems. This shows that the elastic systems obtained from commercial Na oleate preparations are not due to the presence of an equivalent amount of Na ions nor of other, contaminating fatty acids, e.g. palmitate.

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MATHEMATICS

INFINITELY NEAR POINTS

BY

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(Communicated at the meeting of June 24, 1950)

1. *Introduction.*

It was MAX NOETHER [1]), the father of algebraic geometry, who first introduced the notion of infinitely near points (or, as we now say, neighbour points) of a point P in the plane or on an algebraic surface V. NOETHER's definition is based upon the use of quadratic Cremona transformations, but he does not clearly state what sort of thing an infinitely near point is, nor does he prove that the notion is independent of the choice of the Cremona transformations.

Later, several authors have given "intrinsic" definitions of infinitely near points, i.e. definitions within V. The theories of ENRIQUES [2]) and ANCOCHEA [3]) are based upon the use of power series, whereas ZARISKI [4]) makes use of polynomical ideals and valuations. ANCOCHEA's theory holds in fields of arbitrary characteristic, and so does the theory of ZARISKI, according to a remark of CHEVALLEY [5]).

However, the algebraic expansions used in these theories are rather complicated, whereas NOETHER's original theory was very simple.

The right way to obtain a simple intrinsic definition of neighbour points was celarly indicated by SEVERI in his Trattato di geometria algebrica I, p. 322. I translate SEVERI's own words, because they constitute the best introduction to my paper:

"We are at last in a position to free the notion of infinitely near singularities from the quadratic transformations, which are only an easy means of defining them, and to give the concept of composition of a singularity a substantial geometric contents.

In fact, if a branch γ has only two successive multiple points, they can be defined by means of the intersection multiplicities with linear branches; if a branch γ has three sucessive multiple points, the first two being defined as before, the third may be defined by the intersection

1) M. NOETHER, Nachr. Ges. Wiss. Göttingen, (1871), p. 267; Math. Ann. 9, 166 (1875) and 23, 311 (1883).

2) F. ENRIQUES and O. CHISINI, Teoria geometrica delle equazioni e funzioni algebriche, Vol. II, Libro 4 (Bologna 1918).

3) G. ANCOCHEA, Courbes algébriques. Acta Salamanticensia I § 8 (1946).

4) O. ZARISKI, Polynomial ideals defined by infinitely near points. Amer. J. Math. 60, 151 (1938).

5) See Ancochea, l.c. 3), p. 9, note (5).

multiplicity with a branch of first or second order osculating γ; if γ has four successive multiple points, the first three being defined as before, the fourth may be defined by the intersection multiplicity with a branch of first, second or third order containing these four points, and so on".

I am quite sure I have read these words long ago, but I had forgotten them. Only after having rediscovered the whole theory and written out all the proofs in full, I returned to the Trattato, looking for historical references, and there I found the passage quoted above.

Still, it seems desirable to publish the definitions and proofs in full, in order to show: first, how extremely simple they are, secondly, that they are valid in fields of arbitrary characteristic, thirdly, that the restriction to surfaces is unnecessary. The theory will be developed for the neighbourhood of a simple point on a variety V of arbitrary dimension d.

Instead of considering, as SEVERI does, intersection multiplicities between two curve branches, we make use of intersection multiplicities between curve branches and divisors, i.e. chains of dimension $(d-1)$ on V. A point on a surface is uniquely defined by the set of all divisors passing through it. Just so, a neighbour point will be defined by the set of all divisisors passing through it, and these in turn will be characterized by their intersection multiplicities with certain curve branches.

The definition is recursive: we first define first order neighbour points P_1 of a point P, next second order neighbour points, etc. The proof of the properties used in the definition makes use of an elementary birational transformation, which transforms neighbour points of order h into neighbour points of order $(h-1)$. However, the definition of the notions "neighbour point" and "multiplicity" will not depend on these transformations.

2. *Definition of successive neighbour points.*

Let P be a simple point on an indivisible variety V of dimension d. We want to define the successive neighbour points of P, of finite orders, $P_1, P_2, \ldots$, and the multiplicity of a divisor or of a curve branch at such a neighbour point.

A *divisor* D on V is a sum of indivisible subvarieties of dimension $(d-1)$, in which sum a term may be counted more than once. A divisor on a surface e.g. may be a curve counted twice plus another curve counted once.

Since we are concerned with local properties of the point P only, we agree to leave out of account all parts of D not containing P. If we do this, every divisor D may be considered as the intersection of V with a hypersurface H, i.e. every divisor on V is defined by just one equation $H=0$. We may choose for H the projecting cone of D from a generic S_{n-d-1} of S_n [6]).

[6]) See e.g. my paper Zur algebraischen Geometrie 14, § 5, Math. Ann. 115, 619.

A *curve branch* β on V with P as origin is defined by a set of power series:

(1) $$x_k(t) = p_k + q_k t + z_k t^2 + \dots \qquad (k = 0, 1, \dots, n)$$

such that $x_0(t), \dots, x_n(t)$ satisfy identically in t the equations of V and that the initial values $x_k(0) = p_k$ are just the coordinates of P. The trivial case, in which the $x_k(t)$ are proportional to p_k, will be left out of account, for in this case we have only one point P instead of a curve branch.

We may restrict ourselves to branches of algebraic curves; it makes no difference. If we wanted to stress the elementary algebraic character of our theory, we might even assume the $x_k(t)$ to be polynomials in t.

The *multiplicity of intersection* $(\beta, D)_P$ of a divisor D with a curve branch β at P is defined by substituting the power series (1) into the equation $H = 0$ defining D. If the resulting power series $H(x_0(t), \dots, x_n(t))$ begins with a term t^μ, the multiplicity $(\beta, D)_P$ is μ. If the branch β lies on D, we put $\mu = \infty$. SEVERI has shown [7]) that the resulting multiplicity $\mu = (\beta, D)_P = (\beta, H)_P$ is independent of the choice of the hypersurface H, so we may choose for H a generic projecting cone of D.

The *multiplicity* $(\beta)_P$ of the branch β at the point P is defined as the multiplicity of intersection of β with a generic hyperplane (or hypersurface of any order) passing through P. If P is chosen as the origin of coordinates:

$$p_0 = 1, \text{ all other } p_k = 0$$

we may assume the set of power series to have the special form

$$x_0(t) = 1 + q_0 t + \dots$$
$$x_k(t) = u_k t^r + v_k t^{r+1} + \dots$$

If the terms with $t, t^2, \dots, t^{r-1}$ are missing in all power series $x_k(t)$, but the term t^r not, the multiplicity $(\beta)_P$ is just r. If $r = 1$, the branch β is *simple* at P. The ratios of $u_1, \dots, u_n$ determine the *tangent* of the branch at P.

Similarly, the multiplicity $(D)_P$ of a divisor D at P is defined as the multiplicity of intersection of D with a simple branch in a generic direction at P. The multiplicity of D at P is the same as that of the generic projecting cone H. If we choose P as the origin of coordinates and expand the form H according to terms of increasing order in $x_1, \dots, x_n$:

$$H(x_0, \dots, x_n) = x_0^{h-s} f_s(x_1, \dots, x_n) + x_0^{h-s-1} f_{s+1}(x_1, \dots, x_n) + \dots$$

the order s of the lowest terms is equal to the multiplicity $(D)_P$.

Let r be the multiplicity of β at P and s that of D at P:

$$r = (\beta)_P, \qquad s = (D)_P.$$

We now prove that the intersection multiplicity μ is at least rs:

$$\mu = (\beta, D)_P \geqslant rs.$$

7) F. SEVERI, Abh. Math. Sem. Hamburg 9 (1933).

Proof: If P is the origin of coordinates the power series (2) for $x_k(t)$ will start with the terms t^r, and the form $H(x)$ starts with terms of order s in $x_1, \ldots, x_n$. So $H(x_0(t), \ldots, x_n(t))$ starts with terms of order s in

$$x_1(t), \ldots, x_n(t),$$

and these are of order rs in t. This proves the proposition.

A closer examination of the proof shows that the intersection multiplicity μ exceeds rs if and only the tangent direction $(u_1, \ldots, u_n)$ of β satisfies the equation of the tangent cone $f_s(x_1, \ldots, x_n) = 0$ of H, in other words of β and D have a tangent at P in common. Hence we may define:

Definition A1. A first order neighbour point P_1 of P is the totality of all divisors D having with a given branch β an intersection multiplicity larger than rs, where $r = (\beta)_P$ and $s = (D)_P$.

Thus, to every tangent direction of V at P corresponds just one neighbour point P_1 of P.

We further define:

Definition B1. A branch α passes through P_1, if it has with every divisor D of the set defining P_1 an intersection multiplicity larger than qs, where $q = (\alpha)_P$ and $s = (D)_P$.

If this is the case, the branch α has the same tangent as β. Hence we have

Theorem C1. Every branch α passing trough P_1 can be used instead of β in definition A1 and defines just the same neighbour point P_1.

Next we define:

Definition D1. The multiplicity $r_1 = (\beta)_{P_1}$ of a branch at P_1 is the smallest excess of any intersection multiplicity $(\beta, D)_P$ over rs for all divisors D belonging to P_1:

$$r_1 = \min \{(\beta, D)_P - rs\} \text{ for all } D \text{ belonging to } P_1.$$

Definition E1. The multiplicity

$$s_1 = (D)_{P_1}$$

of a divisor D at P_1 is the smallest excess of any intersection multiplicity $(\beta, D)_P$ over rs for all possible choices of a branch β passing through P_1:

$$s_1 = \min \{(\beta, D)_P - rs\} \text{ for all } \beta \text{ through } P_1.$$

From these definitions we shall prove:

Theorem F1. The intersection multiplicity of β and D at P is at least equal to

$$rs + r_1 s_1 \tag{3}$$

$$\begin{aligned} r &= (\beta)_P & \qquad r_1 &= (\beta)_{P_1} \\ s &= (D)_P & \qquad s_1 &= (D)_{P_1} \end{aligned}$$

The proof will be given in the next section. The same methods of proof may also be used to prove C1 without making use of tangents.

Taking A1 — F1 for granted, we now proceed to the definition of second order neighbour points.

Definition A2. A second order neighbour point P_2, successor of P_1, is the totality of all divisors D belonging to P_1 and having with a given branch β passing through P_1 an intersection multiplicity larger than the expression (3).

Definition B2. A branch α passes through P_2, if it passes through P_1, and has with every divisor D of the set defining P_2 an intersection multiplicity larger than $qs + q_1 s_1$, where

$$q = (\alpha)_P, \; q_1 = (\alpha)_{P_1}, \text{ etc.}$$

On the basis of these definitions, we shall prove

Theorem C2. Every branch α passing through P_2 can be used instead of β in definition A_2 and defines just the same neighbour point P_2.

Definition D2. The multiplicity

$$r_2 = (\beta)_{P_2}$$

of a branch β at P_2 is the smallest excess of any intersection multiplicity $(\beta, D)_P$ over the expression (3) for all divisors D belonging to P_2:

$$r_2 = \min \{(\beta, D)_P - (rs + r_1 s_1)\} \text{ for all } D \in P_2$$

Definition E2. The multiplicity

$$s_2 = (D)_{P_2}$$

of a divisor D at P_2 is the smallest excess of any intersection multiplicity $(\beta, D)_P$ over the expression (3) for all branches β passing through P_2:

$$s_2 = \min \{(\beta, D)_P - (rs + r_1 s_1)\} \text{ for all } \beta \text{ through } P_2.$$

On the basis of these definitions, we shall prove:

Theorem F2. The intersection multiplicity of β and D at P is at least equal to

$$rs + r_1 s_1 + r_2 s_2 \tag{4}$$

where

$$r = (\beta)_P, \; r_1 = (\beta)_{P_1}, \; r_2 = (\beta)_{P_2}$$
$$s = (D)_P, \; s_1 = (D)_{P_1}, \; s_2 = (D)_{P_2}$$

Proceeding in the same way, we may set up definitions A3 and B3, theorem C3, definitions D3 and E3 and theorem F3, etc.

3. *The elementary transformation.*

For the proofs we make use of a birational transformation of V into V' having the following properties:

a) The transformation $V \to V'$ is regular [7a]) everywhere except at P.

[7a]) A rational transformation is called regular at Q, if the denominators of the rational functions defining the transformation do not vanish at Q.

b) The inverse transformation $V' \to V$ is regular everywhere on V'.

c) The point corresponds to a $(d-1)$ dimensional indivisible subvariety E' of V' without singularities.

d) The tangent directions at P are in one-to-one correspondence to the points of E', in the following sense: all curve branches β on V having a certain tangent at P are transformed into branches β' having a point P' of E' in common.

e) All points of E' are simple on V' [8]).

Such a transformation is always possible. Take e.g. for P the origin of coordinates $(1, 0, \ldots, 0)$. Every point $X \neq P$ of V can be projected from P upon the plane $x_0 = 0$. Let Y be the projection. If $(x_0, x_1, \ldots, x_n)$ are the coordinates of X, the coordinates of Y will be

$$y_1 = x_1, \ldots, y_n = x_n.$$

The pairs (X, Y) are in a birational correspondence to the points X of V. The correspondence $X \to (X, Y)$ is regular for $X \neq P$, the inverse correspondence is regular without exception. If X tends to P on a curve branch β, the limiting position of Y is determined by the tangent of the curve branch.

To see this, we write, as before, the power series $x_k(t)$ of the curve branch in the form (2). Since $y_1, \ldots, y_n$ are proportional to $x_1, \ldots, x_n$, we may compute $y_1(t), \ldots, y_n(t)$ by dividing $x_1(t), \ldots, x_n(t)$ by t^r. We obtain

$$\text{(5)} \qquad \begin{cases} x_0(t) = 1 + q_0 t + r_0 t^2 + \ldots \\ x_k(t) = u_k t^r + v_k t^{r+1} + \ldots \qquad (k = 1, \ldots, n) \end{cases}$$

$$\text{(6)} \qquad y_k(t) = u_k + v_k t + \ldots \qquad (k = 1, \ldots, n).$$

These power series define the transformed curve branch β'. The origin of β' is given by

$$x_0(0) = 1, \text{ all other } x_k(0) = 0$$

and

$$y_k(0) = u_k$$

so the tangent direction $(u_1, \ldots, u_n)$ determines in fact the limiting position $Y(0)$ of Y.

The tangents u all lie in the tangent space S_d of V at P, because P was supposed to be a simple point of V [9]). So the corresponding points Y lie in the intersection S_{d-1} of S_d with the hyperplane $x_0 = 0$. This S_{d-1} is a variety E' without singularities. Hence properties a) — d) hold.

To prove e), we consider any point pair (X, Y) of V'. We may suppose that $x_{d+1}, \ldots, x_n$ are regular functions of $x_1, \ldots, x_d$ in a neighbourhood

[8]) A more general type of "elementary transformations", in which a variety takes the place of our point P, has been introduced by L. DERWIDUÉ in his reduction of singularities, Bull. Acad. Roy. Belgique (5) 35, 880 (1949).

[9]) Cf. my Einfuhrung algebr. Geom. § 40, p. 173.

of P [10]), and that $y_1 \neq 0$ for the particular direction Y. Then, in the neighbourhood of the pair (X, Y), we may put $x_0 = 1$ and $y_1 = 1$, and take $x_1, y_1, \ldots, y_d$ as new local parameters $z_1, z_2, \ldots, z_d$.

We then have

$$\begin{array}{ll} x_0 = 1 & \\ x_1 = z_1 & y_1 = 1 \\ x_2 = z_1 z_2 & y_2 = z_2 \\ \ldots & \ldots \\ x_d = z_1 z_d & y_d = z_d \end{array}$$

The remaining coordinates $x_{d+1}, \ldots, x_n$ and $y_{d+1}, \ldots, y_n$ too are regular functions of $z_1, \ldots, z_d$. So all coordinates x_i and y_k are regular functions of d among themselves, which proves that P' is a simple point of V'.

We have seen already, how curve branches β are transformed into curve branches β' on V'. We now study the transormation of divisors D.

A divisor D on V may be defined (locally) by the equation of the projecting cone $H(x) = 0$. Now in the space of point pairs (X, Y), the same equation $H(x) = 0$ defines again a hyperspace H'. The complete intersection C' of H' with V' contains all pairs (X, Y) such that X is on D.

Now suppose that D contains the fundamental point P. Then C' contains the divisor E', maybe even counted a number of times, say s times. Writing

$$C' = sE' + D' \tag{7}$$

we call C' the *total transform* of D, and D' the *reduced transform* of D.

We now prove the following properties:

f) The number s in (7) is equal to the multiplicity $(D)_P$.

g) If a branch β has multiplicity r in P, the transformed branch β' will have intersection multiplicity r with E' at P'.

h) The intersection multiplicity $(\beta, D)_P$ is diminished by rs by the transformation:

$$(\beta, D)_P = rs + (\beta', D')_P$$

The properties f) g) h) derive from one fundamental formula, viz.

$$(\beta, D)'_P = (\beta' C')_{P'} \tag{8}$$

(8) holds in the most general case of a transformation $V \to V'$ having properties a — e), but for our purpose it will be sufficient to prove (8) for the special case of the transformation $X \to (X, Y)$ defined above.

The left side of (8) is computed by substituting the power series $x_k(t)$ of β into the equation $H = 0$ of D. The right side of (8) is computed by substituting the power series $x_k(t)$ and $y_k(t)$ of β into the equation $H = 0$ of C'. But the latter equation does not contain y_k, so we have only to substitute $x_k(t)$ into $H = 0$. So (8) is trivial.

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[10]) Cf. my paper Math. Z. 51, 497 (1948).

From (7) and (8) we have at once

$$(\beta, D)_P = s \cdot (\beta', E')_{P'} + (\beta', D')_{P'} \tag{9}$$

We first take for β a branch not tangent to D and having a simple point at P. Then the left-hand side of (9) is just the multiplicity $(D)_P$; the second term on the right is zero and the coefficient of s is 1. So we get from (9)

$$(D)_P = s \tag{10}$$

which proves f).

Next we take for β an arbitrary branch and for D a generic hyperplane section of V containing P. We then find

$$r = (\beta', E')_{P'} \tag{11}$$

which proves g).

Substituting (11) into (9), we find

$$(\beta, D)_P = rs + (\beta', D')_{P'} \tag{12}$$

which prove h).

Incidentally, the last formula yields a new proof of the inequality

$$(\beta, D)_P \geqq rs$$

Moreover, we see that (β, D) exceeds rs if and only if β' and D' have the point P', origin of β', in common. So the totality of all divisors D, for which $(\beta, D)_P$ exceeds rs, is transformed into the totality of all divisors D' passing through P', in other words:

The first order neighbour point P_1 of P defined by the branch β is transformed into the point P' of V', origin of the branch β'.

According to (12), the excess of $(\beta, D)_P$ over rs is just the intersection multiplicity $(\beta', D')_{P'}$. For a given β', the minimum value of this multiplicity is $(\beta')_{P'}$, and for a given D', the minimum is $(D')_{P'}$. Comparing this with definitions D1 and E1, we find at once.

$$r_1 = (\beta)_{P_1} = (\beta')_{P'}$$

and

$$s_1 = (D)_{P_1} = (D')_{P'}$$

in other words:

The multiplicity of a first order neighbour point P_1 on a branch or on a divisor is equal to the multiplicity of the corresponding point P' on the transformed branch or divisor.

For arbitrary β' and D' passing through P' the intersection multiplicity is at least equal to the product of the multiplicities of β' and D' at P':

$$(\beta', D')_{P'} \geqq r_1 s_1 \tag{13}$$

Combining this with (12), we find at once

$$(\beta, D)_P \geqq rs + r_1 s_1 \tag{14}$$

which proves Theorem F1.

The sign $>$ holds in (14) only if it holds in (13), i.e. (according to definition B1) if D' belongs to the first neighbour point P'_1 of P' on the branch β'.

Combining this with definition A2, we find:

The second order neighbour point P_2 on the branch β is transformed into the first order neighbour point P'_1 of P' on the branch β'.

If another branch α passes through P_2 according to definition B2, the transformed branch α' will pass through P_1, by the same reasoning. By theorem C1, the branch α' defines exactly the same first order neighbour point P'_1 as the branch β' does. Transforming back by the uniform inverse transformation, we conclude that the branch α defines on V the same second order neighbour point P_2 as the branch β does. The two sets of divisors on V defining P_2 on α and on β are identical, because their transforms on V' are identical. This proves theorem C2.

From (12) we have

$$(\beta, D)_P - (rs + r_1 s_1) = (\beta', D')_{P'} - r_1 s_1 \tag{15}$$

Keeping β fixed and varying D, we see that the minimum of the left-hand side is equal to the minimum of the right-hand side, which means

$$(\beta)_{P_2} = (\beta')_{P'_1} \text{ or } r_2 = r'_1$$

In the same way, we find by varying β

$$(D)_{P_2} = (D')_{P'_1} \text{ or } s_2 = s'_1$$

By theorem F1, we have

$$(\beta', D')_{P'} \geqq r's' + r'_1 s'_1 = r_1 s_1 + r_2 s_2$$

Adding rs on both sides, we find

$$(\beta, D)_P \geqq rs + r_1 s_1 + r_2 s_2$$

which proves theorem F2.

In exactly the same way, theorem C3 may be derived from C2, and F3 from F2, by means of the transformation $V \to V'$. And so on until we obtain, after h steps, theorems Ch and Fh.

4. NOETHER's *formula*.

For the case of plane curves, NOETHER has derived a formula expressing the intersection multiplicity as a sum of products of multiplicities of both curves or curve branches in successive neighbour points. This formula may be generalized at once to d dimensions.

NOETHER's *formula. If β and D have $P, P_1, P_2, \ldots, P_h$ in common, but not P_{h+1}, we have*

$$(\beta, D)_P = rs + r_1s_1 + \ldots + r_hs_h$$

If, on the other hand, β and D have P and an infinite series of neighbour points $P_1, P_2, \ldots$ in common, the intersection multiplicity is ∞, i.e. β lies on D.

The first part follows from Theorem Fh and definition A$(h+1)$. Suppose e.g. $h=1$. Theorem F1 yields

$$(\beta, D)_P \geqq rs + r_1s_1$$

and definition A2 says that the sign of equality holds unles D belongs to the second order neighbour point P_2.

If D and β have an infinite series $P, P_1, P_2, \ldots$ in common, we have by Theorem Fh:

$$(\beta, D)_P \geqq rs + r_1s_1 + \ldots + r_hs_h > h$$

since all terms are 1 at least. This implies

$$(\beta, D)_P = \infty$$

which is possible only if β lies on D.

ANTHROPOLOGY

DE STAMBOOM DER VOGELS

DOOR

J. F. VAN BEMMELEN

(Communicated at the meeting of June 24, 1950)

In de jaren 1890 tot 1906 toen ik als leeraar aan Gymnasia en Hoogere Burgerscholen de „natuurlijke historie” had te onderwijzen, trachtte ik wel eens mijn leerlingen de beteekenis van het begrip specialisatie duidelijk te maken, door hun de vraag voor te leggen, wat aan de klasse der zoogdieren veranderd zou moeten gedacht worden, om haar gelijkwaardig te maken met die der vogels. Het antwoord waarop ik hoopte, en dat ik ook een enkele maal van belangstellende toehoorders mocht ontvangen, luidde dat men zich dan alle orden der zoogdieren uitgestorven moest denken op die der vleermuizen na, terwijl deze laatste zich in alle richtingen nog verder hadden gedifferentieerd, zoodat sommige het vliegen weer hadden afgeleerd en enkel waren gaan loopen, andere gaan zwemmen, of springen, enkele zelfs gaan klimmen of graven.

Omgekeerd kan men aan anderen of zichzelf de vraag voorleggen, welke voorstelling men zich mag maken van de voorouders der hedendaagsche vogels, vooral in verband met hun klaarblijkelijke verwantschap met de reptielen. Deze voorstelling kan men alleen baseeren op hetgeen wij sedert 1861 weten van de organisatie van de hagedisstaartvogel (Archaeopteryx) waarvan een tweetal exemplaren in de lithographische steen van EICHSTÄDT zijn ontdekt, die tot de Bovenste of Witte Jura behoort, aangezien noch uit de Trias, noch uit daaraan voorafgaande perioden fossielen bekend zijn, die tot de klasse der vogels gerekend zouden kunnen worden. Men mag echter met zekerheid aannemen, dat in het Triastijdperk voorloopers der vogels bestaan hebben. Toevalligerwijze heeft de term voorloopers hier een dubbelzinnige beteekenis, daar het zeer waarschijnlijk is, dat die voorouders in het oudmesozoïsche tijdperk nog niet konden vliegen, maar alleen konden loopen en springen. Zij liepen vermoedelijk uitsluitend op hun achterpooten, en gebruikten hun voorste ledematen alleen om zich vast te grijpen bij het bereiken van een boomtak op het eind van een grooten sprong. In verband met deze dubbele wijze van voortbeweging: loopen op den grond en springen van tak tot tak, pasten zich de achterpooten uitsluitend aan de beweging op vaste onderlaag aan, en vervormden zich tot den toestand zooals wij dien bij de hedendaagsche vogels, maar ook reeds bij Archaeopteryx aantreffen. Men mag dus aannemen, dat de zeer eigenaardige vervorming der achterpooten, die voor vogels kenmerkend is — vergroeiing

der eerste rij voetwortelbeentjes met het scheenbeen, die der tweede rij met de zich aaneensluitende middenvoetsbeentjes tot het z.g. loopbeen, terwijl zich tusschen de beide rijen een intratarsaalgewicht vormt — voorafgegaan is aan de vervorming der voorpooten tot vleugels. Dit laatste proces toch is bij Archaeopteryx nog slechts in eersten aanleg aan den gang, terwijl de differentiatie van den voetwortel en de onderlinge vergroeiing der middenvoetsbeenderen reeds geheel zijn beslag heeft gekregen.

Stelt men zich nu de vraag van welke reptielen men zich de vogels het best afgeleid kan denken, dan schijnen mij onder de hedendaagsche typen de hagedissen de eenige, die daarvoor in aanmerking komen. Alle eigenaardigheden, die voor de vogels kenmerkend zijn, treffen wij bij enkele leden dezer orde aan, zij het slechts in eersten aanleg. Zoo de tweebeenige gang bij de Australische Chlamydosaurus en andere soorten, de zweefsprong bij het vliegende draakje (Draco volans) dat dit vermogen dankt aan zijdelingsche vliezige huidplooien, die met behulp van naar buiten uitstekende vrije ribben uitgespannen kunnen worden. Tevens bezitten deze diertjes een begin van zangvermogen, daar zij den geheelen dag in de boomen zitten te piepen. Ook de tjitjak's maken piepende geluiden, en hun grootere verwanten de tokkè's (Gecko's) hebben een indrukwekkende spreekstem, wat doet denken aan de stem der raven, beo's en papegaaien, die hen in staat stelt de menschelijke spraak na te bootsen.

Wat hun gedrag en wijze van optreden aangaat, mag men zeggen dat de hagedissen den naam kruipende dieren maar nauwelijks verdienen, veel minder althans dan de slangen, schildpadden en krokodillen, daar zij zich op hun korte pooten toch nog van den grond vermogen op te heffen en vliegensvlug kunnen voortbewegen. Ook voor het boomleven zijn zij zeer geschikt, meer nog dan de slangen, terwijl schildpadden en krokodillen daarvan geheel uitgesloten zijn.

Veeren kan men zich het best ontstaan denken uit zijdelingsche platte dunne schubben, zooals die bij Ptychozoön paradoxum aan de randen van romp, staart en ledematen uitsteken. Denkt men zich zulke schubben geperforeerd door een groot aantal in paralelle rijen gerangschikte spleetvormige gaatjes, dan ontstaat het prototype van een veer, en wel van een contourveer, waaruit dan weer door achteruitgang en vereenvoudiging dons- en haarveeren konden ontstaan.

Daar de zielen der armslagpennen van de huidige vogels bevestigd zijn in groefjes van de ellepijp, en bovendien op een vliezige huidplooi, die de ondereinden hunner schachten met elkaar verbindt, is de onderstelling gewettigd, dat oorspronkelijk zulk een vlies zich ook langs den achterrand van den bovenarm en den zijwand van den romp tot en met de dij uitstrekte, maar behalve langs den onderarm door achteruitgang weder verdween. Bij de voorouders der Jurassische Pterodactyli daarentegen bleef zulk een uitgebreide vliezige huidplooi in haar geheel bestaan, maar gingen de schubben die aanvankelijk daarop bevestigd waren, verloren.

Dat het bezit van een vliegscherm op zichzelf genomen, geen reden kan zijn, om de Pterodactyli nader verwant met de vogels te achten, spreekt vanzelf; het kan daarvoor evenmin dienen als het vliegscherm der vleermuizen.

Maar nu komen de Pterodactyli nog behalve in het vliegvermogen, in zoovele andere opzichten met de vogels overeen, ook in zulke, die niet als aanpassingen aan het vliegen kunnen beschouwd worden, dat zelfs een scherpzinnige deskundige als SEELEY hen uit de klasse der reptielen wilde uitlichten en met de vogels in één klasse (Saurornia) vereenigen. Het valt niet te ontkennen, dat hij daarvoor gewichtige argumenten aanvoert, b.v. de holheid (pneumaciteit) van vele beenderen, de krachtige ontwikkeling van het ravenbeks-sleutelbeen (coracoïd), in tegenstelling met den achteruitgang van het voorste sleutelbeen (clavicula) dat bij de Pterodactyli zelfs geheel ontbreekt.

Maar al die punten van overeenkomst kunnen verklaard worden uit de gelijksoortige afkomst van Pterodactyli en vogels uit hagedis-achtige voorouders, dus uit reptielen, die zich aan dezelfde bewegingswijze (in casu het vliegen) maar op verschillende wijze hebben aangepast. Men kan het vergelijken met de aanpassing aan het waterleven van de walvisschen en de zeehonden onder de zoogdieren, die daardoor op geheel verschillende manier zijn gewijzigd, maar toch in hooge mate op elkaar zijn gaan lijken.

Daarbij komt nog, dat Pterodactyli en vogels (Archaeopteryx) in dezelfde formatie (de lithographische leisteen uit de Boven-Jura), in volledige uitrusting van voor 't vliegen bekwamen lichaamsbouw bijna plotseling te voorschijn treden, al is ook voor de eerstgenoemde de voorgeschiedenis niet zoo duister als voor de laatste.

Een eigenaardigheid der vogels, die onder de reptielen slechts bij de Lepidosauri (hagedissen, slangen en Pythonomorphi) wordt aangetroffen, is de bewegelijkheid van het vierkantsbeen (os quadratum), dat bij hen als kaaksteel fungeert. Alle andere reptielen, uitgestorvene zoowel als recente, hebben een onbewegelijk vierkantsbeen, dat tusschen de aangrenzende schedelbeenderen ingeklemd ligt.

Stelt men zich de vraag wat oorspronkelijker is, een bewegelijk of een onbewegelijk quadratum, en beperkt men zich daarbij tot de onderlinge vergelijking der reptielen, of betrekt hoogstens de Amphibieën in de beschouwing, dan zou men tot het besluit kunnen komen dat het vrij bewegelijke vierkantsbeen uit het vastliggende is ontstaan, door het losraken der verbindingen met de omliggende beenderen, in 't bijzonder met den jukboog. Maar betrekt men ook de visschen in de vergelijking, dan krijgt men den indruk, dat het quadratum oorspronkelijk bewegelijk moet geweest zijn, daar het een onderdeel was van den eersten kieuwboog, en wel van de dorsale helft daarvan, waaruit ook het verhemelte ontstond. Men zou dus kunnen denken, dat het vrijraken van het quadratum der hagedissen en hunne naaste verwanten een terugkeer was tot den primi-

tieven toestand. In allen gevalle echter behoeft aan het verschil tusschen vormen met bewegelijk en onbewegelijk quadratum uit een genetisch oogpunt geen groot gewicht te worden gehecht, daar het slechts een bijzonder geval is van het algemeen verschijnsel dat een gesloten beenpantser in zijn onderdeelen uiteenvalt of geheel verdwijnt. Dergelijke reducties moeten zich in alle tijdperken hebben voorgedaan en zijn dus voor de stamontwikkeling van ondergeschikt belang.

De omgekeerde meening vloeit voort uit de algemeene opvatting, dat de betrekkelijke ouderdom der diergroepen opgemaakt zou mogen worden uit hun eerstbekende voorkomen in de overlevering der aardlagen. Hoe weinig dit veroorloofd is, blijkt juist in het geval der vogels wel bijzonder duidelijk. Kenden wij de twee exemplaren van hagedisstaartvogels uit de lithographische schiefer van Eichstädt niet, dan zouden wij niet weten dat in het Juratijdvak reeds bevederde gewervelde dieren bestonden, die in hun lichaamsbouw in velerlei opzichten grooter overeenkomst met hagedissen vertoonden dan de thans levende vogels. Nu zijn echter de oudst bekende fossiele hagedissen niet uit oudere formaties dan de Boven-Jura bekend, en dus uit hetzelfde tijdvak als Archaeopteryx, terwijl hun naaste verwanten onder de reptielen: de slangen en Pythonomorphen in nog jonger formaties voor 't eerst optreden. Daar deze laatste in hooge mate gespecialiseerd zijn, moeten hun hagedisachtige voorouders in veel oudere perioden geleefd hebben. Als dus Zittel schrijft:

„Die Lacertilia erreichten den Höhepunkt ihrer Entwicklung erst in der Jetztzeit und scheinen überhaupt mit den Schlangen die jüngsten Seitenäste des Reptilienstammes zu bilden", dan mag men beweren, dat deze schijn inderdaad een valsche schijn is, en enkel voortvloeit uit de onvolledigheid der fossiele nalatenschap, die voor de op het land geleefd hebbende fauna's nog aanmerkelijk veel grooter moet zijn dan voor de waterdieren.

Trouwens Zittel zelf plaatst in zijn rangschikking der reptiel-orden de hagedissen, slangen en Pythonomorphen onmiddellijk na de Rhynchocephalen, die tot de oudst bekende reptielen: de Theromorpha behooren, welke reeds uit het Perm, dus uit de palaeozoische periode bekend zijn en daarin door talrijke en zeer verschillende vormen vertegenwoordigd worden, zoodat men mag aannemen, dat hun gemeenschappelijke voorouders in nog veel oudere tijden leefden. Merkwaardigerwijze behoort nu tot die Rhynchocephalia nog een enkele recente soort, de beroemde Tuatara (Hatteria of Sphenodun) die slechts op één eilandje nabij het Noordeiland van Nieuw Zeeland leeft, maar in bijna denzelfden vorm onder die oudste reptielen wordt aangetroffen.

De overeenkomst in organisatie en habitus tusschen dit dier en de hagedissen is zoo groot, dat het door vele zoölogen als een lid der orde van de Lacertilia werd, en wellicht nog wordt beschouwd. Weliswaar is het quadratum van Sphenodon onbewegelijk, daar het door een dubbelen jukboog op zijn plaats wordt gehouden, maar zooals gezegd, kan die

immobiliteit aan secundaire fixatie worden toegeschreven, die bij de hagedissen en hun naaste verwanten uitgebleven zou zijn of, wat waarschijnlijker is, in den loop hunner philogenetische ontwikkeling weer opgeheven door obliteratie van den ondersten jukboog.

Grooter beteekenis voor de beoordeeling van den verwantschapsgraad tusschen hagedissen en Rhynchocephalen mag m.i. toegekend worden aan de bovenvermelde overeenkomst in habitus tusschen beide groepen. Daarvoor pleit dat de hagedis-gestalte kenmerkend is voor alle meer oorspronkelijke viervoetige dieren, en dat alle andere gedaanten zich uit die der hagedissen laten afleiden. Onder de Amphibieën zijn het de salamanders, onder de zoogdieren de muizen en spitsmuizen, die het meest aan het algemeene type: — met korte hals en ledematen, en langen staart — beantwoorden.

Uit dat oogpunt bekeken vinden wij het zelfde verschijnsel terug bij de visschen: haaien zijn oorspronkelijker gebouwd dan roggen, snoeken dan maanvisschen. Daaruit blijkt tevens, dat het niet aangaat, de langgestrekte spoelvormige gedaante van in het water levende gewervelde dieren uitsluitend als gevolg van de aquatiele levenswijze te beschouwen, al moet toegegeven worden, dat deze een versterkenden invloed op den bestaanden aanleg kan uitoefenen, zooals uit de gestalte der Ichthyosauriers en vele andere waterreptielen blijkt, of onder de zoogdieren de walvisschen en robben, onder de vogels de pinguins.

In het bijzonder zijn het de ledematen, die onder dien invloed van gangpooten tot zwemvinnen vervormd worden. Dat dit bij de voorouders van walvisschen, zeeschildpadden, Ichthyosauri en vele andere naar het water teruggekeerde viervoeters (Tetrapoden) heeft plaatsgevonden is niet twijfelachtig, en wordt ook zoover ik weet door niemand ontkend, maar voor de oer-Tetrapoden schijnt bij vele biologen de omgekeerde opvatting te heerschen, daar zij zonder voorbehoud de amphibieën uit de visschen ontstaan denken. Om een enkel voorbeeld aan te halen: in het omvangrijke verzamelwerk van negentien Duitsche biologen onder redactie van G. HEBERER, dat in 1943 in Jena is uitgegeven onder den titel: Die Evolution der Organismen, komt op blz. 166 van J. WEIGELT's verhandeling: Palaeontologie als Stammesgeschichtliche Urkundenforschung, een illustratie voor, waarop de skeletten van drie fossiele Vertebraten naast elkander staan afgebeeld: een visch uit het Devoon, een amphibie uit het Carboon en een reptiel uit het Perm, onder het opschrift: Hauptetappen der Entstehung der Reptilien aus Fischen über die Stegocephalen: I Eusthenopteron, ein devonischer Crossopterygier, II Microbrachium, ein karbonischer Stegocephale, III Seymouria, ein permisches Urreptil (Cotylosauria). Aus Kühn 1937.

De overeenkomst tusschen hen bestaat, behalve in gestalte, bijna uitsluitend in de samenstelling van het schedeldak, dat zooals men weet, vooral bij visschen uiterst variabel is. Ook overigens zou men voor de afbeelding van het visschenstadium met evenveel (beter gezegd met even

weinig) recht de figuur van een recenten haai kunnen teekenen. Met meer recht daarentegen zou men mogen beweren, dat de devonische Crossopterygier uit een tewatergegane oer-hagedis was ontstaan, dan omgekeerd.

Hoe de onbekende praecambrische voorouders van alle gewervelden gebouwd waren, kunnen wij slechts gissen. Waarschijnlijk ademden zij door kieuwen en ontwikkelden de longen zich bij hun overgang tot het landleven. Maar dit zelfde moet ook aangenomen worden voor de voorouders der beenvisschen, wier zwemblaas een vervormde long is, zooals ons door de Dipnoï (Longvisschen) afdoende bewezen wordt. Of dit ook voor de kraakbeenvisschen (haaien en roggen) geldt, is twijfelachtig, maar niet ondenkbaar; zij zouden dan alle sporen van longen weer verloren moeten hebben. Maar dat de ons bekende visschen, ook de allerоudste, niet de voorouders der amphibieën kunnen geweest zijn, evenmin als deze laatste die der reptielen, schijnt mij een uitgemaakte zaak. De geheele voorstelling der opeenvolging van uit hun voorgangers ontstane klassen der gewervelden: visschen, amphibieën, reptielen, vogels, zoogdieren, is een overblijfsel uit de 18e eeuwsche gedachte, dat het geheele dierenrijk, ja zelfs alle levende wezens tezamen, zich in een doorloopende reeks lieten rangschikken. In dien gedachtengang werden de struisvogels als overgang van de vogels tot de zoogdieren beschouwd. Daarbij was echter geen sprake van ontwikkeling der laatste uit de eerste: alle soorten en vormen van levende wezens waren onveranderlijk. Ofschoon aan dit dogma door DARWIN's Origin of Species, nu reeds voor bijna een eeuw de laatste stoot werd toegebracht, spookt de leer der opeenvolging van de diertypen nog steeds voort.

Omdat de oudst bekende visschen uit het Siluur, de oudst bekende amphibieën uit het Devoon, de oudst bekende reptielen uit het Carboon, de oudst bekende vogels uit de Jura stammen, zouden zij in die volgorde uit elkaar ontstaan zijn. Alleen voor de zoogdieren moest een uitzondering gemaakt worden, daar zij onmogelijk als veranderde vogels konden beschouwd worden, maar zij werden dan toch als warmbloedigen tezamen met de vogels tegenover al de andere gesteld. HUXLEY (de oude) was de eerste, die het verband tusschen vogels en reptielen duidelijk in het licht stelde en ze met elkaar vereenigde tot de klasse der Sauropsida. Dat ook de zoogdieren als hooger gedifferentieerde reptielen mochten beschouwd worden, bleek uit de ontdekking van Theromorphe fossielen in Z.-Afrika.

Daarmee kwam de beteekenis der warmbloedigheid als onderscheidingskenmerk te vervallen; zij was trouwens reeds in twijfel getrokken door HAECKEL, die de onderstelling had uitgesproken, dat ook bij de hooger gedifferentieerde reptielen, in 't bijzonder de reusachtige Dinosauriërs, de lichaamstemperatuur zich boven die van de omgeving in zekere mate kon verheffen. De hoogste graad van onafhankelijkheid der lichaamstemperatuur van die der omgeving kon slechts bereikt worden bij volstrekte scheiding van grooten en kleinen bloedsomloop. Hoe die tot stand kwam, leert ons de ontwikkelingsgeschiedenis van het arterieele

vaatstelsel. Deze toont ons, dat zich de vertakking der groote slagaderen bij vogels en zoogdieren het best laat afleiden uit het arterieele vaatstelsel der hagedissen, juist omdat dit den oorspronkelijken symmetrischen segmentalen bouw der Oer-vertebraten het zuiverst bewaard heeft.

Summary.

The phylogeny of birds.

The relation between the classes of birds and mammals can best be understood when we imagine the latter reduced to the order of the bats by extinction of all the other orders, and these handflyers still further differentiated in several directions, some of them having lost the power of flying and adopted a terrestrian locomotion, or even the habit of swimming, climbing or digging.

The organisation of Archaeopteryx clearly proves that the reptilian ancestors of birds must have been lizardlike. Consequently the lizards must have deviated from the common stock of all reptiles at a much earlier date than the Jurassic period, as is also proved by their near relation to the Rhynchocephalia, that already were fully developed in the Perm, and so belong to the oldest known reptiles. The still older common ancestors of the reptiles, as yet completely unknown, cannot be seen in the amphibia and still less in the fishes. Probably they were already landliving animals and existed in much earlier periods than the Perm.

The resemblance between Pterodactyli (Pterosauria) and birds is a consequence of convergence, by adaptation to aviatic movement. Yet they are near-related to each other, as both took their origin from lizard-like ancestors.

The pentadactyle limbs of the Tetrapoda cannot be derived from the fins of fishes; on the contrary the latter in all cases most probably are modifications of terrestrian limbs by the influence of return to the aquatic mode of life.

Résumé.

l'Origine phylogénétique des oiseaux.

La relation entre les classes des oiseaux et des mammifères se comprend le mieux quand on s'imagine que ces derniers fussent réduits aux chauvesouris par l'éteinte complète de tous les autres ordres, tandis qu'au contraire les chiroptères devinrent encore plus différenciés en plusieurs directions: quelques'uns perdant le pouvoir de voler et adoptant la démarche terrestre, mais seulement sur les pattes de derrière, ou même s'habituant à nager ou à s'enfouir sous terre.

L'organisation de l'Archaeopteryx nous apprend, que les ancêtres des oiseaux appartenaient à la classe des reptiles, et spécialement à l'ordre des Lézards. Il en suit que ces derniers doivent avoir pris leur origine de

l'ancêtre commun des reptiles à un époque beaucoup plus éloigné que le Jura, ce qui de plus est aussi bien prouvé par leur connection intime avec les Rhynchocéphales, dont les premiers représentants furent découverts dans le Perm. Au contraire les ancêtres communs de tous les reptiles ne peuvent être vus dans les amphibiens, et encore moins dans les poissons. Quoique inconnus jusqu'à présent, on a le droit de supposer que c'étaient des animaux terrestres, qui vivaient dans la période silurienne ou même encore auparavant.

La ressemblance entre les Pterodactyles et les oiseaux est en premier lieu un cas de convergence, à cause d'adaptation au vol dans l'atmosphère, mais s'explique en outre par leur origine commun de membres de l'ordre des lézards.

Les pattes pentadactyles des Tetrapodes ne peuvent avoir pris leur naissance de nageoires de poissons. Au contraire il est plus probable que ces derniers se dérivaient de pattes terrestres, modifiés par l'influence de l'adoptation secondaire de la vie aquatile.

Zusammenfassung.

Die Stammesgeschichte der Vögel.

Die Beziehungen zwischen den Klassen der Vögel und der Säugetiere verstehen sich am besten wenn man die letzteren vollständig ausgestorben denkt mit alleiniger Ausnahme der Fledermäuse, während man diese in verschiedenen Richtungen noch weiter differenzirt denkt, sodass einige das Flugvermögen wieder verloren und zum Laufen, aber ausschliesslich auf den Hinterbeinen zurück kehrten, andere sich das Schwimmen und Tauchen angewöhnten oder in die Erde wühlten.

Frägt man umgekehrt nach den Voreltern der Vögel, so geht aus der Organisation des Archaeopteryx hervor, dass man sich diese als Eidechsenartige Reptilien vorzustellen hat. Daraus folgt wieder, dass der Eidechsentypus viel älter sein muss als die Obere Jura, worin die ersten Vertreter der Lacertilia zusammen mit Archaeopteryx gefunden wurden. Zu derselben Schlussfolgerung kommt man, wenn man die nahen Beziehungen der Eidechsen zu den Rhynchocephalen in Betracht zieht, die schon im Perm vollständig ausgebildet unvermittelt hervortreten.

Was die Voreltern des Reptilienstammes betrifft, so darf man dieselben weder unter den Amphibien noch unter den Fischen suchen. Vielmehr waren es schon Landtiere, die im Silur oder noch früher lebten. Die pentadactylen Gliedmassen der Tetrapoden können nicht aus den Flossen der Fische hervorgegangen sein, im Gegenteil können die Fischflossen als modifizirte Tetrapodenfüsse betrachtet werden, die sich unter dem Einfluss des Wasserlebens, wozu ihre Voreltern zurückkehrten, umgestalteten.

Die Ähnlichkeit zwischen Pterosauria und Vögel ist in erster Linie als eine Konvergenzerscheinung in Folge der fliegenden Lebensweise zu betrachten, aber daneben als Ergebnis ihrer gemeinsamen Abstammung aus eidechsenartigen Vorfahren.

ZOOLOGY

SACCULINA PULCHELLA, A RHIZOCEPHALAN PARASITE OF THREE DIFFERENT HOSTS

BY

H. BOSCHMA

(Communicated at the meeting of June 24, 1950)

The specific characters of *Sacculina pulchella* were defined in the following manner (BOSCHMA, 1933, p. 223): "Male genital organs in the posterior part of the body, outside the visceral mass. Testes completely separated, more or less globular. Colleteric glands approximately in the central region of the lateral surfaces of the visceral mass, with a moderate number of tubes. External cuticle of the mantle covered with excrescences which consist of a kind of chitin differing from that of the main layers. The excrescences are short papillae with numerous spines at their tops and with a small number of root-like expansions at their bases. The height of the excrescences varies from 12 to 18 μ. Retinacula have not been found."

In the present paper besides some notes on the specimens on *Huenia proteus* de Haan that have been described in previous papers, the chief details are given of one specimen of *S. pulchella* on *Hyastenus brockii* de Man and two specimens on *Egeria arachnoides* Latreille, two hosts on which the parasite previously was not known to occur. The entire material at present available has the following data.

Seychelles, Western Indian Ocean, H. M. S. "Sealark", Sta. F 8, 34 fms., October 20, 1905, 1 specimen (holotype) on *Huenia proteus* de Haan (collection United States National Museum, Washington, D. C.).

Amirante, Western Indian Ocean, H. M. S. "Sealark", Sta. E 11, 25—80 fms., November 10, 1905, 1 specimen on *Huenia proteus* de Haan (collection United States National Museum, Washington, D. C.).

Snellius Expedition, Amboina, September 11—17, 1930, 1 specimen on *Hyastenus brockii* de Man.

Chittagong coast, "Golden Crown", August, 1908, 2 specimens on one specimen of *Egeria arachnoides* Latreille (collection Indian Museum, Calcutta).

With the description of *Sacculina pulchella* a figure was given of the excrescences of the external cuticle of the specimen on *Huenia proteus* from Amirante (BOSCHMA, 1933, fig. 3); this figure is reproduced here as text-figure 1. In a later paper (BOSCHMA, 1937) some more data were mentioned concerning the species, including figured sections of the male organs and of one of the colleteric glands. A longitudinal section of the type specimen and the excrescences of the external cuticle of the same specimen were figured in another paper (BOSCHMA, 1950); this paper

moreover contains a figure of the left side of the type specimen. In the two specimens on *Huenia proteus* the excrescences of the external cuticle are of a strongly similar shape; they consist of small papillae showing

Fig. 1. *Sacculina pulchella*, specimen on *Huenia proteus* from Amirante. Excrescences of the external cuticle, surface view. From BOSCHMA (1933, fig. 5). × 530.

numerous minute spines on their topmost flat surfaces and possessing a few root-like expansions in their lower parts, these roots adhering to the main layers of the cuticle.

The shape of the specimen on *Hyastenus brockii* (fig. 2*a*) is very similar to that of the larger specimen on *Egeria arachnoides* (fig. 2*b*). The two

Fig. 2. *Sacculina pulchella*. *a*, specimen on *Hyastenus brockii*, left side; *b*, the larger specimen on *Egeria arachnoides*, left side. *a*, × 6; *b*, × 3.

specimens are slightly oval, in both of them the mantle opening is a very narrow slit in the central part of the anterior region on the left side of the body. The surroundings of the mantle opening do not protrude above the surface. In contradistinction to these two specimens the type is distinctly kidney-shaped, whilst here the mantle opening is found at the top of a small tube (cf. BOSCHMA, 1950, fig. 1*e*). The dimensions of the type specimen are $5 \times 2\frac{1}{2} \times$ less than 2 mm; those of the specimen on *Hyastenus brockii* $6 \times 4 \times 2$ mm; those of the larger specimen on *Egeria arachnoides* $11 \times 8 \times 3$ mm. The smaller specimen on *Egeria arachnoides*, obviously a young specimen that recently had become external, does not exceed a size of 2 mm in any direction. In the two specimens of fig. 2 the mantle does not show any grooves or wrinkles, with the exception of the median

depression on the right surface caused by pressure of the median ridge of the abdomen of the host.

Parts of longitudinal sections of the specimen on *Hyastenus brockii* are shown in fig. 3. The first section (fig. 3*a*) shows the narrow right vas deferens and of the left male organ the region in which the vas deferens

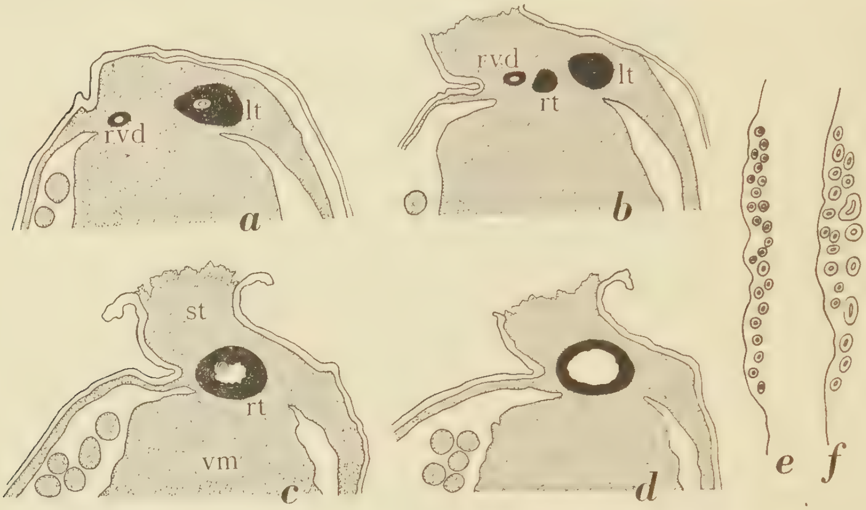

Fig. 3. *Sacculina pulchella,* specimen on *Hyastenus brockii. a—d,* posterior parts of longitudinal sections, *a* from a region ventral from the stalk, each following section from a more dorsal region. *e, f,* longitudinal sections of one of the colleteric glands. *lt,* left testis; *rt,* right testis; *rvd,* right vas deferens; *st,* stalk; *vm,* visceral mass. *a—d,* × 36; *e, f,* × 64.

is passing into the testis; here the cavity is lined with a distinct layer of chitin, a character found in all the species of the genus in which the vasa deferentia more or less abruptly pass into the testes. From this region the left testis does not appreciably increase in size, it soon reaches its dorsal end, so that in the next figured section (fig. 3*b*) the closed dorsal part of the left testis is visible. Here moreover the ventral part of the right testis appears. The latter has obtained its largest size in a slightly more dorsal region (fig. 3*c*), showing here moreover the narrow chitinous canal which forms the connexion between the vas deferens and the testis. Towards a still more dorsal region the right testis does not become appreciably larger, though its cavity becomes distinctly wider (fig. 3*d*). The figures show that in this specimen the right testis is found slightly behind the left.

Two sections of one of the colleteric glands are drawn in the same figure, the one (fig. 3*e*) from a more peripheral region than the other (fig. 3*f*). The glands contain a rather flat mass of branched canals, the latter have a comparatively thick layer of chitin. In the most strongly branched region of these glands a longitudinal section contains 24 canals.

The external cuticle of the mantle, which is comparatively thin (thickness 9 to 18 μ), bears excrescences of a similar shape as those of the type

specimen. The excrescences are small papillae which at least in their outer layers consist of a hyaline kind of chitin, differing from that of the main layers by its lack of affinity for stains. In sections of the cuticle the internal part of the excrescences often is slightly stained, proving that here the chitin is of a softer consistency. The excrescences have a height of 18 to 24 μ, at their upper surface they possess a great number of spines, and at their sides they show a few larger spines (fig. 4*a*, *b*). In surface

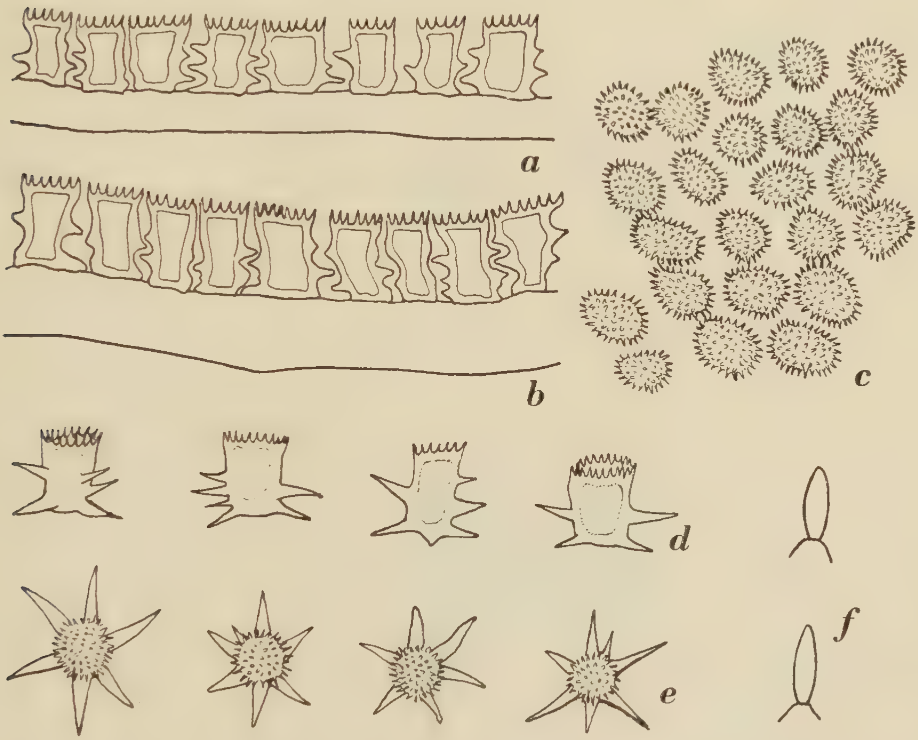

Fig. 4. *Sacculina pulchella*, specimen on *Hyastenus brockii*. *a*, *b*, sections of the external cuticle; *c*, excrescences of the external cuticle in surface view (top parts only); *d*, isolated excrescences in side view; *e*, isolated excrescences in surface view; *f*, retinacula. × 530.

view it appears that the excrescences vary in diameter from 9 to 18 μ (fig. 4*c*; here the lateral spines of the excrescences have been omitted). The structure of the excrescences is best understood in isolated specimens scraped from the cuticle (fig. 4*d*, *e*). Then it appears that the large spines are not confined to the basal parts of the excrescences only, as the roots of the excrescences of the type specimen, but they project freely between the neighbouring excrescences; these spines may develop to a fairly large size, reaching a length of about 15 μ.

On the internal cuticle of the mantle a number of retinacula were found, which seem to have a fairly regular distribution over this cuticle. Each retinaculum (fig. 4*f*) consists of a small basal part and a single

spindle, the latter has a length of 18 to 24 μ. These spindles seem to be devoid of barbs.

In the larger specimen on *Egeria arachnoides* the male genital organs are very similar to those of the type specimen. The vasa deferentia have a narrow lumen, which in the region in which they pass into the testes shows a distinct layer of chitin (visible in the central part of the left male organ in fig. 5*a*). Towards the dorsal region the testes become distinctly larger (fig. 5*b*), but their cavities remain narrow. One of the testes is

Fig. 5. *Sacculina pulchella*, larger specimen on *Egeria arachnoides*. *a*, posterior part of a longitudinal section in the region of the vasa deferentia; *b*, posterior part of a longitudinal section in the region of the testes; *c*, longitudinal section of one of the colleteric glands. *lt*, left testis; *rt*, right testis. *a*, *b*, $\times$ 36; *c*, $\times$ 64.

slightly larger than the other, but both male organs are fully developed.

The colleteric glands of this specimen again are rather flattened. They contain a slightly larger number of canals than those of the specimens dealt with above; in a longitudinal section of the most strongly branched region of these glands the number of canals may amount to 38 (fig. 5*c*). In this specimen again the canals possess a well developed layer of chitin.

The excrescences of the external cuticle in this specimen consist of cylindrical papillae composed of an outer layer of hard chitin showing no affinity for stains and an internal mass of softer structure that in the sections is slightly stained (fig. 6*a*, *b*). The height of the excrescences

varies from 12 to 21 μ, the main layers of the cuticle have a thickness varying from 9 to 20 μ. The excrescences at their tops are covered with numerous small spines, of which the marginal ones are somewhat larger than those in the centre of the upper surface. The diameter of the papillae

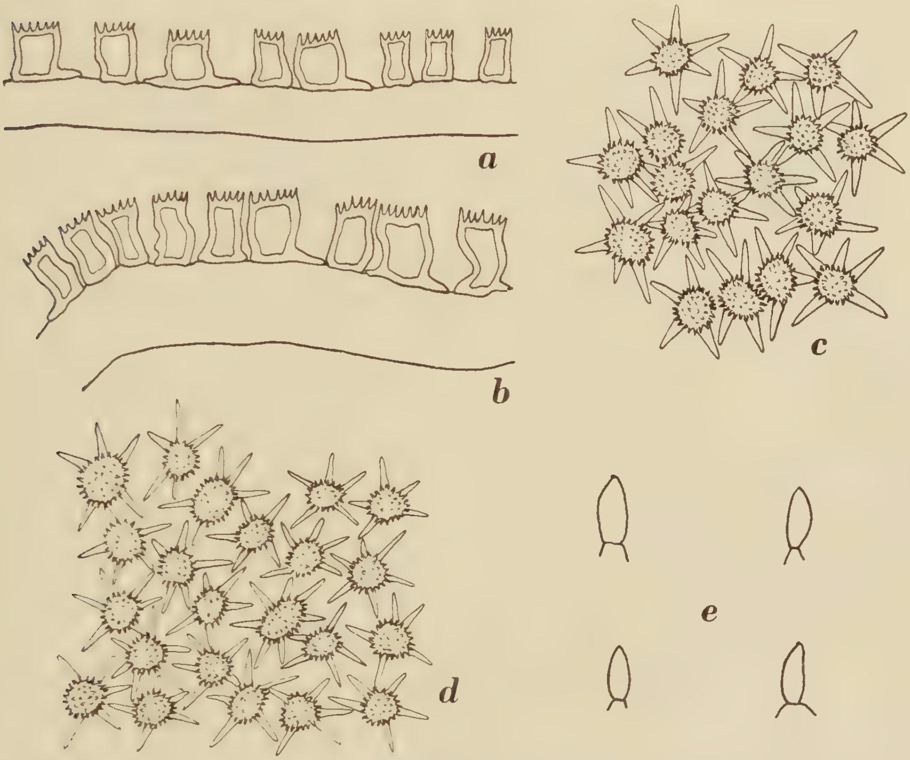

Fig. 6. *Sacculina pulchella*, larger specimen on *Egeria arachnoides*. *a*, *b*, sections of the external cuticle; *c*, *d*, excrescences of the external cuticle in surface view; *e*, retinacula. × 530.

varies from 7 to 14 μ (fig. 6*c*, *d*). In their basal parts the excrescences possess a variable number of root-like expansions, which are adhering to the main layers of the cuticle. In some parts of the mantle these roots are rather strong (fig. 6*c*), in other parts they are much thinner (fig. 6*d*). The length of these roots as a rule is not more than 12 μ. In many parts of the cuticle the excrescences are more crowded than those represented in fig. 6*c* and *d*.

On the internal cuticle of the mantle there are a fairly large number of retinacula, which seem to be rather regularly distributed over the surface of this cuticle. These retinacula consist of a small basal part and a single spindle each. The spindles vary in length from 13 to 18 μ, they seem to possess no barbs.

The small parasite found on the same crab of the species *Egeria arachnoides* as the larger specimen has a more or less globular shape, slightly flattened laterally. Its larger diameter is about 2 mm. From this specimen

a series of longitudinal sections was made, which shows that the animal is very young and still immature. The two male organs occupy exactly the same position as those of the specimens dealt with above (fig. 7). The two

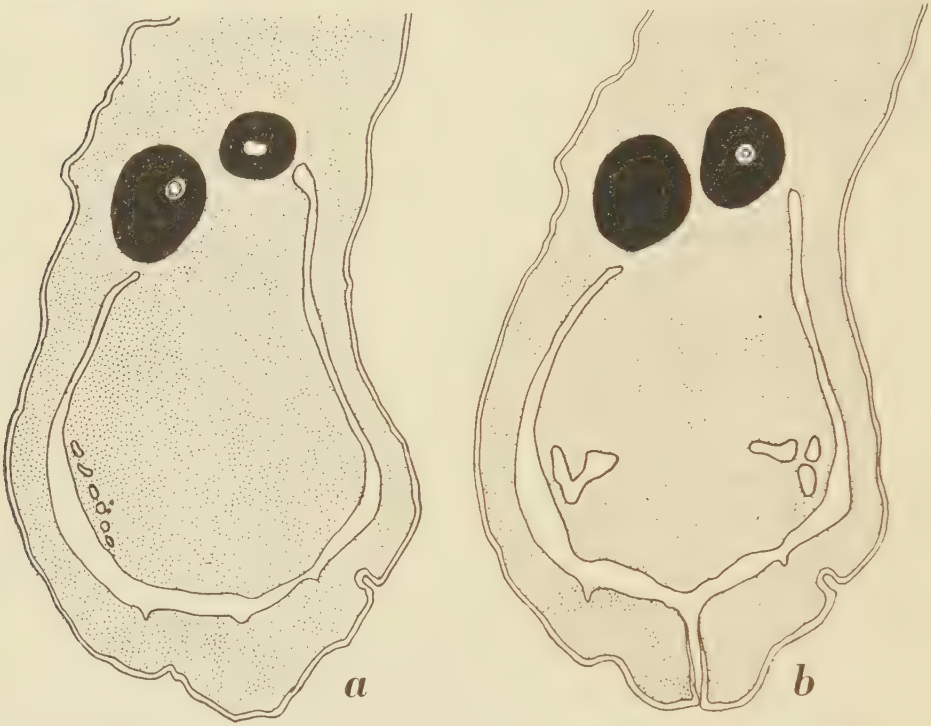

Fig. 7. *Sacculina pulchella*, smaller specimen on *Egeria arachnoides*, longitudinal sections, *a* from a more ventral region than *b*. × 54.

male organs have an approximately equal size and a similar shape. In the two male organs the chitinous canal that forms the transition between the vas deferens and the testis is already distinctly visible, in one of the male organs this canal is found in a more ventral plane than in the other. Fig. 7*b* represents a median section showing the narrow mantle opening and the central parts of the colleteric glands, which here have a few rather wide canals only. In the more ventral section of fig. 7*a* one of the colleteric glands is present; though the section shows this gland in its most strongly divided part there are six canals only.

In this young specimen the external cuticle of the mantle is a thin layer of chitin (thickness approximately 6 μ), which has a smooth surface, without any excrescences.

It is interesting that one specimen of *Egeria arachnoides* is infested by two parasites of extremely different sizes, the larger diameter of the one being 11 mm, that of the other about 2 mm. As the external cuticle of the smaller specimen does not bear excrescences it is not absolutely certain that the two specimens both belong to *Sacculina pulchella*. As, however, the

male organs in the two specimens correspond in every detail, it is highly probable that the two are conspecific. The differences found in the colleteric glands then may be explained by the fact that the small specimen is still immature, so that the colleteric glands have remained undeveloped.

As it is improbable that the young parasite is the result of an infection of the host in its fully adult state, the young external sac must have arisen as a recent bud from the root system of the older parasite. However this may be, the present case remains exceptional. When more than one Sacculinid parasite are found on one crab as a rule these parasites have reached about the same stage of development though they may markedly differ in size, usually on account of lack of sufficient space for a normal development.

The various specimens described above in all probability all belong to the species *Sacculina pulchella*. Among each other they show differences, but these seem to be too insignificant as peculiar specific characters. In the specimens on the three different crabs the male organs have a corresponding structure. There are slight differences, as in one the testes are found the one behind the other, and in the others more or less side by side, but this may be explained by individual variation. The same holds for the differences in the canal system of the colleteric glands. Leaving aside the very small specimen there is a difference between the larger specimen on *Egeria* and those on the other hosts. Whilst in the latter the number of canals is not much over 20, in the former this number is nearly 40. But this difference may have come about in connexion with the differences in size, the larger specimen on *Egeria* being much larger than those on the other crabs.

The excrescences of the external cuticle of the larger specimen on *Egeria* in every respect are similar to those of the specimens on *Huenia*. On the other hand the excrescences of the specimen on *Hyastenus* differ from those of the parasites of the other crabs by having spines freely extending laterally, not only in the extreme basal region and adhering to the main layers of the cuticle. This difference again may be the result of individual variation.

It is an important fact that in the specimen on *Hyastenus* as well as in the larger specimen on *Egeria* there are distinct retinacula of a peculiar shape. This decidedly points to the fact that these parasites are conspecific. In the specimens on *Huenia* no retinacula were found, but we may safely conclude that one of the specific characters of *Sacculina pulchella* is the possession of retinacula, each of which bears a single spindle only.

It is interesting to note that the three hosts of *Sacculina pulchella*, viz., *Huenia proteus*, *Hyastenus brockii*, and *Egeria archnoides*, all belong to the family Maiidae. As far as at present known the parasite therefore seems to be restricted to crabs of one family.

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THE CHEMISTRY OF ACETYLENIC ETHERS

BY

J. F. ARENS AND P. MODDERMAN

(Communicated at the meeting of Sept. 30, 1950)

PART. I.

Review of known reactions.

Relatively few publications deal with the reactivity of acetylenic ethers.

a. The acetylenic hydrogen atom is mobile and can be exchanged for metal atoms by means of a GRIGNARD compound or another suitable reagent [1]). The magnesium and lithium derivatives behave like ordinary GRIGNARD reagents and react for instance with carbonyl compounds [1, 2]).

$$HC\equiv COC_2H_5 + C_2H_5MgBr \rightarrow BrMgC\equiv COC_2H_5 + C_2H_6$$

$$R-\underset{\underset{R'}{|}}{C}=O + BrMgC\equiv COC_2H_5 \xrightarrow[\text{2 steps}]{} R-\underset{\underset{R'}{|}}{\overset{OH}{C}}-C\equiv COC_2H_5$$

I

b. The acetylenic ethers are easily hydrated by means of acidified water yielding esters [3a]).

$$HC\equiv COC_2H_5 + H_2O \xrightarrow{H^+} H_3C-C\begin{matrix} \nearrow O \\ \searrow OC_2H_5 \end{matrix}$$

This property belongs also to the derivatives without an acetylenic hydrogen atom. So I can be transformed into an α, β-unsaturated ester [3b]).

$$R-\underset{\underset{R'}{}}{\overset{OH}{C}}-C\equiv COC_2H_5 \xrightarrow[(H_2O)]{H^+} R-\underset{\underset{R'}{|}}{C}=CH-C\begin{matrix} \nearrow O \\ \searrow OC_2H_5 \end{matrix}$$

c. Alcoholic HCl and ethoxyacetylene give ethylacetate and ethylchloride [4]).

$$HCl + C_2H_5OH + HC\equiv COC_2H_5 \rightarrow C_2H_5Cl + CH_3COOC_2H_5$$

d. Dry hydrogen chloride in ether gives ethyl-α-chlorovinyl-ether [1c]).

$$HC\equiv COC_2H_5 + HCl \rightarrow H_2C=C\begin{matrix} OC_2H_5 \\ Cl \end{matrix}$$

This substance reacts violently with water to yield ethylacetate.

e. Alcohol can be added to the triple bond under the influence of boronfluoride.

The primary product is an ortho ester which decomposes under the influence of the catalyst into an ether and an acetate [1b]).

$$HC \equiv COC_2H_5 + 2\,C_2H_5OH \xrightarrow{BF_3} H_3C{-}C(OC_2H_5)_3$$

$$C_2H_5OC_2H_5 + CH_3COOC_2H_5 \xleftarrow{BF_3}$$

f. The acetylenic triple bond can be partially hydrogenated by means of a Pd catalyst giving a vinyl ether which in turn can be hydrolysed to an aldehyde (compare ref. [2])).

$$R{-}C \equiv COC_2H_5 \xrightarrow[Pd]{H_2} R{-}CH = C\begin{matrix}\diagup H \\ \diagdown OC_2H_5\end{matrix} \xrightarrow[H^+]{H_2O} R{-}CH_2{-}C\begin{matrix}\diagup H \\ \diagdown\diagdown O\end{matrix} \quad - C_2H_5OH$$

g. On storing the acetylenic ethers polymerise slowly yielding complex mixtures [1, 5]). On heating the ethers may explode [1]).

h. For halogen derivatives of phenoxyacetylene see ref. 6.

With the aim of extending the knowledge of the chemistry of the acetylenic ethers, we are engaged in the study of the action of various organic compounds on ethoxyacetylene.

PART II.

Formation of anhydrides in the reaction of ethoxyacetylene with anhydrous carboxylic acids.

Summary.

Ethoxyacetylene $HC \equiv C - OC_2H_5$ reacts with saturated and unsaturated aliphatic and aromatic acids under the formation of the anhydride of the acid and ethylacetate. With formic acid ethylacetate and carbon monoxide are formed. In these reactions ethoxyacetylene thus behaves like a powerful dehydrating agent.

Considering the reaction mentioned in part I point *d*, it might be expected that upon addition of organic acids to ethoxyacetylene one would obtain vinylderivatives like II.

$$HC \equiv COC_2H_5 + HOOC{-}R \rightarrow \underset{\text{II}}{H_2C = C\begin{matrix}\diagup OC_2H_5 \\ \diagdown OCOR\end{matrix}} \qquad \text{(Reaction A).}$$

We observed however that each molecule of ethoxyacetylene reacts with two molecules of the acid. The net result is the formation of ethylacetate

and the anhydride of the acid. This transformation proceeds without the addition of a catalyst.

$$HC \equiv COC_2H_5 + 2\ RCOOH \rightarrow CH_3COOC_2H_5 + (RCO)_2O$$

In the case of formic acid, the reaction takes a somewhat different course as the anhydride of this acid, is unstable. We observed a vigorous evolution of carbon monoxide upon the addition of ethoxyacetylene to an ethereal solution of 2 moles of anhydrous formic acid. One mole of the acid remained unchanged. The only other product of the reaction was ethylacetate.

$$HC \equiv COC_2H_5 + HCOOH \rightarrow CH_3COOC_2H_5 + CO$$

It is probable that the reactions with the acids occur in two steps. The first (Reaction *A*) results in the formation of a vinyl compound (II) a derivative of ketene, which, except in the case of formic acid, reacts further with another molecule of the acid according to reaction *B*, the hypothetical intermediate substance III being unstable.

$$\underset{\text{II}}{H_2C = C\begin{matrix} \diagup OC_2H_5 \\ \diagdown OCOR \end{matrix}} + HOOC-R \rightarrow \underset{\text{III}}{\left[H_3C-C\begin{matrix} \diagup OC_2H_5 \\ -OCOR \\ \diagdown OCOR \end{matrix} \right]} \rightarrow$$

$$CH_3COOC_2H_5 + RCO-O-COR$$

Reaction B

In order to obtain the substance II ($R=CH_3$) as the main product of the reaction, we slowly added acetic acid to a large excess of ethoxyacetylene. However we again obtained ethylacetate and acetic anhydride, while a considerable part of the ethoxyacetylene remained unchanged. This result can be explained by assuming that the reaction rate of *B* is much faster than of *A*, so that any II formed immediately reacts with acetic acid. The reactions as described above, are analogous to the reactions of acetylene with organic acids [7]) altough these require catalysts like boronfluoride and mercuric oxide or strong acids and a mercuric salt.

$$HC \equiv CH + CH_3COOH \rightarrow H_2C = CH-OCOCH_3 \qquad \text{IV}$$

$$\text{IV} + CH_3COOH \rightarrow H_3C-C\begin{matrix} \diagup H \\ -OCOCH_3 \\ OCOCH_3 \end{matrix}$$

$$\rightarrow H_3C-C\begin{matrix} \diagup H \\ \diagdown O \end{matrix} + CH_3CO-O-COCH_3.$$

The formation of anhydrides in the reaction of ethoxyacetylene with carboxylic acids seems to be general, as also aromatic and unsaturated acids give similar results. Even non carboxylic acids such as picric acid

react with ethoxyacetylene, though the reaction takes a somewhat different course. This will be published in a following paper.

Our thanks are due to Mr Yo Kim Tek for valuable assistance in the laboratory.

Experimental part [1].

a. Reaction of ethoxyacetylene with 2 moles of acetic acid.

To 13 g of glacial acetic acid was added under stirring a solution of 7.5 g of ethoxyacetylene in 20 cm^3 of absolute ether. The mixture became warm. After the complete addition it was refluxed for ½ hr. and subsequently distilled fractionally by means of a 20 cm column packed with glass helices. After a fore running of ether we obtained 7½ g of ethylacetate (bp. 74°) and 7 g of acetic anhydride (bp. 125 — 133°). Residue 3½ g.

b. Reaction of acetic acid with a large excess of ethoxyacetylene.

4,5 g of glacial acetic acid dissolved in 15 cm^3 of abs. ether was added very slowly to 14,5 g of ethoxyacetylene. The temperature was kept at 40°. After standing for 1 night at roomtemperature (20 — 25°) the solution was fractionally distilled using a 20 cm column packed with glass helices. We obtained the following fractions:

33 — 60°	16,0 g	mixture of ether and ethoxyacetylene (b.p. ethoxyacetylene 51°).
60 — 72°	1 g	ethylacetate.
		The distillation was continued in vacuo (25 mm); between the pump and the receiver a cold trap was used.
52 — 53° (25 mm)	2,0 g	acetic anhydride.

residue 0,5 g. The cold trap contained 2½ g of ethylacetate.

c. Reaction of ethoxyacetylene with butyric acid.

To 36,3 g of butyric acid was slowly added 14½ g of ethoxyacetylene in 25 cm^3 of ether. Heat was evolved. After standing for ¾ hr. the mixture was fractionally distilled using a 20 cm column packed with helices. After a fore running of ether we obtained 13 g of ethylacetate bp. 65 — 74° (main part distilling at 74°), then, in vacuo, 18 g of a fraction bp. 85 — 105° (34 mm) consisting of butyric acid and butyric anhydride and finally 12½ g of butyric anhydride bp. 105 — 106° (34 mm). The residue weighed 1½ g. The cold trap between the pump and the receiver contained 2½ g of ethylacetate.

d. Reaction of ethoxyacetylene with benzoic acid.

An absolute ethereal solution of 7 g of ethoxyacetylene in 35 cm^3 of ether was added to 24,4 g of benzoic acid. The mixture was refluxed for 3 hrs. The acid gradually dissolved and the odour of the ethoxyacetylene

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[1]) Unless otherwise stated all distillations were carried out at 700 mm, the ordinary pressure at Bandung.

disappeared. A rather large amount of benzoic acid was however unchanged. On cooling to — 10° it separated (7½ g). The filtrate was slowly distilled through a 20 cm column packed with glass helices. After a large forerunning of ether we obtained a fraction bp. 70 — 72 consisting of ethyl acetate (1½ g). This yield is rather low but can be explained by considerable losses during the distillation. A part of the acetate is entrained with the fore runnings and a part remains in the residue. The residue was dissolved in ether and washed with ice cold 5 % sodiumhydroxide. (Upon the addition of hydrochloric acid the aqueous layer yielded 6,7 g of benzoic acid). The ethereal extract was dried and evaporated and left a residue of 9½ g of benzoic anhydride (mp. 41 — 42°).

e. Reaction of ethoxyacetylene with formic acid.

To a solution of 19 g of 98 — 100 % formic acid in 20 cm³ of abs. ether was dropped under stirring but without cooling, a mixture of 14 g (0,2 mole) of ethoxyacetylene and 35 cm³ of ether. A large amount of gas evolved (about 5 liters). This proved to be carbon monoxide. After the reaction was complete a sample was titrated with 0,1 n sodium hydroxide. 0,263 mole of acid had remained, so that 0,150 mole had reacted with the ethoxyacetylene. The reaction mixture was washed with water, neutralised with sodium carbonate and distilled. We obtained 12½ g of ethylacetate (bp. 74°).

f. Reaction of ethoxyacetylene with cinnamic acid.

A solution of 7½ g of ethoxyacetylene in 12 cm³ of ether was added to 10 g of dry, finely powdered cinnamic acid. The mixture was refluxed under stirring for 2 hrs. The acid gradually dissolved. After standing overnight a crystalline solid was formed. This had a mp. of 130 — 134° and gave no depression of the mp. with the anhydride of cinnamic acid (mp. 136°). Yield 7¼ g. The filtrate was distilled and yielded a fore-running of ether and then 2,3 g of ethylacetate. The residue crystallised and weighed 2 g.

g. Reaction of ethoxyacetylene with (trans) crotonic acid.

A solution of 7½ g of ethoxyacetylene in 12 cm³ of ether was added to 14 g of crotonic acid. The reaction started after heating to 40°. The mixture was refluxed for 1½ hrs. and then fractionally distilled by means of a 20 cm column packed with glass helices. After a fore running of ether we obtained 5¾ g of ethylacetate bp. 71 — 74°. Distillation was continued in vacuum (25 mm). Between the pump and the receiver we used a trap cooled in ice salt, in order to condense the remainder of the ethylacetate. The boiling point at once rose to that of crotonic anhydride (135 — 140°, 25 mm). Yield 10¾ g. Residue 1 g. The trap contained 2 g of ethylacetate.

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July 1950.

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CHEMISTRY

INFLUENCE OF ORGANIC COMPOUNDS ON SOAP- AND PHOSPHATIDE COACERVATES — XIV [1]

The action of fatty acids, alcohols and esters on coacervates of a sulfate soap

BY

H. L. BOOIJ AND MISS E. S. VAN CALCAR [2] [3] [4]

(Communicated by Prof. H. G. BUNGENBERG DE JONG at the meeting of June 24, '50)

1. *Introduction.*

The influence of many series of organic compounds on soap and phosphatide coacervates have already been investigated by BUNGENBERG DE JONG and collaborators (see H. L. BOOIJ, 1949). For the investigation of the esters, however, there are some objections against the usual substrata.

1°. With oleate we work at a high p_H, so saponification may occur and no reliable results will be obtained.

2°. With laurylsulfate we can work at low p_H but to obtain coacervates of this soap we must use a certain amount of menthol or another salt sparing substance which complicates the method (see BOOIJ and VREUGDENHIL, 1950).

3°. Coacervates of cetylsulphate can only be obtained at rather high temperatures (66° C) so that here too saponification may occur (see BOOIJ and VREUGDENHIL, 1950).

So what we want is a sulfate soap (possibility to work at a low p_H) of which coacervates can be obtained at low temperatures without adding some salt sparing substance.

Now all these restrictions are fulfilled in the case of a sulfate soap called T-pol of the following composition: 21 % sodium sec. alkylsulfate, 5 % sodiumsulfate and 74 % water [5]). The general formula of the sodium sec. alkylsulfate is

$$\begin{array}{c} \mathrm{H} \\ | \\ \mathrm{R_1{-}C{-}R_2} \\ | \\ \mathrm{OSO_3Na} \end{array}$$

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[1]) Publication no. XIII of this series will be found in Proc. Kon. Akad. Wetensch. Amst. 53, 882 (1950).

[2]) We are much indebted to the Koninklijke/Shell Laboratorium for a generous gift of T-pol.

[3]) We wish to express our sincere thanks to the N.V. POLAK & SCHWARZ's Essencefabrieken (Hilversum), who were so kind to send us a great number of esters for our investigations.

[4]) Publication no. 9 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

[5]) We are much indebted to Dr Ir S. L. LANGEDIJK for the informations about the structure and composition of the sample of T-pol.

R_1 and R_2 are alkylgroups with a normal chain, R_1 being in the main CH_3 or C_2H_5 while the total number of carbon atoms of the molecule varies from 8 — 18 (for further particulars see MELSEN, 1950).

Furthermore the solubility in this soap of a large number of esters was rather high which made it easy to test them. The objection against the use of T-pol is, that it consists of a mixture of various carbon chains and *a priori* no micelles with good arrangement, as with oleate, laurylsulfate etc. are to be expected.

We used the following method:

T-pol was diluted 1 : 1 with a sodiumacetate — acetic acid buffer composed of 6 parts 0,2 mol. acetic acid and 94 parts 0,2 mol sodiumacetate ($p_H = 5{,}8$). This provides us with the so-called "blank" solution. 5 ml of this solution was added, in a calibrated tube, to a mixture of x ml 4,5 n $MgCl_2$ solution and $(10 - x)$ ml aqua dest., whereby x is varied.

Hitherto KCl solutions were commonly used to bring about coacervation, but in the case of the T-pol no coacervation could be obtained with whatever KCl solution, reason why a $MgCl_2$ solution was used. At a certain $MgCl_2$ concentration coacervation appears. Now, five different concentrations of $MgCl_2$ were taken. One whereat just no coacervation appears and further those concentrations that are exactly 0,015 mol l higher every time. The tubes were kept in a thermostate of 25° for 24 hours; unmixing appears into two layers of which the coacervate layer is the top one.

The concentration of the $MgCl_2$ in the tube is plotted against the volume of the coacervate layer. This supplies us with the so-called "blank" curve. Now the substance whose influence on the coacervation we want to know, is added to the "blank" solution in two different concentrations. Repeating the above experiments, this supplies us with two curves which are shifted with regard to the "blank" one over a certain distance. Now a substance is said to exert a salt-sparing action (here $MgCl_2$-sparing) if at a certain $MgCl_2$ concentration the volume of the coacervate layer is smaller than in the case of no substance being present. In the reverse case we call a substance to exert a salt-demanding action.

In a diagram the number of mmols substance being present in the tube is plotted against the shift of the curve with regard to the blank, measured in differences in concentration at a coacervate percentage of 60 %.

Before we start the investigation of the esters we will first look at the "components" viz. the acids and the alcohols.

2. *Fatty acids.*

Acids were investigated from valeric acid up to and including myristic acid. First the influence of hydrochloric and acetic acid were investigated in order to be sure that the established effect could not be due to shifts of the p_H. But no influence at all was found for hydrochloric and acetic acid within the range of concentrations where the investigation took place.

In fig. 1 the action at a concentration of 1 mmol/l is plotted against the number of carbon atoms. It is seen that the action gradually increases with increasing number of carbon atoms. This increase, however, becomes relatively smaller at about eight carbon atoms. If we compare these results with those obtained with oleate (Booij and Bungenberg de Jong, 1949) we find an altogether different behaviour. In the case of the oleate a salt-demanding influence was found, whilst here we find a salt-sparing influence. With oleate, however, a high p_H prevailed so that

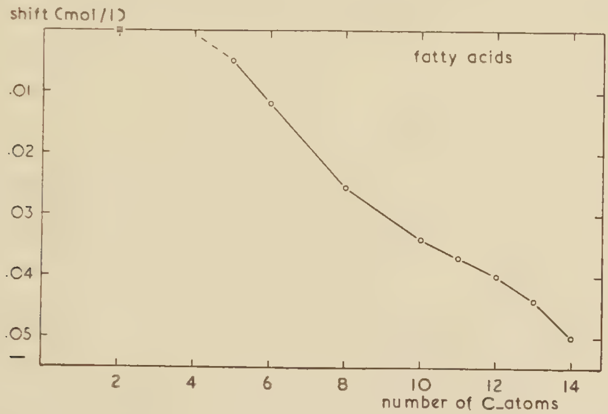

Fig. 1. Action of fatty acids (1 mmol/l) on a coacervate of T-pol. The activity is expressed as the shift of the coacervation curve in mol/l $MgCl_2$.

the fatty acid was dissociated for the most part. Consequently the action of the fatty acid anion was measured whilst we measured the action of the molecule. Moreover the oleate micelle is an ordered one while presumably the T-pol micelle is not, as we have mentioned already. With lauryl- and cetylsulfate a KCl-sparing influence was found (Booij and Vreugdenhil, 1950). As here too a low p_H prevailed the differences in the course of the curves must be sought herein, that the lauryl- and cetylsulfate micelles are ordered whilst the T-pol micelle is not.

3. *Alcohols.*

Again in fig. 2 the action at a concentration of 1 mmol/l is plotted against the number of carbon atoms. Here too it is seen that the action gradually increases with increasing number of carbon atoms, however, only up till 9 carbon atoms whereafter no further increase was observed.

Comparing these results with those obtained with oleate (Booij, Vogelsang and Lycklama, 1950) and laurylsulfate (Booij and Vreugdenhil, 1950) we observe for the former that no influence was found from 2 up till and including 4 carbon atoms whereafter an increase in action was observed up till and including 14 carbon atoms; ethyl alcohol, showed a slight salt-demanding action. With the latter, however, ethylalcohol also showed a slight salt-sparing action whereafter an increase in

action was found up till and including 12 carbon atoms. The curve reaches its minimum here and proceeds in an upward direction again. We think

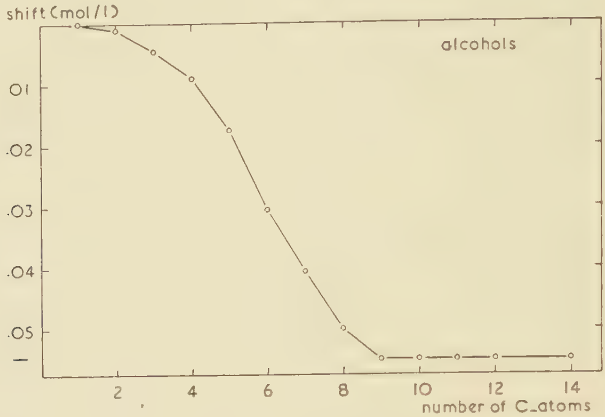

Fig. 2. Influence of normal alcohols (1 mmol/l) on a coacervate of T-pol. Here — as in the case of the fatty acids — we find a salt-sparing action.

that the flat part of the curve with T-pol as substratum finds its explanation in the fact that there is no proper order in the interior of the micelles. A pronounced maximum of activity in a homologous series is only possible if the substratum is very regularly built.

4. *Esters.*

In fig. 3 the action at a concentration of 1 mmol/l is plotted against the number of carbon atoms. It is seen that with the acetates, butyrates and the esters of ethylalcohol the action gradually increases with increasing number of carbon atoms on the understanding that from about six carbon atoms this increase becomes relatively smaller. The curves

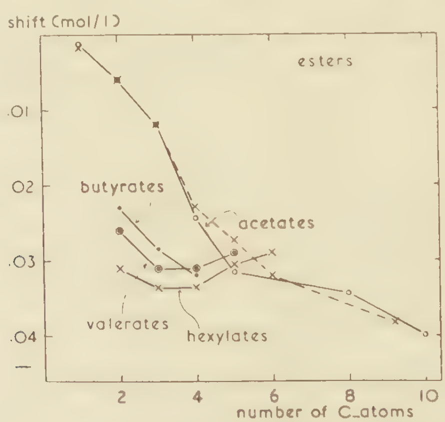

Fig. 3. Action of some series of esters on the T-pol coacervate. The abscissa gives the number of carbon atoms of the alcohol part of the ester. The dotted line shows the influence of the ethylesters (here e.g. 6 = ethylhexylate). Obviously the curves of the acetates and the ethylesters agree rather well.

for the valerates and hexylates show a minimum at 3 — 4 carbon atoms whereafter they proceed again in an upward direction.

The comparison of these esters with the corresponding acids and alcohols may be carried out in two ways viz. we may compare the esters with the acids and alcohols out of which they are composed or we may compare them with those compounds which have the same total number of carbon atoms. If we compare the curves for the esters with the one for the acids we find that they always remain below the latter, which means that the esters all exert a stronger action than the acids out of which they are composed and even than the acids with a same total number of carbon atoms. This is not surprising as the polar nature of the acid group is much stronger than that of the esters.

If we compare the curves for the esters with the one for the alcohols we find that the curves intersect at a value of about 6 carbon atoms. This was to be expected as the curve for the alcohols runs below the one for the acids and moreover the former has a steeper course. The intersection happens at a lower value of the number of carbon atoms for the curve of the hexylates than for the curve of the acetates as a result of the fact that the former displays a minimum at 3 — 4 carbon atoms.

In fig. 4 the action of esters with the same number of carbon atoms at the alcohol as well as at the acid side is plotted against this number. The curve shows a minimum at 4 carbon atoms indicating that the phenomenon

Fig. 4. The action of esters with an equal number of carbon atoms at the acid and the alcohol side of the molecule on a coacervate of T-pol (here 6 means hexylhexylate etc.). A pronounced maximum of activity is observed in this homologous series.

— decrease of the salt-sparing influence with increasing number of carbon atoms — only occurs if the carbon chain has reached a certain length as well at the alcohol as at the acid side of the ester molecule.

A similar series of experiments was carried out with the benzoates, phenylacetates and phenylpropionates, the results of which are plotted in fig. 5. The same phenomenon was observed as with the long aliphatic esters. The minimum is displaced towards lower numbers of carbon atoms

at the alcohol side, going from the benzoates to the phenylpropionates. In accordance with earlier investigations (BUNGENBERG DE JONG, SAUBERT and BOOIJ, 1938) it was found that the benzene ring has the

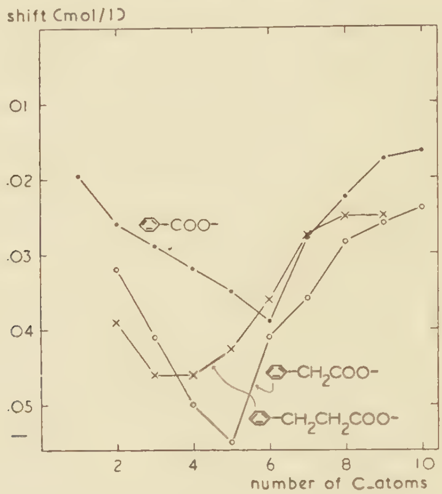

Fig. 5. Influence of esters of aromatic acids on a T-pol coacervate, plotted against the number of carbon atoms of the alcohol chain.

same influence as a normal aliphatic chain of about 4 carbon atoms. For methylvalerate and ethylvalerate exert the same action as respectively methylbenzoate and ethylbenzoate. A cyclohexyl ring has the same influence as a normal aliphatic chain of about 5 — 6 carbon atoms, cyclohexylacetate exerting an influence which is somewhat greater than that of amylacetate. This too is in accordance with earlier experiments (BUNGENBERG DE JONG, SAUBERT and BOOIJ, 1938).

5. *Discussion.*

The most important phenomenon is the fall in activity with increasing number of carbon atoms, which was found with the higher esters. All the other phenomena which were observed are in accordance with earlier experiments. There are indications (BOOIJ, 1949) that in coacervates micelles are present in which the molecules are arranged in a parallel manner. The salt-sparing influence (of alcohols etc.) is owing to an increase in the London-v. d. Waals attraction forces between the hydrocarbon chains. The salt-demanding influence (fatty acid anions) is due to an increase of the repulsive Coulomb forces of the charged polar groups without a proportional increase of the London-v. d. Waals forces. When a certain amount of the substance is added to the soap solution an equilibrium will arise whereby part of the substance will be taken up by the micelle and part of it will be restricted in the medium. Now when the carbon chain is lengthened one must discern two effects:

With increasing length of the carbon chain the substance becomes

less soluble in water (medium) and so the equilibrium will be shifted in favour of the micelle. In this way we could already explain the ever increasing salt-sparing effect which occurs when the carbon chain is lengthened. A second effect, however, takes place with molecules where the polar groups are very weak or absent. Then it is supposed that these molecules are taken up within the micelle (between the CH_3-end groups of the carbon chains of the substratum). This may eventually result in a salt-demanding action (BOOIJ, 1949).

A. If an ester has a short and a long carbon chain (e.g. decylacetate) one would suppose that the polar group is in a line with the heads of the soap molecules. In such a homologous series an ever increasing (salt-sparing) effect would be expected. Fig. 6*A* shows how these esters are presumably arranged in the micelle.

The problem arises what place an ester with two long carbon chains will take in the micelle. Both chains will be pushed out of the water and several possibilities (fig. 6*B*, *C* and *D*) are coming to the fore.

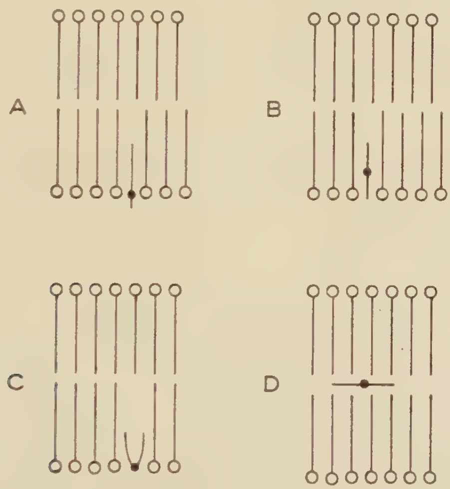

Fig. 6. Possible places of esters within the soap micelle (see text).

B. The molecule is pushed into the micelle, the weak polar group losing its anchorage to the water. It is possible that this would result in a lower salt-sparing action than case *A*.

C. The polar group remains in contact with the water, but the two chains are extended towards the interior of the micelle. It is expected that this would result in a strong decrease of the salt-sparing action as the disturbance of the arrangement in the micelle is very great. This disturbance, however, would become less as the carbon chains of the ester are much lengthened, which would be seen once more as an increase of the salt-sparing activity. This effect in a homologous series (increase of salt-sparing activity, then decrease and finally increase again) we did not find (see fig. 4). Thus scheme *C* does not seem likely to us.

D. It was supposed that aliphatic alkanes — when long enough — would be concentrated in the inside of the micelle. This leads to the view that even a substance with a polar group might go to this place, provided that *a*) the polar group is not situated at the end of the molecule and *b*) that the distance between the polar group and the nearest end of the molecule is at least four carbon atoms. This situation would result in a salt-demanding activity. It might be that the very slight salt-sparing action of long esters which have the polar group in the middle of the molecule (fig. 4) is due to this salt-demanding factor. Then in this homologous series the action would show a maximum, as the partition of the molecules between the places *A*, *B* and *D* (fig. 6) would shift ever more to the right, when the carbon chains of the esters are lengthened.

ALEXANDER and SCHULMAN (1937) have already found with their experiments on the spreading in monolayers of the esters, that an ever increasing energy is necessary to push a longer carbon chain into the water. Long esters viz. cetylpalmitate form a film with the two chains adjacent to one another and both chains standing up from the water surface. This does not implicate that for our experiments schema *C* (fig. 6) is realised as with lengthening of the carbon chain, the anchorage of the water attracting groups is diminished, leading to the possibilities *B* and *D* (see ADAM 1941).

The investigation will be extended with experiments on the speed of saponification of the esters and their spreading in monolayers.

Summary.

1. The influence of fatty acids, normal alcohols and esters on a coacervate of T-pol has been measured. T-pol was selected because with $MgCl_2$ it gives a coacervate at low temperature and at slightly acid p_H.

2. The activity of fatty acids and alcohols increases with increasing number of carbon atoms (up till 9 carbon atoms for alcohols). The fact that the value of the increase diminishes with the higher homologues might be due to the fact that the T-pol micelle is not as regularly built up as e.g. the oleate micelle.

3. In homologous series of esters two different types were found:
a) the activity increases with increasing length of the carbon chain (this was found with esters which have a short and a long chain while the length of the long chain was varied);
b) the activity shows a maximum (this we found with those esters where both chains have at least four carbon atoms).

4. We discussed the possibilities of the situation of the ester molecules in the soap-micelles (fig. 6).

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MATHEMATICS

COMPACT SPACES WITH A LOCAL STRUCTURE DETERMINED BY THE GROUP OF SIMILARITY TRANSFORMATIONS IN E^n

BY

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(Communicated by Prof. W. VAN DER WOUDE at the meeting of June 24, 1950)

§ 1. *The problem.*

The spaces X we want to consider are n-dimensional manifolds with a local structure which we shall define with the help of the similarity space Es^n as a fixed reference space. Es^n is the Euclidean n-dimensional space assigned with the group I of similarity transformations ("similarities") i.e. the group generated by the Euclidean motions and the geometrical multiplications.

A *reference system* is a topological mapping Φ of a (open) domain V in X onto a domain $\Phi(V)$ in Es^n. It carries the local structure (e.g. angles, straight lines) of Es^n in V. Two reference systems (Φ, V) and (ψ, W) are said to agree, if for every component U of the intersection $V \cap W$, the topological mapping $\psi\Phi^{-1} : \Phi U \to \psi U$ is a similarity in Es^n. If all reference systems of a set agree mutually, then they determine a unique local structure in the covered pointset of X. The set is then called *a set of preferred reference systems*. Any additional reference system is preferred if it agrees with all others. A set of preferred reference systems, covering X, is called complete if it contains any reference system that agrees with it. We now define: An n-dimensional manifold X is called (*locally*) *sim E* [1]), if it is covered by a complete set of preferred reference systems, the reference being with respect to Es^n.

Our problem is analogous to the space problem of CLIFFORD—KLEIN ([2], [4]) and consists in *the examination of all compact* (= X is covered by a finite subset of any set of neighbourhoods that cover X.) *connected locally sim E spaces* X. The results are stated in the theorem at the end of the paper.

§ 2. *Preliminaries.*

If X is sim E and connected, then so is the universal covering space $\check{X}$ of X. It is well known that in $\check{X}$ a discrete group of topological transformations without fixed point, the covering group D, operates. Any $d \in D$ leaves the sim E structure of $\check{X}$(!) invariant. A point of X can be

[1]) In a theory of spaces with generalised displacements, the name "similarity-flat" would be appropriate.

considered as a point set $D\tilde{x}(\tilde{x} \in \tilde{X})$ of $\tilde{X}$. D is isomorphic to the fundamental or POINCARÉ group of X.

For simply connected sim E spaces, e.g. $\tilde{X}$, the following theorem holds.

Lemma 1. If $\tilde{X}$ is simply connected and sim E, then a mapping $\Phi : \tilde{X} \to \tilde{X}^* \subset Es^n$ exists, such that any point $\tilde{x} \in \tilde{X}$ has a neighborhood $N(\tilde{x})$ for which $(\Phi, N(\tilde{x}))$ is a preferred reference system. Any two such mappings Φ, ψ obey a relation $\Phi = g \cdot \psi$, where $g \in I$ is a similarity.

We omit the proof of this lemma which is a consequence of an analogous theorem on conformally flat spaces (KUIPER [5], th. 4; analogous theorems with analogous proofs hold for a large class of spaces, the locally homogeneous spaces [3], [7]).

If Φ is fixed according to lemma 1, and $d \in D$, then Φd is another mapping with the same properties as Φ. Hence $d^* \in I$ exists such that $\Phi d = d^* \Phi$ (lemma 1). It is easily seen that the correspondence $d \to d^*$ is a homomorphism of D onto a subgroup $D^* \subset I$. If $\Phi : \tilde{X} \to \tilde{X}^*$ is topological then D and D^* are isomorphic.

From now on X denotes a compact connected sim E space. X then admits a finite triangulation, and this yields in the natural way a triangulation of $\tilde{X}$. It is possible to choose an open *fundamental domain* F in $\tilde{X}$, composed of simplices of the triangulation, with the following properties: The covering mapping $f : \tilde{X} \to X$, restricted to F, is topological $f : F \longleftrightarrow f(F) \subset \subset X$; $f(\bar{F}) = X$ ($\bar{F}$ is the closure in $\tilde{X}$ of F); the boundary $\bar{F} - F$, as well as its image $f(\bar{F} - F)$ is the union of a finite number of closed n—1-dimensional simplices, called faces: If d_1 and d_2 are different elements of the covering group D, carrying F onto the fundamental domains $d_1(F)$ and $d_2(F)$ respectively, then $d_1(F) \cap d_2(F)$ is void: However the union of all $d(\bar{F})$ for $d \in D$ is $\tilde{X}$. From now on F denotes a fixed fundamental domain, and Φ or (*) a fixed mapping according to lemma 1.

§ 3. *Φ has no boundary points.*

Definition: A boundary point of $\Phi : \tilde{X} \to \tilde{X}^* \subset Es^n$ is the end point $\Phi(0)$ of a continuous curve $\Phi(t)$, $0 \leq t \leq 1$, in Es^n with the property: There exists a curve $\tilde{x}(t) \subset \tilde{X}$ such that $\Phi(\tilde{x}(t)) = \tilde{x}^*(t) = \Phi(t)$ for $0 < t \leq 1$; there does not exist a point $\tilde{x}(0) \subset \tilde{X}$, such that moreover $\tilde{x}(t)$ is a continuous curve for $0 \leq t \leq 1$.

If Φ has no boundary points then: $\tilde{X}^* = Es^n$; $(\Phi, \tilde{X})$ is a covering-space (in the topological sense) of $\tilde{X}^* = Es^n$; and as Es^n is simplyconnected, Φ is a topological mapping and D and D^* are isomorphic. Suppose $d^* \in D^*$ is a similarity transformation which is not a Euclidean motion. Then d^* has a "factor of multiplication of distances" $f < 1$ (if $f > 1$, we take $(d^*)^{-1}$), and an invariant point, say A, in Es^n. Let $d \in D$ correspond

with d^* under the isomorphism $D \longleftrightarrow D^*$; Let $\tilde{x}(t)$, $f \leqq t \leqq 1$, be a continuous curve with endpoints $\tilde{x}(1)$ and $\tilde{x}(f) = d\,\tilde{x}(1)$. Finally let $\tilde{x}(t)$, $0 < t \leqq 1$, be defined by

$$\tilde{x}(t) = d^k \cdot \tilde{x}(f^{-k} \cdot t) \quad \text{for} \quad f^{k+1} \leqq t \leqq f^k,\ k = 1, 2, \ldots$$

Then $\lim_{t\to 0} \Phi(\tilde{x}(t)) = A$ is a boundary point in contradiction with the assumptions. Hence D^* contains only motions. Therefore it is possible to introduce a locally Euclidean metric in X, and we have:

Lemma 2. If X is compact and sim E, and $\Phi : \tilde{X} \leftarrow \tilde{X}^*$ has no boundary points, then Φ is topological, and X can be considered as a compact locally Euclidean space of which only the local sim E structure is taken into account.

§ 4. *Boundary points of* Φ.

Let B be a boundary point of $\Phi : \tilde{X} \to \tilde{X}^* \subset Es^n$, defined by $\tilde{x}(t)$ and $\Phi(\tilde{x}(t)) = \Phi(t)$, $0 < t \leqq 1$, as above. Suppose $\tilde{x}(t)$ meets for t running from 1 to 0 successively the closed fundamental domains $d_\nu(\bar{F})$, $\nu = 1, 2, \ldots$. If this were only a finite number of pointsets, with a compact sum (!), then a point $\tilde{x}_0 \in \tilde{X}$ would exist, in any neighborhood of which points $\tilde{x}(t)$, $t < \varepsilon$, $\varepsilon > 0$ and arbitrary, would occur. The inverse of the topological mapping Φ of some neighborhood of $\tilde{x}_0$ in Es^n, then would show that B is not a boundary point with respect to $\tilde{x}(t)$. Contradiction. Hence $\tilde{x}(t)$ meets an infinite number of closed fundamental domains $d_\nu(\bar{F})$. Also if $\varepsilon > 0$, then a number $N(\varepsilon)$ exists, and $\tilde{x}(t)$ respectively $\Phi(t)$, $t < \varepsilon$, does meet the pointset $d_\nu(\bar{F} - F)$ respectively $\Phi d_\nu(\bar{F} - F) = d^*_\nu(\bar{F}^* - F^*)$ for all $\nu > N(\varepsilon)$. (1)

Similarities g of which the factor of multiplication of distances, denoted by $|g|$, is bounded: $0 < p \leqq |g| \leqq q < \infty$, p, q constants, form a compact subset of the group space I. Hence if the factors $|d^*_\nu|$ are bounded in this way, then some subsequence $d^*_{\nu(\mu)}$ of d^*_ν will converge to a similarity say d^*. If μ is large, then $d^*_{\nu(\mu)}$ transforms some point of $\Phi(\bar{F}) = \bar{F}^*$ into a point close to B. Without restriction we may then assume F to be such that B lies in the interior of $d^*(F^*)$. This implies that B has a neighborhood which does not contain any of the points of $d^*_{\nu(\mu)}(\bar{F}^* - F^*)$ for μ sufficiently large; in contradiction with (1).

Because the factors $|d^*_\nu|$ are not bounded, a subsequence $d_{\nu(\mu)}$ of d_ν will exist for which $\lim_{\mu\to\infty} |d^*_{\nu(\mu)}| = \infty$ or $= 0$. In the first case $\lim_{\mu\to\infty} |d^*_{\nu(\mu)})^{-1}| = 0$. If μ is large, then $(d^*_{\nu(\mu)})^{-1}$ transforms some point very near to B (in the intersection of $\Phi(t)$ and $d^*_{\nu(\mu)}\bar{F}(!)$) into $\bar{F}^*$. Because moreover $|(d^*_{\nu(\mu)})^{-1}|$ is very small, $(d^*_{\nu(\mu)})^{-1} \cdot \Phi(t)$ will prove to have a limit point for $t \to 0$, which is not a boundary point with respect to $(d^*_{\nu(\mu)})^{-1} \cdot \tilde{x}(t)$. But then $\lim_{t\to 0} \Phi(t) = B$ is not a boundary point with respect to $\tilde{x}(t)$ either.

Finally we come to the conclusion that d^*_ν contains a subsequence $d^*_{\nu(\mu)}$,

with factors, that tend to zero. Because moreover the pointsets $d_\nu^*(F^*)$ contain points that tend to B, the invariant points of the similarities $d^*_{\nu(\mu)}$ also tend to B. This proves

Lemma 3. If B is a boundary point, $N(B)$ a neighborhood of B, U a bounded pointset in Es^n, then D^* contains an element d^*, such that $d^*(U) \subset N(B)$.

§ 5. *The set of boundary points.*

Lemma 4. *Let V be a closed subset of the Euclidean space E^n, not contained in any E^{n-1}; $n \geqq 0$. Let G be a subgroup of the similarity group operating in E^n, such that for any $g \in G : gV = V$.*

Moreover suppose that for any point B with neighborhood $N(B)$ in V and any bounded pointset U in E^n, there exists an element $g \in G$ such that $g \cdot U \subset N(B)$.

Then $V = E^n$.

(Roughly: small parts of V are similar to large parts).

Proof:

The lemma is trivial for $n = 0$. We suppose $n > 0$.

First we agree that any neighborhood under consideration will be the interior of a hyper sphere. Let $U(A)$ and $N(A) \subset U(A)$ $(N(A) \neq U(A))$ be neighborhoods of the point $A \in V$. Let $g \in G$ be such that $g \cdot N(A) \subset \subset g \cdot U(A) \subset N(A)$. Straightforward geometrical consideration as well as the Brouwer fixed point theorem then show that $N(A)$ contains a point A^* invariant under g. Let the factor of multiplication of distances of g be $|g| < 1$. The set of points $g^m \cdot A$ $(m = 1, 2, \ldots)$ all belonging to the closed set V, converges to A^*, which therefore also belongs to V.

Let $|g|$ also denote the geometrical multiplication with centre A^* and with the same factor of multiplication of distances as g. $h = g \cdot |g|^{-1}$ is then a rotation about A^*. Suppose the point $B \neq A$ is in $V \cap U(A)$. The set of points $h^m B (m = 1, 2, \ldots)$ has at least one limit point on the hypersphere with centre A^* and radius $=$ distance (A^*, B). Therefore if $\varepsilon > 0$, integers m_1 and $m_2 = m_1 + m > m_1$ exist, for which

$$\text{angle } (h^{m_1} B, A^*, h^{m_2} B) = \text{angle } (B, A^*, h^m B) = \text{angle } (B, A^*, g^m B) < \varepsilon.$$

Because $g^m \cdot B = C$ is a point of V as well as of $N(A)$ we have:

Statement 1. If A and B are different points in V, $N(A)$ is a neighborhood of A, $\varepsilon > 0$, *then* there exist points A^* and C in $V \cap N(A)$, for which angle $(B, A^*, C) < \varepsilon$.

This statement allows us to verify:

Statement 2. There exists a set of points $A_1, A_2^*, A_3^*, \ldots, A_{n+1}^*$, R, all in V, such that R is interior point of the nondegenerated simplex (Euclidean, not only topological) with the other points as vertices.

The existence of $n + 1$ points $A_1, A_2, \ldots, A_{n+1}$ in V, but not in any E^{n-1}, follows from the conditions in the lemma. We replace A_2 by two

points A_2^* and C_1 both in V, near to A_2 and such that the angle (A_1, A_2^*, C_1) is so small that C_1 lies on the same side of the hyperplanes spanned by $A_2^*, A_3, A_4, \ldots, A_{n+1}$ and by $A_1, A_3, \ldots, A_{n+1}$ as the inside of the simplex with vertices $A_1, A_2^*, A_3, \ldots, A_{n+1}$ does. Statement 1 allows this choice.

In the next step we replace A_3 and C_1 by two points A_3^* and C_2 in V, near to A_3, and such that C_2 lies on the same side of the hyperplanes $(A_2^*, A_3^*, A_4, \ldots, A_{n+1})$, $(A_1, A_3^*, A_4, \ldots, A_{n+1})$ and $(A_1, A_2^*, A_4, \ldots, A_{n+1})$ as the inside of the simplex $(A_1, A_2^*, A_3^*, A_4, \ldots, A_{n+1})$. Statement 1 allows this choice.

Continuing with analogous steps we finally obtain the required set of points $A_1, A_2^*, A_3^*, \ldots, A_{n+1}^*$ and $C_n = R$.

Let $C(T, \psi)$ be a solid hypercone in E^n, that is the locus of halflines which in their fixed endpoint T meet one of these half lines under angles $\leqq \psi < \pi/2$. Let us denote the part of a $C(T, \psi)$ which is between or in two concentric hyper spheres with centre T and radii r and $r \cdot f (r > 0, 0 < f \leqq 1)$ by $CS(T, \psi, f)$. Then we are able to formulate:

Statement 3. There exists a point $R \in V$, an angle $\psi < \pi/2$ and a number $f > 0$, such that any $CS(R, \psi, f)$ contains at least one point of V.

Consider a configuration $A_1, A_2, \ldots, A_{n+1}, R$ of points in V, for which R lies in the interior of the simplex $A_1, \ldots, A_{n+1}$. Then there exists a number $\psi < \pi/2$, such that any solid cone $C(R, \psi)$ contains at least one of the points $A_1, \ldots, A_{n+1}$. Even ψ can be choosen such that the same is true if we replace R by any point in a sufficiently small neighborhood $N(R)$ of R. So we do. Now let $g \in G$ transform a large neighborhood $U(R)$ of R onto $g \cdot U(R) \subset N(R)$. g has an invariant point in $N(R) \cap V$. Without restriction we may assume that this point is R. Let $|g| < 1$ be the factor of multiplication of distances of g, and let $p \leqq 1$ be the ratio of the minimum and the maximum of the distances $(A_1, R), (A_2, R), \ldots, (A_{n+1}, R)$. Then the property of statement 3 holds for any

$$CS(R, \psi, f) = CS(R, \psi, |g| \cdot p/2)$$

(Consider the set of points $g^m A_i \subset V$; $m = 1, \ldots; i = 1, \ldots, n+1$.)

Statement 4. If SS is a solid hyper sphere in E^n, $\psi < \pi/2$, $0 < f \leqq 1$, then for any point P sufficiently near to SS, there exists a $CS(P, \psi, f)$ which is completely contained in the interior of SS.

This follows immediately from the analogous statement for the Euclidean plane E^2.

Assumption. Now suppose $V \neq E^n$. Then because V is closed, there exists a solid sphere SS in E^n, the inside of which has no point in common with V, but the boundary of which intersects V in at least one point T. (V is not void by assumption !). There exists a transformation $h \in G$, which transforms the point R of statement 3 into a point so near to the point T of the boundary of SS, that there exists a $CS(hR, \psi, f)$ completely contained in the interior of SS (Conditions in the lemma; statement 4;

ψ and f are the constants obtained in the proof of statement 3). That particular $CS(hR, \psi, f)$ *does not contain any point of* V, because the interior of SS does not. On the other hand h is a $1 \longleftrightarrow 1$ similarity transformation which transforms all $CS(R, \psi, f)$ onto all $CS(hR, \psi, f)$ and which leaves V invariant. Because all $CS(R, \psi, f)$ did contain points of V, so do *all* $CS(hR, \psi, f)$. Hence the assumption leads to a contradiction, and $V = E^n$ q.e.d.

As a corollary of lemma 4 we have

Lemma 5.

Let V be a closed subset of E^n; $n \geqq 0$. Let G be a subgroup of the similarity group operating in E^n, and for any $g \in G : gV = V$. Moreover suppose that for any point B in V with neighborhood $N(B)$ and any bounded pointset U in E^n, there exists a $g \in G$ for which $g \cdot U \subset N(B)$.

Then $V = E^m$, $m = -1, 0, 1, \ldots,$ or n.

Proof: Apply lemma 4 to the linear space E^m of smallest dimension which contains V.

Lemma 5 can be applied to our problem: The closure of the set of boundary points in Es^n of the fixed mapping Φ, is invariant under the similarity transformations $d^* \in D^*$, operating in Es^n. Lemma 3 presents the other necessary conditions for application of lemma 5. Hence:

Lemma 6. *The closure of the set of boundary points in* Es^n, *under the fixed mapping* $\Phi : \tilde{X} \rightarrow \tilde{X}^*$, *is a linear space* E^m, $m = -1, 0, 1, \ldots, n$.

§ 6. *The set of boundary points. Continued.*

Assumption 1. In lemma 6 is the number $m = n$.

Let A be the invariant point of the similarity $a \in D^*$ with factor of multiplication of distances $|a| < 1$. Let $\tilde{x}^* \in \tilde{X}^*$ and let the pointset $\Phi^{-1}(\tilde{x}^*)$ consist of the points $\tilde{x}_\lambda (\lambda = 1, 2, \ldots)$. Let, under the homomorphism $D \rightarrow D^*$, d correspond with a. As in § 3 we construct with *each* point $\tilde{x}_\lambda$ as initial point $\tilde{x}_\lambda(1)$, a continuous curve $\tilde{x}_\lambda(t)$ $(0 < t \leqq 1)$ for which $\lim_{t \to 0} \Phi(\tilde{x}_\lambda(t)) = A$ is a boundary point! From that it follows that $\Phi^{-1}(A)$ is void. The set of invariant points like A is dense in E^n (consequence of assumption 1). Assumption 1 leads to the contradiction: $\tilde{X}^*$ is void.

Assumption 2: In lemma 6: $0 < m < n$.

Let A and B be two different invariant points of similarities a and b respectively; $a, b \in D^*$, $|a| < 1$, $|b| < 1$. Consider the set of numbers $|a|^r \cdot |b|^s$ $(r \neq 0,\ s \neq 0$ integers). This set has the number 1 as a limit point. Therefore a sequence $c_\nu = a^{r_\nu} b^{s_\nu} \in D^*$ exists for which $\lim_{\nu \to \infty} |c_\nu| = 1$. From the activity of c_ν on the point B we deduce that the set $(c_\nu)^\mu (\nu = 1, \ldots; \mu = \text{integer } \mu \neq 0)$ has in the similarity group space the identity *not* as an accumulation point.

Let ε be a small positive number. We determine a number $N = N(\varepsilon)$

so large that for any rotation r in a centered Euclidean vector space of dimension n and a vector $\vec{v}$, the angle between $\vec{v}$ and at least one of the vectors $r^{\mu}\vec{v}$, $0 < \mu < N$, is smaller then ε. This is possible because the sphere of unit vectors is compact and has an invariant measure (angle).

Next we choose $c = a^r \cdot b^s$ obeying (c exists!)

$$1 - \varepsilon < |c^{\mu}| = |a^{\nu} c^{\mu} a^{-\nu}| < 1 + \varepsilon \text{ for all } 0 < \mu < N,\ \nu \text{ integer.}$$

We denote the set of boundary points of Φ by E^m. $A, B \in E^m$. Let the line (P^*, A) be perpendicular to E^m, and let $P \in \tilde{X}$, $\Phi(P) = P^* \in \tilde{X}^*$. The points $(a^{\nu} c^{\mu} a^{-\nu})\, A$ are different from A, though for $\nu \to \infty$ they converge to A. There exists a value $\nu = p$, so that the distance $(A - (a^p c^{\mu} a^{-p}) A)$ is smaller then ε for $0 < \mu < N$. Idem for any $\nu < p$.

The points $(a^p c^{\mu} a^{-p})\, P^*$ $(0 < \mu < N)$ are contained in the linear space of dimension $n - m$ perpendicular to E^m in the point $(a^p c^{\mu} a^{-p}) A$.

The angle between the line $\{A \to P^*\}$ and the line

$$\{(a^p c^{\mu} a^{-p}) A \to (a^p c^{\mu} a^{-p}) P^*$$

is, for a particular choice: $\mu = u$, smaller then ε (see above).

Because ε was arbitrary, it now follows (elementary) that any neighborhood $N(P^*)$ of P^* contains points $d^* P^*$ $(d^* \in D^*)$ different from P^* and equivalent with P^*.

If $0 < m < n - 2$ or $m = n - 1$, then $\tilde{X}^*$ determined by E^m is simply connected and the mapping $\varphi : \tilde{X} \to \tilde{X}^*$ is topological. The points aequivalent with P^* under the covering group *cannot* converge to P^* in this case. Contradiction.

If $m = n - 2$, then Φ is not topological and $(\Phi, \tilde{X})$ is the universal covering space of $\tilde{X}^*$. Every point $\tilde{x} \in \tilde{X}$ can be described by the point $\tilde{x}^* = \Phi(\tilde{x})$ and an angle of rotation about the E^{n-2} of boundary points. We call the angle: argument. Let under the homomorphism $D \to D^*$: $a \to a$, $c \to c$. Clearly:

$$\arg\,(a^{\nu} c^{\mu} a^{-\nu})\, P = \arg c^{\mu} P.$$

The points $(a^{\nu} c^{\mu} a^{-\nu}) P$, $\nu < p$, $0 < \mu < N$, are then different and aequivalent under the covering group D in $\tilde{X}$, and they are contained in a compact bounded set of $\tilde{X}$. Hence they have a limit point in $\tilde{X}$: contradiction.

Assumptions 1 and 2 being false, Lemma 6 yields: $m = -1$ or 0.

$m = -1$ was considered in lemma 2. If $m = 0$, $n > 2$, then Φ is topological; the simplyconnected space $\tilde{X}^*$ can be identified with $\tilde{X}$. If $m = 0$, $n = 2$, then $\tilde{X}$ is the wellknown universal covering space of a plane with exception of a point. If $m = 0$, $n = 1$, $\tilde{X}^*$ is a Euclidean half line. (The case $m = 0$, $n > 2$ was considered in KUIPER [6] section 5*b*). We finally state our results in the

Theorem. *The universal covering space* (*with preservation of the sim E structure* !) *of a compact connected locally sim E space of dimension* $n > 0$ *is either the Euclidean space* Es^n, *or a*) *for* $n > 2$: *the same with exception of one point, b*) *for* $n = 2$: *the universal covering space of the Euclidean plane with exception of one point, c*) *for* $n = 1$: *the Euclidean half line. In the first case the original space is determined* (*identification*) *by a* (*covering-*)*group of Euclidean transformations* (*lemma* 2). *In the second case a*) *idem by a group generated by a subgroup* Ω *of rotations about the excluded point and a similarity* b (b *is not a rotation*) *with the excluded point as invariant point, and which commutates with* Ω: $\Omega b = b\Omega$. *In the second case b*) *we use polar coordinates*: (*radius, angle*) $= (r, \omega)$ *in the space* $\tilde{X}$. *The covering group is generated by two transformations*: (1) $r' = r,\ \omega' = \omega +$ constant and (2) $r' = K \cdot r\ (0 < K < 1),\ \omega' = \omega +$ *constant. In the third case c*) *the covering group is generated by a geometrical multiplication*: $r' = K \cdot r\ (0 < K < 1)$.

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MATHEMATICS

A FAMILY OF PARAMETERFREE TESTS FOR SYMMETRY WITH RESPECT TO A GIVEN POINT. II

BY

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(Communicated by Prof. D. VAN DANTZIG at the meeting of June 24, 1950)

6. *Introduction.*

6. 1. In a previous paper on this subject [1]) an exact test has been given for the hypothesis H_0, that n random variables $\mathbf{z}_i (i = 1, \ldots, n)$ are distributed independently, each with a probability distribution, which is symmetrical with respect to zero. We shall now give a generalisation of this test by describing a family of tests for H_0, which contains this one as a special case. The computations involved in the application of the test are described in section 11 and an example is given at the end of this paper in section 12.

6. 2. These tests will be based on the simultaneous application of the sign test, which depends on the number of positive and negative values among $z_1, \ldots, z_n$, and on the application of a parameterfree two sample test to the two groups of values $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ defined in section 3.

A two sample test is a test for the hypothesis H', that two random samples $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ have been drawn independently from the same population. We shall mainly be concerned with a "family" of two-sample tests, consisting of those two-sample tests, which are based on the fact, that, assuming H' to be true, all partitions of the $n_1 + n_2$ values x_j and y_k of the two samples, taken together, into two samples of n_1 and n_2 values respectively, have the same probability. This fact may also be expressed by saying, that, if H' is true and the samples are drawn in a fixed order, all permutations of the obtained values are equally probable.

6. 3. Several two-sample tests have been developed on this basis, e.g. by E. J. PITMAN (1937), N. SMIRNOFF (1939) (using a theorem developed by A. KOLMOGOROFF (1933)), A. WALD and J. WOLFOWITZ (1940) and F. WILCOXON (1945). Wilcoxon's test was studied in detail by H. B. MANN and D. R. WHITNEY (1947).

7. *The main theorem.*

7. 1. Let T be a two sample test of the type described above and let

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1) These Proceedings 53, 941—955 (1950).

$\mathbf{u}_1, \ldots, \mathbf{u}_\nu$ be the statistics, on which T is based. These statistics are known functions of the random variables $\mathbf{x}_1, \ldots, \mathbf{x}_{n_1}$ and $\mathbf{y}_1, \ldots, \mathbf{y}_{n_2}$ and n_1 and n_2 are given numbers. Usually $\nu = 1$, but this is by no means necessary. We shall therefore give the main theorem in the more general form with $\nu \geqq 1$.

Since we are using two-sample tests, which have been developed previously, we may assume the simultaneous distribution of $\mathbf{u}_1, \ldots, \mathbf{u}_\nu$, under the assumption that H' is true, to be known. We shall denote by

$$G^* (u_1, \ldots, u_\nu) \tag{11}$$

the conditional simultaneous distribution function [2]) of $\mathbf{u}_1, \ldots, \mathbf{u}_\nu$, under the condition (denoted by the asterisk), that the two samples, taken together, assume the set of values $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$ and under the assumption, that H' is true.

7. 2. If, instead of the two samples, we take the values $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ defined in section 3, it follows from lemma 3, that, if H_0 is true, if $\mathbf{n}_1 = n_1$ and if condition Z is satisfied, the conditions indicated by the asterisk in (11) are satisfied too, and that (11) is the conditional distribution function of $\mathbf{u}_1, \ldots, \mathbf{u}_n$. We express this fact by changing the notation of this distribution function into

$$G (u_1, \ldots, u_\nu | Z; \mathbf{n}_1 = n_1; H_0) \tag{12}$$

where $u_1, \ldots, u_\nu$ are derived from $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$, the group of positive values and the group of negative values (taken positively) of the original observations $z_1, \ldots, z_n$, which are available to test the symmetry of the variables $\mathbf{z}_1, \ldots, \mathbf{z}_n$.

For $n_1 = 0$ and $n_1 = n - m$ (i.e. $n_2 = 0$) the statistics $\mathbf{u}_1, \ldots, \mathbf{u}_\nu$ have not yet been defined, since one of the groups $x_1, \ldots, x_{n_1}$ or $y_1, \ldots, y_{n_2}$ is empty in that case. Defining for this case $\mathbf{u}_1 = \ldots = \mathbf{u}_\nu = 0$, we find from lemma 2 and 3:

Theorem III: *If H_0 is true, the conditional simultaneous probability distribution of $\mathbf{n}_1$ and $\mathbf{u}_1, \ldots, \mathbf{u}_\nu$, under the condition Z, is given by*

$$\left\{ \begin{array}{l} \mathrm{P}\,[\mathbf{n}_1 = n_1; \mathbf{u}_1 \leqq u_1; \ldots; \mathbf{u}_\nu \leqq u_\nu | Z; H_0] = \\ = 2^{-n+m} \binom{n-m}{n_1} G (u_1, \ldots, u_\nu | Z; \mathbf{n}_1 = n_1; H_0), \end{array} \right. \tag{13}$$

with $0 \leqq n_1 \leqq n - m$.

Remarks: 1. If we want to test the hypothesis H_0', that all $\mathbf{z}_i$ are distributed independently according to the *same* symmetrical probability distribution, T need not be restricted to the family of tests described in 6. 2. For it is easy to prove, that under the hypothesis H_0' and under the conditions $\mathbf{n}_1 = n_1$ and $\mathbf{m} = m$ the values $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ may

[2]) We use the term "distribution function" in the sense sometimes denoted by the term "cumulative distribution function".

be regarded as independent random samples from a common population. This may be of importance, if additional information about the common probability distribution of the $\mathbf{z}_i$ is available, or is contained in the hypothesis to be tested, since we may then use any two sample test based on this information.

2. Theorems I and III enable us to give a test for H_0, based on the statistics $\mathbf{n}_1$ and $\mathbf{u}_1, \ldots, \mathbf{u}_\nu$. Since a family of tests T may be used (cf. section 6. 2), we have a family of tests for H_0. The exact test, described in part I of this paper, is a member of this family as may be seen from remark 3 of section 4. 2. T is then a two sample test based on the statistic $\mathbf{u}$.

8. *The critical region.*

8. 1. In section 7. 1 we have supposed the conditional probability distribution of $\mathbf{u}_1, \ldots, \mathbf{u}_\nu$, under the conditions Z, $\mathbf{n}_1 = n_1$ and hypothesis H_0, to be known, since T is a known two sample test. For the same reason we now assume a critical region for $\mathbf{u}_1, \ldots, \mathbf{u}_\nu$ to have been chosen already. We shall, however, want to make a distinction between *bilateral* and *unilateral* critical regions. To make this clear, the critical regions of some of the two-samples tests mentioned in 6. 3 will be described.

8. 2. WILCOXON's test depends on the number of pairs (x_j, y_k) $(j = 1, \ldots, n_1;\ k = 1, \ldots, n_2)$ with $x_j > y_k$. This statistic, usually denoted by $\mathbf{U}$, can take all values $0, 1, \ldots, n_1 + n_2$. A unilateral critical region has either the form

$$\mathbf{U} - \frac{n_1 n_2}{2} \geqq U_a$$

or

$$\mathbf{U} - \frac{n_1 n_2}{2} \leqq -U_a$$

where U_a depends on n_1, n_2 and the chosen significance level a. The first of these critical regions is suitable for testing the hypothesis H', that $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ are random samples taken independently from the same population, against the alternative (composite) hypothesis, that $x_1, \ldots x_{n_1}$ are independent observations of a random variable $\mathbf{x}$ and $y_1, \ldots, y_{n_2}$ of a random variable $\mathbf{y}$, with

$$\mathrm{P}\,[\mathbf{x} < \mathbf{y}] < \tfrac{1}{2}$$

and the second critical region is suitable for testing H' against the alternative hypothesis, that

$$\mathrm{P}\,[\mathbf{x} < \mathbf{y}] > \tfrac{1}{2}\ ^{3)}$$

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3) This has been proved recently by Prof. D. VAN DANTZIG as a generalisation of MANN and WHITNEY's theorem, according to which WILCOXON's test is consistent against alternatives with $P\,[\mathbf{x} \leqq a] < P\,[\mathbf{y} \leqq a]$ for all a, if the first of the above mentioned unilateral critical regions is used, and consistent against alternatives with $P\,[\mathbf{x} \leqq a] > P\,[\mathbf{y} \leqq a]$ if the second critical region is used. Cf. D. VAN DANTZIG (1947—1950), Chapter 6, § 3.

A symmetrical bilateral critical region for $\boldsymbol{U}$ has the form

$$\left| \boldsymbol{U} - \frac{n_1 n_2}{2} \right| \geq U_{\frac{1}{2}\alpha}$$

and is suitable for testing H' against the alternative hypothesis, that

$$\mathrm{P}\,[\boldsymbol{x} < \boldsymbol{y}] \neq \tfrac{1}{2}.\ ^{3})$$

The probability distribution of $\boldsymbol{U}$ can be computed exactly with the aid of a recursion formula given by MANN and WHITNEY, under the assumption, that H' is true and that $\boldsymbol{x}$ and $\boldsymbol{y}$ have a continuous probability distribution. It has been tabulated by them for $n_1 \leq 8$ and $n_2 \leq 8$ and for larger values the normal distribution with mean $\frac{n_1 n_2}{2}$ and variance $\frac{1}{12} n_1 n_2 (n_1 + n_2 + 1)$ (which is the asymptotic distribution of $\boldsymbol{U}$ for $n_1 \to \infty$ and $n_2 \to \infty$, n_1/n_2 and n_2/n_1 being bounded) is a good approximation.

8. 3. The statistic of PITMAN's test, which we shall also denote by $\boldsymbol{U}$, is the difference of the means of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n_1}$ and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{n_2}$:

$$\boldsymbol{U} = \frac{1}{n_1} \sum_{j=1}^{n_1} \boldsymbol{x}_j - \frac{1}{n_2} \sum_{k=1}^{n_2} \boldsymbol{y}_k.$$

The unilateral critical regions

$$\boldsymbol{U} \leq -U'_\alpha$$

and

$$\boldsymbol{U} \geq U'_\alpha$$

where U'_α depends on the observed values $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$ and the chosen significance level α, are suitable for testing H' against the alternative hypotheses

$$\mathcal{E}\boldsymbol{x} < \mathcal{E}\boldsymbol{y}$$

and

$$\mathcal{E}\boldsymbol{x} > \mathcal{E}\boldsymbol{y}$$

respectively.

A bilateral critical region

$$|\boldsymbol{U}| \geq U'_{\frac{1}{2}\alpha}$$

is suitable for testing H' against the alternative hypothesis

$$\mathcal{E}\boldsymbol{x} \neq \mathcal{E}\boldsymbol{y}$$

The probability distribution of $\boldsymbol{U}$ can be derived exactly from the values $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$. This, however, is only practicable, if n_1 and n_2 are very small. For larger values of n_1 and n_2 PITMAN has given an approximation for the distribution of $\boldsymbol{U}$. The assumption of continuity is not necessary.

8. 4. The test of WALD and WOLFOWITZ is based on the number of runs in the sequence of values x_j and y_k ($j = 1, \ldots, n_1$; $k = 1, \ldots, n_2$)

when arranged according to decreasing magnitude. Small values of this statistic are critical. Its probability distribution is known exactly as well as asymptotically, under the assumption that H' is true and that the probability distribution of $\boldsymbol{x}$ and $\boldsymbol{y}$ is continuous. F. S. SWED and C. EISENHART (1943) have given tables of this distribution for $n_1 \leqq n_2 \leqq 20$.

For this test we shall not try to make a distinction between unilateral and bilateral critical regions, since the class of alternative hypotheses, for which the test is consistent contains nearly all possible alternative hypotheses. It is difficult to see how this class could be divided into two mutually exclusive classes of a kind similar to the two classes of alternatives for Wilcoxon's test or Pitman's test, which have been described in 8. 2 and 8. 3. As far as our application of the test of Wald and Wolfowitz is concerned, its critical region can therefore be taken to be a *bilateral* one.

8. 5. The probability distribution of the statistic of the test of KOLMOGOROFF–SMIRNOFF is known asymptotically only. An exact method for determining the confidence limits for an unknown distribution function (the problem, which had been solved asymptotically by A. KOLMOGOROFF (1933)) has been given by A. WALD and J. WOLFOWITZ (1939). Possibly the method applied by Smirnoff to derive a two sample test from Kolmogoroff's theorem might be applied to this theorem of Wald and Wolfowitz and give an exact two sample test of this type.

So far, however, we have no knowledge either of the exact probability distribution of the statistic of this test, nor of the amount of the divergence between this exact distribution and the asymptotical one, derived by Smirnoff. Apart from this the remarks, made above about the critical region of the test of Wald and Wolfowitz, apply to this test also. No attempt will be made to make a distinction between unilateral and bilateral critical regions. The only difference is, that in this case *large* values of this statistic are critical, and that no continuity of the probability distribution of $\boldsymbol{x}$ and $\boldsymbol{y}$ is needed.

8. 5. We shall now consider the choice of a *critical region* for testing H_0, *if no alternative hypothesis is specified*. In order to simplify the notation, we confine ourselves to $\nu = 1$, i.e. to the case, that the two sample test T is based on one statistic $\boldsymbol{U}$. Denoting the bilateral critical region for T with size ε by $R_{n-m,n_1}(\varepsilon)$, we propose the following construction of a critical region R_1^* with size $\leqq \alpha$ for testing H_0, applicable if $\alpha \geqq 2^{-n+m+1}$:

A. Let k be the largest positive integer $\leqq \frac{n-m}{2}$, for which the relation

$$2^{-n+m}\binom{n-m}{k} \leqq \frac{\alpha}{n-m+1} \tag{14}$$

holds (where m is the value of $\boldsymbol{m}$ following from the observations $z_1, \ldots, z_n$) or, if no positive integer satisfies (14), $k = 0$.

B. Put

$$\beta = \beta(n-m, \alpha) = 2^{-n+m+1} \sum_{i=0}^{k} \binom{n-m}{i} \tag{15}$$

and

$$\varepsilon = \varepsilon(n-m, n_1, \alpha) = \frac{\alpha-\beta}{n-m-2k-1} \cdot \frac{2^{n-m}}{\binom{n-m}{n_1}} \tag{16}$$

C. Then the critical region R_1^* consists of those points (n_1, U), for which at least one of the following two conditions holds:

$$C_1: \quad n_1 \leqq k \text{ or } n_1 \geqq n-m-k$$
$$C_2: \quad U \in R_{n-m, n_1}(\delta)$$

where $\delta \leqq \varepsilon$, and $\varepsilon - \delta$ is as small as possible. It is clear, that the size of R_1^* is then $\leqq \alpha$.

8. 6. For $n-m = 12$ and $\alpha = 0{,}10$ (α has been chosen rather large to get better diagrams) R_1^* has been outlined in fig. 3 for the case that

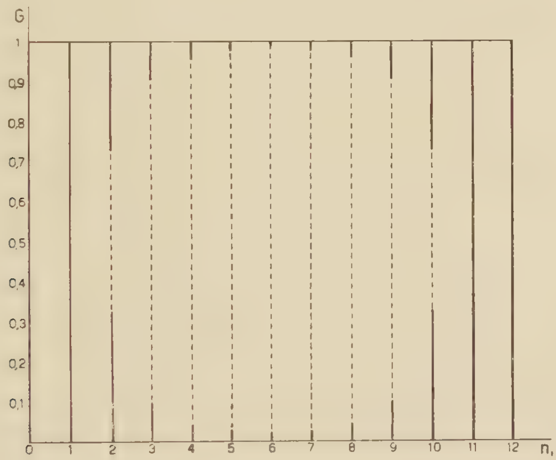

Fig. 3. Critical region R_1^*, when no alternative hypothesis is specified and when T is WILCOXON's test or PITMAN's test; $\alpha = 0{,}10$.

Wilcoxon's test or Pitman's test is used for T and in fig. 4 if the test of Wald and Wolfowitz is used. In these figures $G(U|Z;\ \mathbf{n}_1 = n_1;\ H_0)$, the conditional distribution function of $\boldsymbol{U}$, has been plotted on vertical lines above the points $n_1 = 1, \ldots, n_1 = 11$. R_1^* consists of the points (n_1, U) on those parts of these lines, which have been *drawn*. The points with $n_1 = 0$ and $n_1 = 12$ belong to the critical region according to 8. 5. *A*. This has been indicated by drawing the vertical lines above these points. The *broken* parts of the vertical lines indicate the region, where H_0 is not

rejected. The reader should bear in mind, that in reality G is discontinuous.

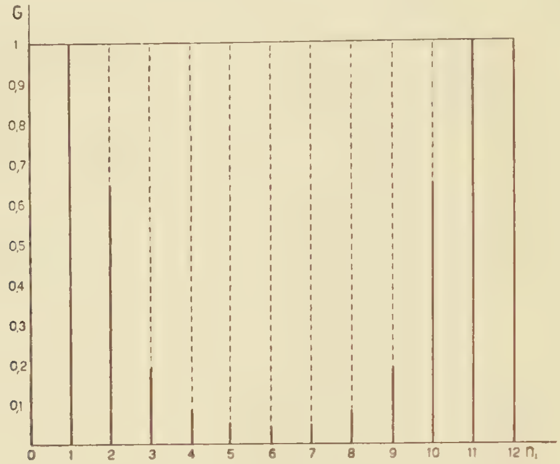

Fig. 4. Critical region R_1^*, when no alternative hypothesis is specified and T is the test of Wald and Wolfowitz; $\alpha = 0{,}10$.

Remark: The critical region R_1^* for the case, that the test of Kolmogoroff–Smirnoff is used for T, can be constructed in an analogous way, large values of G being critical instead of small values. Strictly speaking, however, we do not know much about α in that case.

8. 7. As a *special alternative hypothesis*, against which H_0 can be tested, we consider the hypothesis H (cf. section 5. 3) of a *displacement* of one or more of the variables $\mathbf{z}_i$ in one direction along the z-axis. In this case we restrict the "family" of tests T to those tests, where a *unilateral* critical region can be indicated (cf. 8. 2, 8. 3 and 8. 4). We shall denote unilateral critical regions of the types $\boldsymbol{U} \leqq U_1$ and $\boldsymbol{U} \geqq U_1'$ with size ε by $R'_{n-m,n_1}(\varepsilon)$ and $R''_{n-m,n_1}(\varepsilon)$ respectively. These critical regions may also be defined by the relations

$$G(\boldsymbol{U}|Z;\mathbf{n}_1 = n_1; H_0) \leqq G_1 = G(U_1|Z;\mathbf{n}_1 = n_1; H_0)$$

and

$$G(\boldsymbol{U}|Z;\mathbf{n}_1 = n_1; H_0) \geqq G_1' = G(U_1'|Z;\mathbf{n}_1 = n_1; H_0).$$

For reasons given in section 5. 3 we exclude, if $n - m$ is even, the points (n_1, U) with $n_1 = \frac{n-m}{2}$ from the critical region R_2^* for testing H_0 against H. We further remark, that for Wilcoxon's test and Pitman's test the probability of small (large) values of $\boldsymbol{U}$ increases if some of the variables $\mathbf{z}_i$ are shifted towards the left (right) and decreases, if the displacement is towards the right (left) along the z-axis (cf. section 5. 3). We therefore propose for these cases the following construction for R_2^* (with size $\leqq \alpha$), applicable if $\alpha \geqq 2^{-n+m+1}$:

A. Let k be the largest positive integer $< \frac{n-m}{2}$, for which the relation (14) holds, or, if no positive integer satisfies (14), $k = 0$.

B'. Define β by (15) and (cf. (16)):

$$\varepsilon' = \varepsilon'(n-m, n_1, a) = \varepsilon(n-m, n_1, a) \tag{16'}$$

if $n-m$ is odd, and

$$\varepsilon' = \varepsilon'(n-m, n_1, a) = \frac{a-\beta}{n-m-2k-2} \frac{2^{n-m}}{\binom{n-m}{n_1}} \tag{16''}$$

if $n-m$ is even.

C'. Then R_2^* consists of those points (n_1, U), for which at least one of the following three conditions holds:

$$C_1:\quad n_1 \leqq k \text{ or } n_1 \geqq n-m-k$$

$$C_2':\quad n_1 < \frac{n-m}{2} \text{ and } U \in R'_{n-m,n_1}(\delta)$$

$$C_3':\quad n_1 > \frac{n-m}{2} \text{ and } U \in R''_{n-m,n_1}(\delta)$$

where $\delta \leqq \varepsilon$, and $\varepsilon - \delta$ is as small as possible.

For $n-m = 12$ and $a = 0{,}10$ R_2^* has been given in figure 5 for the case that Wilcoxon's test or Pitman's test has been used for T.

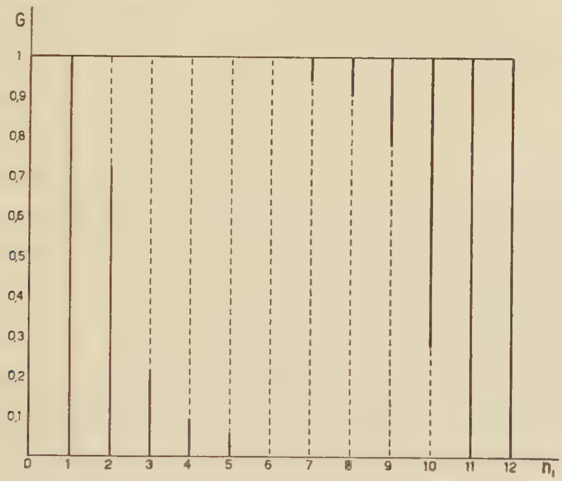

Fig. 5. Critical region R_2^*, when the alternative is a displacement of at least one of the distributions in one direction along the z-axis; $a = 0{,}10$.

If the direction of the displacement is specified in the alternative hypothesis, the critical region may be confined either to the left or to the right half of the diagram only, using $2a$ instead of a in (14) and (15).

8. 8. The computations are now comparatively simple. A table of k, 2^{n-m} and of the quantities

$$\gamma = \frac{a-\beta}{n-m-2k-1} \cdot 2^{n-m} \tag{17}$$

and, for even values of $n-m$, of

$$\gamma' = \frac{\alpha-\beta}{n-m-2k-2}\cdot 2^{n-m} \tag{18}$$

has been computed by the Computing Department of the "Mathematisch Centrum" for $\alpha = 0{,}025$; $0{,}05$ and $0{,}10$ and for $n-m = 10(1)50$ (cf. section 10). From this table the value of $\varepsilon(n-m, n_1, \alpha)$ or $\varepsilon'(n-m, n_1, \alpha)$ is easily computed with the aid of a table of the binomial coefficients (cf. 5. 2). If then condition C (or C') is satisfied, the result is significant with significance level $\leqq \alpha$.

Moreover, if $n_1 \neq 0$ and $\neq n-m$, an upper bound for the size of the smallest critical region of type R_1^* or R_2^*, which contains the point (n_1, U) following from the observations, may be found as follows:

Let η be the size of the smallest critical region for $\boldsymbol{U}$, given $n-m$ and n_1 (either bilateral or unilateral), which contains the observed value U, then

$$\alpha^* = 2^{-n+m}(n-m+1)\binom{n-m}{n_1}\cdot\eta \geqq \alpha. \tag{19}$$

8. 9. Sections 8. 5 and 8. 6 may be applied to the special case, described in sections 4 and 5. According to remark 3 of section 4. 2, $\boldsymbol{u}$ has, if H_0 is true, for given n_1 and under the condition Z, a hypergeometric distribution. For this distribution we have

$$\mathcal{E}(\boldsymbol{u}\,|\,Z;\boldsymbol{n}_1 = n_1; H_0) = \frac{rn_1}{n_1+n_2} \tag{20}$$

and

$$\sigma^2_{\boldsymbol{u}\,|\,Z;\,\boldsymbol{n}_1=n_1;\,H_0} = \frac{r\,n_1\,n_2\,(n_1+n_2-r)}{(n_1+n_2)^2\,(n_1+n_2-1)} \tag{21}$$

with $n_1+n_2 = n-m$. A normal probability distribution with (20) and (21) as mean and variance respectively is a good approximation of this probability distribution of $\boldsymbol{u}$, especially if a continuity correction is applied.

If $n-m$ is so large, that the exact method of section 5 becomes too laborious, this approximate method may be used instead. It should, however, be born in mind, that the construction of the critical regions R_1^* and R_2^* is different from the construction of R_1 and R_2, and that R_1^* and R_2^* should therefore not be regarded as approximations of R_1 and R_2, but as an approximate method using a slightly different form of critical regions.

On the other hand, if the number of observations is small and T is a test, such that the exact distribution of $\boldsymbol{U}$ is known, the critical region may, for the general case, be chosen according to a system analogous to the method described for the special case in section 5. We shall not go into the details of this method for other special cases, since the principle remains unchanged for every T.

9. *Remarks.*

Of the two-sample tests, mentioned in 6. 2, the tests of Wilcoxon and

of Wald and Wolfowitz can only be applied to our problem if $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ are all different. This is not required for the tests of Pitman and Kolmogoroff–Smirnoff. On the other hand, the latter is not an exact test, as has been pointed out already in section 8. 5. Since small values of n_1 or n_2 will often occur in the application of the test, this is a serious drawback. The same applies, to a certain extent, to Pitman's test, since the computation of the exact distribution of its statistic is impracticable for values of n_1 and n_2, which are at all large. Furthermore little is known about the accuracy of the approximation to the distribution of $\boldsymbol{U}$, given by Pitman, especially in the case of discontinuous random variables.

So far the only exact test for H_0, developed untill now, which is valid if there are equal values among $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$, and which can be used for reasonably large values of $n_1 + n_2$, seems to be the one described in sections 4 and 5. Moreover for large values of $n_1 + n_2$ the accuracy of the approximate method, described in 8. 9 is independent of the number of equal values among $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$.

If no equal values occur among the x_j and y_k, Wilcoxon's test seems a very suitable one for T, especially when the alternative hypothesis is the hypothesis H of a displacement along the z-axis.

The number of values z_i, which are equal to zero, is of no consequence whatever as far as the choice of T is concerned.

11. *Explanation of the table and of the practical application of the test.*

The use of the table in applying the test may be facilitated by the following indications:

n denotes the number of observations $z_1, \ldots, z_n$ and m the number of values z_i which are equal to zero;

n_1 denotes the number of positive values $x_1, \ldots, x_{n_1}$ among $z_1, \ldots, z_n$.

If $n_1 \leqq k$ or $n_1 \geqq n - m - k$, H_0 is rejected with significance level $\leqq \alpha$. If $k < n_1 < n - m - k$, two cases are to be distinguished:

I. If no alternative hypothesis to H_0 is specified, the chosen two sample test T is applied to the two sets of values $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ (the y_k are the negative values among $z_1, \ldots, z_n$ taken positively). This results in a value U of the statistic $\boldsymbol{U}$ of T. Let η be the size of the smallest bilateral critical region for $\boldsymbol{U}$, belonging to T, which contains U. If then

$$\eta \leqq \gamma \binom{n-m}{n_1}$$

H_0 is rejected with significance level $\leqq \alpha$; cf. (17) for γ.

In case Wilcoxon's test is used for T, we have

$$\eta = 2G\,(U\,|\,Z;\, \mathbf{n}_1 = n_1;\, H_0)$$

if $U < \frac{n_1 n_2}{2}$ and

$$\eta = 2\,\{1 - G\,(U\,|\,Z;\, \mathbf{n}_1 = n_1;\, H_0)\}$$

if $U > \frac{n_1 n_2}{2}$.

10. Table of k, 2^{n-m}, γ and γ'.

$a = 0,025;\ 0,05;\ 0,10$

$n-m$	2^{n-m}	$a = 0,025$			$a = 0,05$			$a = 0,10$		
		k	γ	γ'	k	γ	γ'	k	γ	γ'
10	1024.10^{0}	0	$2,622.10^{0}$	$2,950.10^{0}$	0	$5,467.10^{0}$	$6,150.10^{0}$	0	$1,116.10^{1}$	$1,255.10^{1}$
11	2048.10^{0}	0	$4,919.10^{0}$		0	$1,004.10^{1}$		1	$2,260.10^{1}$	
12	4096.10^{0}	0	$9,126.10^{0}$	$1,004.10^{1}$	1	$1,987.10^{1}$	$2,235.10^{1}$	1	$4,262.10^{1}$	$4,795.10^{1}$
13	8192.10^{0}	1	$1,768.10^{1}$		1	$3,816.10^{1}$		1	$7,912.10^{1}$	
14	1638.10^{1}	1	$3,450.10^{1}$	$3,796.10^{1}$	1	$7,175.10^{1}$	$7,892.10^{1}$	2	$1,585.10^{2}$	$1,783.10^{2}$
15	3277.10^{1}	1	$6,560.10^{1}$		1	$1,339.10^{2}$		2	$3,035.10^{2}$	
16	6554.10^{1}	1	$1,234.10^{2}$	$1,337.10^{2}$	2	$2,730.10^{2}$	$3,003.10^{2}$	2	$5,709.10^{2}$	$6,280.10^{2}$
17	1311.10^{2}	2	$2,475.10^{2}$		2	$5,205.10^{2}$		3	$1,144.10^{3}$	
18	2621.10^{2}	2	$4,776.10^{2}$	$5,175.10^{2}$	2	$9,817.10^{2}$	$1,064.10^{3}$	3	$2,203.10^{3}$	$2,424.10^{3}$
19	5243.10^{2}	2	$9,091.10^{2}$		3	$1,991.10^{3}$		3	$4,175.10^{3}$	
20	1049.10^{3}	3	$1,809.10^{3}$	$1,959.10^{3}$	3	$3,825.10^{3}$	$4,144.10^{3}$	4	$8,405.10^{3}$	$9,246.10^{3}$
21	2097.10^{3}	3	$3,521.10^{3}$		3	$7,267.10^{3}$		4	$1,622.10^{4}$	
22	4194.10^{3}	3	$6,749.10^{3}$	$7,231.10^{3}$	4	$1,473.10^{4}$	$1,596.10^{4}$	4	$3,086.10^{4}$	$3,344.10^{4}$
23	8389.10^{3}	3	$1,285.10^{4}$		4	$2,840.10^{4}$		5	$6,248.10^{4}$	
24	1678.10^{4}	4	$2,624.10^{4}$	$2,812.10^{4}$	4	$5,421.10^{4}$	$5,807.10^{4}$	5	$1,205.10^{5}$	$1,306.10^{5}$
25	3355.10^{4}	4	$5,053.10^{4}$		5	$1,101.10^{5}$		5	$2,299.10^{5}$	
26	6711.10^{4}	4	$9,657.10^{4}$	$1,026.10^{5}$	5	$2,125.10^{5}$	$2,278.10^{5}$	6	$4,679.10^{5}$	$5,069.10^{5}$
27	1342.10^{5}	5	$1,970.10^{5}$		5	$4,068.10^{5}$		6	$9,019.10^{5}$	
28	2684.10^{5}	5	$3,804.10^{5}$	$4,043.10^{5}$	6	$8,281.10^{5}$	$8,874.10^{5}$	6	$1,723.10^{6}$	$1,845.10^{6}$
29	5369.10^{5}	5	$7,291.10^{5}$		6	$1.600.10^{6}$		7	$3,523.10^{6}$	
30	1074.10^{6}	6	$1,488.10^{6}$	$1,582.10^{6}$	6	$3,068.10^{6}$	$3,260.10^{6}$	7	$6,785.10^{6}$	$7,269.10^{6}$
31	2148.10^{6}	6	$2,878.10^{6}$		7	$6,264.10^{6}$		7	$1,298.10^{7}$	
32	4295.10^{6}	6	$5,528.10^{6}$	$5,837.10^{6}$	7	$1,210.10^{7}$	$1,286.10^{7}$	8	$2,663.10^{7}$	$2,853.10^{7}$
33	8590.10^{6}	7	$1,130.10^{7}$		7	$2,323.10^{7}$		8	$5,125.10^{7}$	
34	1718.10^{7}	7	$2,187.10^{7}$	$2,307.10^{7}$	8	$4,755.10^{7}$	$5,053.10^{7}$	8	$9,808.10^{7}$	$1,042.10^{8}$
35	3536.10^{7}	8	$4,412.10^{7}$		8	$9,184.10^{7}$		9	$2,019.10^{8}$	
36	6872.10^{7}	8	$8,611.10^{7}$	$9,092.10^{7}$	8	$1,765.10^{8}$	$1,864.10^{8}$	9	$3,883.10^{8}$	$4,126.10^{8}$
37	1374.10^{8}	8	$1,667.10^{8}$		9	$3,623.10^{8}$		10	$7,934.10^{8}$	
38	2749.10^{8}	9	$3,376.10^{8}$	$3,565.10^{8}$	9	$6,993.10^{8}$	$7,383.10^{8}$	10	$1,534.10^{9}$	$1,630.10^{9}$
39	5498.10^{8}	9	$6,581.10^{8}$		10	$1,424.10^{9}$		10	$2,951,10^{9}$	
40	1100.10^{9}	9	$1,273.10^{9}$	$1,337.10^{9}$	10	$2,765.10^{9}$	$2,918.10^{9}$	11	$6,052.10^{9}$	$6,430.10^{9}$
41	2199.10^{9}	10	$2,590.10^{9}$		10	$5.339.10^{9}$		11	$1,169.10^{10}$	
42	4398.10^{9}	10	$5,040.10^{9}$	$5,291.10^{9}$	11	$1,090.10^{10}$	$1,151.10^{10}$	11	$2,248.10^{10}$	$2,373.10^{10}$
43	8796.10^{9}	10	$9,755.10^{9}$		11	$2,115.10^{10}$		12	$4,623.10^{10}$	
44	1759.10^{10}	11	$1.988.10^{10}$	$2,088.10^{10}$	11	$4,083.10^{10}$	$4,287.10^{10}$	12	$8,920.10^{10}$	$9,415.10^{10}$
45	3518.10^{10}	11	$3,867.10^{10}$		12	$8,363.10^{10}$		13	$1,825.10^{11}$	
46	7037.10^{10}	11	$7,487.10^{10}$	$7,825.10^{10}$	12	$1,621.10^{11}$	$1,702.10^{11}$	13	$3,536.10^{11}$	$3,732.10^{11}$
47	1407.10^{11}	12	$1,530.10^{11}$		13	$3,302.10^{11}$		13	$6,820.10^{11}$	
48	2815.10^{11}	12	$2,972.10^{11}$	$3,107.10^{11}$	13	$6.420.10^{11}$	$6,744.10^{11}$	14	$1,400.10^{12}$	$1,477.10^{12}$
49	5629.10^{11}	13	$6,040.10^{11}$		13	$1,244.10^{12}$		14	$2,708.10^{12}$	
50	1126.10^{12}	13	$1,177.10^{12}$	$1,232.10^{12}$	14	$2,541.10^{12}$	$2,668.10^{12}$	14	$5,222.10^{12}$	$5,483.10^{12}$

According to continental usage the comma designates the decimal sign (e.g. $0,5 = \frac{1}{2}$).

II. If H_0 is tested against the alternative hypothesis H of a displacement along the z-axis in one direction (which one not being specified) of some of the variables $\mathbf{z}_i$, a two sample test T is applied to $x_1, \ldots, x_{n_1}$ and $y_1, \ldots y_{n_2}$, which allows a distinction between two unilateral critical regions. Taking e.g. Wilcoxon's test, the size η' of the smallest unilateral critical region containing the value U, found from the observations, is computed, using the unilateral critical region of the form

$$\mathbf{U} - \frac{n_1 n_2}{2} \geqq U_a$$

if $n_1 > \frac{n-m}{2}$, and the unilateral critical region of the type

$$\mathbf{U} - \frac{n_1 n_2}{2} \leqq - U_a$$

if $n_1 < \frac{n-m}{2}$ (cf. 8. 2).

In the first case, we have

$$\eta' = 1 - G(U | Z; \mathbf{n}_1 = n_1; H_0)$$

and in the second case

$$\eta' = G(U | Z; \mathbf{n}_1 = n_1; H_0).$$

If $n - m$ is odd, and

$$\eta' \leqq \gamma \binom{n-m}{n_1}$$

or if $n - m$ is even, and

$$\eta' \leqq \gamma' \binom{n-m}{n_1}$$

H_0 is rejected; cf. (17) and (18) for γ and γ'.

The values of 2^{n-m} have been included in the table to facilitate the computation of the size a^* of smallest critical region (either of unilateral or bilateral type), which contains the point (n_1, U), following from the observations. This computation has been described in 8. 8.

12. *Example.*

Let us consider a set of observed values $z_1, \ldots, z_{22}$:

$-8{,}0$; $-5{,}0$; $-4{,}5$; $-3{,}0$; $-2{,}7$; $-2{,}3$; $-2{,}1$; $-1{,}3$; $-1{,}2$; $-1{,}0$; $-0{,}9$; $-0{,}5$; $-0{,}2$; 0; 0; $1{,}8$; $2{,}5$; $3{,}5$; $6{,}2$; $7{,}3$; $7{,}4$; $9{,}5$.

We then have $n = 22$, $m = 2$, $n_1 = 7$. From the table of section 11 we find (for $a = 0{,}05$)

$$\varepsilon = \gamma \Big/ \binom{20}{7} = \frac{3825}{77520} = 0{,}049$$

and $k = 3$. Therefore $k < n_1 < n - m - k$ and a two sample test must be applied. Let us take Wilcoxon's test for this. The number of pairs (x_j, y_k) with $x_j > y_k$ is 73. According to section 8. 2 we have

$$\mathcal{E}\,\mathbf{U} = \frac{7 . 13}{2} = 45{,}5$$

and

$$\sigma_u = \sqrt{\tfrac{1}{12}\, 7.13 (7 + 13 + 1)} = 12{,}62.$$

Applying a correction for continuity, we find

$$\frac{U-\frac{1}{2}-\mathcal{E}\boldsymbol{U}}{\sigma_u}=\frac{72,5-45,5}{12,62}=2,14.$$

From a table of the normal distribution we find therefore, that

$$\eta=\mathrm{P}\left[|\boldsymbol{U}-\mathcal{E}\boldsymbol{U}|\geqq|73-45,5|\,|Z\;\boldsymbol{n}_1=n_1;H_0\right]=0,032.$$

Since $\eta<\varepsilon$, H_0 is rejected with significance level 0,05.

If, however, H_0 is tested against the alternative hypothesis H of a displacement of some of the variables $\mathbf{z}_i$ in one direction along the z-axis, H_0 is *not* rejected, since

$$n_1<\frac{n-m}{2}=\mathcal{E}(\boldsymbol{n}_1|H_0)$$

and

$$U>\frac{n_1n_2}{2}=\mathcal{E}(\boldsymbol{U}|\boldsymbol{n}_1=n_1;H_0)$$

thus $G(U|Z;\,\boldsymbol{n}_1=n_1;\,H_0)$ having the value $1-0,016=0,984$. The point (n_1, U) corresponding with this result is not contained in the critical region R_2^* (cf. figure 5). This means, that the observations do not indicate a displacement of some of the $\mathbf{z}_i$ in one direction along the z-axis. They do, however, suggest displacements in both directions, or asymmetry of some of the distributions or a combination of displacements and asymmetry. This follows from the fact, that H_0 is rejected if no special alternative hypothesis is specified.

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ON CERTAIN LOCAL PROPERTIES OF A TOPOLOGICAL SPACE ASSOCIATED WITH A PSEUDO-METRIC. I

BY

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1. *Introduction.* In a previous paper [1]) the author has given a new proof of the well-known theorem of POINCARÉ and VOLTERRA that a multiform analytic function assumes at a given point of the complex plane only a countable number of distinct values. He has announced there that he will generalize the reasoning to a theory of local hereditary properties of metric spaces and more generally to that of certain local hereditary properties of uniform spaces. He also made there a short sketch of the chief features of this generalization. It is this programme which shall be carried out in the work bearing the above title and of which the present paper is the first part.

We consider in this Part I not a metric in a metrizable space, but a pseudo-metric and more generally what we call a semi-pseudo-metric in an arbitrary T-space (cf. § 2). The chief rôle is played by what we call *element-invariants*, in particular by those termed basic integers (cf. § 8). It will probably be impossible to find precise set-theoric topological equivalents for the element-invariants. Consequently our theory, though very general and simple, seems to be of purely metric nature.

2. *Local spherically hereditary properties of a T-space R associated with a semi-pseudo-metric in R.* Let R be a T-space (cf. [2, 27], [1, 37] [2])). If to every ordered pair of points x, y of R a non-negative finite real number $f(x, y)$ is assigned such that

1° $f(x, x) = 0$;

2° $f(x, y) = f(y, x)$ (Symmetry axiom);

3° $f(x, z) \leqslant f(x, y) + f(y, z)$ (Triangle axiom);

4° for every $x \in R$ and $\eta > 0$, $\{y \mid f(x, y) < \eta\}$ is an open set;

then the function $f(x, y)$ is termed, following J. W. TUKEY, a *pseudo-metric in R* [2, 50]. A function $f(x, y)$ in R satisfying 1°, 2°, 4° only (but *not* the triangle axiom) will be called a *semi-pseudo-metric in R*.

The set $\{x \mid f(a, x) < \varepsilon\}$ $(\varepsilon > 0)$ is termed the *sphere of radius* ε and

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[1]) "On a theorem of Poincaré and Volterra", accepted for publication by the London Math. Soc. This paper will be referred to in the sequel as P.V.

[2]) References to the bibliography at the end of the paper are given in brackets. The first number designates the entry and subsequent numbers the pages.

of centre a (or *about a*) in the semi-pseudo-metric f in R (cf. [2, 51]) and is denoted by $S_f(\varepsilon|a)$ or, where no ambiguity is possible, by $S(\varepsilon|a)$.

The condition 4° can be then expressed by saying that the spheres in a semi-pseudo-metric f in R are open in R. This gives together with 1°

(2. A) *Every sphere about a in the semi-pseudo-metric f in R is an open neighbourhood of a in R.*

From (2. A), 2°, 3° is readily deduced

(2. B) *A pseudo-metric f in R is continuous in both variables together* [2, 50].

That is, given a, b in R and $\eta > 0$, there are open neighbourhoods $U(a)$, $U(b)$ in R of a, b respectively such that if $x \in U(a)$ and $y \in U(b)$, then $|f(a, b) - f(x, y)| < \eta$.

If $S(\varepsilon|a)$, $S(\eta|b)$ are two spheres in the semi-pseudo-metric f in R, then we say that $S(\eta|b)$ is *contained* in $S(\varepsilon|a)$ if and only if $f(a, b) < \varepsilon$ and $\eta < \varepsilon - f(a, b)$ (cf. § 3). For a semi-pseudo-metric f this does not imply in general that $S(\eta|b)$ is a subset of $S(\varepsilon|a)$; if, however, f is a pseudo-metric, then from the triangle axiom follows that $S(\eta|b) \subset S(\varepsilon|a)$.

As is well known a property Π of subsets of a T-space R is called *hereditary* (*cogredient* in the nomenclature of Hausdorff, cf. [1, 34]) provided that: if a subset A of R has the property Π, every subset B of A also has the property Π.

A property Π of substes of R shall be termed *spherically hereditary with respect to the semi-pseudo-metric f in R* if and only if (2. α) *if a sphere $S_f(\varepsilon|a)$ has the property Π, then every $S_f(\eta|b)$ contained in $S_f(\varepsilon|a)$ also has the property Π.*

If f is a *pseudo-metric* in R, then every hereditary property of R is spherically hereditary with respect to f. The converse is not true.

We call a property Π of subsets of R a *local property of R associated with the semi-pseudo-metric f in R* and say that R is *locally Π with respect to f* if and only if (2. β) *for every point x of R there exists a sphere $S_f(\varepsilon|x)$ of centre x which has the property Π.*

If Π satisfies both (2. α) and (2. β), then Π is termed a *local spherically hereditary property of R associated with f.*

3. *Elements.* If f is a semi-pseudo-metric in a T-space R and A a subset of R, then the supremum of $\{f(x, y)|x, y \in A\}$ is termed the *diameter of A in f* and is denoted by $d_f(A)$ or merely $d(A)$. If the diameter of the whole space R in f is *finite*, then f is said *bounded.*

We consider in the present paper only bounded semi-pseudo-metrics. In the sequel "semi-pseudo-metric" means always "bounded semi-pseudo-metric".

Let Π be a local spherically hereditary property of R associated with the semi-pseudo-metric f in R. For a given point a of R consider the set of all $S_f(\varepsilon|a)$ having the property Π and a radius ε not greater than the diameter $d_f(R) = d$ of R, and denote by ϱ the supremum of their radii.

From the boundedness of f follows that $\varrho(\varrho \leqq d)$ is finite, from (2. β) follows $\varrho > 0$. Then $S_f(\varrho|a)$ will be termed a *Π-element of R in f*, or, where no ambiguity is possible, i.e. where a single property Π and a single semi-pseudo-metric f is considered, merely an *element* of R, a its *centre* and ϱ its *radius of validity.*

Two elements of R will be said equal (notation $=$) if and only if they have equal centres. We will show in the following §§ that the set of all distinct elements of R is an analogue of the set S of all distinct power series with a non vanishing radius of convergence considered in P.V.

Let us first make some simple remarks on elements of R.

From the definition of supremum and from (2. α) follows

(3. A) *If $S(\varrho|a)$ is an element of R, then every sphere $S(\varepsilon|a)$ with the same centre a and with a radius $\varepsilon < \varrho$ has the property Π.*

From (3. A) and (2. α) we get the following generalization of (3. A)

(3. B) *If $S(\varrho|a)$ is an element of R, then every sphere $S(\varepsilon|b)$ which is contained in $S(\varrho|a)$ has the property Π.*

Proof. By hypothesis $0 < \varepsilon < \varrho - f(a, b)$ (cf. § 2), and hence we can put $\varrho - f(a, b) - \varepsilon = 2\eta > 0$. Then $f(a, b) < \varrho - \eta$ and $\varepsilon < (\varrho - \eta) - f(a, b)$. Hence the sphere $S(\varepsilon|b)$ is contained in the sphere $S(\varrho - \eta|a)$, which, by (3. A), has the property Π. (2. α) gives then (3. B).

We consider in (3. C), (3. D), (3. E) a pair of elements of R, namely $S(\varrho|a)$, $S(\sigma|b)$, *such that* $f(a, b) < \varrho$.

From (3. B)

(3. C) *The radius of validity of $S(\sigma|b)$ satisfies the inequality $\sigma \geqslant \varrho - f(a\, b)$.*

From (3. C)

(3. D) *The radius of validity of $S(\sigma|b)$ satisfies the inequality $\varrho + f(a, b) \geqslant \sigma$.*

Proof. If $\varrho + f(a, b) < \sigma$, then $f(b, a) = f(a, b) < \sigma$ and (3. C) can be applied to the pair $S(\sigma|b)$, $S(\varrho|a)$. We obtain $\varrho \geqslant \sigma - f(b, a) = \sigma - f(a, b)$, which is in contradiction to $\varrho + f(a, b) < \sigma$.

From (3. C), (3. D)

(3. E) *The absolute value of the difference of σ and ϱ satisfies the inequality $|\sigma - \varrho| \leqslant f(a, b)$.*

From (3. E) and (2. A) follows

(3. F) *The radius of validity of elements of R considered as function on R is continuous on R.*

Already the above considerations show that elements of R have properties very similar to that of power series elements of an analytic function. The analogy will become even more striking in the subsequent §§.

4. *a direct continuations.* If $S(\varrho|a), S(\sigma|b)$ is an ordered pair of elements of R, then the quotient $A = \frac{f(a, b)}{\varrho}$ will be termed the *relative distance* of the ordered pair $S(\varrho|a)$, $S(\sigma|b)$ and denoted by $S(\varrho|a) - S(\sigma|b) = A$ or by $S(\varrho|a) \overset{A}{-} S(\sigma|b)$. If $A \leqslant a$, then we write $S(\varrho|a) - S(\sigma|b) \leqslant a$ or $S(\varrho|a) \overset{\leq a}{-\!-}$

$-S(\sigma|b)$; if $A < a$, $A > a$, $A \geqslant a$, then we replace in the previous formulas the sign $\leqslant$ by $<$, $>$, $\geqslant$ respectively.

$S(\sigma|b)$ will be called *direct continuation* of $S(\varrho|a)$ and written $S(\varrho|a) \rightarrow \rightarrow S(\sigma|b)$ if the relative distance of $S(\varrho|a)$, $S(\sigma|b)$ is less than 1 (i.e. $S(\varrho|a) - S(\sigma|b) < 1$). The formulas $S(\varrho|a) \rightarrow S(\sigma|b) = A$, $S(\varrho|a) \xrightarrow{A} S(\sigma|b)$ etc. have then a sense analogous to the above, the sign $-$ being replaced by $\rightarrow$.

$S(\sigma|b)$ will be said *a direct continuation* of $S(\varrho|a)$ if both $0 < a \leqslant 1$ and $S(\varrho|a) \rightarrow S(\sigma|b) < a$. Remark that in this definition the sign of equality is not valid, i.e. we have really $f(a, b) < a\,\varrho$, but not $f(a, b) = a\,\varrho$. The direct continuations defined before are, in this new nomenclature, 1 direct continuations. Remark that if $S(\sigma|b)$ is a direct continuation of $S(\varrho|a)$ and if $0 < a \leqslant \beta \leqslant 1$, then $S(\sigma|b)$ is also β direct continuation of $S(\varrho|a)$.

The configuration $S(\varrho|a) \rightarrow S(\sigma|b)$ will be termed a 2-*chain*. We consider more exactly the 2-chain

$$S(\varrho|a) \xrightarrow{A} S(\sigma|b). \tag{4.1}$$

From (3. C), (3. D)

(4. A) *The radius of validity σ of $S(\sigma|b)$ satisfies the inequality* $(1+A)\varrho \geqslant \geqslant \sigma \geqslant (1-A)\varrho$.

From (4. A) and from the symmetry axiom

(4. B) $$S(\sigma|b) - S(\varrho|a) \leqslant \frac{A}{1-A}.$$

Proof. $f(b,a) = f(a,b) = \dfrac{f(a,b)\,(1-A)\,\varrho}{(1-A)\,\varrho} = \dfrac{A}{1-A}(1-A)\,\varrho \leqslant \dfrac{A}{1-A}\,\sigma.$

In exactly the same way we prove

(4. C) $$S(\sigma|b) - S(\varrho|a) \geqslant \frac{A}{1+A}.$$

From (4. B) follows that a sufficient condition for $S(\varrho|a)$ to be direct continuation of $S(\sigma|b)$ is $\dfrac{A}{1-A} < 1$, i.e. $A < \frac{1}{2}$. Thus

(4. D) *If $S(\sigma|b)$ is a direct continuation of $S(\varrho|a)$ and if $a \leqslant \frac{1}{2}$, then $S(\varrho|a)$ is $\left(\dfrac{a}{1-a}\right)$ direct continuation of $S(\sigma|b)$.*

5. *Simple configurations.* In §§ 5, 6 we assume that f is a pseudo-metric. The results of §§ 5, 6 are in general not valid for semi-pseudo-metrics. A consequence is that our theory can be fully developed only for a local spherically hereditary property H associated with a pseudo-metric f in R. We have considered semi-pseudo-metrics because they are far more general than pseudo-metrics and because all the results of the present Part I with the sole exception of those of §§ 5, 6 and of Theorem (12. B) are valid also for them.

We consider first the configuration

$$S(\tau|c) \xleftarrow{B} S(\varrho|a) \xrightarrow{A} S(\sigma|b). \tag{5. 1}$$

From (4. A) and from the triangle axiom

(5. A) $$S(\tau|c) - S(\sigma|b) \leqslant \frac{A+B}{1-B}.$$

Proof. By (4. A) we have $\tau \geqslant (1-B)\varrho$. From $f(a, b) = A\varrho$, $f(a, c) = B\varrho$, from the symmetry axiom and from the triangle axiom we get $f(c, b) \leqslant (A + B)\varrho$, and hence

$$f(c, b) \leqslant \left(\frac{A+B}{1-B}\right)(1-B)\varrho \leqslant \left(\frac{A+B}{1-B}\right)\tau.$$

From (5. A)

(5. B) *To every pair a, a' with $0 < a < a' \leqslant 1$ it is possible to determine β with $0 < \beta < 1$ such that if $S(\sigma|b)$, $S(\tau|c)$ are respectively a, β direct continuations of $S(\varrho|a)$, then $S(\sigma|b)$ is a' direct continuation of $S(\tau|c)$.*

Proof. In the configuration

(5. 1′) $$S(\tau|c) \xleftarrow{\beta>} S(\varrho|a) \xrightarrow{<a} S(\sigma|b).$$

where $0 < a < a' \leqslant 1$ we have to consider $S(\varrho|a) \rightarrow S(\sigma|b) = A < a$ as fixed and to determine β such that $S(\sigma|b)$ be a' direct continuation of $S(\tau|c)$. It suffices to take $\beta = \frac{a'-a}{1+a'}$, for then, by (5. A), we get $S(\tau|c) - S(\sigma|b) < \frac{a+\beta}{1-\beta} = a'$.

Second we consider the configuration

(5. 2) $$S(\varrho|a) \xrightarrow{A} S(\sigma|b) \xrightarrow{C} S(\omega|d).$$

(5. C) $$S(\varrho|a) - S(\omega|d) \leqslant A + C(1 + A).$$

Proof. From the triangle axiom and from (4. A) follows

$$f(a, d) \leqslant A\varrho + C\sigma \leqslant A\varrho + C(1 + A)\varrho.$$

(5. D) *To every pair a, a' with $0 < a < a' \leqslant 1$ it is possible to determine γ with $0 < \gamma < 1$ such that if $S(\sigma|b)$ is a direct continuation of $S(\varrho|a)$ and $S(\omega|d)$ is γ direct continuation of $S(\sigma|b)$, then $S(\omega|d)$ is a' direct continuation of $S(\varrho|a)$.*

Proof. In the configuration

(5. 2′) $$S(\varrho|a) \xrightarrow{<a} S(\sigma|b) \xrightarrow{<\gamma} S(\omega|d)$$

where $0 < a < a' \leqslant 1$, we have to consider $S(\varrho|a) \rightarrow S(\sigma|b) = A < a$ as fixed and to determine γ such that $S(\omega|d)$ be a' direct continuation of $S(\varrho|a)$. We can take $\gamma = \frac{a'-a}{1+a}$, for by (5. C) we get

$$S(\varrho|a) - S(\omega|d) < a + \gamma(1 + a) = a'.$$

6. *Some other configuration-invariants.*

(5. B), (5. D) allow to overlook the configuration

$$S(\tau|c) \overset{\beta>}{\longleftarrow} S(\varrho|a) \overset{<\alpha}{\longrightarrow} S(\sigma|b) \overset{<\gamma}{\longrightarrow} S(\omega|d). \tag{6. 1}$$

(6. A) *To every pair α, α' with $0 < \alpha < \alpha' \leqslant 1$ it is possible to determine β, γ with $0 < \beta$, $\gamma < 1$ such that if, in* (6. 1), *$S(\tau|c)$ is β direct continuation of $S(\varrho|a)$ and $S(\omega|d)$ is γ direct continuation of $S(\sigma|b)$, then $S(\omega|d)$ is α' direct continuation of $S(\tau|c)$.*

Proof. Choose α'' such that $\alpha < \alpha'' < \alpha'$. Determine, by (5. D), $\gamma (0 < \gamma < 1)$ such that

$$S(\varrho|a) \to S(\omega|d) < \alpha'', \text{ i.e. take } \gamma = \frac{\alpha'' - \alpha}{1 + \alpha}.$$

Consider the configuration $S(\tau|c) \overset{\beta>}{\longleftarrow} S(\varrho|a) \overset{<\alpha''}{\longrightarrow} S(\omega|d)$ and determine, by (5. B, $\beta (0 < \beta < 1)$ such that

$$S(\tau|c) \to S(\omega|d) < \alpha', \text{ i.e. take } \beta = \frac{\alpha' - \alpha''}{1 + \alpha'}.$$

We are now able to prove

(6. B) *To every pair α, α' with $0 < \alpha < \alpha' \leqslant 1$ it is possible to determine β with $0 < \beta < 1$ such that if $S(\varrho|a)$ $S(\sigma|b)$ are α direct continuations of one another and $S(\tau|c)$, $S(\omega|d)$ are β direct continuations of $S(\varrho|a)$, $S(\sigma|b)$ respectively, then $S(\tau|c)$, $S(\omega|d)$ are α' direct continuations of one another.*

Proof. (6. B) is an immediate consequence of the application of (6. A) to the configuration

$$S(\tau|c) \overset{\beta>}{\longleftarrow} S(\varrho|a) \overset{<\alpha}{\underset{\alpha}{\rightleftarrows}} S(\sigma|b) \overset{<\beta}{\longrightarrow} S(\omega|d). \tag{6. 2}$$

From the proof of (6. A) follows that it suffices to take for β the less of the numbers

$$\frac{\alpha'' - \alpha}{1 + \alpha}, \; \frac{\alpha' - \alpha''}{1 + \alpha'}.$$

The real numbers connected with the configurations considered in §§ 4, 5, 6 never depend on the particular elements of the configuration. For this reason every such number will be called an *element-invariant* or a *configuration-invariant.*

7. *Derived sets. Element-symmetric properties.* If $S(\varrho|a)$ is an element of R, the set of all distinct elements of R which are α direct continuations of $S(\varrho|a)$ will be called the *first α derived set* of $S(\varrho|a)$ and will be denoted, as in P.V., by ${}_\alpha\{S(\varrho|a)\}$. The *n-th α derived set* of $S(\varrho|a)$ is defined by induction and is denoted by ${}_\alpha\{S(\varrho|a)\}^n$ $(n = 2, 3, \ldots)$. The *infinite α derived set* of $S(\varrho|a)$ is the union of *all* finite n-th α derived sets of $S(\varrho|a)$, $n = 1, 2, \ldots$, and is denoted by ${}_\alpha\{S(\varrho|a)\}^\infty$. Derived sets are infinite con-

figurations, while those considered in § 4, 5, 6 were finite configurations.

Obviously $\alpha \leqslant \beta$ implies

$$(7.1) \qquad {}_{\alpha}\{S(\varrho|a)\} \subset {}_{\beta}\{S(\varrho|a)\},$$

and hence, by complete induction

$$(7.2) \qquad {}_{\alpha}\{S(\varrho|a)\}^n \subset {}_{\beta}\{S(\varrho|a)\}^n,\ n = 2,3,\ldots,\ {}_{\alpha}\{S(\varrho|a)\}^{\infty} \subset {}_{\beta}\{S(\varrho|a)\}^{\infty}.$$

In § 8 we shall introduce certain infinite-configuration-invariants, namely certain *derived-set-invariants*, which will play a fundamental rôle in our theory. To this effect we first give in the present § the following definition.

A property Φ of ordered pairs of elements of R will be termed *element-symmetric* if and only if

(7. α) *(Identity). If $S(\varrho|a) = S(\sigma|b)$, then the pair $S(\varrho|a)$, $S(\sigma|b)$ has the property Φ.*

(7. β) *(Symmetry). If the pair $S(\varrho|a)$, $S(\sigma|b)$ has the property Φ, then the "inverse" pair $S(\sigma|b)$, $S(\varrho|a)$ has also the property Φ.*

Here are two examples of element-symmetric properties: the property of (7. a) being t direct continuations of one another, t fixed (i.e. $S(\varrho|a)$, $S(\sigma|b)$ has the property Φ if and only if $S(\varrho|a) \overset{<t}{\underset{t>}{\rightleftarrows}} S(\sigma|b)$, $0 < t \leqslant 1$); (7. b) having intersecting (= non disjoint) n-th t derived sets, n, t fixed.

The property (7. a) has a great importance for our theory and will be called the *t direct continuation property* or simply the *t property*. Of course the noun "t property" in reality denotes an uncountable infinity of distinct element-symmetric properties, namely one for each fixed t of the uncountable set $0 < t \leqslant 1$.

8. *Basic integers.* If Φ is an element-symmetric property, then we say that an integer $m \geqslant 2$ is a *Φ basic integer* (where no ambiguity is possible merely *basic integer*) if and only if

(8. α) *there exists a positive integer n and a real number $a (0 < a \leqslant 1)$ such that*

(8. β) *for every element $S(\varrho|a)$ of R in every sequence of m elements of ${}_{a}\{S(\varrho|a)\}^n$, say $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$, $\ldots$, $S(\varrho_m|a_m)$, there is a pair of elements with distinct indices, say*

$$S(\varrho_\mu|a_\mu),\ S(\varrho_\nu|a_\nu),\ \mu \neq \nu\ (\mu, \nu = 1, \ldots, m),$$

which has the property Φ.

The integer m is said then to *correspond* to n, a, Φ and is denoted by $m(n, a, \Phi)$. An integer m corresponding to n, a, Φ is an invariant (in the sense of § 6) of the n-th a derived set.

In the above definition the elements of the sequence $S(\varrho_1|a_1), \ldots, S(\varrho_m|a_m)$ in (8. β) need not be all distinct. From the identity property (7. α) of Φ follows, however, that m corresponds to n, a, Φ if and only if, for every

$S(\varrho|a)$ of R, in every set of m *distinct* elements of ${}_a\{S(\varrho|a)\}^n$ there is a pair of *distinct* elements with the property Φ.

If Φ is the t property, then the integer $m = m(n, a, \Phi)$ is said, as in P.V., to correspond to n, a, t and is denoted by $m(n, a, t)$.

Obviously

(8. A). *If m corresponds to n, a, Φ, then every integer m' greater than m also corresponds to n, a, Φ.*

From (7. 2)

(8. B) *If m corresponds to n, a, Φ, and if $n_1 \leqslant n$, $a_1 \leqslant a$, then m corresponds also to n_1, a_1, Φ (of course $n_1 =$ positive integer, $a_1 > 0$).*

For the t property we have even the stronger form of (8. B)

(8. B′) *If m corresponds to n, a, t, and if $n_1 \leqslant n$, $a_1 \leqslant a$, $t_1 \geqslant t$, then m corresponds also to n_1, a_1, t_1.*

9. *Bases.* The chief importance of the Φ basic integers is that they lead to the existence of what we call *Φ bases.*

Let W be a set of distinct elements of R and Φ an element-symmetric property. We say that the element $S(\varrho|a)$ of R *has the property Φ with respect to the set W* if and only if

(9. α) *there exists an element $S(\sigma|b)$ of W such that the pair $S(\varrho|a)$, $S(\sigma|b)$ has the property Φ.*

If U is a set of distinct elements of R, then a subset V of U will be called a *Φ basis* of U if and only if

(9. β) *V consists of distinct elements*;

(9. γ) *no pair of distinct elements of V has the property Φ*;

(9. δ) *every element of U has the property Φ with respect to V.*

If Φ is the t property, then we say, as in P.V., simply *t basis.*

We now prove

(9. A) *Theorem. If m corresponds to n, a, Φ, then for every element $S(\varrho|a)$ of R there exists a Φ basis of ${}_a\{S(\varrho|a)\}^n$ consisting of less than m distinct elements.*

Proof. Let $S(\varrho_1|a_1)$ be an arbitrary element of ${}_a\{S(\varrho|a)\}^n$. Then either (1) every element of ${}_a\{S(\varrho|a)\}^n$ has the property Φ with respect to $S(\varrho_1|a_1)$ or (2) there exists an element $S(\varrho_2|a_2)$ of ${}_a\{S(\varrho|a)\}^n$ which has not the property Φ with respect to $S(\varrho_1|a_1)$; in this last case $S(\varrho_2|a_2)$ is distinct from $S(\varrho_1|a_1)$ (identity property (7. α) of Φ!) and the set $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$ satisfies (9. γ) (symmetry property (7. β) of Φ!). If (1) is valid, $S(\varrho_1|a_1)$ constitutes already a Φ basis of ${}_a\{S(\varrho|a)\}^n$. If (2) is valid, then again either (1. 2) every element of ${}_a\{S(\varrho|a)\}^n$ has the property Φ with respect to $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$ or (2. 2) there exists an element $S(\varrho_3|a_3)$ of ${}_a\{S(\varrho|a)\}^n$ distinct from $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$, such that the set $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$, $S(\varrho_3|a_3)$ satisfies (9. γ). Thus again either we obtain a Φ basis of ${}_a\{S(\varrho|a)\}^n$ or we continue the operation. Since m corresponds to n, a, Φ, the iteration of this process must end after not more than $m - 1$ steps.

Remark. In the definitions $(8.\alpha)$, $(8.\beta)$, $(9.\alpha)$, $(9.\beta)$, $(9.\gamma)$, $(9.\delta)$ the symmetry property $(7.\beta)$ does not occur. Hence it is possible to define basic integers and bases also for properties of ordered pairs of elements of R which satisfy $(7.\alpha)$ only but not $(7.\beta)$. An example of such a property is the property of being t direct continuation, t fixed, i.e. $S(\varrho|a)$, $S(\sigma|b)$ has the property Φ if and only if $S(\varrho|a) \overset{\leq t}{\to} S(\sigma|b)$, $0 < t \leqslant 1$. But then the theorem (9. A) is not more valid, i.e. for such "element-assymmetric" properties the existence of a basic integer does not imply the existence of a finite basis. For this reason we consider in the present paper only element-symmetric properties. Φ denotes in the sequel always an element-symmetric property.

10. *The main problem.* In the present § we show that the following situation is of particular interest.

$(10.\alpha)$ *There is a β such that all the basic integers $m(n, \beta, \Phi)$, $n = 1, 2, \ldots,$ exist.*

Every β satisfying $(10.\alpha)$ will be called a *remarkable value* of Φ (of course, to be quite exact, it is necessary to add: *with respect to the local property Π considered*). If a remarkable value of Φ exists, Φ is termed *remarkable* (*with respect to Π*).

It is easy to show

(10. A) *Theorem. If α is a remarkable value of Φ, then, for every element $S(\varrho|a)$ of R, every subset V of ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$ that satisfies the conditions* $(9.\beta)$, $(9.\gamma)$ *is countable.*

Proof. Associate with every element $S(\sigma|b)$ of V one and only one positive integer n indicating that $S(\sigma|b)$ belongs to ${}_{\alpha}\{S(\varrho|a)\}^{n}$, ${}_{\alpha}\{S(\varrho|a)\}^{n+1}, \ldots,$ but not to ${}_{\alpha}\{S(\varrho|a)\}^{n-1}$, and call it the *order of $S(\sigma|b)$ with respect to $S(\varrho|a)$*. Since $m(n, \alpha, \Phi)$ exists, there is only a finite number of elements of V of order $n (n = 1, 2, \ldots)$, and therefore the elements of V can be enumerated according to their order.

From (10. A) follows that, if α is a remarkable value of Φ, then every Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$ is countable. Another question is whether such a basis always exists.

(10. B) *Theorem. If α is a remarkable value of Φ, then, for every element $S(\varrho|a)$ of R, there exists a (countable) Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$.*

Proof. For an arbitrarily fixed positive integer n construct, by (9. A), a finite Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{n}$, say $S(\sigma_1|b_1)$, $S(\sigma_2|b_2) \ldots, S(\sigma_k|b_k)$. Consider then the subsequent derived set ${}_{\alpha}\{S(\varrho|a)\}^{n+1}$ and, by iterating the operation of the proof of (9. A), complete the just constructed Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{n}$ to a finite Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{n+1}$, say $S(\sigma_1|b_1)$, $S(\sigma_2|b_2), \ldots, S(\sigma_k|b_k)$, $S(\sigma_{k+1}|b_{k+1}), \ldots, S(\sigma_{k+l}|b_{k+l})$, the elements $S(\sigma_{k+1}|b_{k+1}), \ldots, S(\sigma_{k+l}|b_{k+l})$ being of order $n+1$ with respect to $S(\varrho|a)$ (cf. the proof of (10. A)). By continuing infinitely this process obtain a countable set V which obviously satisfies $(9.\beta)$, $(9.\gamma)$. Since every element of ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$ belongs to a certain

finite derived set ${}_a\{S(\varrho|a)\}^n$, V satisfies also (9. δ) with $U = {}_a\{S(\varrho|a)\}^\infty$.
(10. C) *The theorems* (10. A), (10. B) *remain valid if we replace in them* ${}_a\{S(\varrho|a)\}^\infty$ *by any one of its subsets.*

The theorems (10. A), (10. B) show that, if a is remarkable value of Φ, then the infinite a derived sets have a particularly simple "structure with respect to Φ". This leads to consider the following problem as a fundamental one for our theory.

(10. β) *The main problem. To find necessary and sufficient conditions for a given element-symmetric property* Φ *to be remarkable* (*with respect to a given* Π).

We will give in § 12 a partial solution of the main problem for the t property, namely two sufficient conditions for the t property to be remarkable. The topological importance of such a solution becomes at once evident if one remarks the following fact which is an immediate consequence of (2. A), (4. D).

(10. D) *For every* $a(0 < a \leqslant 1)$ *and for every element* $S(\varrho|a)$ *of* R, *the union of the centres of all elements* $S(\sigma|b)$ *of* ${}_a\{S(\varrho|a)\}^\infty$ *is both open and closed in* R.

From (10. D) follows that for a *connected* R the union of the centres of all elements of ${}_a\{S(\varrho|a)\}^\infty$ is already the whole space R. If then the t property is remarkable, it is possible, by (10. B), to cover R by a countable number of elements.

The case of a non-connected R is more complicated and shall be discussed in detail in Part II.

The more important of the sufficient conditions of § 12 (namely (12. B)) will be derived from § 6 and from a general reasoning which will be developed in § 11.

11. *The condensation principle.* Suppose that m corresponds to n, a, Φ. Consider an indexed set of m^2 (not necessarily distinct) elements of ${}_a\{S(\varrho|a)\}^n$ arranged as an array of m rows, say

$$S_{k1}, S_{k2}, \ldots, S_{km} \qquad (k = 1, 2, \ldots, m) \tag{11. 1}$$

where S_{kl} is written for $S_{kl}(\varrho_{kl}|a_{kl})$ $(k, l = 1, \ldots, m)$. By (8. β) in every row there is a pair of elements with distinct indices which has the property Φ. Suppose the arrangement is such that in each row the pair

$$S_{k1}, S_{k2} \qquad (k = 1, 2, \ldots, m) \tag{11. 2}$$

has the property Φ. Again a pair of elements with distinct indices of the first column of (11. 2), say S_{11}, S_{21}, has the property Φ. Then both pairs S_{11}, S_{12} and S_{11}, S_{21} have the property Φ. Thus we have proved: in every indexed set of m^2 elements of ${}_a\{S(\varrho|a)\}^n$ there are three elements with distinct indices, say S_0, S_1, S_2, such that the both pairs S_0, S_1 and S_0, S_2 have the property Φ. More generally we say that the elements $S_1, S_2, \ldots, S_l$

can be Φ condensed to the element S_0 if and only if the l pairs S_0, S_1; S_0, S_2; ...; S_0, S_l all have the property Φ, and we prove, by complete induction, in exactly the same way as before.

(11. A) *Theorem. If m corresponds to n, a, Φ, then in every indexed set of m^l elements* ($l \geqslant 1$) *of* ${}_a\{S(\varrho|a)\}^n$ *there are* ($l+1$) *elements with distinct indices such that l among them can be Φ condensed to the remaining* ($l+1$)*st.*

The reasoning by which (11. A) was proved and sometimes (11. A) itself shall be called the *condensation principle.*

For the t property the condensation principle has the following important consequence.

(11. B) *If m corresponds to n, a, t and l corresponds to* 1, t, u, *then* m^l *corresponds to n, a, u.*

Proof. By (11. A) in every indexed set of m^l elements of ${}_a\{S(\varrho|a)\}^n$ there are ($l+1$) elements with distinct indices, say $S_0, S_1, \ldots, S_l$, such that $S_1, S_2, \ldots, S_l$ can be "t condensed" to S_0. Since l corresponds to 1, t, u, there are in the sequence $S_1, \ldots, S_l$ two elements with distinct indices which are u direct continuations of one another.

Remark. In §§ 7 — 11 the triangle axiom has not been used. Hence all the definitions and results of these §§ are valid also for a local spherically hereditary property Π associated with a *semi*-pseudo-metric in R.

12. *Sufficient conditions for the t property to be remarkable.*

(12. A) *Theorem. If a basic integer* $m(2, a, t)$ *with* $0 < t \leqslant a \leqslant 1$ *exists, then all basic integers* $m(n, a, t)$, $n = 1, 2, \ldots,$ *exist.*

Proof. We proceed by complete induction by assuming that the basic integers $m(1, a, t)$, $m(2, a, t)$, ..., $m(n-1, a, t)$ ($n - 1 \geqslant 2$) all exist and we prove the existence of the "subsequent" basic integer $m(n, a, t)$. Since $m = m(n-1, a, t)$ exists, there exists, by (9. A), a finite t basis of ${}_a\{S(\varrho|a)\}^{n-1}$, say $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$, ..., $S(\varrho_k|a_k)$, ($k < m$). Since $t \leqslant a$, we have, by (7. 1)

$$ {}_a\{S(\varrho|a)\}^{n-1} \subset {}_a\{S(\varrho_1|a_1)\} \cup {}_a\{S(\varrho_2|a_2)\} \cup \ldots \cup {}_a\{S(\varrho_k|a_k)\} $$

and hence

$$ {}_a\{S(\varrho|a)\}^n \subset {}_a\{S(\varrho_1|a_1)\}^2 \cup {}_a\{S(\varrho_2|a_2)\}^2 \cup \ldots \cup {}_a\{S(\varrho_k|a_k)\}^2. \tag{12. 1} $$

From (12. 1) follows that in any indexed set of km (not necessarily distinct) elements of ${}_a\{S(\varrho|a)\}^n$ there are m elements with distinct indices, say $S'_1, S'_2, \ldots, S'_m$, which belong to the second a derived set of a single element of the just constructed t basis, say to ${}_a\{S(\varrho_1|a_1)\}^2$. Since $m = m(n-$ $-1, a, t)$ ($n-1 \geqslant 2$), m corresponds (by (8. B)) also to 2, a, t, and hence in the sequence $S'_1, S'_2, \ldots, S'_m$ there are two elements with distinct indices which are t direct continuations of one another. Thus km corresponds to n, a, t, and (12. A) is proved.

Corollary. If Φ_t is a given t property (t fixed), then a n·a·s·c· for $a \geqslant t$ to be a remarkable value of Φ_t is the existence of the basic integer m $(2, a, t)$.

Notice that (12. A) is valid also for a Π associated with a semi-pseudometric. The triangle axiom (namely (6. B)) allows to establish the following result which can be considered as an improvement on (12. A).

(12. B) *Theorem. If a basic integer $m(1, a, t)$ with $0 < t < a \leqslant 1$ exists, then it is possible to determine β with $0 < \beta < a$ such that all basic integers $m(n, \beta, t)$, $n = 1, 2, \ldots,$ exist.*

Proof. By (6. B) it is possible to determine to the pair t, a in (12. B) a β with $0 < \beta < a$ such that if the elements S_1, S_2, S_3, S_4 form the configuration

$$S_1 \overset{\beta>}{\longleftarrow} S_2 \underset{t>}{\overset{<t}{\rightleftarrows}} S_3 \overset{<\beta}{\longrightarrow} S_4 \tag{12. 2}$$

then S_1, S_4 are a direct continuations of one another. By hypothesis $l = m(1, a, t)$ exists, and hence, by (8. B), $m(1, \beta, t)$ exists also. Therefore we assume that $m(1, \beta, t), \ldots, m(n-1, \beta, t)$ all exist and we prove the existence of $m(n, \beta, t)$. Let $m = m(n-1, \beta, t)$ and consider any sequence of m elements of ${}_\beta\{S(\varrho|a)\}^n$, say $S'_1, S'_2, \ldots, S'_m$. From the definition of the derived sets follows that there exists a sequence of m elements of ${}_\beta\{S(\varrho|a)\}^{n-1}$, say $S''_1, S''_2, \ldots, S''_m$, such that S'_k is β direct continuation of S''_k for $k = 1, 2, \ldots, m$. Since m corresponds to $n-1, \beta, t$, there are two elements with distinct indices of this last sequence, say S''_1, S''_2, which are t direct continuations of one another. Then the elements S'_1, S''_1, S''_2, S'_2 form the configuration (12. 2). Hence S'_1, S'_2 are a direct continuations of one another. Thus we have proved that m corresponds to n, β, a. Since, by hypothesis, $l = m(1, a, t)$ exists, m^l corresponds, by (11. B), to n, β, t, and (12. B) is proved.

The existence of $m(2, a, t)$ in (12. A) and that of $m(1, a, t)$ in (12. B) (a, t fixed) are properties "in the small" of the T-space R considered, more exactly perhaps "properties in the small of R with respect to Π". The existence of all basic integers $m(n, a, t)$, $n = 1, 2, \ldots$ (a, t fixed) is a property "in the large", whose topological importance becomes particularly evident in the case of a connected R (cf. the end of § 10). We can say that (12. A), (12. B) derive from the above properties "in the small" of R the above property "in the large" of R. But, as will be discussed more in detail in Part II, it will be probably very difficult to find a precise set-theoric topological equivalent for the above properties "in the small", in other words, our theory is of purely metric nature.

Copenhagen, July 18, 1949.

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A NOTE ON THE THEORY OF STELLAR DYNAMICS. II

BY

JAAKKO TUOMINEN

(*Mededeling no. 68b uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft*)

(Communicated by Prof. J. M. BURGERS at the meeting of June 24, 1950)

Abstract. — The components of average velocity and the values of $\overline{U^2}$, $\overline{V^2}$, $\overline{UV}$ have been derived for a flat, not rotationally symmetrical stellar system with the mass concentrated in the centre. It is found that in the system considered $\overline{U^2}$, $\overline{V^2}$ and $\overline{UV}$ are of the same order of magnitude.

1. *Introduction.* — In the present note we are going to consider a stellar system which in other respects is similar to the system of the preceding note (1), except that the longitudes of the apcentra (ϑ_0) are not distributed at random. Instead we suppose them for one half of the orbits to be equal to zero, and for the other half to be equal to π. In this way the most "oval" system possible for a given distribution of excentricities is obtained. In order to simplify the mathematical treatment we divide the whole system into two sub-systems, one with $\vartheta_0 = 0$, the other with $\vartheta_0 = \pi$. First we consider the two systems separately and then combine them in order to obtain the final results.

2. *The Motion of Individual Stars.* — The motion of an individual star in the system is described by the same equations as in the previous note [equations (2, 11), (2, 12) and (2, 16)]. For the system with $\vartheta_0 = 0$, we have

$$(2,1)\quad \begin{cases} T = \sqrt{\dfrac{\mu}{r}} \cdot \sqrt{1-(1-s^2)\cos\vartheta} = \sqrt{\dfrac{\mu}{r}}\, s \sqrt{1+\dfrac{2\,(1-s^2)}{s^2}\sin^2\tfrac{1}{2}\vartheta} \\[2ex] R = -\sqrt{\dfrac{\mu}{r}}\, \dfrac{(1-s^2)\sin\vartheta}{\sqrt{1-(1-s^2)\cos\vartheta}} = -\sqrt{\dfrac{\mu}{r}}\,\dfrac{1-s^2}{s}\,\dfrac{\sin\vartheta}{\sqrt{1+\dfrac{2\,(1-s^2)}{s^2}\sin^2\tfrac{1}{2}\vartheta}} \end{cases}$$

while for the system with $\vartheta_0 = \pi$

$$(2,1)'\quad \begin{cases} T = \sqrt{\dfrac{\mu}{r}} \cdot \sqrt{1+(1-s^2)\cos\vartheta} = \sqrt{\dfrac{\mu}{r}}\, s \sqrt{1+\dfrac{2\,(1-s^2)}{s^2}\cos^2\tfrac{1}{2}\vartheta} \\[2ex] R = \sqrt{\dfrac{\mu}{r}}\, \dfrac{(1-s^2)\sin\vartheta}{\sqrt{1+(1-s^2)\cos\vartheta}} = \sqrt{\dfrac{\mu}{r}}\,\dfrac{1-s^2}{s}\,\dfrac{\sin\vartheta}{\sqrt{1+\dfrac{2\,(1-s^2)}{s^2}\cos^2\tfrac{1}{2}\vartheta}} \end{cases}$$

In both systems, neglecting third order terms of $1 - s^2$,

$$D = -r/s \tag{2, 2}$$

3. *Density and Components of Mean Velocity.* — Let

$$\tfrac{1}{2} f(r_0, T_0, t_0)\, dr_0\, dT_0\, dt_0 \tag{3, 1}$$

denote the number of stars with apcentric distances between r_0 and $r_0 + dr_0$, apcentric velocities between T_0 and $T_0 + dT_0$, and the times of passage through the apcentron between t_0 and $t_0 + dt_0$, in the two systems, $\vartheta_0 = 0$ and $\vartheta_0 = \pi$, separately. The integral of the expression (3, 1) over all values of r_0, T_0 and t_0 gives the total number of stars in each of the two systems separately.

In order to obtain the density per unit area, we change the variables

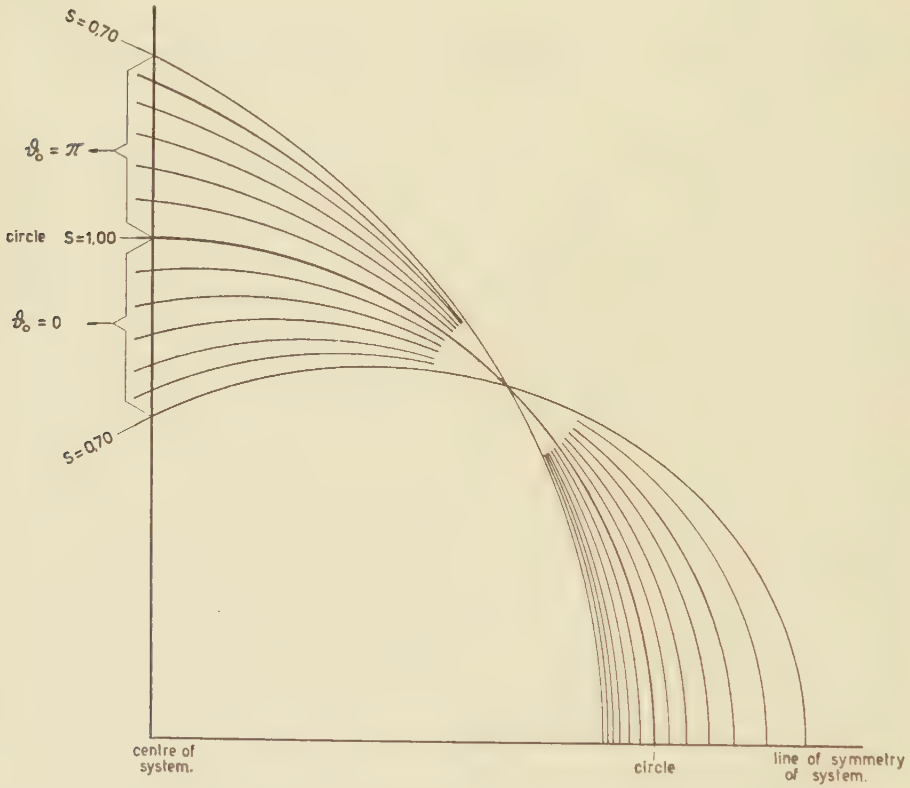

Fig. 1. The figure illustrates the ellipses going through one point, in the system treated in the second note ($s = 0.70, 0.75, \ldots, 0.95, 1.00$; $\vartheta = \pi/4$). The greater axis of each ellipse coincides with the line of symmetry of the system.

r_0, T_0, t_0 in (3, 1) into r, ϑ, s, then divide the expression by $dr \cdot r d\vartheta$, and finally integrate over s. First we have

$$dr_0\, dT_0\, dt_0 = -D \cdot dr\, d\vartheta\, ds.$$

It will be seen that in the present case again the assumption

$$(3,2)\qquad \begin{cases} f(r_0, T_0, t_0) = N, \text{ from } s = 1 \text{ to } s = s_0 \\ f(r_0, T_0, t_0) = 0, \quad \text{for } s < s_0, \end{cases}$$

where $N =$ const. and $1 - s_0 \ll 1$, leads to a constant density, in the limits of accuracy. We have, for each separate system,

$$(3,3)\qquad \varrho = \tfrac{1}{2} N \int_{s_0}^{1} (1/s)\, ds$$

i.e.

$$\varrho = \tfrac{1}{2} N \ln (1/s_0).$$

In the complete system obtained by superposing the two systems with $\vartheta_0 = 0$ and $\vartheta_0 = \pi$ we have accordingly

$$(3,4)\qquad \varrho = N \ln (1/s_0),$$

or introducing $1 - s = \sigma$; $1 - s_0 = \sigma_0$, which are small quantities,

$$(3,5)\qquad \varrho = N \sigma_0 (1 + \tfrac{1}{2} \sigma_0 + \tfrac{1}{3} \sigma_0^2 + \ldots)$$

The terms written down in this expression are not influenced by the terms neglected in equation (2, 2). We see accordingly that the density is practically independent of r and ϑ, when the function $f(r_0, T_0, t_0)$ is defined by (3, 2).

The next task is to calculate $\overline{T}$ and $\overline{R}$. We have

$$(3,6)\qquad \begin{cases} \varrho \overline{T} = \tfrac{1}{2} N \sqrt{\dfrac{\mu}{r}} \displaystyle\int_{s_0}^{1} \sqrt{1 + \frac{2(1-s^2)}{s^2} \sin^2 \tfrac{1}{2}\vartheta}\; ds \\ \varrho \overline{R} = -\tfrac{1}{2} N \sqrt{\dfrac{\mu}{r}} \sin\vartheta \displaystyle\int_{s_0}^{1} \frac{\dfrac{1-s^2}{s^2}}{\sqrt{1 + \dfrac{2(1-s^2)}{s^2} \sin^2 \tfrac{1}{2}\vartheta}}\, ds \end{cases}$$

for the system with $\vartheta_0 = 0$, and

$$(3,6)'\qquad \begin{cases} \varrho \overline{T} = \tfrac{1}{2} N \sqrt{\dfrac{\mu}{r}} \displaystyle\int_{s_0}^{1} \sqrt{1 + \frac{2(1-s^2)}{s^2} \cos^2 \tfrac{1}{2}\vartheta}\; ds \\ \varrho \overline{R} = \tfrac{1}{2} N \sqrt{\dfrac{\mu}{r}} \sin\vartheta \displaystyle\int_{s_0}^{1} \frac{\dfrac{1-s^2}{s^2}}{\sqrt{1 + \dfrac{2(1-s^2)}{s^2} \cos^2 \tfrac{1}{2}\vartheta}}\, ds \end{cases}$$

for the one with $\vartheta_0 = \pi$. The expressions can also be written in the form

$$\varrho\,\overline{T} = \tfrac{1}{2}\,N\sqrt{\frac{\mu}{r}}\int\limits_{s_0}^{1}\left[1+\frac{1-s^2}{s^2}\sin^2\tfrac{1}{2}\vartheta-\tfrac{1}{2}\,\frac{(1-s^2)^2}{s^4}\sin^4\tfrac{1}{2}\vartheta+\ldots\right]ds$$

$$\varrho\,\overline{R} = -\tfrac{1}{2}\,N\sqrt{\frac{\mu}{r}}\sin\vartheta\int\limits_{s_0}^{1}\left[\frac{1-s^2}{s^2}-\frac{(1-s^2)^2}{s^4}\sin^2\tfrac{1}{2}\vartheta+\ldots\right]ds$$

for the former system, and

$$\varrho\,\overline{T} = \tfrac{1}{2}\,N\sqrt{\frac{\mu}{r}}\int\limits_{s_0}^{1}\left[1+\frac{1-s^2}{s^2}\cos^2\tfrac{1}{2}\vartheta-\tfrac{1}{2}\,\frac{(1-s^2)^2}{s^4}\cos^4\tfrac{1}{2}\vartheta+\ldots\right]ds$$

$$\varrho\,\overline{R} = -\tfrac{1}{2}\,N\sqrt{\frac{\mu}{r}}\sin\vartheta\int\limits_{s_0}^{1}\left[-\frac{1-s^2}{s^2}+\frac{(1-s^2)^2}{s^4}\cos^2\tfrac{1}{2}\vartheta+\ldots\right]ds$$

for the latter. For the system as a whole we obtain after some calculation:

$$\varrho\,\overline{T} = N\sqrt{\frac{\mu}{r}}\int\limits_{s_0}^{1}\left[1+\tfrac{1}{2}\,\frac{1-s^2}{s^2}-\tfrac{1}{8}\,\frac{(1-s^2)^2}{s^4}\,(1+\cos^2\vartheta)+\ldots\right]ds$$

$$\varrho\,\overline{R} = -N\sqrt{\frac{\mu}{r}}\sin\vartheta\int\limits_{s_0}^{1}\left[\tfrac{1}{2}\,\frac{(1-s^2)^2}{s^4}\cos\vartheta+\ldots\right]ds.$$

Here we may write

$$\frac{1-s^2}{s^2} = 2\,(1-s)+3\,(1-s)^2+\ldots\;;\qquad \frac{(1-s^2)^2}{s^4} = 4\,(1-s)^2+\ldots$$

With $1-s=\sigma$ we then have:

$$\varrho\,\overline{T} = N\sqrt{\frac{\mu}{r}}\int\limits_{0}^{\sigma_0}\left[1+\sigma+(1-\tfrac{1}{2}\cos^2\vartheta)\,\sigma^2+\ldots\right]d\sigma$$

$$\varrho\,\overline{R} = -N\sqrt{\frac{\mu}{r}}\sin 2\,\vartheta\int\limits_{0}^{\sigma_0}\left[\sigma^2+\ldots\right]d\sigma.$$

Integration gives:

$$(3,\,7)\qquad \begin{cases} \varrho\,\overline{T} = N\sqrt{\mu/r}\,\left[\sigma_0+\tfrac{1}{2}\,\sigma_0^2+\tfrac{1}{3}\,(1-\tfrac{1}{2}\cos^2\vartheta)\,\sigma_0^3+\ldots\right] \\ \varrho\,\overline{R} = -N\sqrt{\mu/r}\,\tfrac{1}{3}\sin 2\,\vartheta\,\left[\sigma_0^3+\ldots\right] \end{cases}$$

from which, dividing by (3, 5):

$$(3,\,8)\qquad \begin{cases} v = \overline{T} = \sqrt{\mu/r}\,\left[1-\tfrac{1}{6}\cos^2\vartheta\cdot\sigma_0^2+\ldots\right] \\ u = \overline{R} = -\sqrt{\mu/r}\,\tfrac{1}{3}\sin 2\,\vartheta\cdot\sigma_0^2+\ldots \end{cases}$$

Accordingly the deviation of the mean motion from a purely circular motion is of the second order of magnitude as regards to σ_0.

4. *The Quantities* $\overline{U^2}$, $\overline{V^2}$, $\overline{UV}$. — In order to calculate the quantities $\overline{U^2}$, $\overline{V^2}$ and $\overline{UV}$, the individual velocities may be expressed for the first system in the form

$$T = \sqrt{\frac{\mu}{r}}\, s \left[1 + \frac{1-s^2}{s^2} \sin^2 \tfrac{1}{2}\vartheta + \ldots\right] = \sqrt{\frac{\mu}{r}} \left[1 - (1-s) \cos\vartheta + \ldots\right]$$

$$R = -\sqrt{\frac{\mu}{r}} \sin\vartheta \, \frac{1-s^2}{s} + \ldots$$

and for the second system in the form

$$T = \sqrt{\frac{\mu}{r}}\, s \left[1 + \frac{1-s^2}{s^2} \cos^2 \tfrac{1}{2}\vartheta + \ldots\right] = \sqrt{\frac{\mu}{r}} \left[1 + (1-s) \cos\vartheta + \ldots\right]$$

$$R = \sqrt{\frac{\mu}{r}} \sin\vartheta \, \frac{1-s^2}{s} + \ldots$$

For the first system we accordingly obtain, neglecting higher powers of $1 - s$,

$$(4,1) \quad V = T - \overline{T} = -\sqrt{\mu/r}\,(1-s)\cos\vartheta; \quad U = R - \overline{R} = -2\sqrt{\mu/r}\,(1-s)\sin\vartheta,$$

and for the second system:

$$(4,2) \quad V = T - \overline{T} = +\sqrt{\mu/r}\,(1-s)\cos\vartheta; \quad U = R - \overline{R} = +2\sqrt{\mu/r}\,(1-s)\sin\vartheta.$$

The squares of U and V as well as the product UV are accordingly the same in both systems. We obtain:

$$(4,3) \quad \begin{cases} \varrho\,\overline{U^2} = \frac{4}{3} N \sigma_0^3 \frac{\mu}{r} \sin^2\vartheta\,; \quad \varrho\,\overline{V^2} = \frac{1}{3} N \sigma_0^3 \frac{\mu}{r} \cos^2\vartheta\,; \\ \varrho\,\overline{UV} = \frac{1}{3} N \sigma_0^3 \frac{\mu}{r} \sin 2\vartheta \end{cases}$$

In order to obtain the values of $\overline{U^2}$, $\overline{V^2}$ and $\overline{UV}$ themselves, these expressions have to be divided by $N\sigma_0$. The following list summarizes the values found in sections 3 and 4, up to the lowest power of σ_0:

$$(4,4) \quad \begin{cases} u = -{}^{1}\,\sqrt{\mu/r}\,\sigma_0^2 \sin 2\vartheta; \quad v = \sqrt{\mu/r}\,(1 - \frac{1}{6}\sigma_0^2\cos^2\vartheta); \\ \overline{U^2} = {}^{4}\,\frac{\mu}{r}\sigma_0^2\sin^2\vartheta; \quad \overline{V^2} = \frac{1}{3}\frac{\mu}{r}\sigma_0^2\cos^2\vartheta; \quad \overline{UV} = \frac{1}{3}\frac{\mu}{r}\sigma_0^2 \sin 2\vartheta \end{cases}$$

5. *Conclusions as Regards to Lindblad's Theory of the Formation of Spiral Arms.* — The hydrodynamical equations of motion have already been written down in the first note, but we repeat them here:

$$(5,1) \quad \begin{cases} \varrho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\vartheta} - \frac{v^2}{r}\right) + \frac{\partial(\varrho\overline{U^2})}{\partial r} + \frac{1}{r}\frac{\partial(\varrho\overline{UV})}{\partial\vartheta} + \frac{\varrho\overline{U^2}}{r} - \frac{\varrho\overline{V^2}}{r} = \varrho\frac{\partial\Phi}{\partial r} \\ \varrho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\vartheta} + \frac{uv}{r}\right) + \frac{\partial(\varrho\overline{UV})}{\partial r} + \frac{1}{r}\frac{\partial(\varrho\overline{V^2})}{\partial\vartheta} + 2\frac{\varrho\overline{UV}}{r} = 0 \\ \frac{\partial\varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial r} + \frac{1}{r}\frac{\partial(\varrho v)}{\partial\vartheta} + \frac{\varrho u}{r} = 0. \end{cases}$$

The last equation is, of course, the equation of continuity. These equations are satisfied by the values (4, 4), similarly as they are satisfied by the values (4, 4) of the previous note. Now equations (4, 4) show that by making the system sufficiently "oval", the quantities $\overline{U^2}$, $\overline{V^2}$, $\overline{UV}$ will be of equal order of magnitude. Accordingly $\overline{UV}$ cannot be neglected. Besides, there is a certain ratio between $\overline{U^2}$ and $\overline{V^2}$, so that we are not allowed to choose for instance $\overline{U^2} = \overline{V^2}$. If, however, we should choose $\overline{UV} = 0$ and $\overline{U^2} = \overline{V^2}$, this would be equivalent to introducing extra forces on the right hand side of the equations. The consequence would be that $\partial u/\partial t$ and $\partial v/\partial t$ would generally not remain zero. The mass elements can then be expected to follow spirals around the centre. Starting from a slightly oval system, we apparently could obtain spiral arms.

One might think that this conclusion does not necessarily have relation to LINDBLAD's theory of the formation of spiral arms, because we have considered only the particular case in which the whole mass is concentrated in the centre. Spiral nebulae, according to observation, rotate nearly as rigid bodies (2), and must therefore have a quite different distribution of density. However, we have been able to show that at least in one special case spiral arms can be expected to appear precisely with the assumptions which have been made by LINDBLAD, but without any disturbance in the density distribution.

It should moreover be observed that if we calculate the orbit of a star in the system, whatever is the distribution of the density, we calculate $u + U$ and $v + V$ as a whole. A priori there should be no special conditions for U and V (such as $\overline{UV} = 0$, $\overline{U^2} = \overline{V^2}$) as long there are no collisions.

I should like to express my gratitude to Prof. J. M. BURGERS for helpful advice in the treatment of the problem of the above two notes, and to Prof. J. H. OORT, Prof. B. LINDBLAD, Dr R. COUTREZ and Prof. M. SCHÜRER for inspiring discussions.

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PERIODIC PATTERNS OF RIPPLED AND SMOOTH AREA'S ON WATER SURFACES, INDUCED BY WIND ACTION

BY

L. M. J. U. VAN STRAATEN

(Communicated by Prof. F. A. VENING MEINESZ at the meeting of June 24, 1950)

In the course of the years 1948 and 1949 the author carried out marine geological researches in the Dutch Wadden sea [1]). This Wadden sea is a tidal flat area, the bottom of which is uncovered for the greater part at every ordinary low tide. During the researches two phenomena were observed almost invariably whenever the relevant conditions as to wind velocity, depth etc. were fulfilled. Both phenomena, though quite different as to their mechanism of development, are connected with the presence of small quantities of contaminations on the surface of the water. By the action of the wind (either direct or indirect) these contaminations become concentrated in strips or streaks. The first phenomenon consists in a longitudinal concentration, a streak pattern being formed parallel to the direction of the wind; it has already been the subject of several investigations [2]). By the other phenomenon elongated concentrations are produced transversely to the wind direction.

I. *The longitudinal phenomenon: foam streaks, lines of smooth water.*

When strong winds blow there appear streaks of foam on the water surface, running parallel to the direction of the wind. Usually these streaks do not continue very far: they soon split up in two, or combine with each other, or interfinger. Nor, as a rule, do the distances between these streaks present much regularity. Sometimes however, especially when viewed from some height, there can be detected a certain dominating interspace value. And under special conditions (mostly a moderate wind and small depths of water) remarkably regular spacing of the streaks may be observed.

The foam of these streaks is ordinary sea foam, which originates in different ways, the most important being:

a. The breaking of the wave crests in strong winds and the consequent inclusion of atmospheric air bubbles;

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[1]) The costs of these researches were defrayed by a grant from the "Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek" (Netherlands Organization for Pure Scientific Research).

[2]) See references 1, 2, 3, 5.

b. The breaking and splashing of waves against solid objects and beaches;
c. The escape of air and gas from bottom sediment that has been uncovered at low tide and is again inundated by the rising tide (foam front of flood water);
d. The contact of water masses of different temperatures and salinities (current junctions). It may often be clearly seen (especially from the air) that from such current junctions, orientated at any angle to the wind, there is formed a set of smaller foam streaks parallel to the direction of the wind.

Together with foam there are concentrated in these streaks all kinds of other materials: weeds, bird's feathers, leaves of eelgrass etc. Also, when small floating objects such as matches, are scattered at random over the foam streaked water, one may note how they are gradually driven together in these lines, to remain there with their long axes preferably parallel to the wind.

Another property of these streaks, although not so conspicuous as the white foam, is their smooth surface. Big waves pass unhindered through the streaks, but the smaller wavelets and ripples, which give the water surface between the streaks its rough appearance, are obstructed. And when there is little or no foam available, in calmer weather and away from current junctions or beaches, this smoothness of the water surface is often the only feature by which the phenomenon may be detected (photo 1).

Langmuir, in 1938, pointed out that "the effect of the wind is to produce a series of alternating right and left helical vortices in the water, having horizontal axes parallel to the wind". The materials floating on the water surface: foam, algal detritus, oil etc. are taken by this movement from both sides to the zones of descent, where they remain in streaks as long as the water keeps converging in these same streaks.

Later, similar phenomena have been investigated by Woodcock (1941) and have been interpreted in the first place as due to thermal instability in the upper layers of the water. It was found that the water was cooler at the surface than below. This cooling of the surface water is probably caused mainly by the wind itself (evaporation). The cold water at the surface will tend to exchange places with the underlying warmer and specifically lighter water. The result is convection, whereby, owing to the wind induced flow of the water, a pattern of horizontal longitudinal rolls evolves. As mentioned above the paths described by the water particles in these rolls have the shape of screw threads, alternately right and left.

Often it is difficult to observe directly this screwing motion in the water itself. Exceptionally clear cases however were observed by the author in the Easter Scheldt (province of Zealand) near Bergen op Zoom (Oct. 1949). The water which had depths of about 10 to 40 cm flowed off

with the ebb tide in a N.W. direction, parallel to a moderate wind from the S.E. On the surface there were regularly spaced sets of smooth streaks, in which the scarce foam present was concentrated. The motion in the water itself was rendered directly visible by the presence of large quantities of rather coarse, suspended mud flakes. This suspended mud was homogeneously distributed in the zones of rising water, halfway between the streaks of smooth water. Below these latter streaks, however, the lines of descent themselves were conspicuous by their relatively clear water [1]).

The velocity of the ebb current at the surface in the smooth streaks was about twice that in the zones of upwelling water. The upwelling water showed little more than the small nearbottom current velocity, whereas the water arriving in the smooth streaks had been near the level of maximum velocity and in direct contact with the downstream wind for the maximum length of time.

The velocity of the convective flow itself was difficult to measure. In one of the Easter Scheldt cases the following approximative data were obtained:

Depth of water	9 cm
Distance between smooth streaks	40 cm
Ebb flow velocity at surface in smooth streaks . .	35 cm/sec
Ebb flow velocity at surface in zones of upwelling water	18 cm/sec
Time in which mud flakes covered distance along surface from zones of ascent to streaks of descent . .	4 — 5 sec
∴ Average velocity of convective flow at surface. . .	4 — 5 cm/sec.

Unfortunately the observer was not equiped to measure temperatures or wind velocities. According to data of synoptic weather stations in the vicinity the wind velocity during the observation was about 12 knots; the temperature of the air was 9° C; that of the bottom probably 11° C or more.

In shallow water, from 1 cm to a few meters deep, there is often an obvious relation between depth and distance between the lines of convergence (see photo 1). This distance is from ca. 2 to ca. 4 times, usually about 3 times the depth. The relation is maintained even when the depth changes considerably over short distances, as for example along beaches.

The regularly spaced parallel foam streaks observed by Seilkopf (as communicated by Neumann (5)) in the Baltic near the German coast seem to be of the same kind. He found distances from 8 to 10 m at a water depth of 2 m, that is a ratio from 4 to 5 between mutual distance and depth.

Theoretical investigations by Pellew and Southwell, and others (Ref. 6, 7), have yielded a numerical value for this ratio, for the first cells which are formed when the thermal instability of the water layer

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[1]) This peculiarity has a nice parallel in many cloud systems (cf. reference no. 4).

is just sufficient to give rise to maintained convective motion. The simplifying assumptions made in these treatments, however, detract somewhat from their value in applying them to the case in question. So e.g. the displacement of the water mass as a whole and the velocity shear in the wind direction are not taken into account, and in each of both horizontal boundaries of the water layer considered the temperature is supposed to be a constant. The mentioned ratio between mutual distance and depth should be 2.83 if both horizontal boundaries are supposed to be "free", that means, if besides the vertical convection velocities the horizontal shearing stresses owing to the convection vanish there. It should be 2.34 if the upper boundary is supposed to be free and the lower one to be "rigid", that means, if all components of the convection velocities vanish there. This condition will apply if the convection extends to the bottom.

In the above mentioned Easter Scheldt case the ebb current had the same direction as the wind. The streaks of smooth water travelled downstream in their own direction. Usually the tidal currents diverge from the direction of the wind. The series of streaks are then seen to travel with the current, preserving their orientation parallel to the wind. This is conceivable, because it is to be expected that the direction of the convection rolls is determined by the direction of the velocity shear beneath the surface, as induced by the wind.

II. *The transverse phenomenon: intermittent rippling.*

The transverse phenomenon appears only in very shallow water. It consists in the succession of more or less ellipse-shaped patches of rippled water in an environment of entirely smooth water or, more frequently, in the formation of series of ellipses of smooth water in an environment of rippled water (see photo 2 and fig. 1). Instead of ellipses other shapes

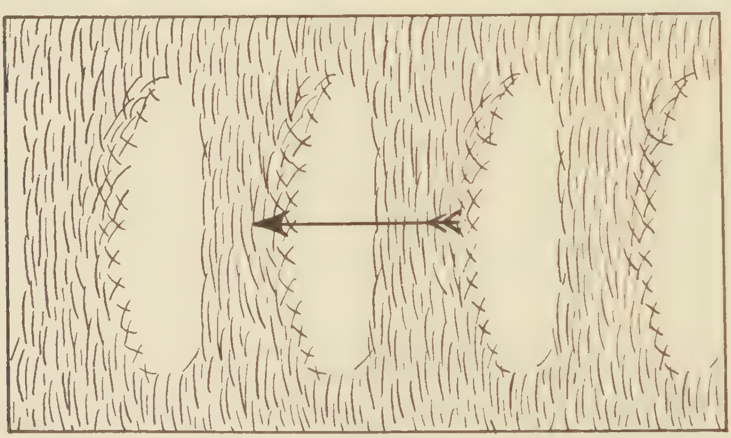

Fig. 1. Train of patches of smooth water travelling from right to left.

may appear, but always with a long axis at right angles to the wind. The long axes of these isolated patches may be no more than one decimeter, but are usually longer: a few decimeters to a few meters. The maximal breadth (i.e. in the wind direction) of the ellipses is usually about 20 – 30 cm and is mostly approximately equal to the minimal breadth of the interspaces. However, when the long axes of the patches are smaller than a few decimeters, are the breadths reduced correspondingly. During very strong winds there may form, instead of trains of ellipses, series of strips, extending over lengths of 2 to 20 m, transversely to the wind.

The phenomenon is observed only in water with average depths of about 1.5 cm (usually great areas of sand flats are covered with bottom ripple marks with heights of ca. 1 cm; an average water depth of 1.5 cm means therefore a depth of 1 cm above the crests of the ripple marks). Only exceptionally may distinct intermittent rippling phenomena be observed in deeper water with depths up to ca. 5 cm. The condition of the bottom: ripple marked (if not too coarsely) or smooth is without any influence.

Furthermore the phenomenon is only seen when the wind surpasses a certain velocity. This critical velocity seems to be about 6 m/sec at a height of 2 m above the water surface. Finally a necessary condition is that the water is flowing. In pools of the required shallowness where the water is stagnant, intermittent rippling was never observed, however large the dimensions of the pools might be. The trains of alternately rippled and smooth patches resp. strips travel downstream.

Owing to the very small depths in which the phenomenon occurs, the wind is usually the main factor determining the direction of flow.

The intermittent rippling may be observed during both ebb and flood stages. During the rising of the water with the flood the duration at a given locality is usually very short. The water then rises with a velocity differing but little from that in deeper parts. During ebb however the sinking of the water level above the flats is retarded, due to the friction of the off-flowing water with the bottom. This friction increases with decrease of depth and it is not to be wondered at, that the range of depths necessary for the occurrence of intermittent rippling is maintained relatively long. Another favourable factor is the wind. The wind may itself ensure a long maintenance of a depth of about 1.5 cm over certain areas by the supply of water it brings along from elsewhere. Strong winds may even push water from deeper reservoirs upwards over sloping banks onto level flats. Thus a continuous flow of shallow water from one channel, over a flat, to another channel may be maintained during the whole period of "emergence" of this flat.

A list of measured data concerning one case is given below:

Date of observation	17 — IX — 1948
Hour	14.h.00 — 16.h.00
Tide	Ebb tide
Locality	Tidal flat immediately W. of jetty at Nes (Ameland)
Bottom	Sand with *Arenicola marina*
Ripple marks on bottom	Asymmetrical ripples, steep towards the East
Orientation of crests of bottom ripple marks	N.N.W.
Wave length of ,, ,, ,, ...	6.7 cm
Ripple height of ,, ,, ,, ...	1.0 cm
Index asymmetry of ,, ,, ,, ...	1/2
Depth of water	1.0 — 2.0 cm
Wind velocity at 1 m above water [1])	9.7 m/sec
,, ,, ,, 3 cm ,, ,, ...	7.1 m/sec
Wind direction	towards E
Direction of flow of water	E
Approximate velocity of water at surface [2])	23 cm/sec
,, velocity of water at 5 mm below surface [2])	12 cm sec
Velocity of patches of smooth water	31 cm/sec
Velocity, parallel to current, of water ripples	43 cm/sec
Velocity, diag. to current, of water ripples	25 cm/sec
Wave length of water ripples — larger ones	5.0 — 6.0 cm
— smaller ones	0.5 — 1.0 cm
Height of water ripples	larger ripples ca. 0,5 cm
Profile of water ripples	very asymmetrical, especially the larger ripples (see fig. 2)
Dimension of smooth patches and of rippled interspaces parallel to wind	ca. 25 cm
Dimension of smooth patches at right angles to wind	ca. 100 cm
Shape of smooth patches	front (downstream) convex, back straight or concave
Diverse observations	Downstream, where the water flows into a channel, the direction of flow deviates considerably from the wind direction (max. about 45°). The long axes of the smooth patches remain at right angles to the wind, but the patches themselves move with the water.

[1]) Determined with the aid of a simple hand anemometer.

[2]) Determined by means of small objects, floating upon and in the water (e.g. small pieces of dry resp. wet paper).

While testing various hypotheses to explain the observed phenomenon the following properties were also ascertained.

1. At a given moment the wind, a few centimeters above the water surface, has the same velocity above smooth patches and the adjacent rippled interspaces. At different times, shortly after each other, these velocities sometimes differed greatly, without influence on the surface pattern.

2. The depth of the water of the smooth patches and the average depth of the rippled strips were equal (with an accuracy of about 1 mm).

3. Experiments with a pipette with capillary opening, from which milk was allowed to escape into the flowing water, showed that the flow in both smooth and rippled patches was turbulent.

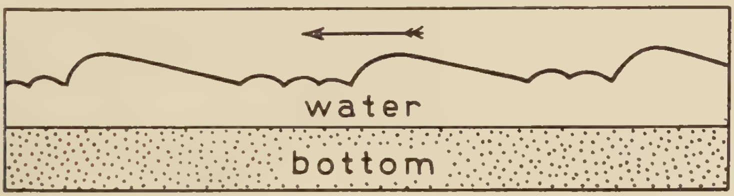

Fig. 2. Profile of water ripples (intermittent rippling phenomenon).

The following hypothesis appears to agree with all known data. On the water there must be an invisible, monomolecular layer. For convenience the material of this layer is referred to as oil although proteins or other capillary active substances may be present as well. When there is no wind and no convection in the water, the thickness of this layer is about equal over large surfaces in consequence of the spreading tendency of the substances composing the film.

Wind, however, when reaching a certain velocity, is able in some way to tear up this layer and to accumulate the oil as a coherent film in between the rents. The rents present water practically freed from oil in which therefore ripples and wavelets immediately appear. It may be observed how these ripples form at the leeward side of a smooth patch, as simple parabolic disturbances (photo 3), then gradually interfere with one another to give parallel wave fronts at right angles to the wind, and then, at the beginning of the next smooth patch abruptly disappear. These smooth surfaces present the places where the oil is pushed up and through which therefore only bigger (solitary) waves may pass.

It is natural that the oil patches travel downstream with the underlying water, also when this flows at an angle to the wind. Nevertheless, there may be some dependance also on the wind, as the velocity of the patches seems always to be a little higher than that of the surface water (see table below).

Velocity of	1948 15–7	21–7	26–8	26–8	6–9	17–9	17–9	28-7-'49
Wind at 2 m above water . . .	—	—	—	—	—	—	—	1080
,, ,, 1 m ,, ,, . . .	—	—	960	960	890	970	970	1020
,, ,, 3 cm ,, ,, ...	—	—	760	760	650	730	710	770
Wind "at watersurface" [1]) . . .	500	1000	—	—	—	—	—	710
Smooth patches	17	47	25	25	20	29	31	—
Surface water.	—	—	16	17	10	23	23	—

All values are expressed in cm/sec.

[1]) I.e. the velocity of the dark ripple shadows shooting over the water which may usually be observed during strong and gusty winds (cf. the longitudinal depressions in the water surface as shown by Photo 4).

It is only when the water is very shallow and no larger waves can arise, that the phenomenon becomes conspicuous. The smaller waves which are the only ones formed cannot pass across the film of oil.

In a paper to be published later in these Proceedings R. DORRESTEIN will give more detailed theoretical considerations based upon the above hypothesis.

In order to test the hypothesis in the field, some simple experiments were carried out.

1. When vigorously sweeping aside the uppermost sheet of water from the smooth patches (by means of a board, for instance) there soon appear ripples in the formerly unrippled spaces.

2. On a smooth water surface, where an oil film can be assumed preventing the formation of ripples while the wind is strong enough to cause the phenomenon elsewhere, artificial rents in that oil film can be made. This may be done by holding a small board vertically above the water, at right angles to the wind, with the lower edge for instance 1 dm above the surface. Behind the board wind vortices then form and below these vortices the wind velocity is much higher. Here a rippled area appears. When taking the board away (and only then) the rippled area starts moving downstream with about the velocity of the surface water itself. This patch however gradually diminishes in size by the spreading tendency of the oil and after covering some 10 meters distance the oil layer finally closes again over the gap and the whole water surface is smooth again.

3. In opposition to the above mentioned circumstances, it also often

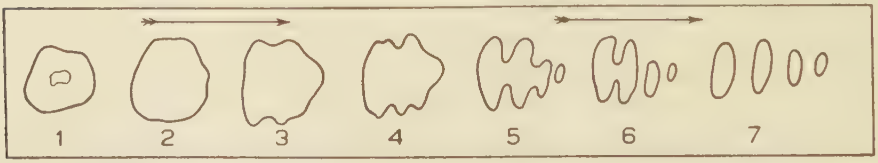

Fig. 3. Intermittent rippling phenomenon developed from solitary smooth patch of gasoline: 1—7 subsequent stages.

occurs, especially during strong winds, that big area's with water of suitable depth, present an evenly rippled surface without any smooth patches. When in such cases a few drops of low grade gasoline are poured on the water, the following series of events can be observed (see fig. 3).

a. A large patch appears on the water, in which all ripples are completely smoothed out. In the centre of this patch there is a coloured area (colours of thin films); the outer parts remain entirely uncoloured (fig. 3 : 1).

b. After a few seconds the coloured centre has disappeared, obviously due to rapid evaporation of the most volatile constituents of the gasoline mixture; the size of the patch has not increased much (fig. 3 : 2).

c. From the sides of the patch the ripples of the surrounding water penetrate in wedge-shaped areas. At the same time small patches detach themselves from the front of the main patch (fig. 3 : 3 — 6).

d. Finally the wedges of ripples penetrating the smooth patch from both sides join together and a series of smooth ellipses has been formed, indistinguishable from natural series (fig. 3 : 7).

Appendix 1. *Origin of the film.*

As to the origin of the substances forming the monolayers two sources are apparent. In the first place oil may come from ships, but perhaps a more important part of the material of the monolayers may be derived from organisms. It has been stated already that the smooth patches occurring on rippled water or the rippled patches occurring on smooth water, have a certain constant length at right angles to the wind (see photo 2). The patches are arranged in parallel trains of a constant width. At first it was thought that these trains, separated by belts of either evenly smooth or evenly rippled water had something to do with longitudinal convective rolls in the air (cf. the longitudinal convective rolls in water treated in section I of this paper). This however was disproved in two ways. Some experiments with smoke showed that the distance between the parallel trains was independent of the motion of the air. Further it was observed that the space between the trains always remained constant, even when the trains curved round with the water towards a tidal channel and thus deviated from their original direction parallel to the wind.

On the other hand it was frequently noted that trains of smooth patches or longitudinal belts of smooth water separating trains of rippled patches, originated at small clumps of mussels (Mytilus edulis). Probably oil is given off by these clumps, although it is difficult to ascertain whether the mussels themselves are responsible, or micro-organisms concentrated in these clumps, or even the mud, containing organic matter, deposited by the mussels (faecal pellets).

Also it is a common observation that during ebb tide narrow streaks

of smooth water appear on the evenly rippled surface of the water in deeper channels, starting at the mouth of minor gullies incised in mud banks. These gullies carry water, often derived from mud flats without any mussels to speak of, but usually containing large quantities of freshly eroded mud in suspension. From chemical analyses it is known that the mud of the tidal flats is always relatively rich in organic matter. So it is not unlikely that most of the material for the monolayers of the intermittently rippled trains is derived from bottom deposits.

Appendix 2. Bottom ripple marks.

It was mentioned already that the presence of ordinary sand ripple marks with heights of $\pm$ 1 cm does not interfere with the intermittent rippling phenomenon.

Meanwhile it is sometimes seen that together with the occurrence of intermittent rippling of the water new bottom ripple marks are formed, preferably when the bottom surface was originally smooth. These ripple marks may form in sand, but are much more prominent when composed of mud. In the few observed cases they formed systems of diagonally crossing ridges with angles of about 35° to each side of the direction in which the water flowed (see photo 4). It is not clear whether the flow of the water alone is responsible or whether a part is also played by the water ripples, which originate at the windward side of the ripple spaces as more or less parabolic disturbances (see photo 3 and fig. 1).

Summary.

Two phenomena are described in this paper which may be observed very frequently in the Wadden sea. The first phenomenon is the development of a system of convective rolls in the water due to the action of the wind blowing over the water. The rolls are parallel to the wind. The convective movement of the water brings about the concentration of floating objects, such as seaweeds, foam and oil, above the lines of descent between adjacent rolls. The concentration of oil results in the formation of approximately equidistant streaks of relatively smooth water parallel to the wind. In shallow water there is often a relation of the distance between these streaks (which is equal to the transverse horizontal diameter of pairs of two adjacent convective rolls) and the water depth, the former being usually 2 — 4 times the latter.

The other phenomenon has not been described before in print. The author proposes to refer to it under the name of *intermittent rippling.* It consists in the alternation of patches of smooth water and rippled interspaces or vice versa. The patches have their long axes at right angles to the wind, and are arranged in trains which run parallel to the wind. The phenomenon appears only in very shallow water (average depth normally no more than 1.5 cm) which flows off, mainly under influence of the wind.

Photo 1. Set of smooth streaks on water surface in "zwin" ("low", longshore depression of sandbeach). Low tide, North Sea coast of Ameland. Note the narrow spacing of the streaks in the shallower water at both sides.

Photo 2. Trains of ellipses of smooth water travelling against observer. (Tidal Flat South of Ameland).

Photo 3. Detail of train of smooth water patches, travelling against observer (Tidal Flat, Ameland).

Photo 4. Detail of train of smooth water strips travelling from right to left. Direction of (strong) wind is marked by narrow longitudinal depressions in water surface. Dark ridges of faecal pellet mud (from Molluscs) are formed diagonally to direction of water flow, on flat sand bottom. Mussels at left (Tidal Flat, Ameland).

The patches move downstream with the water, even when the direction of flow deviates from that of the wind. It is assumed that there is an invisible film of oil and/or other substances on the water surface. When the wind surpasses a critical velocity this film is torn up rhythmically. In the rents the water is rippled and between the rents the surface remains smooth. This hypothesis was supported by some simple experiments in the field.

Groningen, March 1950.

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ON THE EFFECT OF LIGHT OF VARIOUS SPECTRAL REGIONS ON THE SPROUTING OF POTATO-TUBERS

BY

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(Communicated by Prof. A. J. KLUYVER at the meeting of June 24, 1950)

Introduction.

It is well-known that potato-tubers, some time after being harvested, develop long sprouts in darkness at suitable temperatures. It has been known since a long time that in light, already at low intensities, the length of the developed sprouts is strongly decreased (1, 2, 3). In practice, this knowledge has been applied by the construction of store-houses with windows. In view of storage of potatoes in cellars and other rooms with artificial illumination, it was of importance to know something about the spectral sensitivity of the mentioned effect in relation with the choice of the most suitable light sources.

Moreover, a study of the sensitivity of the sprouting appeared of considerable physiological interest. The observation that light inhibits sprouting indicates that a photochemical process is at the base of the observed effect, and it appeared required to attempt an identification of the pigment, active as light absorber. Since stored potatoes do not contain large, obviously manifest amounts of pigments, which might, beforehand, be visualized as responsible for photochemical activities, it is probable that the photoactive pigment is present only in small amounts, and that the process is of of the 'stimulus' type, *i.e.*, that the energy for the reaction is for the greater part energy derived from the cell, directed only by a small amount of light energy (4).

In connection with experiments on the influence of temperature on sprouting of potatoes (5), the second author, in 1947, made some observations on inhibition of sprout elongation using a 'red' and a 'blue' fluorescent tube, and a HPW 75W (high pressure mercury vapour lamp) with Wood's glass bulb, transmitting chiefly ultra violet light. The 'red' and the 'blue' light were found to be more active in inhibiting the growth of the potato-sprouts than the ultra violet.

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Preliminary observations along another line, using different light qualities, obtained by glass filters were made by the first mentioned author in collaboration with Dr R. van der Veen, who published some preliminary results (6). A more detailed study, however, appeared worth while.

Observations and Discussion.

A first series was started in the equipment described (7). In each compartment 6 horizontal gauze plates were mounted, much like a staircase, yielding different intensities. The intensities as measured are collected in Table I, in which also the amounts of energy at wave lengths below 1 μ are indicated.

TABLE I

Energies incident upon the respective gauze plates (1—6) in the various compartments. A: radiation with wavelength $< 3\,\mu$; B: radiation with wavelength $< 1\,\mu$.

A

	1	2	3	4	5	6
violet	1150	600	450	250	120	60
blue	2850	1600	1500	750	250	100
green	3700	2000	1200	630	250	100
yellow.	10000	5400	2900	1450	540	160
red	4500	2750	1650	900	325	100
infrared	7500	3950	2750	1500	750	300

B

	1	2	3	4	5	6
violet	805	420	315	175	84	42
blue	2622	1472	1380	690	230	92
green	3108	1680	1008	529	210	84
yellow.	5000	2700	1450	725	270	80
red	3825	2337	1402	765	276	85
infrared	2100	1106	770	420	210	84

The light intensity was measured at each gauze plate in the various compartments, with the aid of a selenium photronic cell, calibrated for each spectral region. For the infrared region a standardized thermopile was used. It has been shown in (7) that none of the radiations used was completely free from infrared of wavelengths beyond 1 μ. In some compartments, *e.g.*, yellow and infrared, the amount of this radiation was considerable. Its magnitude was established as expounded in (7); for the greater part, probably, it may be considered as photochemically inactive in the process studied (*cf.* below). Moreover, some compartments contained radiation which was not the one desired, but which could not be considered to be inactive. This held especially for violet and red which contained a non-negligible amount of radiation between ~ 0.7 and 1 μ. It must be considered as "impurity" of our spectral light, and as an inadequacy still inherent in our present experiments, which we hope to master more completely in the near future.

Experiments are now in progress to check the activity of the radiation $> 1\,\mu$ directly.

On each gauze plate 30 tubers of the same size-class of the early potato variety 'Eersteling' were exposed to the light. The tubers had been desprouted once and had been stored at 2° C. A dark control was added. The experiment was started Nov. 11, 1949. The temperature was 17° C, the light burnt continuously. Temperature readings in the compartments were taken twice a day; at each reading they did not differ more than 1° C for 5 of 6 gauze plates (the upper one excepted) in all compartments. A small ventilator in each compartment caused air circulation. On Dec. 6, Dec. 20, 1949 and Jan. 3, 1950, 10 tubers of each plate were examined, and the sprouts measured. In Table II, the length of the longest sprout,

TABLE II

Sprout lengths in mm (longest sprout, average of 10 tubers) at various dates reached upon the irridiations summarized in Table I. Start of the experiment: Nov. 11, 1949.

6.12.'49	1	2	3	4	5	6	dark
violet	11.3	11.5	11.4	12.6	13.6	16.4	
blue	10.8	11.7	11.1	13.0	12.2	14.8	
green	16.0	20.7	26.2	29.5	36.9	48.1	200.0
yellow	19.7	22.6	34.1	39.0	44.4	40.5	
red	9.8	10.2	10.0	10.5	11.0	11.2	
infrared	8.8	10.8	11.2	11.2	11.7	11.6	
20.12.'49							
violet	15.0	17.0	16.1	17.8	17.1	38.4	
blue	15.1	14.3	16.9	15.8	19.0	18.9	
green	39.8	40.2	62.1	106.9	104.2	124.8	400.0
yellow	35.5	49.5	54.8	79.1	92.3	98.7	
red	12.6	14.6	14.0	15.0	14.2	15.2	
infrared	12.3	13.4	14.2	13.2	15.1	15.2	
3.1.'50							
violet	19.3	22.7	22.0	22.7	33.6	75.3	
blue	21.6	20.8	21.1	23.0	23.7	35.4	
green	63.0	88.5	113.0	15.5	15.9	20.8	680.0
yellow	56.0	62.0	99.2	131.0	130.0	166.0	
red	18.2	19.7	18.6	20.6	21.0	26.3	
infrared	16.5	18.6	17.5	18.3	22.1	22.3	

taken as a measure, averaged over 10 tubers, is recorded in mm. Photographs of representative tubers are presented on Plate 1 (one tuber of each gauze plate, decreasing intensity from bottom to top of the photograph).

It is seen that all of the intensities used in violet, blue, red and infrared yield a nearly complete suppression of sprouting. Only the last harvest shows slightly weaker inhibition in the lowest intensities blue and violet; nevertheless, the procentual inhibition does not decrease considerably. Yellow and green, in the intensities used, show longer sprouts but also here, the weakest inhibition still is of the order of 70 % (*cf.* Table III,

TABLE III

Percentage inhibition of sprouting (as compared with dark control) by exposure to various radiations. Experiment started Nov. 11, 1949.

6.12.'49	1	2	3	4	5	6
violet	94.3	94.2	94.3	93.7	93.2	91.8
blue	94.6	94.1	94.4	93.5	93.9	92.6
green	92.0	89.6	86.9	85.2	81.5	75.9
yellow.	90.1	88.7	82.9	80.5	77.8	79.8
red	95.1	94.9	95.0	94.7	94.5	94.4
infrared	95.6	94.6	94.4	94.4	94.1	94.2
20.12.'49						
violet	96.2	95.7	96.0	95.5	95.7	90.4
blue	96.2	96.4	95.8	96.0	95.2	95.3
green	90.0	89.9	84.5	73.3	73.9	68.8
yellow.	91.1	87.6	86.3	80.2	76.9	75.3
red	96.8	96.3	96.5	96.2	96.4	96.2
infrared	96.9	96.6	96.4	96.7	96.2	96.2
3.1.'50						
violet	97.2	96.7	96.8	96.7	95.0	88.9
blue	96.8	96.9	96.9	96.6	96.5	94.8
green	90.7	87.0	83.4	77.2	76.6	69.4
yellow.	91.8	90.9	85.4	80.7	80.9	75.6
red	97.3	97.1	97.3	97.0	96.9	96.1
infrared	97.6	97.3	97.4	97.3	96.7	96.7

showing the percentage inhibition of sprouting). In order to evaluate the inhibiting effect of the various regions the experiment has to be extended to lower intensities or shorter exposures which is now in progress.

The preliminary results shown reveal a few interesting facts. First, the activity of the red region of the spectrum may be deemed to rule out the possibility that a carotenoid is the photosensitive pigment. The weaker action of yellow and green, as compared with red, blue and violet would be in accordance with the assumption that a chlorophyllous pigment is in play. If this is true, it must, already in low concentrations, be able to cause the observed effect, since only during harvesting, the tubers may have been exposed to light, and, accordingly, showed only very slight amounts of chlorophyll. In the colours violet, blue, green, yellow, and red, however, considerable amounts of chlorophyll are formed both in the tubers and in the sprouts during the expositions (*cf.* Table IV, fig. 1.).

The determinations of chlorophyll were made separately in the tubers and the sprouts. Of each group 4 discs, 15 mm in diameter and 3 mm thick out of the surface part of the tuber, and 3 sprouts of comparable size and known weight were extracted. The tissues were killed in boiling water, and extracted till colourless in 25 cc ethanol 90 % at 60° C. The optical density of the extracts was determined with a "lumetron, model 400A", colorimeter, using the red filter.

TABLE IV

Relative chlorophyll contents ("optical densities") of tubers and sprouts from various irradiations. Experiment started Nov. 11, 1949. Analysis of Dec. 20, 1949, and Jan. 3, 1950. *Cf.* text.

	Tuber						
20.12.'49	1	2	3	4	5	6	dark
violet	0.17	0.20					0.04
blue	0.27	0.20					
green	0.55	0.77				0.07	
yellow	1.14	1.00				0.15	
red	1.55	1.60				0.60	
infrared	0.05						
3.1.'50							
violet	0.22	0.17	0.13	0.08	0.07	0.07	0.03
blue	0.26	0.23	0.22	0.16	0.09	0.09	
green	1.43	1.06	0.75	0.34	0.20	0.18	
yellow	1.69	1.21	0.88	0.62	0.38	0.24	
red	2.03	2.11	1.67	1.25	0.58	0.48	
infrared	0.08	0.04	0.12	0.06		0.04	
	Sprout						
3.1.'50	1	2	3	4	5	6	dark
violet	1.23						0.15
blue	2.00	1.56	1.60	1.40	1.08	0.85	
green	3.15						
yellow	2.85						
red	3.40	3.70	3.30	3.57	2.47	2.03	
infrared	0.18						

Chlorophyll formation was found strongest in the red light, less in blue and violet, intermediate in yellow and green. In the infrared no increase in optical density as compared with darkness was detectable. A comparison of data obtained on Jan. 3, with those of Dec. 20, showed that in most groups the chlorophyll content of the tubers still had increased.

It might be assumed that protochlorophyll is the photosensitive pigment involved in the process inhibiting sprouting. In this connection the results obtained in infrared, are of considerable interest. In this region the inhibition of sprouting is not less complete than in corresponding intensities of the visible red, but very little chlorophyll is formed. One might assume that nevertheless protochlorophyll might be responsible for the inhibition of sprouting with a much higher quantum efficiency than for the formation of chlorophyll. However, the behaviour in yellow and green seems to contradict this, for, here, less inhibition of sprouting is found than in the infrared, whereas much more chlorophyll has been formed.

A second possibility to be considered is that chlorophyll itself is the

pigment involved in the inhibition of sprouting. The fact that the tubers and sprouts, after exposition to our infrared showed practically no greening (the optical density of the extract was of the same order as

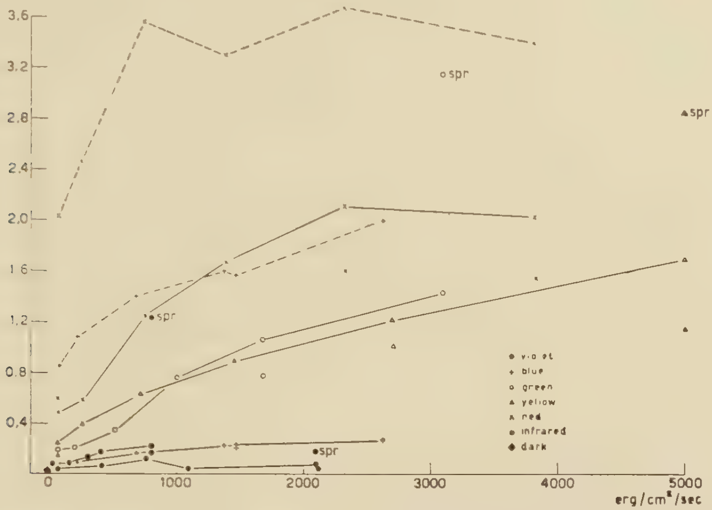


Fig. 1. Relative chlorophyll contents of tubers and sprouts from various irradiations. Experiment started Nov. 11, 1949. Records of Dec. 20, 1949 and Jan. 3, 1950 Abscissa: Intensity of irradiation of wavelengths $< 1\ \mu$, in ergs/cm² sec. Ordinate: Optical density of extract. Solid line: tuber; – – – sprout; connected points: Jan. 3, 1950; separate points: Dec. 20, 1949.

that found after sprouting in darkness) does not exclude that the considerable inhibition found in the infrared might be due to the presence of traces of chlorophyll formed during harvesting.

In this connection it appeared of importance to try to establish the long wave length limit of the activity in the infrared.

To this purpose separate cases were built, of $128 \times 35 \times 30$ cm, each with a loose roof, in which, at one end, in the middle of a short side, a square window was made, provided with coloured filters. The following filters were used: SCHOTT RG 2, RG 5, RG 8, RG 5 + BG 3, RG 10, RG 7 and the combination no. 21 + no.1, of our own glasses, being used for the infrared in the previous experiment. All were combined with our glass no. 7, yielding less further infrared radiation. Incandescent lamps of 100 and 60 Watt were used as light sources, mounted directly above the filters. Fig. 2 shows the amounts of radiation transmitted in the various wave lengths regions (*cf.* also 7). From this graph the amount of radiation $> 1\ \mu$ was computed. Direct checks were made using the filter RG 7 as differential filter which yielded results agreeing within 10 % with those of the above method.

Ten rows of 5 tubers each were placed parallel to the short sides on the case floor, at suitable distances. The light intensities were measured at a number of places in each case with the aid of a thermopile and the values for the other places were interpolated. The intensities in the various cases are given in Table V (total radiation, and radiation $< 1\ \mu$).

The results are given in Plate 2*A*, which shows very clearly that the region transmitted by RG 10 and RG 7 is much less active than that

transmitted by the other filters, so that the activity is concentrated in the near infrared, at wavelengths chiefly lower than 0.96 μ. Nevertheless, in

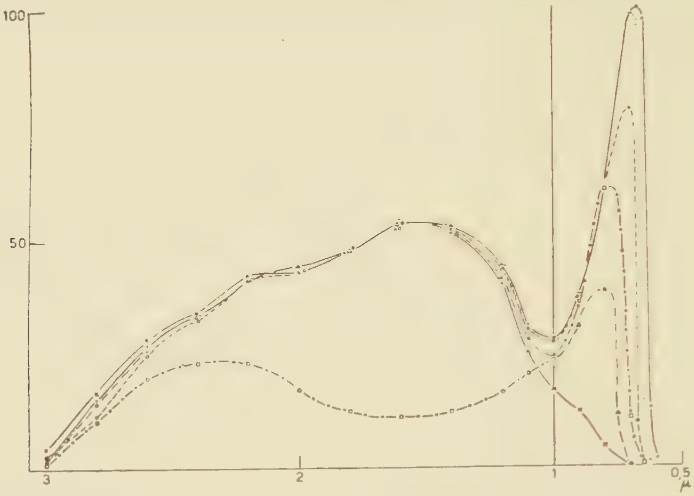


Fig. 2. Spectral distribution of energy of incandescent radiation as transmitted by various filters. From right to left: RG 2 + 7; RG 5 + 7 (weak dots); RG 8 + 7; BG 3 + RG 5 + 7; RG 10 + 7; RG 7 + 7. Beyond 1μ: All equal except BG 3 + RG 5 + 7.

TABLE V

Energy incident upon the respective rows (1 — 10) in the various cases. A: radiation with wavelength $< 3\ \mu$; B: radiation with wavelength $< 1\ \mu$. Experiment started Feb. 6, 1950.

A	1	2	3	4	5	6	7	8	9	10
RG 2 + 7	9294	7609	981	947	700	507	400	313	255	200
RG 5 + 7	10055	6919	918	722	530	388	318	260	195	150
RG 8 + 7	8855	3708	1008	832	620	466	372	301	248	194
21 A + 1 + 7. . .	3587	2770	272	204	150	107	92	74	60	45
BG 3 + RG 5 + 7	8334	6424	3706	1176	740	467	370	291	218	157
RG 10 + 7	6258	1432	418	270	222	180	126	91	66	48
RG 7 + 7	8305	5675	1850	702	570	463	335	254	192	139
B										
RG 2 + 7	2170	1780	230	222	164	119	93.5	73.5	60	47
RG 5 + 7	2140	1474	196	154	113	83	68	55.5	41.5	32
RG 8 + 7	1630	681	184	153	114	86	68	55.5	46	36
21 A + 1 + 7. . .	1076	831	81.5	61	45	32	27.5	22	18	13.5
BG 3 + RG 5 + 7	2425	1870	1080	342	215	136	108	84.5	63.5	45.5
RG 10 + 7	662	152	44.5	28.5	23.5	19	13	9.5	7	5
RG 7 + 7	267	182	59	22.5	18.5	15	11	8	6	4.5

this series, the light transmitted by RG 10 shows some inhibitory effect in the highest intensities used. The sprout lengths and the percentage inhibition found in the various regions is given in Table VI.

TABLE VI

A: Sprout lengths in mm (longest sprout, average of 5 tubers) for the irradiations of Table V.

B: Percentage inhibition of sprouting as compared with dark controls, from the data under A. Experiment started Feb. 6, 1950. Record of March 6, 1950.

A	1	2	3	4	5	6	7	8	9	10
RG 2 + 7	15.2	14.0	14.1	14.8	13.7	16.8	15.8	16.8	16.4	14.8
RG 5 + 7	14.0	13.0	14.8	14.0	13.2	15.0	15.0	16.2	15.9	16.0
RG 8 + 7	14.4	13.3	12.9	15.2	14.4	12.8	15.3	14.7	15.9	16.6
21 A + 1 + 7. . .	14.8	14.7	16.5	16.2	17.4	16.2	17.8	45.4	73.0	85.6
BG 3 + RG 5 + 7	14.0	13.1	14.0	15.0	16.2	16.9	16.8	18.2	18.8	22.4
RG 10 + 7	93.4	125.6	154.2	197.8	164	194	136	167	178	187
RG 7 + 7	158.0	177.5	179.5	167	156.5	191	174	169	186	173
dark	148.0		147		153	-	137		148	139
B										
RG 2 + 7	92.4	93.0	93.0	92.6	93.1	91.6	92.1	91.6	91.8	92.6
RG 5 + 7	93.0	93.5	92.6	93.0	93.4	92.5	92.5	91.9	92.0	92.0
RG 8 + 7	92.8	93.3	93.3	92.4	92.8	93.6	92.3	92.6	92.0	91.7
21 A + 1 + 7. . .	92.6	92.6	91.7	91.9	91.3	91.9	91.1	77.3	63.5	57.2
BG 3 + RG 5 + 7	93.0	93.4	93.0	92.5	91.9	91.5	91.6	90.9	90.6	88.8
RG 10 + 7	53.3	37.2	22.6	1.1	18.0	3.1	32.0	16.3	10.8	6.5
RG 7 + 7	21.1	11.3	10.3	16.4	21.7	4.6	13.0	15.7	7.0	13.6

TABLE VII

Relative chlorophyll contents ("optical densities") of tubers and sprouts from various irradiations. Experiment started Feb. 6, 1950. Record of March 6, 1950.

	Tuber									
	1	2	3	4	5	6	7	8	9	10
RG 2 + 7	1.06		0.53		0.38					0.28
RG 5 + 7	0.47		0.34							0.08
RG 8 + 7	0.36		0.16							0.07
21 A + 1 + 7. . .	0.16						0.08			0.13
BG 3 + RG 5 + 7	0.07		0.06		0.03					
RG 10 + 7	0.03									
RG 7 + 7	0.04									
dark	0.09									
	Sprout									
RG 2 + 7	2.41		1.48		0.91					0.47
RG 5 + 7	1.70		0.71							0.17
RG 8 + 7	0.68		0.28							0.17
21 A + 1 + 7. . .	0.15		0.09				0.06			0.05
BG 3 + RG 5 + 7	0.15		0.10		0.06					
RG 10 + 7	0.03									
RG 7 + 7	0.04									
dark	0.03									

Chlorophyll determinations were also made (Table VII, fig. 3), yielding results which, in general, agreed with those obtained before.

In a following series the light intensities applied were decreased in the cases with the filters RG 2 — (RG 5 + BG 3) included, by substituting 15 Watt bulbs for the 60 Watt ones, and increased in those with RG 10 and RG 7 by using 200 Watt instead of 100 Watt lamps. These substitutions yielded changes in the relation of the radiations $< 1\,\mu$ and $> 1\,\mu$ of only a few per cents which where neglected.

The results of this series are shown in Tables VIII and IX, and fig. 3.

TABLE VIII

Energy incident upon the respective rows (1—10) in the various cases. *A*: radiation with wavelength $< 3\,\mu$; *B*: radiation with wavelength $< 1\,\mu$. Experiment started March 14, 1950.

A	1	2	3	4	5	6	7	8	9	10
RG 2 + 7	2620	2150	276	224	178	125	87	63	46	37
RG 5 + 7	2670	2000	256	207	161	109	76	58	43	30
RG 8 + 7	2120	1686	266	234	168	118	87	61	51	38
21 A + 1 + 7. . .	412	53	37	29	15	11	8	5	4	3
BG 3 + RG 5 + 7	921	645	90	79	56	44	25	19	18	11
RG 10 + 7	23924	9094	1600	1358	1004	670	492	341	263	210
RG 7 + 7	22200	13390	2270	1790	1368	917	648	475	362	267
B										
RG 2 + 7	612	502	64.5	52.5	41.5	29	20.5	14.5	11	8.5
RG 5 + 7	568	426	54.5	44	34	23.5	16	12.5	9	6.5
RG 8 + 7	390	310	49	43	31	21.5	16	11	9.5	7
21 A + 1 + 7. . .	123	16	11	8.5	4.5	3.5	2.5	1.5	1.2	0.9
BG 3 + RG 7 + 7	267	187	26	23	16.5	13	7	5.5	5	3
RG 10 + 7	2540	964	170	142	107	71	52	36	28	22
RG 7 + 7	710	428	72	57.5	44	29.5	20.5	15	11.5	8.5

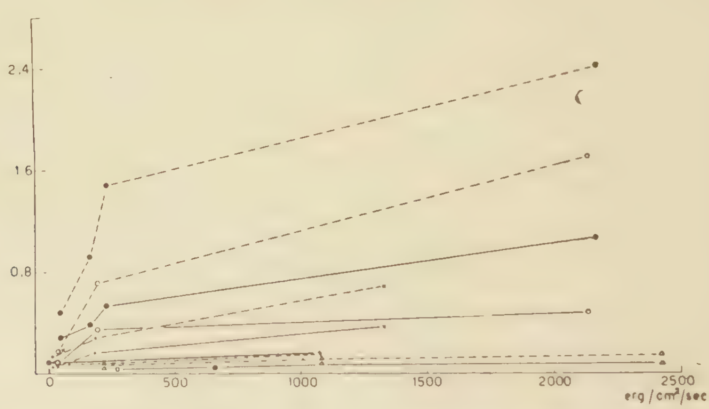
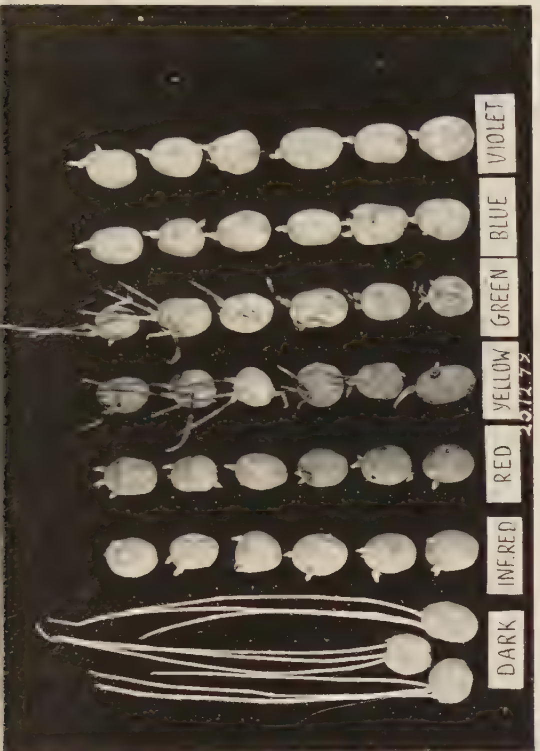


Fig. 3. Relative chlorophyll contents of tubers and sprouts from various irradiations. Experiment started Feb. 6, 1950. Record of March 6, 1950. Abscissa: Intensity of irradiation of wavelength $< 1\,\mu$, in ergs/cm² sec. Ordinate: Optical density of extract. Solid line: tuber; - - - sprout. ● RG 2 + 7; ○ RG 5 + 7; × RG 8 + 7; △ BG 3 + RG 5 + 7; (+) 21 A + 1 + 7; ⊗ RG 10 + 7; □ RG 7 + 7. Four last ones: all values below 0.2, *cf.* Table VII.



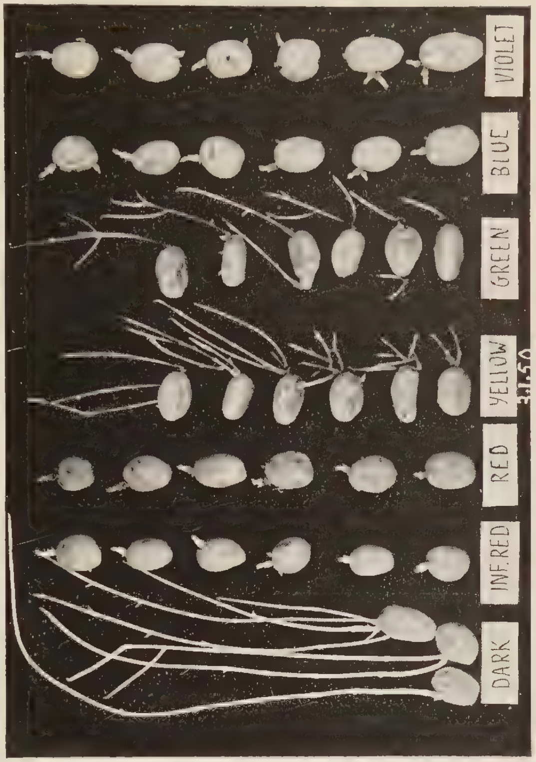
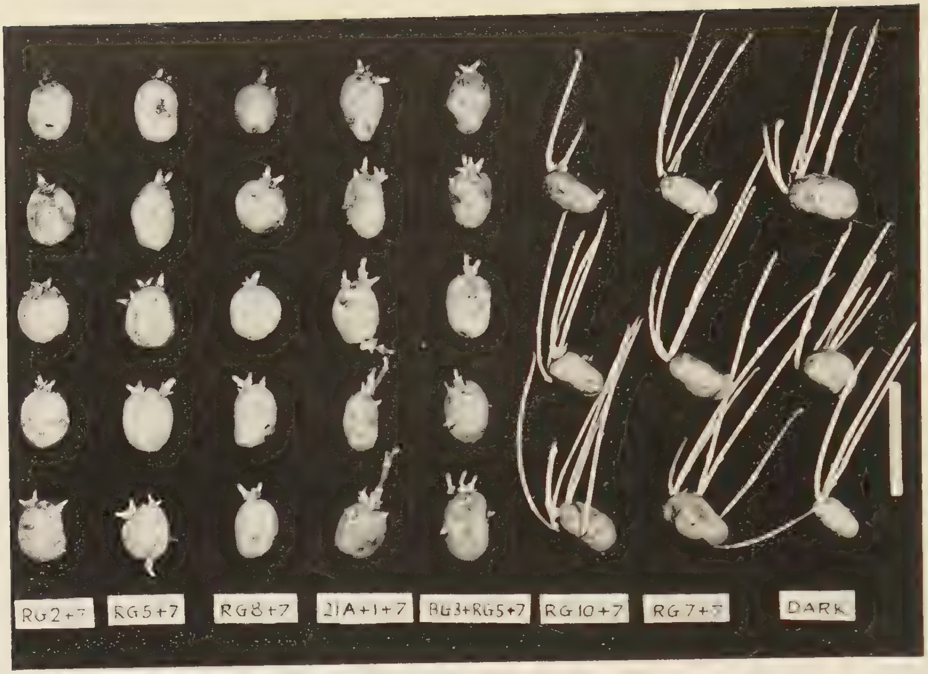


Plate 1. Representative tubers of each gauze plate from the various coloured irradiations. Decreasing intensity from bottom to top of each photograph. Experiment started Nov. 11. 1949. Photographs of Dec. 6 (left), Dec. 20 (right, top), Jan. 3, 1950 (right, bottom).

A



B

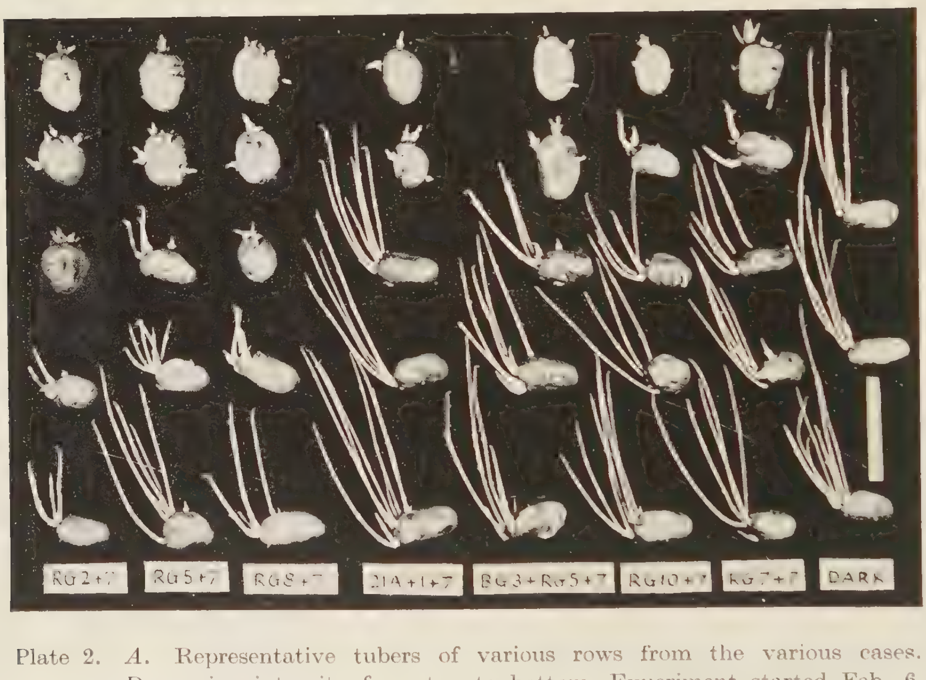


Plate 2. *A*. Representative tubers of various rows from the various cases. Decreasing intensity from top to bottom. Experiment started Feb. 6, 1950. Photograph: March 7, 1950. Bar at right side: 10 cm. From top to bottom of photograph: For the filters RG 2 to BG 3 + RG 5 included: tuber from rows 1, 4, 6, 8, 10 respectively. For RG 10: tuber from rows 1, 3, 5. For RG 7 from rows 1, 4, 6. For dark from rows 5, 7, 9.

B. Representative tubers of various rows from the various cases. Decreasing intensity from top to bottom. Experiment started March 14. Photograph: April 13, 1950. Bar at right side: 10 cm. From top to bottom of photograph: For the filters RG 2, RG 8: tuber from rows 1, 3, 5, 6, 8 respectively. For RG 5: tuber from rows 1, 4, 5, 7, 9. For 21 A c.s., BG 3 c.s.: tuber from rows 1, 2, 3, 5, 7. For RG 10, RG 7 tuber from rows 1, 2, 3, 4, 5. For dark tuber from rows 1, 5, 10.

TABLE IX

A: Sprout lengths in mm (longest sprout, average of 5 tubers) for the irradiations of Table VIII.

B: Percentage inhibition of sprouting as compared with dark controls, from the data under *A*. Experiment started March 14, 1950. Record of April 12, 1950.

A	1	2	3	4	5	6	7	8	9	10
RG 2 + 7	14.0	15.0	15.2	17.5	16.5	40.5	58.0	70.5	85.2	83.5
RG 5 + 7	15.0	14.8	17.2	18.8	44.5	50.8	55.5	97.2	105.3	107
RG 8 + 7	14.8	15.8	16.5	17.5	17.8	50.2	73.2	107.8	102.2	105.2
21 A + 1 + 7. . .	19.8	19.2	126.8	128.0	175.8	141.2	163	206	207	179
BG 3 + RG 5 + 7	18.0	17.0	76.0	93.5	113.2	146.8	152.2	155	178	187.5
RG 10 + 7	23.8	37.5	99.2	138	163	166	138	147	155	180
RG 7 + 7	36.8	44.8	97.2	128	157	179	144.5	195	150	177
dark	183		191		196		178		150	172
B										
RG 2 − 7	93.0	92.5	92.4	91.2	91.7	79.7	71.0	64.7	57.4	58.2
RG 5 + 7	92.5	92.6	91.4	90.6	77.7	74.6	72.2	51.8	47.3	46.5
RG 8 + 7	92.6	92.1	91.7	91.2	91.1	74.9	63.4	46.1	48.9	47.4
21 A + 1 + 7. . .	90.1	90.4	36.6	36.0	12.1	29.4	18.5	0	0	10.5
BG 3 + RG 5 + 7	91.0	91.5	62.0	53.2	43.4	26.6	23.9	22.5	11.0	6.2
RG 10 + 7	88.1	81.2	50.4	31.0	18.5	17.0	31.0	26.5	22.5	10.0
RG 7 + 7	81.6	77.6	51.4	36.0	21.5	10.5	27.7	2.5	25.0	11.5

The intensities of irradiation are collected in Table VIII; Plate 2, *B* gives a survey of the results. The range of slight inhibitions has been reached in most cases, whereas with RG 10 and RG 7 now considerable inhibitions have been obtained. We calculated the percentages inhibition, on the basis of sprout length of the dark control tubers (table IX), and established the energy at which 50 % inhibition was reached. Calculated upon the total energy $< 3\,\mu$, we arrived at the figures collected in table X, column I.

TABLE X

Energy/cm² sec. required for 50 % inhibition of sprout elongation in red and infrared radiation (continuous illumination). Experiment of March 14—April 12, 1950.

Filter (+ no. 7)	Energy ($\lambda < 3\,\mu$)/cm²sec. I	Energy ($\lambda < 1\,\mu$)/cm²sec. II
RG 2, RG 5, RG 8 . . .	~ 45 ergs	~ 10 ergs
RG 5 + BG 3	~ 55 ,,	~ 20 ,,
RG 10	~ 1600 ,,	~ 170 ,,
RG 7	~ 3000 ,,	~ 100 ,,

There is a considerable gap in the energy required for 50 % sprout inhibition between the filters RG 2, RG 5, RG 8, RG 5 + BG 3 on the one hand, and RG 10, RG 7 on the other hand. This indicates that the effectiveness of the energy, thus, the absorption of the photoactive pigment

strongly declines in this region. Provisionally, we also calculated the energy required in the various regions to cause 50 % inhibition under the assumption that only energy of wavelengths $< 1\ \mu$ are active. This assumption, of course, is arbitrary, but the slight response to the energy passing RG 10 and RG 7 indicates that, at least, the energy with $\lambda > 1\ \mu$ is not very active. We thus arrive at the numbers of column II, Table X.

The energies indicated by VAN DER VEEN (6) for red light are considerably lower than those given in Table X. So far we have no explanation for this discrepancy.

It is interesting to compare sprout inhibition with chlorophyll formation under the various filters. In Table XI, *A* some data on chlorophyll formation and sprout inhibition are compared, effected by about equal energy in various spectral regions. In Table XI, *B* energies are chosen in a region showing a clear range of sprout inhibition (in this experiment no chlorophyll determinations were made). It is seen that chlorophyll formation shows a sharp decline in the region between RG 2 and BG 3 *c. s.*, and a slow decline between BG 3 *c. s.* and RG 7; sprout inhibition shows the reverse behaviour. This is additional evidence that protochlorophyll is not the pigment involved in sprout inhibition.

TABLE XI

Decline in activity as to inhibition of sprout elongation, and chlorophyll formation in the near infrared. A: Experiment of Feb. 6—March 6, 1950. *B*: Experiment of March 14—April 12, 1950.

	A					*B*		
Filter (+ no. 7)	Energy /cm²sec $\lambda < 1\ \mu$	Chloroph. content	Ratio	% Inhib. sprout elong.	Ratio	Energy cm²sec $\lambda < 1\ \mu$	% Inhib. sprout elong.	Ratio
RG 2	230	1.48		93		29	80	
			25		1.01			1.3
BG 3 c.s.	215	0.06		92		26	62	
			1.5		4.3			4.0
RG 7	267	0.04		21		29.5	15	

In (7), fig. 2, a curve representing the absorption of a chlorophyll-protein complex has been drawn. It extends into the infrared at least up to 0. 8 μ. In how far straylight may have influenced the curve in this region is difficult to say. As such this curve doesn't contain definite evidence against the supposition that the chlorophyll-protein complex acts as light absorber in the process of photo-inhibition of sprout elongation.

Summary.

Sprouting of potatoes was studied in light of various spectral regions. For the quality of these regions, *cf.* (7). Continuous exposure with various energies was applied. In green and yellow light inhibition of sprout elong-

ation is less effective than in blue, violet and red. Near infrared also causes strong inhibition. In this region the sprouts and tubers do not form appreciable amounts of chlorophyll, whereas in the other regions they do. A steep fall in chlorophyll formation occurs between the filters SCHOTT RG 2 and RG 5 + BG 3, a steep fall in inhibition of sprout elongation between RG 5 + BG 3 and RG 7. The inhibitions in the regions between 0.3 and 0.7 μ suggests that a chlorophyllous pigment is active in absorbing the light energy inducing the inhibitory action. It seems improbable that protochlorophyll is the active pigment. It is possible that low concentrations of chlorophyll — bound to protein — are responsable for the observed effect.

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PETROGRAPHY

RÖNTGENOLOGISCH ONDERZOEK VAN DE ONDOORZICHTIGE ZWARE FRACTIE VAN ENKELE NEDERLANDSE ZANDEN

DOOR

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(Communicated by Prof. J. H. F. UMBGROVE at the meeting of June 24, 1950)

1. *Inleiding.*

De mineralogische analyse van sedimenten ten behoeve van stratigrafisch of palaeogeografisch onderzoek is in de regel gebaseerd op de zgn. "zware fractie" dier sedimenten; deze fractie bevat die mineralen, die een soortelijk gewicht groter dan 2,9 hebben. F. A. VAN BAREN (1934) vulde de zo verkregen petrologische kennis van ons land aan door een overzicht te geven van de samenstelling der lichte fracties. Daarbij bleek, dat afzettingen van verschillende herkomst ook langs deze weg mineralogisch kunnen worden onderscheiden. Toch is over het algemeen een voorkeur blijven bestaan voor de kenschetsing van een sediment op grond van zijn zware fractie, en dit is ongetwijfeld een gevolg van het feit, dat het aantal karakteristieke mineralen in de zware fractie groter en hun onderlinge verhouding sterker gevariëerd is dan in de lichte fractie. Dit is vooral van voordeel bij het onderzoek van mengassociaties. Daarenboven is de kans op nieuwe karakteristieke korrels eveneens groter in de zware fractie.

Bij de gebruikelijke "routine-telling" beperkt men zich tot de determinatie van de doorzichtige zware mineralen. De opake bestanddelen lenen zich uit den aard der zaak niet voor nadere determinatie; zij worden verenigd onder de rubriek "opaak" en in percenten t.o.v. het totaal der zware fractie weergegeven. Op deze wijze is men dus in zekere zin onderricht over de "dichtheid" der opake bestanddelen in het preparaat.

Betrekkelijk zelden is gepoogd hun samenstelling qualitatief en/of quantitatief nader te onderzoeken. Bekend zijn in dit opzicht de vermeldingen in de literatuur over zware strandzanden, die belangrijke hoeveelheden magnetiet, titanomagnetiet, ilmeniet e.a. ertsen bevatten. Zo bv. de door L. M. R. RUTTEN (1927) en R. W. VAN BEMMELEN (1949) vermelde zware zanden bij Tjilatjap; voorts een strandzand bij Goeree met hoog gehalte aan zware mineralen en beschreven door SLOTBOOM–CROMMELIN (1944). Dikwijls wordt bij de beschrijving van zulke zware zanden de gehele groep van opake mineralen eenvoudigheidshalve als "erts" samengevat, en als zodanig tegenover de groep der doorzichtige mineralen geplaatst. Een recente publicatie van V. DE VRIES behandelt aldus de "granaat-erts verhouding" van Nederlandse kustzanden; kenne-

lijk ziet deze auteur de ertsfractie voornamelijk voor magnetiet aan. Reeds aan RETGERS (1891) was, dank zij zijn uitvoerig mineralogisch onderzoek, de aanwezigheid van kleine hoeveelheden magnetiet en ilmeniet in het Nederlandse duinzand bekend. Ten slotte zij vermeld, dat ook de mijnbouwkundige afdeling der Technische Hogeschool te Delft zich met het vraagstuk betreffende de samenstelling der opake bestanddelen heeft bezig gehouden: helaas is over dit onderzoek niets gepubliceerd. Schrijvers beogen thans een systematisch onderzoek van alle in een sediment als opaak aangemerkte korrels en vermelden onderstaand als *voorlopig* resultaat de desbetreffende waarnemingen aan een aantal Nederlandse zanden.

2. *Selectie-methoden.*

Algemeen wordt de opvatting gehuldigd, dat de voor "opaak" doorgaande korrels in een zware mineralen preparaat, naast enkele gesteenteresten, bestaan uit ertsen, t.w. oxyden en sulfiden van zware metalen.

Uitgaande van deze veronderstelling zouden voor een afscheiding der opake bestanddelen uit het totaal der zware fractie in aanmerking komen:

a. decanteermethoden ("panning"),

b. magnetische scheiding,

c. electrostatische scheiding.

Proefondervindelijk blijkt, dat bovenvermelde "mechanische" scheidingen geen afdoende selectie teweeg brengen.

a. Zo zijn bv. de onderlinge verschillen in s.g. onvoldoende om een zuivere scheiding door middel van decanteren te bewerkstelligen. Dit wordt begrijpelijk als naderhand zal blijken, dat "opaak" niet altijd behoeft te betekenen "erts".

b. Het gelukt om met een gewone staafmagneet de sterk magnetische mineraalkorrels, als magnetiet en magnetische ilmeniet, af te scheiden. Het was verrassend te constateren, dat over 't algemeen de concentratie dezer mineralen in de onderzochte monsters zeer onbeduidend is. Slechts een strandzand (Goeree), voor ruim 80 % uit zware fractie bestaande, bleek op deze magnetische test zichtbaar te reageren. Later röntgenologisch onderzoek wees uit, dat het hier magnetiet en magnetische ilmeniet betrof. Wenst men ook een afscheiding der minder magnetische fracties (ijzerrijke amfibolen, pyroxenen, toermalijn, rutiel, etc.) dan is dit met een electromagneet mogelijk. Hiervan werd echter afgezien; bovendien is door de vele ijzerrijke insluitsels in talrijke andere mineralen de scheiding verre van zuiver.

c. De electrostatische scheiding berust, zoals bekend, op het verschil in electrisch geleidingsvermogen der diverse mineralen. Goede geleiders zijn magnetiet, hematiet, ilmeniet, pyriet, spinellen en ijzerrijke rutiel, dus bijna uitsluitend opake bestanddelen. De overige mineralen zijn slechte tot matige geleiders. Het gelukte nu met eenvoudige middelen, bv. een gewreven ebonietstaaf, wel de concentratie der opake korrels op te

voeren van bv. ca. 30 % tot 60 %; betere resultaten werden niet bereikt. Deze mislukking is ten dele te wijten aan de nadelige invloed, die het verschil in korrelgrootte bij de onderzochte monsters uitoefent. Men kiest voor sediment-petrologisch werk liefst de korrelgroottefractie van omstreeks 50 — 500 mu. Het verschil in diameter tussen de kleinste en grootste korrels (max. dus een factor 10) komt in het door het electrostatische veld te overwinnen gewicht van de korrels uit met een overeenkomstige factor in de derde macht (d.i. dus maximaal 10^3), en kan daardoor oorzaak zijn, dat een kleine korrel van een mineraal met matig geleidingsvermogen even snel wordt aangetrokken als een grote korrel met sterk geleidingsvermogen. Nodig zou dus zijn het sediment van te voren in een aantal fracties te zeven. Veiliger is het evenwel, geheel af te zien van mechanische selectiemethoden en toe te passen de hieronder volgende scheidingswijze.

d. scheiding onder het binoculair.

Tegenover het nadeel, dat deze werkwijze veel tijdrovender is, staat nu het voordeel, dat men zeker is, ook alle opake korrels te elimineren, en niet alleen dat gedeelte, dat door zijn physische eigenschappen reageert op magnetisme, electrische velden, etc. Noodzakelijk is bij deze scheidingswijze de korrels (m.b.v. een prepareernaald) te elimineren, terwijl zij ingebed zijn in een immersievloeistof, liefst van dezelfde brekingsindex als canadabalsem (bv. tetralinum, met $n = 1{,}539$). Strooit men namelijk de korrels droog op een glasplaat, dan zal men spoedig bemerken, dat verschijnselen der totale reflectie tot geheel foutieve resultaten leiden. Bij een proef bleek, dat vooral toermalijn, dank zij zijn gewoonlijk sterke afronding, zich als opaak voordeed en het afgescheiden materiaal dientengevolge bij later onderzoek toermalijn als hoofdbestanddeel bevatte! Een tweede belangrijk punt is, de korrels te wentelen alvorens ze te elimineren. Daarbij blijkt, dat in vele gevallen een korrel in 't geheel niet opaak is, doch bij wenteling doorschijnend wordt, bv. donkerbruin of zelfs lichtbruin. In sommige sedimenten kan dit aantal korrels wel ca. de helft bedragen van de oorspronkelijk als opaak gebrandmerkte mineraaldeeltjes. Wij concluderen hieruit, dat het waargenomen percentage opake bestanddelen in canadabalsem-preparaten veelal te hoog is en een functie is van de toevallige ligging der korrels in het preparaat.

3. *Wijze van onderzoek.*

Verschillende wegen staan nu open om een inzicht te verkrijgen in het op bovenstaande wijze uitgezochte materiaal. Ertsmicroscopisch onderzoek zou niet alleen tot determinatie kunnen leiden, maar ons bovendien nog inlichten over eventueel aanwezige structuurverschijnselen (vergroeiingen, lamellen, aantastingen, e.d.). Daar staat als nadeel tegenover het zeer tijdrovende werk om alle mineraalkorrels aan te slijpen (vooral bij serieonderzoek), ten einde toch een enigszins quantitatief beeld der componenten te verkrijgen.

Daar juist dit laatste de opzet van dit onderzoek vormde, werd gebruik gemaakt van de röntgen-analyse. Een honderdtal korrels wordt uitgezocht, zeer fijn gepoederd en hiervan een poederopname gemaakt volgens de methode Debye–Scherrer. De determinatie berust eenvoudig op een identificatie der aldus verkregen diagrammen met "standaarddiagrammen", die men van bekende mineralen gemaakt heeft, met dezelfde camera en onder gebruikmaking van dezelfde soort straling, die voor het verder onderzoek gebruikt wordt. Voor dit doel voldoet zeer goed een ijzerbuis met mangaanfilter (Fe $K\alpha$-straling) en een camera met een radius van $4\frac{1}{2}$ cm. De lijnen op de röntgenfoto's hebben dan voldoende spreiding, om de identificatie betrouwbaar te kunnen uitvoeren. De veelal gebruikelijke Cu-straling is hier niet aan te bevelen; niet alleen door zijn kleinere diffractiehoeken, maar vooral door de hinderlijke zwarting op de film bij bestraling van ijzerhoudende mineralen. Evenzo kan ook aan de hand van standaardopnamen van mengsels de quantitatieve verhouding der verschillende componenten van de onderzochte monsters enigszins geschat worden.

4. *Voorlopige resultaten.*

Bij bestudering der op bovenvermelde wijze verkregen diagrammen valt direct op, dat zij allen een vrij eenvoudig patroon vertonen. De afbuigingslijnen blijken toe te behoren aan 1 — 3 componenten. Slechts deze hebben een aandeel van enige betekenis in de onderzochte fractie, terwijl alle andere eventueel aanwezige mineralen niet meer dan een zeer ondergeschikte rol kunnen spelen. Een zo eenvoudige samenstelling van de opake fractie was bij het begin van dit onderzoek niet te voorzien!

Onderstaand staatje, (en de bijbehorende diagrammen, fig. 1) geven de mineraalcombinaties van enkele der onderzochte monsters:

No.	monster uit	herkomst	milieu	röntgenol. aan te tonen miner.	Opmerkingen	n.v. [2])
236	dekzand	N. Brabant	aeol.	rutiel + weinig ilmeniet	ZM: [1]) A-prov., practisch zonder rutiel	—
257	Hoogterraszand	N. Limburg	fluv.	rutiel en ilmeniet in ongeveer gelijke verhouding	ZM: saussuriet-prov.	1
249	strandzand	Goeree	mar.	idem	80 % zware fractie!	—
258	strandzand (magnetische fractie)	Goeree	mar.	magnetiet + weinig ilmeniet	Als no. 249, doch slechts de magnet. fractie geröngt	—
246	Oud-Holoceen zeezand	Nootdorp (Z.H.)	mar.	pyriet	zeer slibrijk en fijnk. zand, met organ. resten	5

[1]) Z.M. = gewone zware mineraal-analyse.

[2]) n.v. = aantal der lijnen in het diagram, dat niet zijn verklaring vindt in de in kolom 5 genoemde mineralen.

Fig:1 Diagrammen van enkele opake fracties (Fe - K_α - straling).

De hoogte der lijnen geeft de relatieve intensiteit der overeenkomstige lijnen op de röntgenfoto weer.
is de hoek tussen de invallende röntgenstraal en de reflecterende netvlakken.

Een en ander kan men op onderstaande wijze samenvatten:

1. Hetgeen bij de gebruikelijke routinetelling voor "opaak" doorgaat, bestaat slechts voor een deel uit opake mineralen s.str. (d.w.z.: mineralen met zeer sterke absorptie en sterke reflectie: ertsmineralen). Sommige korrels lijken opaak bij een bepaalde ligging of bij sterke afronding (pleochroïsme · totale reflectie!). Dit is geconstateerd bv. bij toermalijn en bij rutiel. Weer andere korrels zijn ondoorzichtig, doordat ze als zeer fijn

aggregaat voorkomen (interne totale reflectie). Voorbeelden zijn sommige gesteenteresten en een deel der zgn. "saussurieten". Voorts kunnen hiertoe behoren een deel der door ons als rutiel gedetermineerde mineralen in de zware fractie (zie punt 4).

2. Er zijn gewoonlijk niet meer dan 2 hoofdcomponenten.

3. Magnetiet speelt in de opake fractie meestal een geringe rol.

4. Rutiel en ilmeniet nemen vaak een voorname plaats in de "opake fractie" in. Opvallend is, dat zij in den regel in elkanders gezelschap voorkomen. Het is de vraag, of wij met afzonderlijke, homogene korrels van beider soort te maken hebben, ofwel met één inhomogene korrel, waarbij rutiel en ilmeniet op bepaalde wijze met elkaar vergroeid zijn. Men kan zich daarbij voorstellen, dat oorspronkelijke ilmenietkorrels ten dele omgezet zijn in leucoxeen, welk mineraal volgens de thans geldende opvattingen vaak bestaat uit microkristallijne rutiel. Het röntgendiagram kan ons daarover niet nader inlichten; dunne doorsneden, die zowel bij doorvallend als met opvallend licht te bestuderen zijn, moeten hier de oplossing brengen.

5. Men mag verwachten, dat ook diagenetische processen, zoals autigene mineraalvormingen (o.a. pyriet), zich in de opake fractie weerspiegelen.

Zusammenfassung.

Röntgenologische Untersuchung der undurchsichtigen schweren Fraktion einiger niederländischer Sande.

Die übliche Schweremineral-untersuchung von Psammiten beschränkt sich in fast allen Fällen auf die unter dem Mikroskop durchsichtigen Bestandteile; die undurchsichtigen Körner werden als Ganzes, als "opake" Fraktion zwar gezählt, aber nicht näher bestimmt. Verfasser stellten sich zum Ziele, mittels der röntgenologischen Gemischanalyse die wichtigeren Bestandteile der opaken schweren Fraktion einiger niederländischer Quartärsande zu determinieren (Methode Debye–Scherrer, Fe $K\alpha$-Strahlung, Kamera mit Durchmesser von 9 cm). Aus der gemäss üblichem Vorgehen erhaltenen Fraktion mit einer Dichte grösser als die von Bromoform (ca. 2,9) wurden unter dem Binokularmikroskop mit Hilfe einer Präpariernadel in jeder Probe ca. 100 undurchsichtige Körner herausgesucht und abgesondert und zwar waren dabei die Körner in einer Immersionsflüssigkeit eingebettet. Verwendet wurde Tetralin mit $n = 1{,}539$ dessen Lichtbrechung der von Kanadabalsam nahekommt. Bevor zu dieser ziemlich umständlichen und auch zeitraubenden Methode geschritten wurde, haben Verfasser einige physikalische Trennungsmethoden geprüft (elektrostatische, gravimetrische Verfahren), jedoch ohne Erfolg, u.a. weil, wie sich zeigte, unter den mikroskopisch zur "opaken" Fraktion gezählten Körnern neben eigentlich opaken, sulfidischen und oxydischen Erzmineralien auch feinkörnige Aggregate von nicht-opaken Mineralien vorkommen (Gesteinsreste; Rutil-Aggregate?). Ferner ergab sich, dass

stark gerundete Körner von Rutil und Turmalin "opak" erscheinen können bei bestimmten Orientierungen der Körner im Präparat.

Trotz der geringen Menge (100 Körner!) wurden in allen Fällen gute Röntgen-Pulverdiagramme erhalten, die durchwegs recht einfach waren und deren Beugungslinien durch 1 bis 3 Komponenten erklärt werden konnten. Es zeigte sich, dass meist *Rutil* und *Ilmenit* Hauptkomponenten der undurchsichtigen schweren Fraktion sind. *Magnetit* war in keiner Probe Hauptbestandteil; er konnte nur in der magnetischen Fraktion eines an schweren Mineralien besonders reichen Strandsandes bestimmt werden, ist aber auch da in viel geringeren Mengen als Rutil und Ilmenit vorhanden. In einem marinen altholozänen, feinkörnigen Sande war *Pyrit* Hauptbestandteil.

Ein besonderes Problem stellt sich im Vorkommen von Rutil in grösseren Mengen in der opaken Fraktion der untersuchten Sande. Weitere (erzmikroskopische) Untersuchungen müssen zeigen, ob Rutil als "opake" Einzelkristall-körner, oder (was wahrscheinlicher ist) als sehr feinkörnige Aggregate ausgebildet ist. Im letzteren Falle kann ein leukoxenartiges Umwandlungsprodukt von Ilmenit vorliegen, wofür die häufige Assoziation Rutil-Ilmenit spricht. Leukoxen ist nach neueren Arbeiten oft Rutil, dann auch Titanit und Anatas.

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TRANSGRESSIEVE VARIABILITEIT EN TRANSGRESSIE-SPLITSING. II

DOOR

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We zullen nu, — aansluitend aan het overzicht van F_5-1936, dat we gaven —, ascenderend F_4-1935, F_3-1934 en F_2-1933 en descenderend F_6-1937 op het voorkomen van transgressieve variabiliteit nagaan.

F_4-1935. Transgressieve plusvariaties (tab. 5).

De *lengte* biedt geen voorbeelden van transgressieve variabiliteit. Wel de *breedte*. *Pl. 119*. De uitgangsboon is van pl. 159, F_3-1934 en heeft een grote breedte (b = 9.8 mm). Van de bonenopbrengst van pl. 159 staat aangetekend "peulen iets breed; bonen type I". De gemidd. breedte van de bonenopbrengst van pl. 119 toont transgr. variabiliteit (fig. 9). *Pl. 123*. Ook van pl. 123 is de uitg. boon van pl. 159, F_3-1934. De gemidd. breedte en de distributiekromme tonen transgr. variabiliteit. Een grensgeval is *pl. 121*, waarvan de uitg. boon ook van pl. 159 is. De grootste breedte van indiv. bonen van pl. 119, 121 en 123 is zo groot als die van indiv. bonen van de I-lijn van 1935 (tab. 5).

De *dikte*. De grootste gemidd. dikte is van *pl. 123* (tab. 5c). Ze is iets kleiner dan de grootste gemidd. dikte van bonenopbrengsten van de II-lijn van 1935. De grootste dikte van indiv. bonen van pl. 123 is zo groot als de grootste dikte van indiv. bonen van de II-lijn van 1935.

Het *gewicht*. De *pl. 119, 123, 121, 143, 300* en *133* tonen transgr. variabiliteit van het gemidd. gewicht en meestal van het grootste gewicht van indiv. bonen, vergeleken met de I-lijn.

De pl. 119, 123 en 124 hebben ook zeer grote gemidd. lengten en transgrediërende grote breedten; pl. 123 heeft bovendien een zeer grote gemidd. dikte. Op deze grote waarden van de gemidd. afmetingen is de transgressieve variabiliteit van het gewicht terug te voeren.

Transgressieve variabiliteit van minusvariaties. F_4-1935 (tab. 6).

De *lengte*. Transgressieve variabiliteit van minusvariaties van de gemiddelde lengte, treffen we in het materiaal van F_4-1935 niet aan. De kleinste gemidd. lengten tonen een regelmatige stijging van 11.0 mm af. Er zijn in het materiaal van de I-lijn van 1935 3 bonenopbrengsten met kleinere gemidd. lengten.

De *breedte*. Er is geen bonenopbrengst, waarvan de gemidd. breedte kleiner is dan de kleinste gemidd. breedte van bonenopbrengsten van de II-lijn van 1935. Toch is er wel verschil, vergeleken met wat we voor de lengte vinden. In 2 gevallen (*pl. 315* en *319*) is de gemidd. breedte kleiner dan de op één na kleinste gemidd. breedte van bonenopbrengsten van de II-lijn. Volgens een tabel, die wij maakten, zijn er in de bonenopbrengst van pl. 319, 23 bonen met een breedte, die kleiner is dan 7.7 mm. Voor de bonenopbrengsten van de II-lijn, is dit aantal kleiner. Er is waarschijnlijk enige transgressieve variabiliteit van minusvariaties voor de breedte van pl. 319. Er zijn in het materiaal van F_4-1935 5 bonenopbrengsten met een gemidd. breedte, die kleiner is dan 8.2 mm; bij de II-lijn is dit aantal 2.

De *dikte*. Er is geen bonenopbrengst, waarvan de gemidd. dikte kleiner is dan die van bonenopbrengsten van de I-lijn van 1935.

Het *gewicht*. Er is geen bonenopbrengst, waarvan het gemidd. gewicht kleiner is

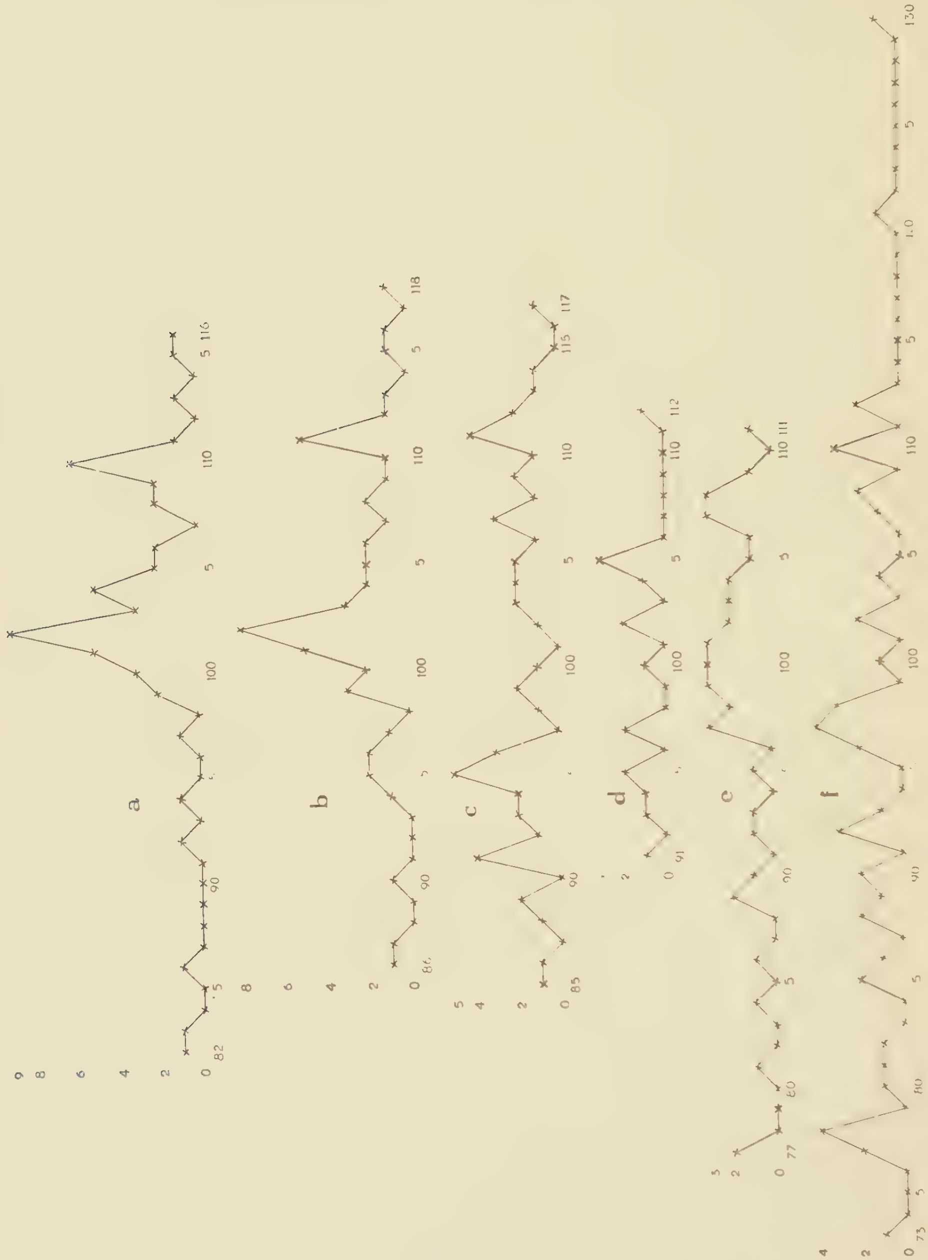

Fig. 9, a—c. F_4-1935. Plus variations. Frequency curves of the breadth. a, pl. 119, n = 51, b_m = 10.3 mm. b, pl. 123, n = 51, b_m = 10.3 mm. c, pl. 121, n = 50, b_m = 10.0 mm.

Fig. 9, d—f. I-line 1935. Frequency curves of comparison beanyields d, pl. 2, n = 15, b_m = 10.0 mm. e, pl. 12, n = 41, b_m = 9.8 mm. f, pl. 16, n = 45, b_m = 9.5 mm. Pl. 16 has many pods with one bean.

dan dat van bonenopbrengsten van de II-lijn van 1935. Van *pl. 315* met een zeer kleine gemidd. breedte (z. boven) is het gemidd. gewicht ook zeer klein.

Alleen voor de breedte vinden we in het materiaal van F_4-1935 een onzeker geval van transgress. variabiliteit van een minusvariatie.

F_3-1934. Transgressieve variabiliteit van plusvariaties (tab. 7).

De *lengte*. Geen transgressieve variabiliteit.

De *breedte*. Er is geen transgressieve variabiliteit van de breedte. Ook uit de distributiekrommen (fig. 10, *pl. 194* en *272*) blijkt ze niet. Er zijn volgens de gemaakte

Fig. 10, a—b. F_3-1934. Frequency curves of the breadth. Plus variations. a, pl. 194, $n = 12$, $b_m = 9.8$ mm. b, pl. 272, $n = 65$, $b_m = 9.8$ mm.

Fig. 10, c—d. I-line 1934. Frequency curves of comparison beanyields. c, pl. 5, $n = 21$, $b_m = 9.9$ mm. d, pl. 4, $n = 40$, $b_m = 9.8$ mm.

tabel 25 bonen met een breedte groter dan 9.8 mm in de bonenopbrengst van pl. 272 ($n = 65$); er zijn er 8 in die van pl. 5 ($n = 21$) van de I-lijn van 1934, d.i. 24 berekend voor $n = 63$ (en 25 voor $n = 65$).

De *dikte*. Er is duidelijke transgr. variabiliteit van de dikte. De distributiekrommen voor de dikte van *pl. 149* en *pl. 115* liggen duidelijk naar rechts (fig. 11).

Het *gewicht*. Van de bonenopbrengsten van *pl. 272* en *pl. 107* vertoont het gemidd. gewicht transgr. variabiliteit. Van pl. 272 zijn de gemidd. lengte en de gemidd. breedte zeer groot; van pl. 107 de gemidd. dikte. De transgressie van het gewicht is terug te voeren tot de zeer grote gemiddelde afmetingen. De bonenopbrengsten van de pl. 573 en 775, die behoren tot een grote groep van bonenopbrengsten, waarvan alleen het gemidd. gewicht bepaald is, tonen duidelijk transgressieve variabiliteit.

F_3-1934 bevat geen bonenopbrengsten met transgr. variabiliteit van de lengte. Van 2 bonenopbrengsten is de gemidd. breedte aan de grens van transgr. variabili-

teit. Twee bonenopbrengsten tonen transgr. variabilitiet van de dikte en vier die van het gewicht.

Transgressieve variabiliteit van minusvariaties van F_3-1934 (tab. 8).

De *lengte* vertoont geen transgr. variabiliteit.

De *breedte*. De kleinste gemidd. breedte is van de bonenopbrengst van *pl. 144* met slechts 15 bonen. De 2 vergelijk-bonenopbrengsten van de II-lijn van 1934, pl. 51

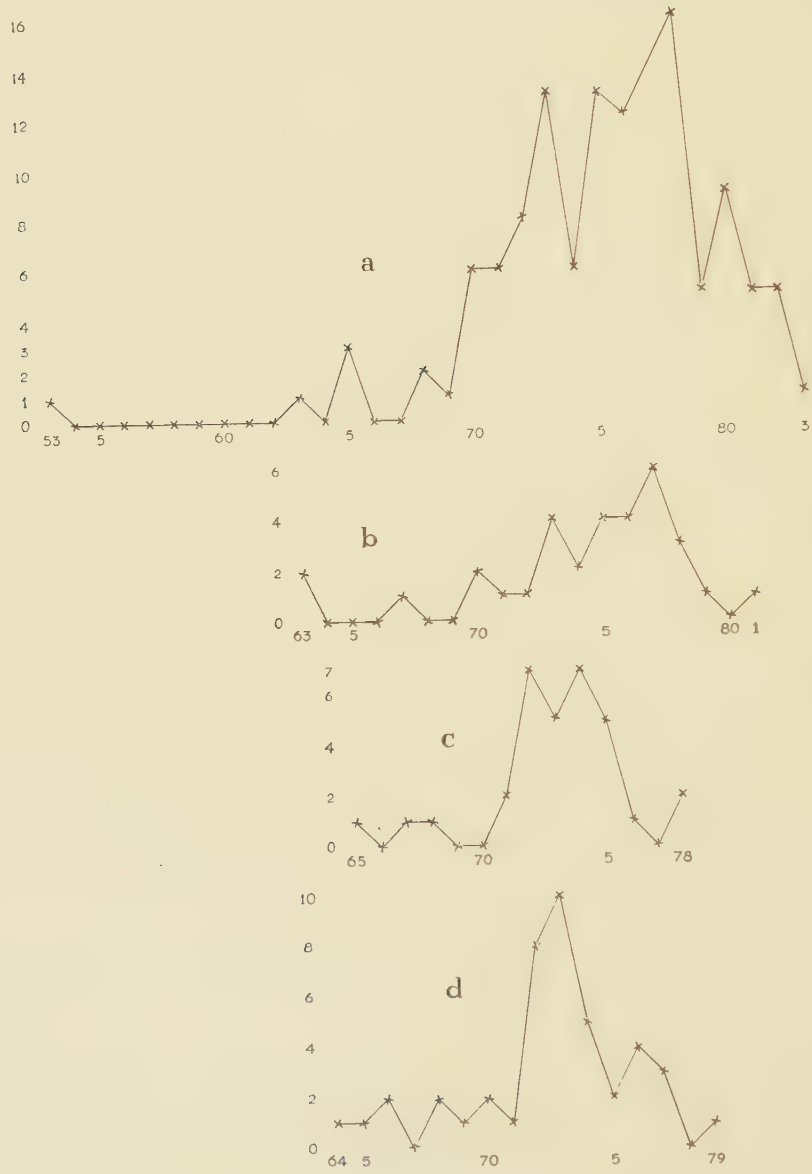

Fig. 11, a—b. F_3-1934. Frequency curves of the thickness. Plus variations. a, pl. 149, n = 126, th_m = 7.6 mm. b, pl. 115, n = 32, th_m = 7.4 mm.

Fig. 11, c—d. II-line 1934. Frequency curves of comparison beanyields. c, pl. 49, n = 32, th_m = 7.3 mm. d, pl. 50, n = 43, th_m = 7.25 mm.

en 51a, hebben ook een kleine bonenopbrengst van resp. 12 en 9 bonen. Deze beide bonenopbrengsten zijn van dezelfde plant, pl. 51 (pl. 51a is de opbrengst van 2 verplicht zelfbestoven bloemen, van 2 peulen dus). De *pl. 140* en *145* met grotere bonenopbrengsten hebben een gemidd. breedte, die even groot is als die van pl. 52 met een grote bonenopbrengst. De gemidd. breedte is aan de grens van transgress. variabiliteit van minusvariaties. De distributiekromme van de pl. 144, 140 en 145 liggen iets meer naar links dan die van pl. 52 (fig. 12).

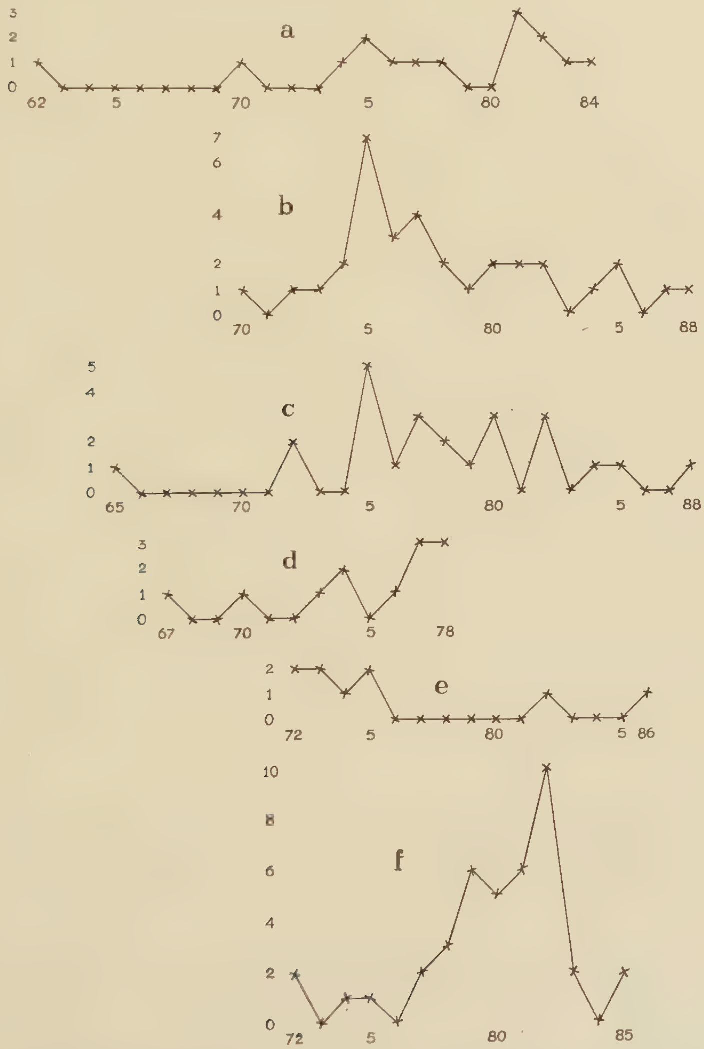

Fig. 12, a—c. F_3-1934. Frequency curves of the breadth. Minus variations. a, pl. 144, n = 15, b_m = 7.7 mm. b, pl. 140, n = 23, b_m = 7.8 mm. c, pl. 145, n = 24, b_m = 7.8 mm
Fig. 12, d—f. II-line 1934. Frequency curves of comparison beanyields. d, pl. 51a, n = 9, b_m = 7.6 mm. e, pl. 51, n = 12, b_m = 7.2 mm. e, pl. 42, n = 40, b_m = 8.0 mm.

De *dikte.* De kleinste gemidd. dikte is van *pl. 105.* Ze is zo groot als de kleinste gemidd. dikte op één na van bonenopbrengsten van de I-lijn van 1934. Er is geen transgr. variabiliteit van de dikte; we zijn er aan de grens van.

Het *gewicht.* Het gemidd. gewicht van *pl. 145* is iets kleiner dan dat van de ver-

gelijkbonenopbrengst van pl. 48 van de II-lijn (de vergelijkbonenopbrengst van pl. 51 en 51a mogen wij waarschijnlijk buiten rekening laten). Er is een geringe transgr. variabiliteit van het gewicht van pl. 145, die afgeleid is uit de kleinste gemidd. afmetingen. *Pl. 695* toont duidelijke transgr. variabiliteit van het gewicht.

F_2-1933. Transgressieve variabiliteit van plusvariaties (tab. 9).

De *lengte*. De grootste gemidd. lengte is belangrijk kleiner dan de grootste gemidd. lengte van bonenopbrengsten van de I-lijn van 1933.

De *breedte*. De grootste gemidd. breedte is kleiner dan de grootste gemidd. breedte van bonenopbrengsten van de I-lijn van 1933. Van *pl. K 82* is de grootste breedte van de indiv. bonen groter dan de grootste breedte van indiv. bonen van vergelijkbonenopbrengsten van de I-lijn van 1933. De breedte is aan de grens van transgr. variabiliteit.

De *dikte*. De grootste gemidd. dikte is zo groot als de grootste gemidd. dikte van bonenopbrengsten van de II-lijn van 1933. We zijn hier aan de grens van transgr. variabiliteit.

Het *gewicht*. Het grootste gemidd. gewicht is belangrijk kleiner dan het grootste gemidd. gewicht van de I-lijn van 1933, doch het is groter dan op één na het grootste gemidd. gewicht van de I-lijn. Er is waarschijnlijk geen transgr. variabiliteit van het gewicht.

Transgressieve variabiliteit van minusvariaties. F_2-1933 (tab. 10).

De *lengte*. De kleinste gemidd. lengte van bonenopbrengsten van de F_2-zaadgeneratie van 1933 is belangrijk groter dan de kleinste gemidd. lengte van bonenopbrengsten van de II-lijn van 1933.

De *breedte*. De kleinste gemidd. breedte is groter dan de kleinste gemidd. breedte van de II-lijn van 1933. De gegevens van de F_2-generatie zijn nog al uiteenlopend.

De *dikte*. De kleinste gemidd. dikte is groter dan die van de I-lijn.

Het *gewicht*. Ook het kleinste gemidd. gewicht is groter.

We vinden voor F_2-1933 geen transgressieve variabiliteit van minusvariaties

Aan het overzicht van de transgressieve variabiliteit van de F_5-zaadgeneratie van 1936 en dat van de ascenderende generaties F_4-1935, F_3-1934 en F_2-1933 sluiten we het overzicht er van aan van de descenderende generatie F_6-1937.

F_6-1937. Transgressieve variabiliteit van plus-variaties (tab. 11).

De *lengte*. De bonenopbrengsten van *pl. 292* en *289* (uitg. boon van pl. 450, F_5-1936) vertonen waarschijnlijk enige transgressieve variabiliteit (fig. 13 en 14). Van pl. 1039, die gegroeid is uit een boon van pl. 666, F_5-1936, (blz. 3) is de gemidd. lengte duidelijk kleiner dan de grootste gemidd. lengte van bonenopbrengsten van de I-lijn van 1937.

Aan het materiaal van F_6-1937 ontlenen we nog fig. 14. Fig. 14a is de kromme van de gemiddelde lengten van 93 bonenopbrengsten van de I-lijn van 1937. Fig. 14b stelt de ligging van de gemidd. lengten van de bonenopbrengsten van de pl. 289 en 292 ten opzichte van de kromme van fig. 15a voor. Fig. 14c is de kromme van de 8 bonenopbrengsten van de pl. 1039—1046, F_6-1937, die alle gegroeid zijn van uitgangsbonen van pl. 666, F_5-1936. Pl. 1039 is één van deze 8 bonenopbrengsten van de pl. 1065—1076 en 1077—1081, die gegroeid zijn van uitgangsbonen van pl. 784 (blz. 1087) en van pl. 785 (blz. 1089), F_5-1936. De kromme van fig. 14b—e zijn erfelijkheid bepaald.

De *breedte*. De bonenopbrengst van *pl. 317* toont transgr. variabiliteit van de mede door breedte.

De *dikte*. De bonenopbrengst van *pl. 374* vertoont duidelijk transgress. variabiliteit van de dikte. De *pl. 373, 402, 427* en *368* vertonen ze in mindere mate.

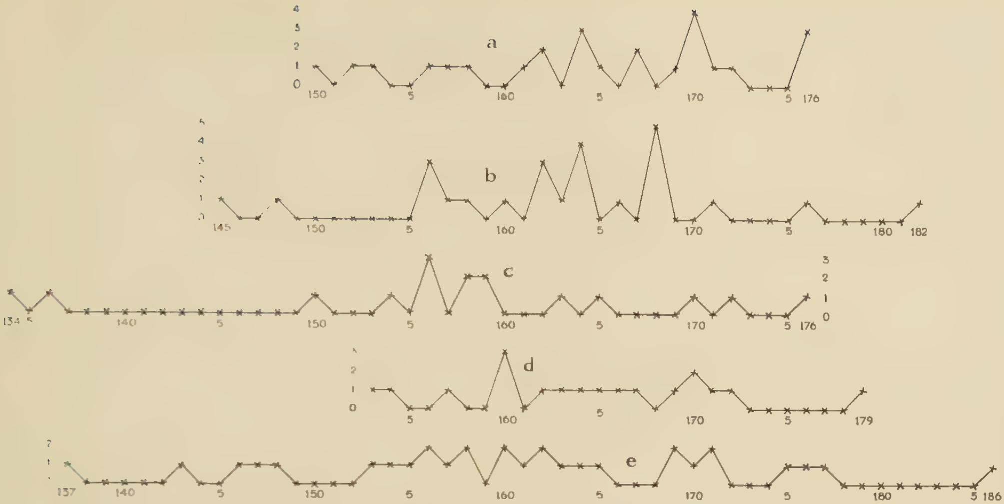

Fig. 13, a—c. F_6-1937. Frequency curves of the length. Plus variations. a, pl. 292, n = 25, l_m = 16.5 mm. b, pl. 289, n = 23, l_m = 16.3 mm. c, pl. 1039, n = 16, l_m = 15.8 mm.

Fig. 13, d—e. I-line 1937. Frequency curves of comparison beaynields. d, pl. 58, n = 18, l_m = 16.5 mm. e, pl. 66, n = 30, l_m = 16.0 mm.

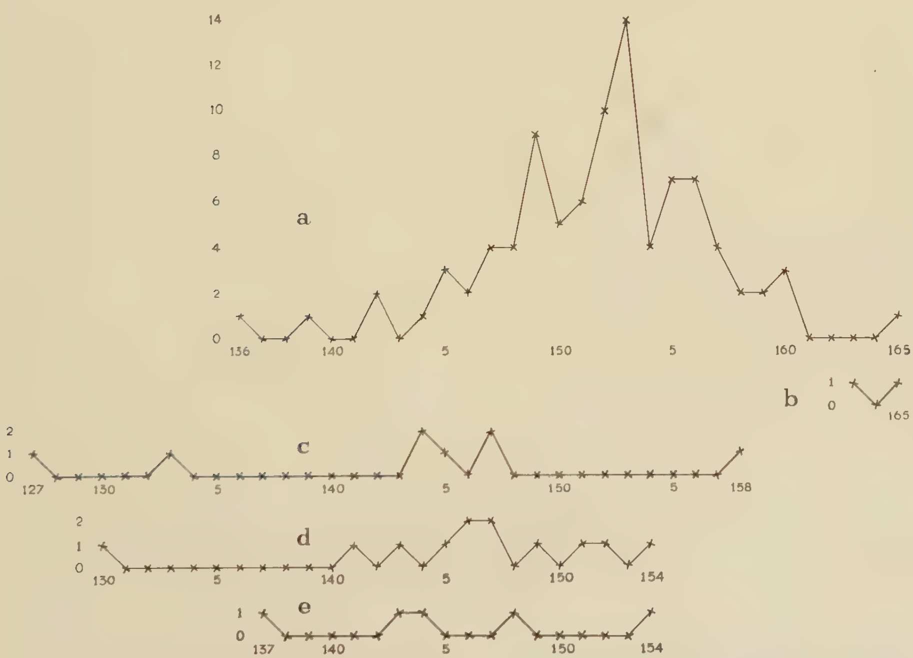

Fig. 14. Fig. 14a. Frequency curve of the mean lengths of 93 beanyields of the I-line of 1937. Fig. 14b, Curve of the mean lengths of pl. 289 and 292, F_6-1937. Fig. 14c, Idem of the 8 beanyields of pl. 1039—1046. Fig. 14d, Idem of the 12 beanyields of pl. 1065—1076. Fig. 14e, Idem of the 5 beanyields of pl. 1077—1081. (See text).

Het *gewicht.* De *pl. 321* en *320* tonen een geringe transgressieve variabiliteit van het gewicht.

F_6-1937. Transgressieve variabiliteit van minusvariaties (tab. 12).

De *lengte.* De bonenopbrengsten van *pl. 459* en *344* tonen duidelijke, de *pl. 338* en *346* tonen een geringe transgr. variabiliteit van de lengte.

De *breedte.* Behalve de in tab. 12 opgenomen *pl. 344, 333* en *343* (fig. 15) tonen nog 12 bonenopbrengsten transgr. variabiliteit van minusvariaties van de breedte. Van de *pl. 334, 338* en *336* is de gemidd. breedte 7.3 mm, van de *pl. 335, 342, 340*

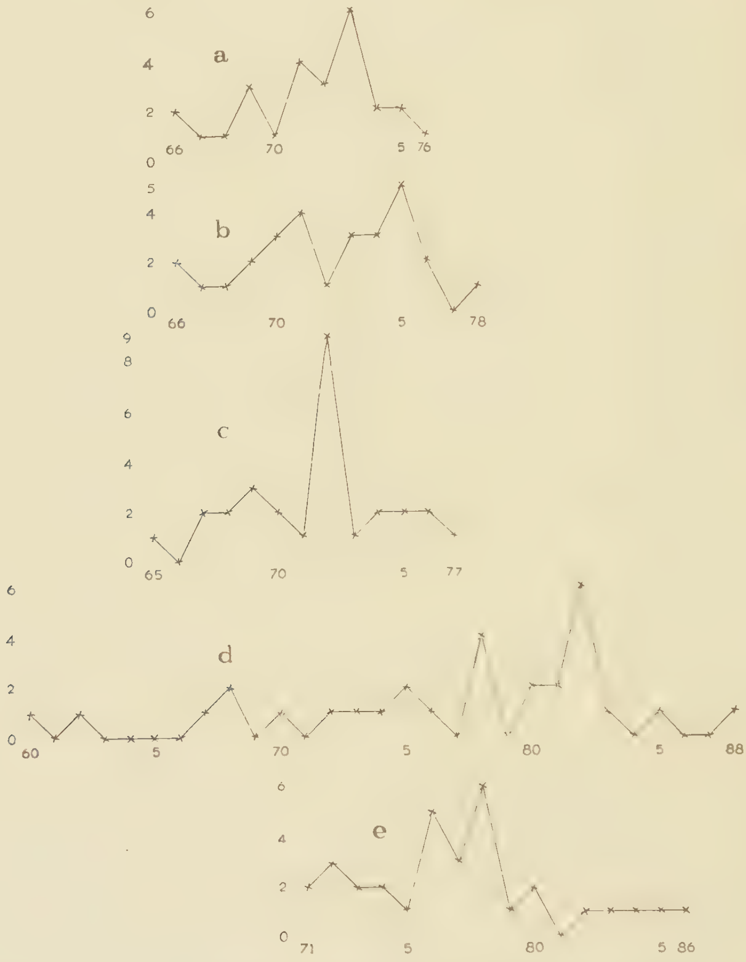

Fig. 15, a—c. F_6-1937. Frequency curves of the breadth. Minus variations. a, pl. 344, n = 26, b_m = 7.1 mm. b, pl. 343, n = 28, b_m = 7.2 mm. c, pl. 333, n = 28, b_m = 7.2 mm.

Fig. 15, d—e. II-line 1937. Frequency curves of comparison beanyields. d, pl. 136. n = 29, b_m = 7.7 mm. e, pl. 117, n = 32, b_m = 7.7 mm.

en *346* is ze 7.4 mm, van de *pl. 349, 459* en *337* is ze 7.5 mm en van de *pl. 345* en *437* is de gemidd. breedte 7.6 mm. Al deze 15 planten, behalve *pl. 459* en *437* zijn gegroeid uit uitgangsbonen van de bonenopbrengst van eenzelfde plant (pl. 492, F_5-1936, 1950). We hebben hier met een zeer goed geval van transgressie-splitsing te doen. Pl. 492, F_5-1936 is de transgressiesplitsing.

De *dikte*. De *pl. 1040, 1045* en *506* tonen in sterke mate transgr. variabiliteit van minusvariaties voor de dikte. De dan volgende *pl. 1065, 507* en *504* tonen ze in geringe mate ($th_m = 5.5$ mm).

Het *gewicht*. Van een groot aantal planten, het zijn er 28, is het gemidd. gewicht kleiner dan het kleinste gemidd. gewicht van bonenopbrengsten van de II-lijn van 1937. (Er zijn er daarbij 3 met $w_m = 34$, 4 met $w_m = 35$, 4 met $w_m = 36$, 8 met $w_m = 37$ en 5 met $w_m = 38$ cg).

De tabellen 13 en 14 geven een samenvatting over de transgressieve variabiliteit resp. van plus- en minusvariaties van F_5-1936 t/m F_2-1933 en van F_6-1937. We zien er uit (tab. 13), dat van F_5-1936, van de afmetingen de breedte het vaakst transgressieve variabiliteit vertoont, van F_4-1935, alleen de breedte, van F_3-1934 alleen de dikte. F_2-1933 toont geen duidelijke transgressieve variabiliteit. Van F_6-1937 toont de dikte het vaakst transgressieve variabiliteit. De lengte toont alleen in F_5-1936 transgressieve variabiliteit.

Transgressieve variabiliteit van minusvariaties (tab. 14) vinden we het vaakst van de breedte. In F_6-1937 zijn er zeer veel gevallen van transgr. variabiliteit, die bijna alle zijn terug te voeren tot uitgangsbonen van een zelfde plant van F_5-1936.

Ook het materiaal van de op elkaar volgende splitsingsgeneraties van kruisingen, die in 1933 verricht zijn, hebben we ten opzichte van de transgressieve variabiliteit onderzocht. We beschikken hier over de generaties F_5-1937 tot en met F_3-1934.

F_5-1937. Transgressieve variabiliteit van plusvariaties (tab. 15).

De *lengte*. De grootste gemidd. lengte van bonenopbrengsten van F_5-1937 is van *pl. 239*. Ze is iets kleiner dan de grootste gemidd. lengte van bonenopbrengsten van de I-lijn van 1937, doch groter dan op één na de grootste (tab. 15 en tab. 11).

De *breedte*. Van *pl. 239* toont de grootste gemidd. breedte transgressieve variabiliteit ten opzichte van de grootste gemidd. breedte van bonenopbrengsten van de I-lijn van 1937 (fig. 16).

We vinden, dat van een bonenopbrengst van F_5-1937 de gemidd. breedte van één der bonenopbrengsten transgressieve variabiliteit vertoont, terwijl dit voor de lengte voor één bonenopbrengst van F_6-1937 het geval is (fig. 13 en 16).

De *dikte*. Van *pl. 211* is de grootste gemidd. dikte even groot als de grootste gemidd. dikte van bonenopbrengsten van de II-lijn van 1937. Van *pl. 223* is ze groter dan op één na de grootste gemidd. dikte van de I-lijn.

Het *gewicht*. De bonenopbrengsten van *pl. 239* en *211* tonen transgressieve variabiliteit van het gewicht. Pl. 239 heeft de grootste gemidd. lengte en grootste gemidd. breedte van het materiaal en pl. 211 heeft de grootste gemidd. dikte. Pl. 239 heeft een grote gemidd. dikte (th_m — 6.6 mm) en pl. 211 heeft een zeer grote gemidd. lengte ($l_m = 15.7$ mm) en grote

gemidd. breedte ($b_m = 9.2$ mm). De transgressieve variabiliteit van het gewicht is terug te voeren tot de zeer grote gemidd. afmetingen.

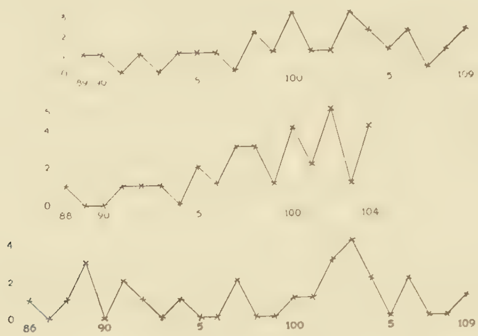

Fig. 16a. F_5-1937. Frequency curve of the breadth. Plus variations. Pl. 239, $n = 25$, $b_m = 10.1$ mm.

Fig. 16, b—c. I-line 1937. Frequency curves of comparison beanyields. b, pl. 66, $n = 20$, $b_m = 9.8$ mm. c, pl. 81, $n = 25$, $b_m = 9.8$ mm.

F_5-1937. Transgressieve variabiliteit van minusvariaties (tab. 16).

De *lengte*. De kleinste gemidd. lengte is belangrijk groter dan de kleinste gemidd. lengte van bonenopbrengsten van de II-lijn van 1937.

De *breedte*. De *pl. 174* en *336* tonen transgressieve variabiliteit. Van de *pl. 161*, *163* en *164* is de gemidd. breedte even groot als de kleinste gemidd. breedte van bonenopbrengsten van de II-lijn van 1937.

De *dikte*. De kleinste gemidd. dikte is groter dan de kleinste gemidd. dikte van bonenopbrengsten van de I-lijn van 1937.

Het *gewicht*. *Pl. 174* toont transgressieve variabiliteit. De gemidd. lengte ($l_m = 7.4$ mm) en de gemidd. breedte van pl. 174 zijn ook zeer klein. Van de *pl. 161* en *229* komt het gemidd. gewicht overeen met het kleinste gemidd. gewicht van bonenopbrengsten van de II-lijn van 1937.

Transgressieve variabiliteit van F_4-1936.

a. Van plusvariaties (tab. 17 en tab. 1).

De *lengte*. De grootste gemidd. lengte van *pl. 331* toont transgressieve variabiliteit. Van *pl. 309* is de grootste gemidd. lengte iets kleiner dan van de bonenopbrengst met de grootste gemidd. lengte van de I-lijn van 1936.

De *breedte*. De grootste gemidd. breedte van *pl. 309* toont transgressieve variabiliteit. Die van de *pl. 291, 298, 194* en *274* is iets kleiner.

De *dikte*. De gemidd. dikte toont geen transgressieve variabiliteit.

Het *gewicht*. Van de bonenopbrengsten van de *pl. 331, 274, 298, 404, 321* en *309* ($w_m = 71$—66 cg) toont het gemidd. gewicht transgressieve variabiliteit. De pl. 331, 274, 298 en 309 hebben een zeer grote gemidd. lengte en breedte; pl. 404 heeft een zeer grote gemidd. dikte en pl. 321 een zeer grote gemidd. breedte.

b. Van minusvariaties (tab. 18 en tab. 3).

De *lengte*. Geen transgressieve variabiliteit.

De *breedte*. De gemidd. breedte van 15 bonenopbrengsten tonen transgressieve variabiliteit. Dan volgen er 4 met een even groot gemidd. breedte als die van de kleinste gemidd. breedte van de II-lijn.

De *dikte*. Geen transgressieve variabiliteit; echter wel zijn er veel bonenop-

brengsten met een even grote gemidd. dikte als de kleinste gemidd. dikte van bonenopbrengsten van de I-lijn.

Het *gewicht.* Van twee bonenopbrengsten toont het gemidd. gewicht transgressieve variabiliteit. Dan volgen er 5 met een gemidd. gewicht, dat overeenkomt met dat van op één na het kleinste gemidd. gewicht van de II-lijn. Van 20 bonenopbrengsten is het gemidd. gewicht kleiner dan 40 cg; van de II-lijn van 1936 zijn het er slechts drie.

Transgressieve variabiliteit van F_3-1935.

a. Van plusvariaties (tab. 19 en tab. 5).

De *lengte* toont geen transgressieve variabiliteit. Ook niet de *breedte.* Van de *pl. 63* en *95* is de gemidd. breedte zo groot als die van bonenopbrengsten met op één na de grootste gemidd. breedte van de I-lijn.

De *dikte* toont geen transgressieve variabiliteit. Van de *pl. 95* en *63* heeft het gemidd. *gewicht* transgressieve variabiliteit. Deze bonenopbrengsten hebben ook een zeer grote gemidd. lengte en gemidd. breedte. Dan volgt *pl. 54* met $w_m = 70$ cg.

b. Van minusvariaties (tab. 20 en tab. 6).

Geen transgressieve variabiliteit van de gemidd. *lengte.* Ook niet van de gemidd. *breedte.* De kleinste gemidd. breedte is van *pl. 99* (echter $n = 6$) en is even groot als de op één na kleinste gemidd. breedte van bonenopbrengsten van de II-lijn van 1935. Geen transgərssievariabiliteit van de *dikte* en van het *gewicht.* Van *pl. 89* is het gemidd. gewicht groter dan het kleinste, doch kleiner dan het op één na kleinste gemidd. gewicht van de II-lijn van 1935. Van pl. 89 zijn alle 3 afmetingen zeer klein.

Transgressieve variabiliteit van F_2-1934.

a. Van plusvariaties (tab. 21 en tab. 7).

TABLE 5. The beanyields of F_4-1935 with the greatest mean dimensions and mean weights and the greatest dimensions and weights of individual beans and some comparison beanyields of the I- and the II-line of 1935

Pl_l	cl_l	pl	n	m	gr.v.	cl_m	pl	n	m	gr.v.
		a. The length. F_4-1935					The I-line 1935			
159	1a	119	51	163	192 [1]	1a	11	39	166	185
159	2b	123	51	160	179	1a	15	36	165	196 [3]
159	1b	121	50	158	190 [2]	1a	16	45	160	216 [4]
		b. The breadth. F_4-1935					The I-line 1935			
159	1a	119	51	103	116	1a	2	15	100	112
159	2b	123	51	103	118	1a	12	41	98	111
159	1b	121	50	100	117	1a	1	36	96	117 [5]
195	2b	285	42	97	111	2a	16	45	95	130 [6]
		c. The thickness. F_4-1935					The II-line 1935			
159	1a	123	51	72	85	1a	22	48	73.5	78
177	3	156	50	72	78	1b	28	66	71.5	85
		d. The weight. F_4-1935					The I-line 1935			
159	1a	119	51	85	108	1a	2a	14	71	84
159	1a	123	51	83	109	1a	12	41	66	85
159	1b	121	50	76	113	1a	4	24	65.5	86
168	1a	143	40	75	92	1b	16	45	61	127 [7]
272	8a	300	44	75	105	1b	1	36	61	105 [8]

[1]) then 18.7. [2]) then 18.4 mm. [3]) then 18.0. [4]) then 19.6 mm.
[5]) then 11.5 [6]) then 12.1 mm. [7]) then 97. [8]) then 95 cg.

De gemidd. *lengte* en de gemidd. *breedte* tonen geen transgressieve variabiliteit. Van de gemidd. *dikte* en van het gemidd. gewicht toont *pl. 70* ze.

b. Van minusvariaties (tab. 22 en tab. 8).

De *lengte* toont geen transgressieve variabiliteit. De gemidd. *breedte* van *pl. 68* is even klein als de kleinste gemidd. breedte van bonenopbrengsten van de II-lijn van 1934. De gemidd. *dikte* en het gemidd. *gewicht* tonen geen transgressieve variabiliteit.

De tab. 23 en 24 geven een samenvatting over de transgressieve variabiliteit resp. van plus- en minusvariaties van F_5-1937 e.v.

TABLE 6. The beanyields of F_4-1935 with the smallest mean dimensions etc.

Pl	cl	pl	n	m	sm.v.	cl	pl	n	m	sm.v.
		The length. F_4-1935					The II-line 1935			
180	5	201	51	110	84	7	30	30	105	86
		The breadth. F_4-1935					The II-line 1935			
275	8c	319	47	77	68	8c	30	30	76	60
245	8c	315	47	77	62 [1])	8b	24	50	79	67
169	4	148	17	79	72	8c	29	56	81.5	69
		The thickness. F_4-1935					The I-line 1935			
194	2a	275	40	54	47	2b	5	50	52	35
195	1b	291	43	56	44	2b	17	31	52	45
		The weight. F_4-1935					The II-line 1935			
168	4	134	16	39	23	8c	30	30	33	15
189	8c	209	45	40	25	8c	24	50	37	25
275	8c	315	47	41	22 [2])	8b	29	56	40	24

[1]) then follows 7.1 mm. [2]) then 34 cg.

TABLE 7. The beanyields of F_3-1934 with the greatest mean dimensions and mean weights and the greatest dimensions and weights of individual beans and some comparison beanyields of the I- and the II-line of 1934

Pl	cl	pl	n	m	gr.v.	cl	pl	n	m	gr.v.
		a. The length. F_3-1934					The I-line 1934			
K 4	2b	194	12	161	177	1a	37	15	168	181
K 33	7	272	65	157	181	1a	9	40	167	195
		b. The breadth. F_3-1934					The I-line 1934			
K 4	2b	194	12	98	107	2a	5	21	99	116
K 33	7	272	65	98	110	1a	4	40	98	105
		c. The thickness. F_3-1934					The II-line 1934			
K 17	4	149	126	76	83	1b	49	32	73	78
K 22	1b	115	31	74	81	1b	50	43	72.5	79
		d. The weight. F_3-1934					The I-line 1934			
K 33	7	272	65	75	106	1a	5	21	72	91
K 23	1a	107	18	74	74	1b	39	36	70	92
K 81		573 [1])	32	78						
K 91		775 [1])	26	77						

[1]) Only the mean weight is known here.

TABLE 8. The beanyields of F_3-1934 with the smallest mean dimensions etc.

Pl	cl	pl	n	m	sm.v.	cl	pl	n	m	sm.v.
		a. The length. F_3-1934					The II-line 1934			
K 17	8b	144	15	109	84 [1]	8b	51	12	97	87
K 17	8c	145	24	110	94	8b	46	38	104	94
		b. The breadth. F_3-1934					The II-line 1934			
K 17	8b	144	15	77	62	8b	51	17	72	59
K 20	1b	140	32	78	70	8c	42	40	80	72
K 17	8c	145	24	78	65	8b				
		c. The thickness F_3-1934					The I-line 1934			
K 23	1b	105	30	58	48	2b	31	21	56	51
K 20	1b	140	32	59	51	8c	7	40	58	42
		d. The weight F_3-1934					The II-line 1934			
K 17	8c	145	24	36	25	8b	51	17	25	14
K 17	8b	144	15	38	28	8b	48	24	37	20
K 20	1b	140	32	39	29	8c	45	24	38	26
K 23		695 [1]	28	25						
K 22		725 [1]	26	31						

[1]) Only the mean weight is known here. [2]) then follows 9.5 mm.

TABLE 9. F_2-1933. The greatest mean dimensions etc. Compar. beanyields 1933

cr	pl	n	m	gr.v.	cl	pl	n	m	gr.v.
	a. The length. F_2-1933					I-line 1933			
I × II	K 5	22	149	158	1b	3b	25	160	183
	b. The breadth. F_2-1933					I-line 1933			
II × I	K 101	40	90	96	1b	3b	25	92	98
II × I	K 82	35	89	102	1b	1	25	89	99
	c. The thickness. F_2-1933					II-line 1933			
II × I	K 99	39	70	74	1b	2a	73	70	77
I × II	K 2	40	69	77	3	1a	40	69	79
	d. The weight. F_2-1933					I-line 1933			
II × I	K 101	40	61.5	77	1b	3b	25	66.5	84
II × I	K 99	39	61	71	1b	5b	30	60	72

TABLE 10. F_2-1933. The smallest mean dimensions, etc. Compar. beanyields 1933

cr	pl	n	m	gr.v.	cl	pl	n	m	gr.v.
	a. The length. F_2-1933					II-line 1933			
I × II	K 40	30	128	118	7	5a	30	116	103
	b. The breadth. F_2-1933					II-line 1933			
II × I	K 15	38	79	69	4	19b	43	76	70
	c. The thickness					I-line 1933			
II × I	K 17	40	62	53	4	2d	24	57	49
	d. The weight					II-line 1933			
II × I	K 17	40	45	29	4	19b	43	37	25

TABLE 11. F_6-1937. The greatest mean dimensions, etc.

pl	cl_i	pl	n	m	gr.v.	cl_m	pl	n	m	gr.v.
		a. The length. F_6-1937						I-line 1937		
450	1a	292	25	165	176	1a	58	18	165	179
450	1a	289	25	163	182	1a	66	30	160	186
666	1b	1039	16	158	176	1a	80	25	153	190
474	2b	321	28	158	178	1a				
		b. The breadth. F_6-1937						I-line 1937		
463	1b	317	19	99	107	1b	66	30	98	104
450	1a	289	25	98	109	1a	81	25	98	109
785	2b	1079	25	98	101	1b	82	25	98	110
474	2b	320	17	97	108	1b	71	25	95	114
		c. The thickness. F_6-1937						II-line 1937		
521	1b	374	25	74	82	1b	127	26	71	77
521	1b	372	25	72	77	1b	110	27	70	75
517	1b	402	26	72	78	1b	132	30	69	73
518	8c	427	27	72	80	1b				
		d. The weight. F_6-1936						I-line 1937		
474	2b	321	28	75	97	1a	58	18	74.5	90
474	2b	320	17	75	105	1b	66	30	73	94
518	8c	427	27	74	96	1b				

TABLE 12. F_6-1937. The smallest mean dimensions, etc.

Pl	cl	pl	n	m	sm.v.	cl	pl	n	m	sm.v.
		a. The length. F_6-1937						II-line 1937		
632	8c	459	25	101	92	8c	114	31	105	89
492	8b	344	26	101	85	8b	124	25	105	93
492	8b	338	28	104	86	8b				
492	8b	346	30	104	80 [1])	8b				
		b. The breadth. F_6-1937						II-line 1937		
492	8b	344	26	71	66	8b	136	29	77	60
492	8b	333	28	72	65	8b	117	32	77	71
492	8b	343	28	72	66	8b				
		c. The thickness. F_6-1937						I-line 1937		
666	2b	1040	20	46	41	4	1	29	54	41
666	8c	1045	18	48	40	8a	13	20	55.5	42
580	8a	506	25	52	35	8a	48	34	56	32
		d. The weight. F_6-1937						II-line 1937		
632	8c	459	25	28	20	8c	136	29	39	18
580	8a	506	25	29	13	8a	128	26	39	15
666	2b	1040	20	31	21	4	118	32	40	31
666	8c	1045	18	31	19	8a				

[1]) then follows 9.1 mm.

TABLE 13. Survey of the cases of transgressive variability of plusvariations in F_5-1936—F_2-1933 and F_6-1937

	The length			The breadth			The thickness			The weight		
	+	+?[1])	±	+	+?	±	+	+?	±	+	+?	±
F_5-1936	450	—	666	783	457	776						
	463			450	784	441			783	783		
	783			463	451	446				463		
	674				785	719				450		
						674				784		
						475				674		
						568				556		
										568		
										719		
F_4-1935	—	—	—	119					123	119		
				123					156	123		
										121		
										143		
										a.o.		
F_3-1934	—	—	—			194	149		107	272		138
						272	115		112	107		196
									279			
F_2-1933	—	—	—			K 101			K 1			K 101
						K 5			K 99			K 99
						K 82			K 2			
									K 50			
F_6-1937		292	289	317		289	374		361	321	427	292
						1079	372		480	320		311
							402					
							427					
							368					

[1]) In these cases the data of the comparison beanyield are uncertain.

TABLE 14. Survey of the cases of transgressive variability of minusvariations in F_5-1936—F_2-1933 and F_6-1936

	The length			The breadth			The thickness			The weight		
	+	+?	±	+	+?	±	+	+?	±	+	+?	±
F_5-1936	—	—	—	664	647	654	759	1066	1061	664	565	654
				788	565	1065	664		673	759	650	656
				1083	650					788		
				492								
F_4-1935	—	—	—	—	319	148	—	—	—	—	—	—
					315							
F_3-1934	—	—	—	—	—	—	—	—	105		145	144
F_2-1933	—	—	—	—	—	K 15	—	—	K 17	—	—	—
									K 31			

TABLE 14. *(Continuation)*

F_6-1937	459	334	344	348	1040	506	1065	459	1048
	344	437	333	a.o.	1045		507	506	
	338	340	343				504	1040	
	346		334					1045	
			338					1050	
			336					507	
			335					344	
			342					504	
			340					350	
			346					450	
			349					457	
			459					451	
			337					1049	
			345					333	
			437					346	
								a.o.	

TABLE 15. F_5-1937. The greatest mean dimensions, etc.

pl	n	m	gr.v.
a. The length. F_5-1937			
239	25	16.2	175
193	30	15.8	180
b. The breadth. F_5-1937			
239	25	101	109
146	14	96	103
c. The thickness. F_5-1937			
211	24	71	77
223	29	70	78
d. The weight. F_5-1937			
239	25	81	95
211	24	77	92

TABLE 16. F_5-1937. The smallest mean dimensions, etc.

pl	n	m	sm.v.	cl_m
a. The length. F_5-1937				
241	31	111	92	8b
161	30	114	98	8c
b. The breadth. F_5-1937				
174	28	74	68	8c
236	29	76	64	8b
c. The thickness. F_5-1937				
207	28	55	43	8c
d. The weight. F_5-1937				
174	28	38	28	8c
161	30	39	24	8c

TABLE 17. F_4-1936. The greatest mean dimensions, etc.

Pl	cl_i	pl	n	m	gr.v.	cl_m
a. The length. F_4-1936						
84	1a	331	25	158	175	1b
73	1a	309	38	154	178	2b
b. The breadth. F_4-1936						
73	1a	309	38	100	113	2b
68	2a	291	15	96	101	2b
c. The thickness. F_4-1936						
54	1b	242	20	70	76	1b
d. The weight. F_4-1936						
84	1a	331	25	74	93	1a
63	1b	274	24	71	86	1b
70	1a	298	24	71	82	1b

TABLE 18. F_4-1936. The smallest mean dimensions, etc.

Pl	cl_i	pl	n	m	sm.v.	cl_m
a. The length. F_4-1936						
98	3	370	27	108	96	8b
b. The breadth. F_4-1936						
62	8c	270	25	70	58	8c
and 14 other ones						
c. The thickness. F_4-1936						
101	4	1033	31	51	44	4
d. The weight. F_4-1936						
62	8c	270	25	31	19	8c
89	2b	340	14	31	24	8a
98	3	370	27	34	27	8b

TABLE 19. F_3-1935. The greatest mean dimensions, etc.

Pl	cl	pl	n	m	gr.v.	cl
a. The length. F_3-1935						
	1b	73	35	159	183	2a
b. The breadth. F_3-1935						
	5	63	50	98	111	1b
	4	95	52	98	105	1b
c. The thickness. F_3-1935						
	7	54	71	71	80	1b
	5	63	50	71	82	1b
d. The weight. F_3-1935						
	4	95	52	75	99	1b
	5	63	50	74	100	1b

TABLE 20. F_3-1935. The smallest mean dimensions, etc.

Pl	cl	pl	n	m	sm.v.	cl
a. The length. F_3-1935						
	7	89	54	117	102	8a
b. The breadth. F_3-1935						
	7	89	54	80	73	8a
c. The thickness. F_3-1935						
	7	89	54	54	42	8a
d. The weight. F_3-1935						
	7	89	54	35	21	8a
	1a	91	61	40	23	4

TABLE 21. F_2-1934. The greatest mean dimensions, etc.

cr	pl	n	m	gr.v.	cl
a. The length. F_2-1934					
$II_2 \times I_5$	83	88	142	163	1b
$II_4 \times I_2$	87	87	139	171	1b
b. The breadth. F_2-1934					
$I_2D \times II_1$	70	51	92	100	1b
c. The thickness. F_2-1934					
$I_2D \times II_1$	70	51	74	80	1b
$II_4 \times I_2$	87	87	72	85	1b
d. The weight. F_2-1934					
$I_2D \times II_1$	70	51	68.5	84	1b
$II_4 \times I_2$	87	87	59	91	1b

TABLE 22. F_2-1934. The smallest mean dimensions, etc.

cr	pl	n	m	sm.v.	cl
a. The length. F_2-1934					
$I_4 \times II_2$	68	66	125	107	8c
$I_2D \times II_1$	72	41	126	95	8c
b. The breadth. F_2-1934					
$I_4 \times II_2$	68	66	77	70	8c
$I_2D \times II_2$	72	41	81	60	8c
c. The thickness. F_2-1934					
$I_4 \times II_2$	68	66	61.5	54	8c
$I_2D \times II_1$	72	41	64	42	8c
d. The weight. F_2-1934					
$I_4 \times II_2$	68	66	40	27	8c
$I_2D \times II_1$	72	41	44.5	15	8c

TABLE 23. Survey of the cases of transgressive variability of plusvariations in F_5-1937 — F_2-1934

	The length			The breadth			The thickness			The weight		
	+	+?	±	+	+?	±	+	+?	±	+	+?	±
F_5-1937	—	—	—	239	—	—	—	211	223	239 211	222	—
F_4-1936	331	309	—	309	291 298 194 274	361	—	—	—	274 298 404 309		
F_3-1935	—	—	—	—	—	63 95	—	—	—	95 63	54 71	84 87 68
F_2-1934	—	—	—	—	—	—	70	—	—	70	—	—

TABLE 24. Survey of the cases of transgressive variability of minusvariations in F_5-1937—F_2-1934

	The length			The breadth			The thickness			The weight		
	+	+?	±	+	+?	±	+	+?	±	+	+?	±
F_5-1937	—	—	—	174	236	161 163 164	—	—	—	174	—	161 229
F_4-1936	—	—	—	270 994 1006 a. 12 o.	—	1048 330 370 979	—	—	1033	270 340 370 a. 18 o.	—	972
F_3-1935	—	—	—	—	—	99	—	—	—	—	89	—
F_2-1934	—	—	—	—	—	68	—	—	—	—	—	—

BOTANY

RAY TERMINOLOGY IN WOOD ANATOMY

BY

CORNELIA A. REINDERS–GOUWENTAK

(Communicated by Prof. G. VAN ITERSON at the meeting of June 24, 1950)

§ 1. *Introduction.*

The present author in an earlier article (9) discussed a classification of rays into homogeneous and heterogeneous rays, both groups containing many rays of distinct structure (see this paper fig. 1). At the same time there appeared a paper prepared by Miss M. M. CHATTAWAY who proposes an alteration of the definition of homogeneous and heterogeneous rays in the Glossary of terms of 1933 (5). The revised definitions differ from those presented by JANSSONIUS (7, vol. IV) and commented by the present writer in her earlier article.

Miss CHATTAWAY's definitions are based on cell shape and cell functions the latter being gathered from cell contents, pitting and staining properties of the wall. The rays in which all the cells, which adjoin the *same* elements, are similar regarding cell shape, contents and pitting, are homogeneous rays. Heterogeneous are the rays in which one or more of these conditions are not fulfilled. However, under such conditions it is evident that the term homogeneous can only seldom be applied. All the conditions required for homogeneity will almost never be present.

To all probability Miss CHATTAWAY is right in assuming functional differences between pitted and unpitted ray cells and between ray cells with different cell contents, but it is not a reason for involving physiology in anatomical classification matters. The stem of a Cactus has a photosynthetic function, but in morphological studies Cactus stems are not classified with the leaves of non succulents. And there is even more to say against Miss CHATTAWAY's concept of homogeneity.

According to Miss CHATTAWAY a homogeneous ray should be a ray composed of procumbent cells or of erect cells only provided all of the cells are also similar in "function" when in contact with the same elements. For example: in a homogeneous ray the cells contiguous to vessels are all of them pitted or all of them not pitted at all, but where the ray cells are adjoining fibres the pitting may be another one than where the ray cells are in contact with vessels.

The question might be asked whether the pits of all the cells which are in contact with a vessel or fibre etc. are all of them of the same shape, dimension, number? Certainly not so. But why not these requirements too? Why draw the limit elsewhere? What, moreover, do such definitions

of homogeneous and heterogeneous rays help in the urgent need for supplying identification features of woods? Miss CHATTAWAY in collecting these cell features highly contributed to the increase of our knowledge about rays but the features mentioned are features in detail and not likely to be of use for ray classification purposes.

No cell details but structure of rays should be used for the classification of rays. A first step was already done by RECORD and CHATTAWAY (8) and one may only wonder why Miss CHATTAWAY did not continue her investigations of *structural* specialization of rays.

From a structural point of view rays have been classified into homogeneous and heterogeneous rays by JANSSONIUS (7) but with the *german* terms: *einfache* and *zusammengesetzte Markstrahlen*. Thanks to our present knowledge of rays JANSSONIUS' work could be supplemented and the results recorded in a key for identification of homogeneous and heterogeneous rays (9).

A key though easy for manipulation is not the form to be used in a glossary of terms. This paper, therefore, presents definitions of homogeneous and heterogeneous rays as covered by the key but which now might be included in a glossary of terms. In connection with Miss CHATTAWAY's recent work a review of the author's previous article will precede the definitions.

§ 2. *Discussion of the terms homogeneous and heterogeneous and allied terms.*

Miss CHATTAWAY in concluding her paper states "that there are good grounds for discontinuing the use of these terms and replacing them with something entirely different". The present author cannot agree with her. To discontinue the use of the terms homogeneous and heterogeneous would lead to too many sequels. The terms *homogeneous type* and *heterogeneous type* of KRIBS (6) would have to disappear too, what with the work of KRIBS well introduced in literature would rather be a disadvantage. Furthermore, KRIBS proposed the terms *homocellular* and *heterocellular* to replace the terms homogeneous and heterogeneous of the Glossary (5). Recently the terms *homogeneous ray* and *heterogeneous ray* (REINDERS-GOUWENTAK (9)) were proposed for various rays constituting the homogeneous types and heterogeneous types of KRIBS: this third type of terms then would have to disappear too. To the present authors opinion all the terms are entirely adequate ones if one can only divorce the words from their proper meaning what in many cases (wood) anatomists appear to have done with other terms.

Properly speaking KRIBS proposed the term *homocellular* for rays formed of procumbent cells only, but extending the usage to rays composed of erect (square) cells seems adequate. The term *heterocellular* has to be applied to a ray composed of both types of cells, the erect (square) and the procumbent cells. Now, according to p. 5 of her recent paper Miss

Chattaway objects to using such terms as erect and procumbent. But how serious is this objection? Is not the point missed? The terms "erect" and "procumbent" surely are not meant to cover transitional stages too. There will be always cells which are neither erect nor procumbent; erect to procumbent would be the proper term for them. Do these difficulties arise only in wood anatomy or has not morphology to deal with the same trouble in applying such terms as circular, elliptical, oblong etc. for the indication of special shapes of leaves?

The terms *homogeneous type of rays* and *heterogeneous type of rays* introduced by Kribs (6) indicate specific ray combinations, each combination a constant feature in genera or families. Leaving the question of terminology to others Kribs enumerated rays constituting the homogeneous and heterogeneous types. For example, the heterogeneous type I in Dillenia. In this wood the ray tissue consists of high uniseriate rays composed of very large elongated cells and of multiseriate rays composed of a middle multiseriate portion of procumbent cells and very long marginal extensions which are composed of cells identical with those of the uniseriate rays.

Reinders–Gouwentak (9) in joining Janssonius' concept of ray structure established that for example Kribs' rays of the heterogeneous type I are represented by the rays which are reproduced in fig. 1 of this paper in *a* and g_1, *a* being identical with the uniseriate extension in g_1. All the other rays of Kribs' combinations and even more than are mentioned by him may be found in fig. 1, 2 (p. 1268 and 1269), 3 (p. 1270) and 4 (p. 1271). In Dillenia there are also present rays identical with *h* (fig. 1). The rays g_1 are, as it were, composed of three individual rays. The uniseriate extensions are identical with the uniseriate ray *a*, the multiseriate portion is similar in structure to a separate multiseriate ray *h*.

Janssonius (7, Vol. IV, p. 403) writing in german designated such rays *a* as *"einfache" rays of the first kind*, the multiseriate rays as *"einfache" rays of the second kind* and rays g_1 as *"zusammengesetzte" rays*. The present writer in her previous article (9) proposed *homogeneous rays of the first kind, homogeneous rays of the second kind* and *heterogeneous rays* as english terms.

It is clear that another meaning is attached now to the terms homogeneous and heterogeneous than in the Glossary. In the latter the terms are used to indicate the presence of a specific cell shape within rays; now they are used for purposes of emphasizing different structural composition. The writer is fully aware of the objections there may be made to an alteration of definitions but retaining of terms. Yet, in her opinion, the advantages of preserving the terms for ray classification are still greater. In the sense now attached to them the terms may prove a success together with the terms homocellular and heterocellular, homogeneous type and heterogeneous type of Kribs. Confusion is already so great that for years to come we cannot do without mentioning author's names in all matters concerning terminology of rays. It may be thought that it will help to reduce confusion or rather prevent.

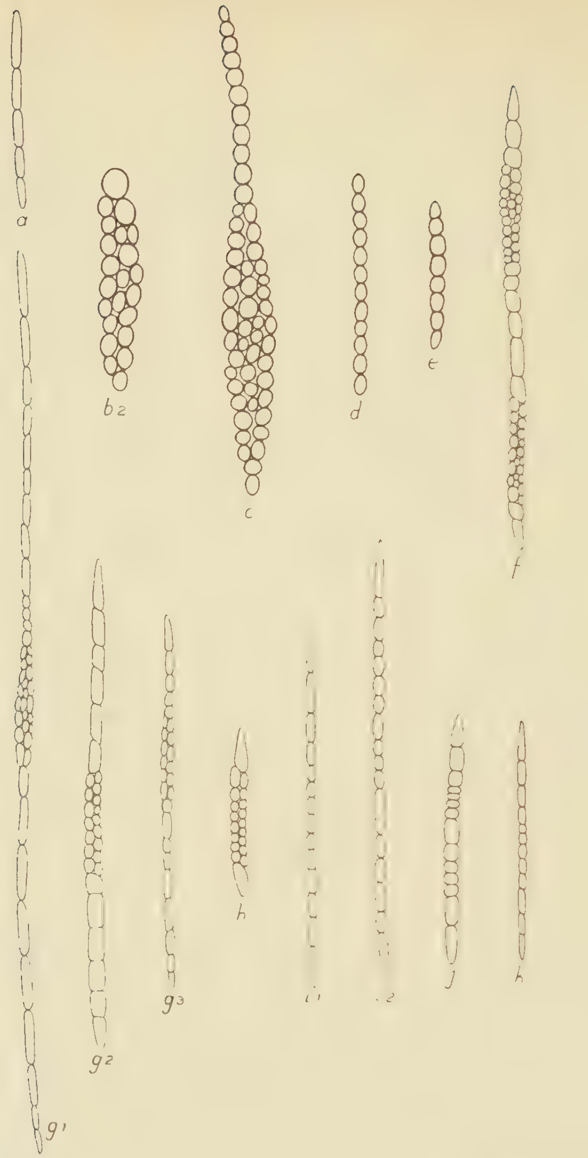

Fig. 1. a, g_1, g_2, g_3, i_1, i_2, j, k: $\times$ 60. b_2, c, d, e: $\times$ 180. f, h: $\times$ 90.

a. Homogeneous ray of the first kind; all cells upright.
b_2. Homogeneous ray of the second kind; all cells procumbent.
c. Heterogeneous ray, with uniseriate tier of procumbent cells.
d. Homogeneous ray of the first kind; all cells procumbent. [1])
e. Homogeneous ray of the second kind; all cells procumbent. [1])
f. Heterogeneous ray composed of 5 tiers.
g_1. Heterogeneous ray composed of 3 tiers; the uniseriate tiers very high.
g_2. Heterogeneous ray composed of 3 tiers; the uniseriate tiers lower.
g_3. Heterogeneous ray composed of 3 tiers; the uniseriate tiers with radial rows of procumbent cells.
h. Homogeneous ray of the second kind. See also fig. 4, *f* on p. [1271].
i_1. Heterogeneous uniseriate ray of 3 tiers. [1])
i_2. Heterogeneous uniseriate ray of many tiers.
j. Heterogeneous uniseriate ray of 5 tiers.
k. Homogeneous uniseriate ray of the first kind [1]) with radial rows of procumbent cells.

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[1]) The rays *d* and *e* are quite identical and so are the rays i_1 and k, but an examination of all ray types in the same sample shows them to belong to different kinds: *d* occurs with *c*, *e* with b_2; *k* with g_3, i_1 not with g_3 (for explanation of kinds, see text).

§ 3. *Homogeneous and heterogeneous rays.*

In many woods there are (among others) rays which are composed of two, three or more vertically arranged stories, alternately composed of uniseriate tiers of erect cells or of erect and procumbent cells, and of multiseriate tiers of mainly procumbent cells. Such rays are examples of the most simple form of heterogeneous rays of 3 stories (fig. 1 : g_1, g_2, g_3), of 2 (fig. 2: *a*, *b*) or of more than 3 stories (fig. 1 : *f*). In a wood containing

Fig. 2. *Sarcocephalus cordatus* Miq. All × 60.
a. Heterogeneous ray composed of two tiers. The uniseriate tier of upright cells is short.
b. As *a*, but the uniseriate tier is longer.
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such heterogeneous rays, there occur always or nearly always uniseriate rays (locally biseriate) too made up of erect cells only (fig. 1 : *a*) or of erect cells and procumbent cells (fig. 1: *k*). These rays are *homogeneous rays of the first kind* (cf. § 2).

Sometimes, for example in Ficus and in Acer (fig. 1: *c*) some or all of the multiseriate rays are made up of a multiseriate storey of procumbent cells and a uniseriate storey of procumbent cells too. To the authors opinion these rays too are *heterogeneous rays*, although the rays are homo-*cellular*. That it is the right thing to do is proved by the presence of multiseriate rays with a uniseriate margin of both erect and procumbent cells or of erect cells only in the same or in allied species (fig. 3), while in that case uniseriate rays occur which are identical with the margins of the multiseriates. These uniseriate rays are all of them *homogeneous rays of the first kind*, although in some of them erect cells occur mingled with procumbent ones and so the rays are heterocellular. But *homocellularity and heterocellularity are not the criteria for homogeneity and heterogeneity*; *only the structural relations between rays are.*

The length of the uniseriate extension is important. When only 1 cell is forming the uniseriate parts whether procumbent (fig. 1: b_2) or erect (fig. 1 : *h*), the ray is considered a *homogeneous ray of the second kind*, even when some or all of the cells of the first (or of the first and second) row of the multiseriate part which are contiguous to the erect uniseriate part of the ray are erect too (fig. 4: *f*). So it is only the number of the cells in the *uni*seriate part which determines whether the ray is to be considered a homogeneous ray of the second kind or a heterogeneous ray, and so ray g_1, g_2 and g_3 of fig. 1 are heterogeneous rays (composed of three stories) and ray *a* and *b* of fig. 2 are heterogeneous rays too (composed of 2 stories) and ray b_2 and *h* of fig. 1 are homogeneous rays of the second kind.

According to JANSSONIUS the length of the uniseriate extension, however, is only important for rays which are composed of two or three parts,

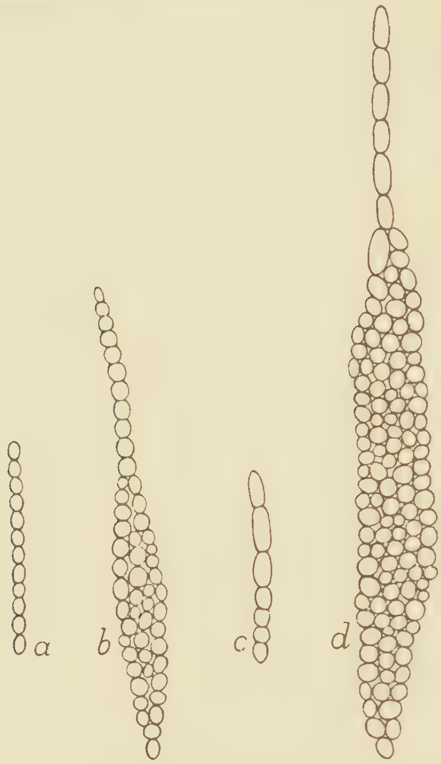

Fig. 3. *Acer campestre* L. All × 180.

a. Homogeneous ray of the first kind but composed of procumbent cells.
b. Heterogeneous ray, the uniseriate tier composed of procumbent cells as the ray of 3*a*.
c. Homogeneous ray of the first kind, composed of upright cells with radial rows of procumbent cells.
d. Heterogeneous ray, the uniseriate tier composed of upright cells.

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in rays composed of more parts (see fig. 1 : *f*) one cell is already forming a storey.

In specific cases a *uniseriate* ray composed of both erect and procumbent cells (arranged in tiers) is itself a *heterogeneous ray* (fig. 1 : i_1, *j*), viz. *if the uniseriate rays are not identical with the uniseriate margins of multiseriate heterogeneous rays present.*

Heterogeneous rays. It may be clear that a heterogeneous ray can be made up of both erect and procumbent cells or of procumbent cells only and is always characterised by showing a marginal extension that is uniseriate or locally biseriate. But for the same reasons a ray made up *entirely* of erect cells may be a heterogeneous ray too provided it is composed of a multiseriate part of erect cells and one or two uniseriate

extensions of erect cells. The present author stated this already in a footnote of her previous paper (9) p. 224 and she agrees here completely with JANSSONIUS (7) who suggested it (Part VI p. 290).

Homogeneous rays of the first or *of the second kind.* A uniseriate ray

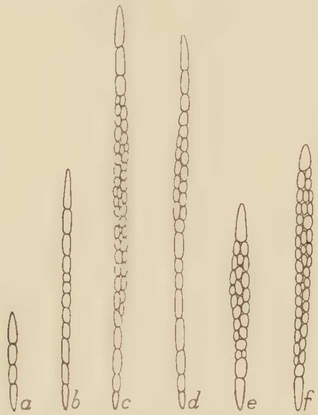

Fig. 4. *Alstonia scholaris* R.Br. All × 60.

a. Homogeneous ray of the first kind. Upright cells only.

b. Homogeneous ray of the first kind. Upright cells and radial rows of procumbent cells.

c. Heterogeneous ray. The uniseriate tiers composed as 4*a*.

d. Heterogeneous ray. The uniseriate tiers with radial rows of procumbent cells (as 4*b*).

e. Homogeneous ray of the second kind with one uniseriate row of upright cells at the margins; see text.

f. Homogeneous ray of the second kind with one uniseriate row of upright cells at the margins; one of the uppermost radial rows of the multiseriate part with an upright cell on tg. face.

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which is composed of either erect (fig. 1: *a*) or procumbent (fig. 1: *d*, *e*) cells is always a *homogeneous ray.* If it is also present as uniseriate margin in another uniseriate ray, as in i_1 (fig. 1) or in a multiseriate ray as in g_1, *c* (fig. 1) of the same wood species, the uniseriate is a *homogeneous ray of the first kind* (*a*, *d*). But, if in the same wood species, it is not present as a margin the uniseriate ray is a *homogeneous ray of the second kind* (fig. 1 : *e*). Consequently homogeneity of uniseriate rays may be reached along two different paths. For purposes of emphasizing the difference in homogeneity JANSSONIUS distinguished the *two kinds of homogeneous rays.* The present writer in suggesting these english terms in her recent publication (9) wrote she would have preferred the word type instead of *kind*, but could not because of KRIBS having the term type already used in another meaning.

Homogeneous rays of the first kind. For reasons of compilation we may now repeat that homogeneous rays of the first kind comprise the rays which in the same wood species are accompanied by heterogeneous rays in the uniseriate extensions of which they are mirrored. As a rule the

homogeneous rays of the first kind are uniseriate but sometimes locally biseriate; they are either composed entirely of erect cells (*Diospyros*, *Dillenia*, *Alstonia* (fig. 4: *a*)) or they are made up of erect cells mingled with procumbent cells (*Sarcocephalus*, *Alstonia* (fig. 4: *b*), *Acer* (fig. 3: *c*)) or they possess procumbent cells only (*Acer* (fig. 3: *a*); *Ficus*).

Homogeneous rays of the second kind. In the second kind of homogeneous rays are placed the uniseriate homogeneous rays which are not accompanied by the heterogeneous rays mentioned. The *uni*seriate homogeneous rays of the second kind may be constituted of either erect or procumbent cells. If made up of erect cells they are more or less similar in shape and size to the homogeneous rays of the first kind as depicted in fig. 1: *a* and possibly only occur in semi-shrubs and herbs (BARGHOORN (1)): these rays had not yet been fitted into the key for identification of homogeneous and heterogeneous rays (9). If constituted of procumbent cells (fig. 1: *c*) the rays are similar in shape to such homogeneous rays of the first kind as depicted in fig. 1: *d*. In the second kind of homogeneous rays are also placed the *multi*seriate rays without a uniseriate extension and those where the uniseriate extension is only 1 cell high (fig. 1: b_2, *h*) or/and those where the first or second upper(lower)most row(s) of the multiseriate part contain erect cells too (fig. 4: *f*). In connection with the multiseriate heterogeneous rays mentioned above which are entirely composed of erect cells the rays which are identical with the body of erect cells of these heterogeneous rays are also homogeneous rays of the second kind (*Pipturus*).

Miss CHATTAWAY's question in the *Newsletter* (2) p. 2 where we are to fit the rays of *Pipturus* etc. has been answered now. In the highly specialised woods of some of the *Urticaceae*, the *Loganiaceae* etc. the homogeneous rays of the second kind are constituted of erect cells. This is probably due to a transformation of ray initials to the more fusiform type of cambial cells and as such a step towards complete elimination of rays (cf. BARGHOORN (1)).

Sheath cells and tile cells do not interfere with the classification of rays, but are merely identification features in the description of wood specimens. The rays of some woods, for example those of the *Flacourtiaceae* and *Rubiaceae*, contain single erect cells or rows of such cells between the procumbent cells of the multiseriate part of a heterogeneous ray or of a homogeneous ray of the second kind; these erect cells must not be taken into account in ray classification matters. But of course they too deserve attention as special identification features in woods. The same may be said about pitting and contents of ray cells.

In all rays mentioned one ray cambium is involved. By the superposition of two or more ray cambia vertically fused rays are the result of their activity (fig. 5: *a* and *b*). The constituent rays of the vertically fused ray must be classified separately. The writer has been able to discern vertically fused rays from heterogeneous rays in most cases studied.

HUBER (4) perhaps did not see enough examples of these kinds of rays to arrive at the same conclusion.

It should be well borne in mind that ray structure has to be studied in those places, where the rays are embedded in libriform or fibertracheid

Fig. 5. *a*, *b* (× 60). Vertically fused rays, each formed by two ray cambia. In *a* the cambia somewhat more fused and forming a 2-storied heterogeneous ray and a homogeneous ray. In *b* both the constituent parts are homogeneous.

tissue, as their cells are often different in shape where contiguous to vessels or to parenchyma. Since the rays in the inner secondary xylem differ often in structure from the rays in the outer, wood samples should not be taken too near to the primary xylem. That because of this rays should not show reliable features for wood identification as BARGHOORN (1) expects has not been evidenced (see also CHOWDHURY (3)).

In a previous paper (REINDERS–GOUWENTAK (9)) the ray characteristics enabling a classification into homogeneous and heterogeneous rays have been recorded in key form. However, definitions suiting more the concise form of a glossary are not an impossibility. But with such far from simple structures definitions cannot be short nor are they easily read and for routine purposes the key mentioned will perhaps prove more useful than the ponderous definitions.

§ 4. *Definitions.*

Heterogeneous rays.

I Multiseriate rays composed of 2 (fig. 2), 3 (fig. 1: *g*) or more (fig. 1: *f*) vertically arranged tiers, alternately uniseriate (locally biseriate) and multiseriate.

The uniseriate tiers composed of:

1. erect cells (fig. 1: g_1; fig. 3: *d*; Urticaceae),
2. procumbent cells (fig. 1: *c*),
3. erect and procumbent cells (fig. 1: g_3).

The multiseriate tiers composed of:

1. procumbent cells (fig. 1: *g*, *c*),
2. procumbent cells with 1 or 2 upper- and/or lowermost rows of erect cells,
3. procumbent cells with erect cells or rows of cells scattered between,
4. erect cells only (Urticaceae etc.).

II Uniseriate rays composed of 2, 3 or more alternating tiers of erect and of procumbent cells and not being present also as the uniseriate marginal tier of another ray (fig. 1: i_1, i_2, j).

Homogeneous rays of the first kind: uniseriate or locally biseriate rays, which are identical with the uniseriate (or locally biseriate) margin of another ray.

Composed of:

1. erect cells (fig. 1: *a*; cf. fig. 1: g_1 and fig. 3: *d*),
2. procumbent cells (fig. 1: *d* cf. fig. 1: *c*),
3. erect cells mingled with procumbent cells (fig. 1: *k*, cf. fig. 1: g_3; fig. 3: *c*).

Homogeneous rays of the second kind: uniseriate (locally biseriate) rays not occurring also as marginal tiers of other rays, and multiseriate rays without uniseriate extension(s) or with a uniseriate extension one cell in height.

If uniseriate (locally biseriate), composed of:

1. erect cells (probably only in semishrubs and herbs),
2. procumbent cells (Leguminosae fig. 1: *e*; Quercus),

If multiseriate, composed of:

1. procumbent cells (admitting local presence of sheath cells or of erect cells scattered between) (fig. 1: b_2),
2. procumbent cells with one uniseriate row of erect cells forming the upper or/and lower margin (fig. 1: *h*) (admitting local presence of sheath cells or of erect cells in the body),
3. as 2, but also in the first and/or second upper and/or lower row(s) of the multiseriate part erect cells (fig. 4: *f*),
4. erect cells (admitting local presence of procumbent cells), e.g. Urticaceae.

Summary.

Kribs' proposal to replace the terms heterogeneous and homogeneous in the Glossary with heterocellular and homocellular is sustained with a slight alteration in application.

The terms homogeneous ray and heterogeneous ray are introduced again but now to indicate different structural composition, the heterogeneous ray though formed by one ray cambium, yet, as it were, composed of two homogeneous rays.

Vertically fused rays consist of several rays which, as viewed tangentially, have not become confluent and are each of them formed by a separate ray cambium.

Kribs' homogeneous and heterogeneous types are formed by two or more single homogeneous or homogeneous and heterogeneous rays.

Definitions of homogeneous and heterogeneous rays are composed and recorded into such a form as renders them suitable for a Glossary of woodanatomical terms.

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ON THE INCREASE OF STATURE AND THE AGE OF MAXIMUM HEIGHT

BY

A. DE FROE

(Communicated by Prof. M. W. WOERDEMAN at the meeting of Sept. 30, 1950)

Since the beginning of this century, anthropological research has drawn the attention to the phenomenon that the height of conscripts has been continuously increasing, during the 19th century.

The first statement of this has, as far as I know, been made by G. DE' ROSSI [1]. In the Netherlands the first paper on this matter appeared in 1906, written by G. W. BRUINSMA [2]. Besides in the Netherlands and in Italy, this fact was also registered in Sweden, Finland, Germany, Switzerland and America [3, 4, 5, 6, 7, 8, 9].

In the Netherlands this phenomenon was further examined and confirmed by BRUINSMA [10], BOLK [11, *a*, *b*, *c*], [12, gives a comparison with the jewish population]; ENKLAAR [13], VAN DEN BROEK [14, 15], BENDERS [16, 17] and MIJSBERG [18, 19, 20].

To understand this phenomenon well, it is necessary to distinguish between the height at a specified age and the maximum height finally reached. Until the maximum height is reached, it regularly increases; after this the height decreases through a variety of factors (such as increased curves in the vertebral column, decrease in height of the intervertebral fibrocartilages and of the articular cartilage, dropping of the arches of the feet).

The age at which the maximum height is reached, varies and is not exactly known.

The increase of the height of the conscripts is generally taken as meaning an increase of the maximum height. Consequently we find the terms "saecular increase of height" and the "increase in height of the population".

Pursuing this idea scholars were confronted with almost ineluctable difficulties to explain this. Inter alia BENDERS [16] has tried to find an explanation in the negative selection during the Napoleonic wars, which excluded during a number of years the men above the minimum height from producing off-spring. MIJSBERG [20] has shown the inadequacy of this attempt. Migration is likewise inefficient, seeing that groups of people without migration show the same phenomenon. An increase in height through mutation would absolutely isolate this phenomenon from what experience has taught us about the consequences of mutations up till now.

E. Schneider [21] is of the opinion that a shifting of types in the direction of leptosomic types, under the influence of civilisation and industrialisation, is to be held responsible. The fact that in rural districts the same phenomenon is also to be observed, militates against this theory.

Against an increase of the maximum height a number of grave arguments tell:

1. This increase as such, has never been demonstrated.
2. It appears inexplainable by the means we have.
3. The age at which the maximum height is reached, is variable and unknown.
4. Bruinsma [2] has already drawn the attention to the fact that the circumference of the head and the length of the feet of the conscripts also increase. Nobody thinks of opining that the maximum head — circumference has also increased. No arguments can be advanced for this.
5. It is not only the 19 years old conscripts that have become taller.

It is the same for all ages under 19, even for new-born babies. This suggests that it is not the maximum height which has increased, but that the age at which it is reached, has become earlier. So from this it appears that the 19 years old conscripts of a century ago, are farther removed from their maximum height than those of the present time. With an acceleration of this kind of the process of growth, the earlier occurrence of the first and second dentition and the earlier occurrence by one or two years of the menarche would be in accord. (Bolk [22, 23]; koch [24]; Mills [25]; Pagliani [26]; Röszle [27]; Weber [28].

Bruinsma [10] has indeed thought of this possibility, but has considered it improbable. Mijsberg [18] too makes mention of this possibility, but rejects it, because he is of the opinion that from earlier measurements it was known that at the age of 19 about 99 % of the total length has been reached and because experience has proved that in a family the children often reach a greater height than their parents.

Both arguments require a thorough verification. Mijsberg [18] goes on to cite Hultkrantz, who had established the increase in height from an investigation, made by Kajava of soldiers between 25—44 years. This investigation is up to now the only one of this kind.

When we consider that the man of the palaeolithic age (at least 3800 years ago), was at the most 6 cm shorter than the man of the present age and in some districts quite as tall, then the increase of 6 cm during the last century becomes somewhat puzzling (Schreiner [29].

G. M. Morant [30] is the only one who champions the supposition of an acceleration of the growth on the basis of still unpublished material. According to him the average height of the male population of England has been, during the last hundred years, constantly 1715 mm.

The age at which the maximum height was reached was, according to MORANT:

about 1900.	26—27 years
,, 1916.	21 ,,
now	19—20 ,,

After reaching the maximum height, there is a decrease in height of about 15 mm every 10 years.

The results of an investigation, made by the present writer, in Bunschoten, in the province of Utrecht, in July 1950, give, as I humbly submit, strong arguments in support of the hypothesis of MORANT.

The local authorities placed at my disposal the registers of the conscripts from 1813 till 1907. These registers contained numerous data of 1221 conscripts, i.a. the height. The figures about the several years were as follows:

1813 : 4	1832 : 16	1851 : 11	1870 : 20	1888 : 15
1814 : 5	1833 : 7	1852 : 14	1871 : 20	1889 : 16
1815 : 11	1834 : 11	1853 : 13	1872 : 17	1890 : 22
1816 : 9	1835 : 15	1854 : 9	1873 : 8	1891 : 19
1817 : 10	1836 : 8	1855 : 12	1874 : 17	1892 : 13
1818 : 12	1837 : 8	1856 : 18	1875 : 9	1893 : 23
1819 : 17	1838 : 9	1857 : 25	1876 : 17	1894 : 29
1820 : 20	1839 : 15	1858 : 10	1877 : 19	1895 : 28
1821 : 8	1840 : 6	1859 : 13	1878 : 14	
1822 : 15	1841 : 9	1860 : 15	1879 : 11	1904 : 15
1823 : 6	1842 : 9	1861 : 9	1880 : 17	1905 : 23
1824 : 10	1843 : 18	1862 : 0	1881 : 20	1906 : 24
1825 : 8	1844 : 10	1863 : 15	1882 : 19	1907 : 33
1826 : 5	1845 : 10	1864 : 11	1883 : 27	
1827 : 9	1846 : 10	1865 : 12	1884 : 15	
1828 : 8	1847 : 8	1866 : 20	1885 : 23	
1829 : 9	1848 : 10	1867 : 13	1886 : 16	
1830 : 10	1849 : 14	1868 : 14	1887 : 21	
1831 : 7	1850 : 13	1869 : 18		

The municipality of Bunschoten, known as a town since 1383, had a progress of population as follows:

1820	860 inhabitants	1890	2380 inhabitants
1830	880 ,,	1900	2900 ,,
1840	1050 ,,	1910	3330 ,,
1850	1200 ,,	1920	4060 ,,
1860	1400 ,,	1930	4920 ,,
1870	1670 ,,	1940	5800 ,,
1880	1900 ,,	1950	7600 ,,

From this we see that the relatively greatest increase of population occurs between 1870 and 1880. It is a remarkable fact that during all these years the number of men exceeds the number of women. Besides the town of Bunschoten, the municipality contains the villages of Spakenburg and Eemdijk.

The proportion of the numbers of inhabitants was in:

	1830	1855	1880
Bunschoten	300	400	470
Spakenburg	450	760	1220
Eemdijk	130	150	210

From this it appears that the population of Bunschoten and Eemdijk, which is chiefly agricultural, has increased relatively much less than that of Spakenburg, whose inhabitants are principally fishermen.

Anthropological differences between the population of Bunschoten and Spakenburg could not be established sofar and are not probable either, as both groups either separately or collectively, show a great measure of inbreeding.

The data derived from the archives, which are subservient to our purpose, were as follows:

1. the year of the registering of conscripts.
2. family- and christian names.
3. date of birth (1794—1888)
4. place of birth.
5. trade.
6. Height, measured at the time of registering,
   until 1862 at the age of 19
   after 1862 at the age of 20.
   from 1813—1825 in antiquated units of measurements, which were reduced to modern units of measurement; after this in mm.
7. Height measured at the registering for the "rustende schuttery" (something like Fencibles)
   from 1813—1822 in the year: 1828, consequently at the age of 34—25 years.
   Further regularly at the age of 25.
8. Colour of the hair (without colour standard)
9. Colour of the eyes (without colour standard)
10. Defects registered at the measuring
11. Name and trade of the father, or the fact of the death.
12. Name of the mother and if she is a widow, her trade or the fact of her death.

The special advantages which our material procures, as compared with the population groups, which have been studied up to now, are as follows:

1. The raw material reaches back to persons born in 1794 and measured in 1813.
2. Practically every year is represented.
3. Immigration is slight and could, in sofar as it was there, be excluded by genealogical investigation.
4. Owing to the inbreeding, the immigration of women is negligeable and has, what immigration there is, little influence.
5. Owing to the inbreeding, we have to do with a thoroughly homogeneous population.
6. Strong influences of milieu or changes in milieu between the members of the population are absent.
7. Owing to the second physical examination of the "rustende schuttery" the height at 2 important ages is known.
8. Differentiation according to trades is possible.
9. Differentiation according to family-circumstances is possible (half orphans, orphans, trade of the parents).
10. Differentiation according to bodily health is possible.
11. Differentiation according to abode (Spakenburg, Bunschoten) is possible.
12. A comparison is possible between the present population and their forbears.

We will limit ourselves in this paper to some principal facts:

I. The average height of the conscripts was:

during the periods:	at the first examin. abt. 19 year olds	at the second examin. abt. 25 year olds
1813—1851	163.6 cm	169.9 cm
1851—1876	166.6 ,,	169.5 ,,
1876—1896	168.4 ,,	167.9 ,,
1904—1908	169.3 ,,	—

From this it follows:

1. that the phenomenon of the increase in height of the 19 years old conscripts of the Bunschoten population is practically the same as that of the entire population of the Netherlands.

According to VAN DEN BROEK this increase is with the Dutch population as follows:

periods	at first examination
1863—1867	164.1 cm
1921—1925	170.7 ,,

In both cases evidently an increase of 6 cm:

2. At the age of 25 years the height is practically the same as that of the 19 years old conscripts after 1900.
3. So the maximum height has not increased, but has come earlier!

II. There is another method to analyse this phenomenon further.

The 19 years old conscripts are actually not of exactly the same age, but may differ as much as practically a year, seeing that though they are being measured on the same day, they were born at different dates, which may lay apart as much as a year. To examine the influence of this difference is so important for this reason, that this year will be of greater influence on the growth than that of later years.

Height at the first examination		
of conscripts, born in the month	during the period 1813—1866	during the period 1875—1908
January	165.7	168,6
February	164.4	168.5
March	165.3	168.6
April.	165.3	170.8
May	165.4	167.8
June	163.8	167.2
July	164.5	167.8
August	162.3	168.2
September	160.8	169.0
October	160.4	167.0
November	160.4	167.9
December	161.1	168.9

During the period 1875—1908 the conscripts appear practically not to have grown during the 19th year.

During the period 1813—1866 the growth still amounted to 4 cm odd.

The difference in height between the conscripts of 1813—1908, who were born in the first half of the year and those who were born in the second half of the year, amounted to circa 6 cm.

With the 25 year old conscripts during the period 1813—1896, this difference was nihil.

This proves that at any rate the greater part of the growth between the 19th and 25th year, takes place in the first year.

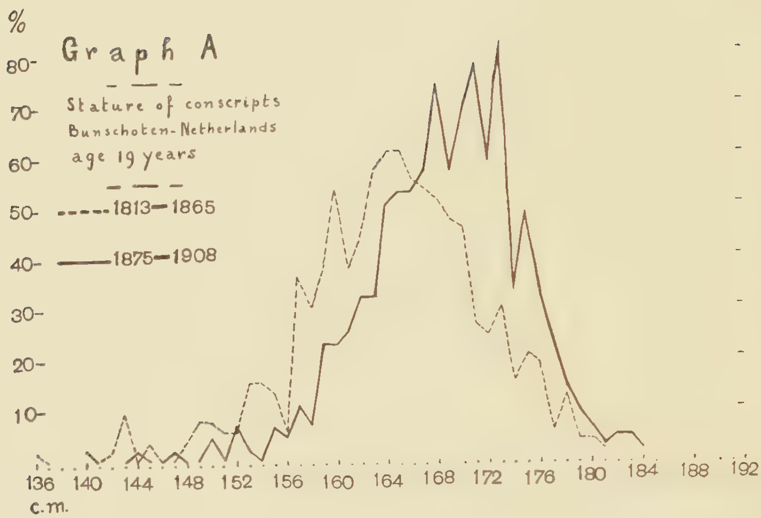
From the graphs *A*, *B* and *C*, too, it appears that the differences in growth

between: 1. the 19 year old conscripts before 1865 and those after 1875
between: 2. the 19 ,, ,, ,, and the 25 year old conscripts
between: 3. the 19 ,, ,, ,, born in the first half of the year and those born in the second half,

show a perfectly analogous process.

An analysis of these graphs with the aid of graphic representation on probability grid, Graph D, proves that:

1. the graphs advance to the higher figures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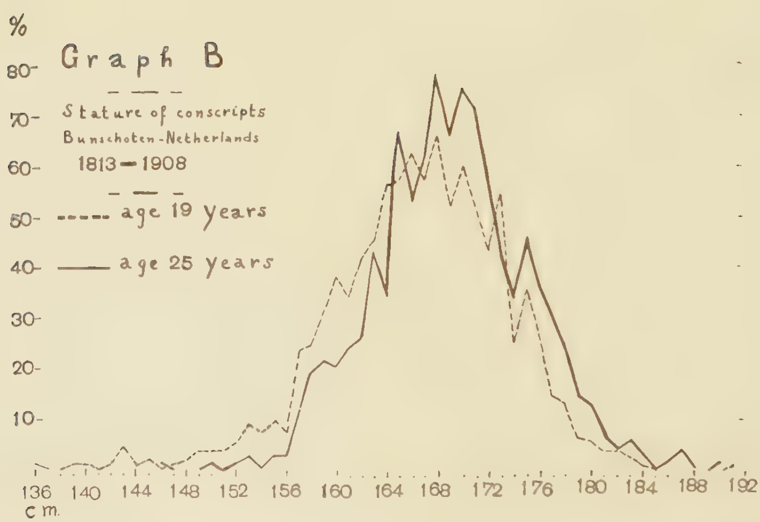
Graph A
Stature of conscripts
Bunschoten-Netherlands
age 19 years
1813—1865
1875—1908
80-
70-
60-
50-
40-
30-
20-
10-
136
140
144
148
152
156
160
164
168
172
176
180
184
188
192
c.m.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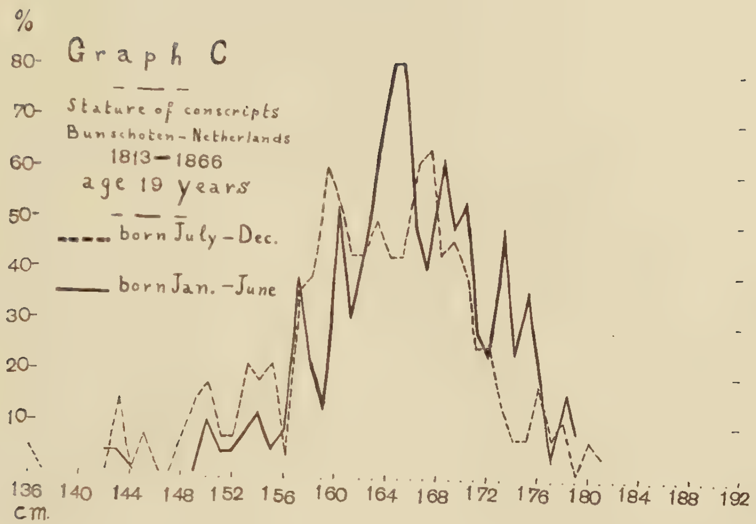
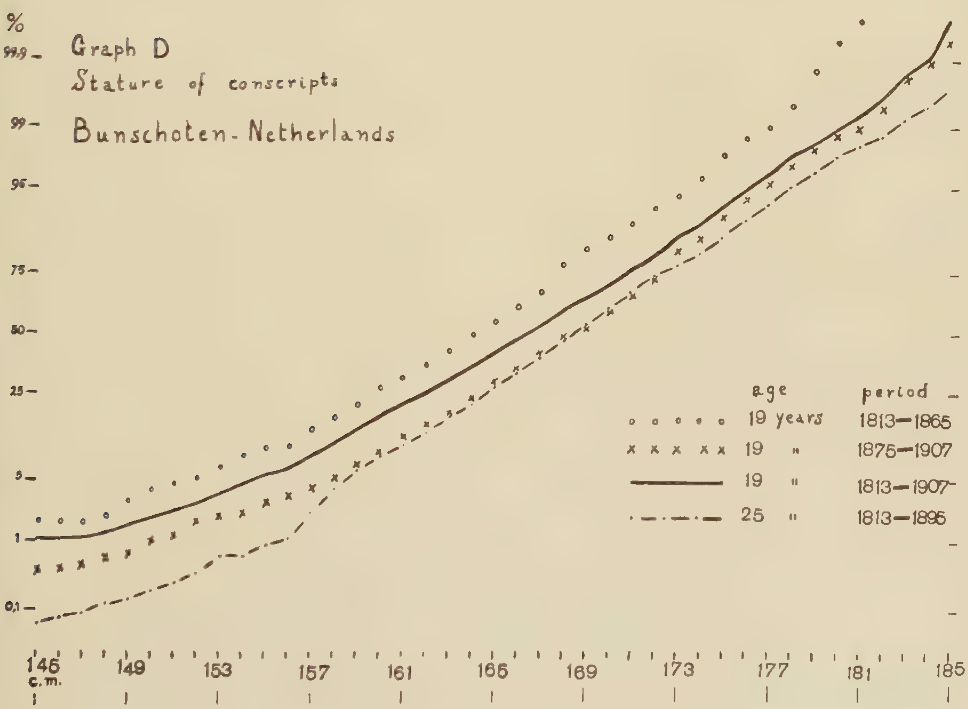
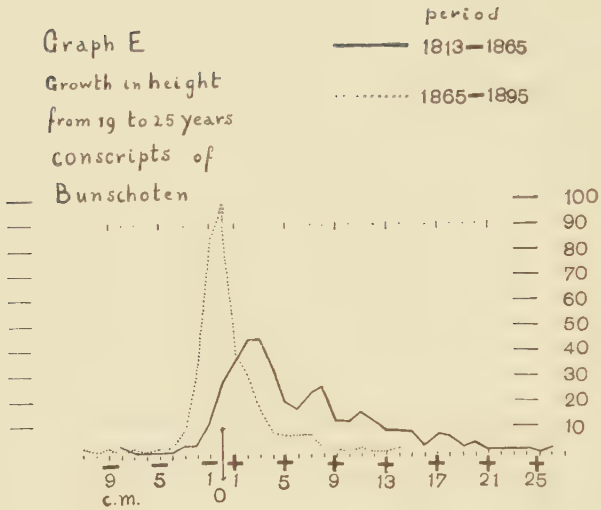
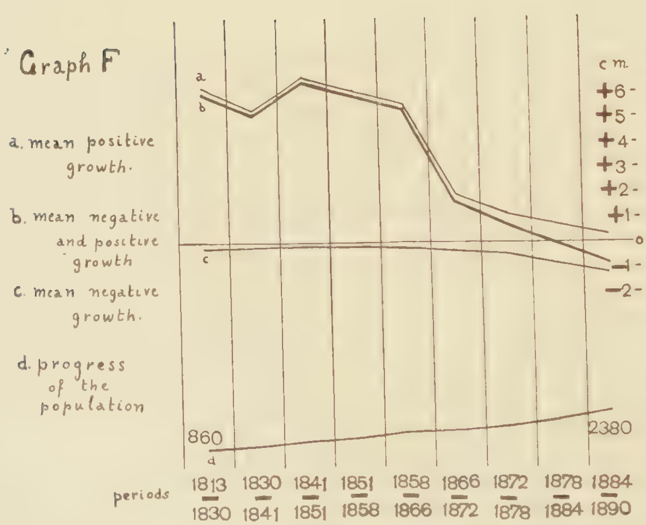
Graph B
Stature of conscripts
Bunschoten-Netherlands
1813—1908
age 19 years
age 25 years
80-
70-
60-
50-
40-
30-
20-
10-
136
140
144
148
152
156
160
164
168
172
176
180
184
188
192
c m.


%
Graph C
Stature of conscripts
Bunschoten - Netherlands
1813—1866
age 19 years
born July – Dec.
born Jan. – June
80
70
60
50
40
30
20
10
136
140
144
148
152
156
160
164
168
172
176
180
184
188
192
cm.


%
Graph D
Stature of conscripts
Bunschoten - Netherlands
99.9
99
95
75
50
25
5
1
0,1
age
period
19 years
1813—1865
19 "
1875—1907
19 "
1813—1907
25 "
1813—1895
145
c.m.
149
153
157
161
165
169
173
177
181
185


Graph E
Growth in height
from 19 to 25 years
conscripts of
Bunschoten
period
1813—1865
1865—1895
100
90
80
70
60
50
40
30
20
10
— 9
— 5
— 1
0
+ 1
+ 5
+ 9
+ 13
+ 17
+ 21
+ 25
c.m.


Graph F
a. mean positive growth.
b. mean negative and positive growth
c. mean negative growth.
d. progress of the population
c.m.
+6
+5
+4
+3
+2
+1
0
—1
—2
860
2380
periods
1813—1830
1830—1841
1841—1851
1851—1858
1858—1866
1866—1872
1872—1878
1878—1884
1884—1890

2. the bases of the graphs grow narrower especially to the left, a phenomenon which MIJSBERG already pointed out.

ad. 1. The advance to the right is also visible in the group-percentages:

	19 year olds	25 year olds
less than 150 cm	1,7 %	0,1 %
150—160 cm	12.6 %	6.3 %
160—164 „	15.9 %	11.2 %
164—167 „	17.7 %	15.4 %
167—170 „	17.5 %	20.5 %
170—180 „	32.9 %	42.9 %
more than 180 cm	1.7 %	3.6 %

ad. 2. The narrowing of the basis is also visible in the limit values.

	19 year olds	25 year olds
minimum value	136	146
maximum value	191	190

When we consider that the 19 year old conscripts of before 1865 were not full-grown and therefore formed a group of men who were not of the same biological age, then the great variability of height is comprehensible.

III. The difference in growth of the 19 year old conscripts of the periods 1813—1865 and 1865—1896 also appears from graph *E*. It stands to reason that this growth can not be assessed at one pCt, as it will be variable.

During the period 1865—1896, however, the one percent is seldom exceeded. In the period preceding it, this excess is, however, the rule and even in a high degree. So in former days it was easy to become a head taller after the 19th year.

When we divide the group of the 19 year olds into 3 sub-groups of the same number, as short, medium and tall, it appears that the group of the short ones, after the 19th year, still grows 9.6 cm, the medium ones 4.9 cm and the tall ones 3,6 cm. So it is especially the short ones who show a later increase. This phenomenon of earlier days is still remembered by some older people. And it is quite probable that the anual second test was founded on it.

IV. That the negative growth between the 19th and 25th year is not insignificant either, likewise appears from the graph *E*.

So this negative growth does not begin before the maximum height has been reached and is therefore before 1865 of small importance.

After 1865 the negative growth is not less significant than the positive growth, both numerically and in degree of strength. After 1875 it is therefore certainly necessary to reckon with it.

V. On the time at which the acceleration of growth sets in, graph F enlightens us.

Already after 1845 a gradual acceleration sets in, but the most important acceleration we see in the years 1865—1875.

Is there any connection with the fact that in 1874 VAN HOUTEN's "Act for the Protection of Children against Excessive Labour" was passed?

Also after 1875 we see a further gradual acceleration of growth.

In this connection we must make mention of the fact that the 19 year olds, who have lost one parent, are on an average 1 cm shorter and those who have lost both parents on an average 2 cm shorter than the 19 year olds of whom both parents were still alive. This adds a fresh argument to those who plead for causal significance of bodily well being.

VI. Besides causes, the accelerated growth also has its consequences. We know that when the growing process stagnates for a long time, a decreased maximum height can be the consequence. We know this from cases of extreme starvation. Presumably, however, every degree of retardation of the growth will have a detrimental effect on the level of the maximum height and contrariwise every degree of acceleration of growth will have a favorable effect on the maximum height.

Consequently we can expect as a consequence of the acceleration of the growth in height, two things:

1. the consequences of the earlier setting in of negative growth;
2. the shifting of the maximum height in the direction of the potential height.

Seeing that we do not know the level of the potential height, we cannot know to what extend the shifting of the maximum height shall go on.

The male full-grown, present population of Bunschoten, has an average height of 172.9, with a maximum of 192.4 cm and a minimum of 158.8 cm.

This is a shifting of ca. 3 cm in 43 years. On the basis of the graphs we cannot expect the potential height to be much higher than 175 cm.

By making use of the rich material which the archives and the living people offer, we may succeed in getting a more clear-cut picture of the phenomenon. For the rest we shall have to abide what the future will teach us.

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THE VALUE OF OBSERVATIONS OF CONTINUOUS SPECTRA IN THE CHROMOSPHERE AND PROMINENCES

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Summary

The intensity of the continuous spectrum for various wavelengths to be expected in the chromosphere and in prominences is computed for electron temperatures of 5000°, 10,000°, 20,000° and 30,000° and electron or proton concentrations of the order 10^{11} (chromosphere) and 10^{10} (prominences). The result contained in Figures 2 to 5 and Table 2 may serve as a guide for future eclipse observations. It is pointed out that present evidence points to a chromospheric electron temperature a good deal lower than REDMAN's kinetic temperature of about 30,000° for atoms and ions. The spectra considered are the Ba_c spectrum, the other recombination and electron switch spectra indicated by C and that by scattering of photospheric light by free electrons marked c. For the chromosphere the determination of electron or proton concentration as well as of electron temperature appears possible. For prominences one can determine concentration and thickness, provided the electron temperature be known. A remark concerning the possible influence of the continuous spectrum due to the formation of negative hydrogen ions is made.

1. *Introduction* *)

Continuous spectra in the chromosphere and in prominences have been treated by various investigators. DAVIDSON, MINNAERT, ORNSTEIN and STRATTON [1]) carried out approximate measurements of relative intensity for slit spectra taken by DAVIDSON and STRATTON at the 1926 eclipse. The continuous spectrum at the head of the BALMER series indicated electron temperature of 3600° for the lower chromosphere, 4000° for a prominence in the flash spectrum and 3200° for a prominence in the coronal spectrum. For the high prominence, the relative distribution in the region λ 4100 to 3800 was equivalent to a black body temperature

*) The considerations set forth in the present paper were presented in an abbreviated form at the Astronomical Colloquium "Problems on Solar Physics" arranged in Dublin by Dunsink Observatory, September 21, 1950.

[1]) C. R. DAVIDSON, M. MINNAERT, L. S. ORNSTEIN and J. F. M. STRATTON, M.N. 87, 536 (1928).

of about 2000°. CILLIÉ and MENZEL [2]), discussing slitless spectrograms taken by CHAPPEL and MENZEL at the 1932 eclipse, found a chromospheric electron temperature of about 10,000° from the relative distribution of the Ba_c spectrum. The absolute intensity of the Ba_c spectrum yielded an electron or proton concentration of 3.8×10^{11} per cm^3 at the base of the chromosphere, likewise from KRAMERS' theory. This agrees, as regards order of magnitude, with PANNEKOEK's [3]) value of 6×10^{11} from the overlapping of the higher lines of the BALMER series due to STARK effect [3]). For prominences observed with the coronagraph LYOT [4]) found for the continuous spectrum in the region λ 5950 to 6400 a degree of polarisation of about 15 per cent, after correction for atmospheric scattering, and concludes that it is mainly produced by scattering of photospheric light by the free electrons in the prominence. On the basis of this assumption, WURM [5]) has recently shown that the ratio of the intensity of the Ba_c spectrum to the continuous spectrum at longer wavelength gives a method for determining the proton or electron concentration per cm^3 for a prominence, if the electron temperature be known, and carried out an approximate estimate on this basis.

The foregoing clearly shows the importance of observations of intensities of the continuous spectrum at various wavelengths in the chromosphere and prominences, preferably from spectra taken during an eclipse.

Thus far the Ba_c spectrum and the scattering from free electrons, which will be indicated by the suffix c, has received most of the attention. For a complete discussion however, also the other recombination spectra due to captures on levels higher than the second plus the electron switch spectra must be considered and will in total be indicated by a capital C. The intensity of this C spectrum, like that of the Ba_c spectrum, follows from the KRAMERS theory. In the following the intensities of these three kinds of spectra to be expected theoretically will be worked out for a few electron temperatures and concentrations, to serve as a future guide for eclipse observations. Also a few provisional conclusions of a very approximate nature will be drawn. The results of the KRAMERS theory, which are given, might be slightly modified by the quantum mechanics, which can be done eventually by the use of GAUNT factors. Another source of continuous spectra might be furnished by captures of free electrons by neutral hydrogen atoms and corresponding electron switches. This will briefly be considered at the end of the paper, but is not thought to be of much importance for the matter on hand.

2. *Intensities of the various continuous spectra to be expected*

In the following it will generally be assumed that most of the electrons

[2]) G. G. CILLIÉ and D. H. MENZEL, Harvard Circular, no. 410 (1935).

[3]) A. PANNEKOEK, M.N. 98, 694 (1938).

[4]) B. LYOT, Comptes Rendus 202, 392 (1936).

[5]) K. WURM, Mitt. Hamb. Sternw. Bergedorf 21, 103 (1948), no. 206.

originate from hydrogen atoms and only very few form negative hydrogen, so that the concentration per cm³ of electrons n_ε becomes equal to that of protons n_1. Moreover the electron temperature T_ε is considered not to vary throughout the chromosphere or the prominence.

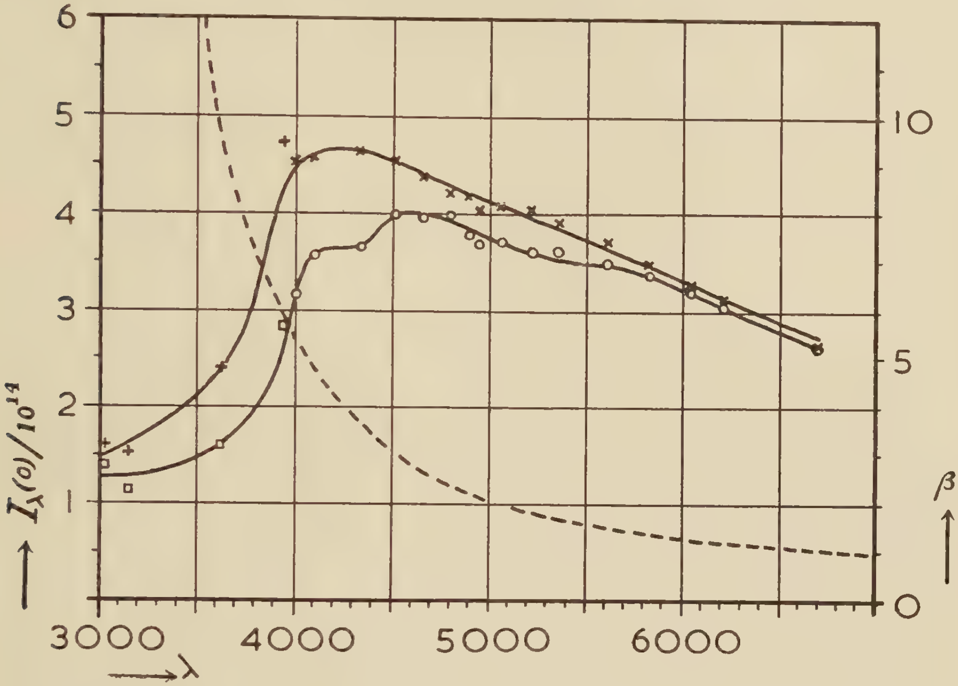

Fig. 1. Values of the intensity at the centre of the sun's disc according to MULDERS. Upper drawn curve: ${}_0I_\lambda(0)$ for the continuous background. Lower drawn curve: ${}_0I_\lambda(0)$ for the spectrum with the FRAUNHOFER lines fuzzed out. Crosses and circles refer to H. H. PLASKETT's observations, vertical crosses and squares to those of FABRY. The dotted curve and righthand scale give the darkening coefficient.

Let $\varepsilon_\lambda\, d\lambda$ be the energy of one of the continuous spectra emitted in a certain direction per unit solid angle per second per cm³, and $E_\lambda\, d\lambda$ the same quantity per cm², or intensity per cm² which is observed or predicted. Then

$$E_\lambda = \int \varepsilon_\lambda\, dl \tag{1}$$

where dl is the element of length along the line of sight.

According to KRAMERS' theory as worked out by CILLIÉ [6]), the emission per cm³ per second per unit frequency for hydrogen is given by

$$J_n(\nu)\, d\nu = n_1 n_\varepsilon \frac{Q}{n^3 T_\varepsilon^{3/2}} e^{-\frac{h\nu - \chi_n}{kT_\varepsilon}} d\nu, \tag{2}$$

$$J'_n(\nu)\, d\nu = n_1 n_\varepsilon \frac{Q'}{T_\varepsilon^{1/2}} e^{-\frac{h\nu}{kT_\varepsilon}} d\nu, \tag{2a}$$

where J_n refers to captures on the level n and J' to electron switches.

[6]) G. G. CILLIÉ, M.N. 92, 820 (1932) and 96, 771 (1936).

The constants in the present paper are based on BIRGE's 1941 values [7]), which give

(2b) $Q = (2.170 \pm 0.005) \times 10^{-32}$, $Q' = (6.88 \pm 0.01) \times 10^{-38}$.

It may be remarked that for objects outside the sun's limb only spontaneous captures should be considered. Forced captures, being negative absorptions, only come into play when the object is observed projected on the disc, strengthening the incident beam without altering its direction.

To obtain ε_λ from J_n or J' one has to multiply by $c/(4\pi\lambda^2)$. Equations (2) and (2a) readily yield the intensity E_{Ba_c} of the Ba_c spectrum at the series limit and of the other spectra added together $E_{\lambda C}$ and their ratio

(3) $$\frac{E_{\lambda C}}{E_{Ba_c}} = \frac{\varepsilon_{\lambda C}}{\varepsilon_{Ba_c}}$$

since T_ε was considered constant. These values for four electron temperatures are plotted in Figures 2, 3, 4 and 5 using the result of BARBIER's [8]) computations, slightly extended, in the full-drawn curves marked with a capital C. Likewise the ratio $E_{\lambda Ba_c}/E_{Ba_c}$ of the intensity of the Ba_c spectrum at the wavelength λ to its value at the series head is represented by the full-drawn curve marked Ba_c.

For $n = 2$, equation (2) yields at the series limit λ 3646.9

(4) $$\varepsilon_{Ba_c} \equiv \varepsilon_{3646\ Ba_c} = (4.86 \pm 0.01) \times 10^{-21}\, n_1 n_\varepsilon \left(\frac{10^4}{T_\varepsilon}\right)^{3/2}.$$

For scattering of photospheric light by free electrons just above the photosphere, the c spectrum, one has

(5) $$\varepsilon_{\lambda c} = I_\lambda(0)\, \frac{\frac{1}{2} + \frac{15}{64}\beta}{1+\beta}\, \sigma\, n_\varepsilon$$

where σ the scattering coefficient per electron

(5a) $$\sigma = (6.67 \pm 0.01) \times 10^{-25}.$$

In this formula $I_\lambda(0)$ represents the intensity at the centre of the solar disc and β the coefficient of darkening for the wave length λ. It may be obtained from equations (3), (4) and (5) of a former paper by the writer [9]) for scattering by an oscillator, which applies equally well to the present case.

The values of $I_\lambda(0)$ were taken from MULDERS, [10]) essentially using a re-plot of his Fig. 2, from his Tables VI and VII. It is represented in our

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[7]) R. T. BIRGE, Reports on Progress in Physics of the Physical Society 8, 90 (1941).

[8]) D. BARBIER, Ann. d'Astrophys. 7, 80 (1944). His J' for 30,000° (p. 104) should be 0.0255.

[9]) H. ZANSTRA, M.N. 101, 250 (1941).

[10]) G. F. H. MULDERS, Aequivalente breedten van FRAUNHOFER-lijnen in het zonnespectrum. Proefschrift Utrecht (1934).

Fig. 1. The upper curve represents ${}_0I_\lambda(0)$, the intensity of the continuous background, which he has obtained by correcting for absorption lines, the lower curve his I_{red} representing the spectrum in which all lines are fuzzed out except the very strongest of ROWLAND intensity larger than 20. We may take I_{red} equal to our $I_\lambda(0)$, since scattering by free electrons will result in a fuzzing out on account of the small mass and consequent high termal velocity. The values of β were obtained from a re-plot of MINNAERT's [11]) values of $I_\lambda(0)/F_\lambda$ based on ABBOT's observations using the expression

$$\frac{F_\lambda}{I_\lambda(0)} = \frac{1+\frac{2}{3}\beta}{1+\beta}$$

as a definition of β, so that this β gives the correct flux. The values of β are given by the dotted curve in Fig. 1, using the right hand scale, or, more accurately by Table 1.

TABLE I

λ 3500	3646	4000	4500	5000	5500	6000	6500	7000
β 14.1	9.6	5.36	3.05	2.05	1.56	1.26	1.06	0.93

From equations (1), (4), (5) and (5*a*) follows the expression for the ratio of intensity per cm² of the scattering spectrum c at wave length λ to the Ba_c spectrum at λ 3646

$$\text{(6)} \qquad \frac{E_{\lambda c}}{E_{Ba_c}} = (0.684 \pm 0.001)\,\frac{1+\dfrac{15}{32}\beta}{1+\beta}\cdot\frac{I_\lambda(0)}{10^{14}}\cdot\left(\frac{T_\varepsilon}{10^4}\right)^{3/2}\cdot\frac{10^{10}}{\bar{n}_1},$$

where

$$\text{(6a)} \qquad \bar{n}_1 = \frac{\int n_1 n_\varepsilon \, dl}{\int n_\varepsilon \, dl}.$$

The quantity $\bar{n}_1$ may be termed the effective concentration of protons for the ratio of c to Ba_c intensity. Equation (6) enables to determine $\bar{n}_1/T_\varepsilon^{3/2}$ from the observed intensity ratio $E_{\lambda c}/E_{Ba_c}$, which is WURM's method. It is however somewhat more refined than the equation given by WURM, since the dependence of scattering upon direction is taken into account and central intensity with darkening coefficient has been used in preference to the black body approximation.

If, as an approximation, the object is assumed to have a thickness L within which the concentration n_1 or n_ε is constant, (1) and (6*a*) reduce to

$$E_\lambda = \varepsilon_\lambda L, \quad \text{(1a)} \qquad \bar{n}_1 = n_1 = n_\varepsilon. \quad \text{(6b)}$$

One might assume this for prominences.

For the chromosphere however one may, following CILLIÉ and MENZEL, assume a concentration n_1 or n_ε falling off with height x as $e^{-\frac{1}{2}a'x}$, where x is very much smaller than the solar radius. Then (6*a*) leads to

$$\text{(7)} \qquad \bar{n}_1 = \frac{n_1(x)}{\sqrt{2}} = \frac{n_\varepsilon(x)}{\sqrt{2}},$$

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[11]) M. MINNAERT, B.A.N. 2, 75 (1942), no. 51, Table IV and Fig. 3.

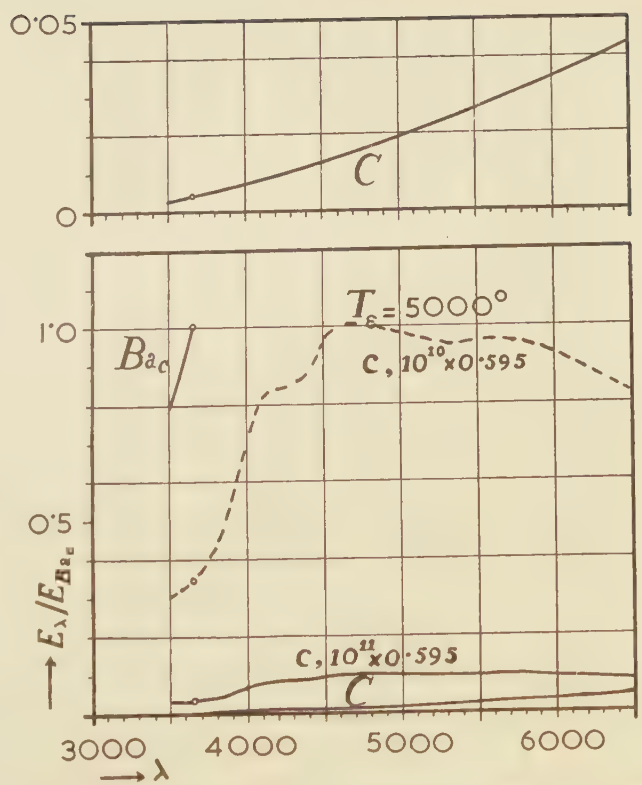

Fig. 2. Theoretical intensities of continuous spectra, expressed in the Ba_c intensity a the head as a unit: C for all recombination and electron switch spectra except Ba_c; c for scattering of photospheric light by free electrons; followed by the effective concentration of protons or electrons. The full drawn c curve is typical for the chromosphere at 1300 km, and the dotted curve for a prominence. Electron temperature 5000°.

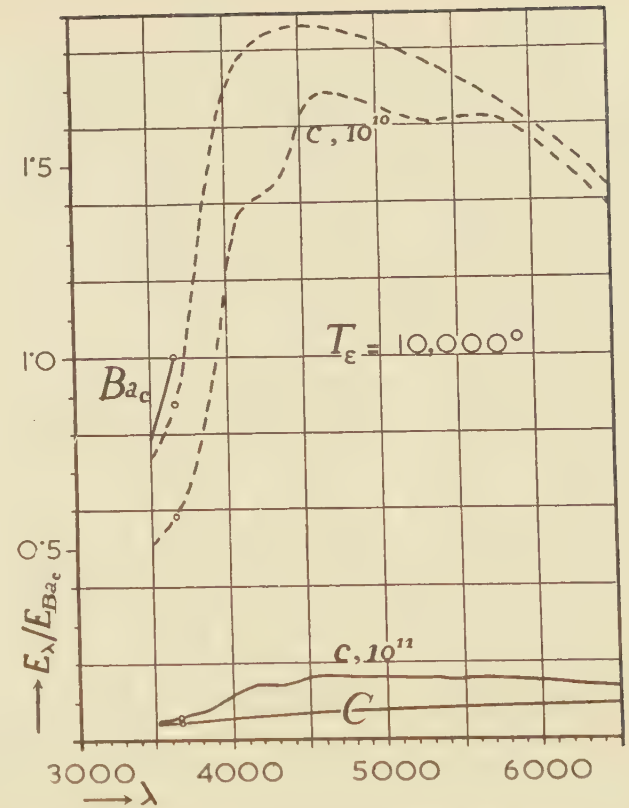

Fig. 3. The same as Fig. 2, but for electron temperature 10.000°. The upper dotted curve refers to scattering without Fraunhofer lines. N.B. *The value of* $E_{\lambda c}/E_{Ba_c}$ *for any* $\bar{n}_1$, *and any* T_ε, *is obtained by multiplying the reading of the lower dotted curve by* $\frac{T_\varepsilon}{10^4} \cdot \frac{10^{10}}{\bar{n}_1}$,

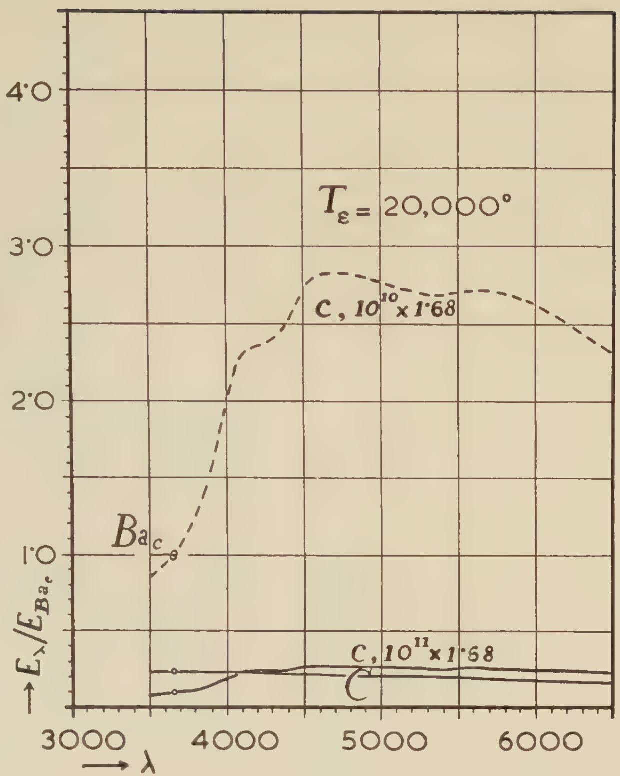

Fig. 4. The same as Fig. 2, but for electron temperature 20,000°.

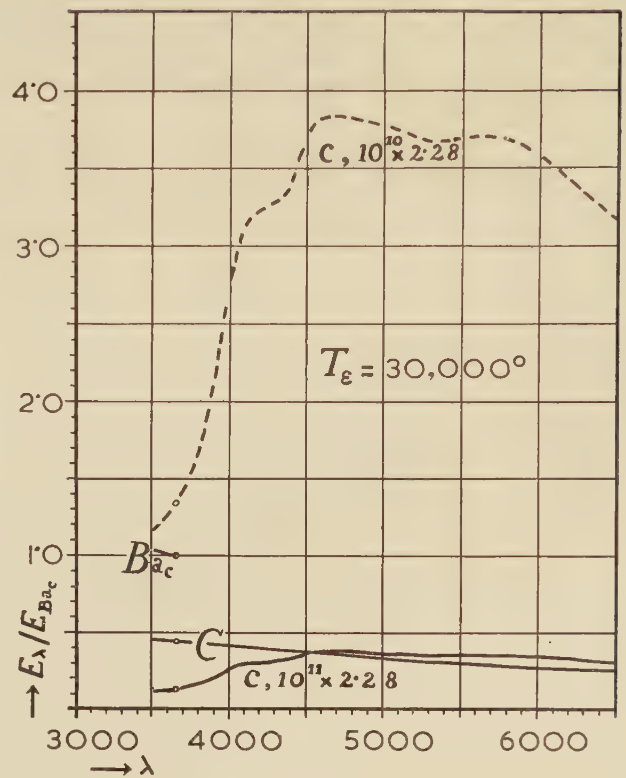

Fig. 5. The same as Fig. 2, but for electron temperature 30,000°.

if x represents the height at which the line of sight approaches closest to the limb.

From their observed absolute values of the intensity of the Ba_c spectrum and KRAMERS' theory then follows

$$(8) \qquad E_{Ba_c} = 3.62 \times 10^{12}\, e^{-a'x}, \qquad n_\varepsilon(x) = n_1(x) = 3.8 \times 10^{11}\, e^{-\frac{1}{2}a'x} \left(\frac{T_\varepsilon}{10^4}\right)^{3/4};$$

C.&M.

where $a' = 1.54 \times 10^{-3}$ km^{-1}. For $x = 1300$ km one gets

$$\bar{n}_1 = n_1(x)/\sqrt{2} = 10^{11} \left(\frac{T_\varepsilon}{10^4}\right)^3.$$

In Figures 2, 3, 4 and 5 the values of $E_{\lambda c}/E_{Ba_c}$ from (6), using $I_\lambda(0)$ and β from Fig. 1, are plotted against λ for four different electron temperatures and these values of $\bar{n}_1$ corresponding to a chromospheric height of 1300 km. The curves are full-drawn and marked with a C, followed by this value of $\bar{n}_1$. They are meant as a guide for future eclipse observations to estimate the intensity of the scattering spectrum c to be expected. Its estimated absolute value follows by multiplication by CILLIÉ and MENZEL's value of E_{Ba_c} given in (8) and can be compared with the continuous background ${}_0I_\lambda(0)$ for the centre of the sun by means of Fig. 1. Once the new eclipse observations are carried out, the values (8) for E_{Ba_c} and $n_1(x)$ viz $\bar{n}_1(x)$ should of course be replaced by new values following from these observations. The value $E_{\lambda C}/E_{Ba_c}$ for the KRAMERS spectrum C is given by the curves marked C and is determined by the electron temperature only, independent of the concentration.

3. *Provisional Conclusions for the Chromosphere*

For wavelengths longer than the series limit λ 3646.9 one should take the sum $E_{\lambda C} + E_{\lambda c}$, and for shorter wavelengths the sum $E_{\lambda Ba_c} + E_{\lambda C} + E_{\lambda c}$. This is done in Table II for the wavelengths λ 5000, λ 3700 and λ 3647 just adjoining Ba_c, as compared with λ 3646 Ba_c. The values in parentheses are the same with the c spectrum left out.

TABLE II

Chromosphere = 1300 km. Values of E_λ/E_{3646} expected for three λ's.

The values in parentheses are those for $E_{\lambda c} \to 0$ or $\bar{n}_1 \to \infty$ and should provide lower limits for any concentration or any height

T_ε	5000°	10,000°	20,000°	30,000°
$\dfrac{E_{5000}}{E_{3646}}$	0.113 (0.0191)	0.219 (0.073)	0.33 (0.170)	0.45 (0.236)
$\dfrac{E_{3700}}{E_{3646}}$	0.040 (0.0046)	0.104 (0.0049)	0.254 (0.187)	0.374 (0.308)
$\dfrac{E_{3647}}{E_{3646}}$	0.038 (0.0042)	0.098 (0.0047)	0.248 (0.187)	0.369 (0.312)

Examining the reproduction of the chromospheric slit spectrum taken

at the 1926 eclipse by DAVIDSON and STRATTON [12]), one cannot escape the impression that at λ 3700 the continuous spectrum, which hardly registers there, is extremely weak as compared with λ 3646, and that the ratio falls considerably below the value 0.37 or 0.31 of the Table required for $T_\varepsilon = 30{,}000°$. This would mean that the electron temperature of the chromosphere falls a good deal below the kinetic temperature of about 30,000° for atoms and ions, which REDMAN [13]) derived from line widths at the 1940 eclipse. Theorists [14]) thus far have mostly been assuming that the two kinetic temperatures of electrons and atoms or ions have the same value of about 30,000°, and for this reason chromospheric observations actually determining the electron temperature would be of great theoretical importance. A much lower electron temperature than 30.000° might perhaps be explained by the electrons losing their thermal energies by exciting collisions, as is the case in nebulae.

For future eclipse observations the best procedure would probably be to determine n_1 by PANNEKOEK's method from the STARK broadening of the higher BALMER lines, preferably by measuring line profiles corrected for instrumental profile, thermal velocities and fine structure. This gives a kind of average n_1, which could serve for $\bar{n}_1$. Assuming STARK effect due to protons only with $\bar{n}_1 = 10^{11}$, one expects for quantum number $n = 25$ a STARK width of about one quarter of the kinetic theory width at 30,000°, while the separation between subsequent lines is four times this kinetic theory width. For $n = 30$, the figures are 1/3 and 2.4. So it appears possible to measure the STARK effect in the wings, since its intensity falls off less rapidly with the distance from the line centre than for the kinetic theory profile.

The ratio E_λ/E_{3646} should further be measured, which for a given λ is a known function of $\bar{n}_1$ and T_ε, and $\bar{n}_1$ being known, T_ε would be determined. [15]) Independent determinations for various λ might provide a check. Provided the observed continuum between lines is strong enough in the ultraviolet, one might find it possible to extrapolate from longer wave lengths to λ 3647 (see the last row in Table II) which would eliminate the effect of differential absorption in the ultraviolet.

If the STARK effect method does not give satisfactory results, it might be replaced or checked by the absolute determination of E_{Ba_c} which yields $n_1/T_\varepsilon^{3/4}$ according to KRAMERS' theory (Cf. equation (8)). However

[12]) C. R. DAVIDSON and F. J. M. STRATTON, Memoirs of the R.A.S. **64**, 105 (1927), Plate 1, DAVIDSON, MINNAERT, ORNSTEIN and STRATTON, loc. cit., Plate 7.

[13]) R. O. REDMAN, M.N. **102**, 140 (1942).

[14]) Cf. R. N. THOMAS, Ap. J. **108**, 142 (1948); R. G. GIOVANELLI, M.N. **109**, 298 (1949).

[15]) $E_{\lambda C}/E_{Ba_c}$ (3), (2), (2a) is known as $f(T_\varepsilon)$.

$E_{\lambda c}/E_{Ba_c}$ (6) is a known constant times $T_\varepsilon^{3/2}/\bar{n}_1$. The extreme case $\bar{n}_1 = \infty$, $E_{\lambda c} = 0$ gives therefore T_ε directly. The other extreme $E_{\lambda C} = 0$ gives $T_\varepsilon^{3/2}/\bar{n}_1$ or T_ε for known $\bar{n}_1$. There should be no difficulty for the general case.

if the chromosphere has a filamentary structure, this method of CILLIÉ and MENZEL should give low values, since it would give a kind of average concentration, while PANNEKOEK's and WURM's methods would refer to the concentration in the filaments.

One might also think of determining T_ε from the relative distribution in wave length in the Ba_c spectrum from KRAMERS' Theory with GAUNT factors, but the results thus far obtained for the chromosphere and the experience of the same method with nebulae seems to show that this method is not very reliable.

It would be very desirable to have the investigation of the continuous spectra leading to an electron temperature carried out at the same eclipse where the kinetic temperature from line widths is determined, if possible at the same spot in the chromosphere, so as to make sure that the two temperatures are determined under the same circumstances.

4. *Prominences*

Individual concentrations in prominences may vary a great deal from case to case. The average derived by UNSÖLD [16]) is $n_\varepsilon = n_1 = 8 \times 10^9$. Let us assume, merely for the purpose of illustration, a concentration n_1 in prominences of one tenth of the n_1 values we took as typical for chromosphere. Then, by the procedure outlined in section 2, the dotted curves of Figures 2, 3, 4 and 5 are obtained for the ratio E_λ/E_{Ba_c} in prominences, which are marked by c, followed by the value of n_1. For the other ratios $E_{\lambda C}/E_{Ba_c}$ and $E_{\lambda Ba_c}/E_{Ba_c}$ the full-drawn curves remain valid. Figure 3 applies to $T_\varepsilon = 10{,}000°$, $n_1 = 10^{10}$ protons or electrons per cm^3 and contains also the curve for scattering by free electrons if the solar spectrum would be free from FRAUNHOFER lines, that is if $I_\lambda(0)$ of Fig. 1 would be replaced by ${}_0I_\lambda(0)$.

Figures 2 to 5 show that probably always E_{λ} is much larger than $E_{\lambda C}$, so that scattering by free electrons predominates for wavelengths longer than the series limit, in agreement with LYOT's observations of polarisation. One may observe therefore $E_{\lambda c}/E_{Ba_c}$ and then equation (6) yields $\bar{n}_1/T_\varepsilon^{3/2}$, [17]) which is WURM's method. However, unless the electron temperature is high, or the concentration a good deal higher than assumed, the C spectrum is relatively weak and a reliable determination of T_ε is not possible on this basis.

Substituting the measured $\bar{n}_1/T_\varepsilon^{3/2}$ or $n_\varepsilon/T_\varepsilon^{3/2}$ (6b) into (5), one obtains $\varepsilon_{\lambda c}/T_\varepsilon^{3/2}$, and then (1$a$) $E_\lambda = (\varepsilon_\lambda/T_\varepsilon^{3/2})\, L T_\varepsilon^{3/2}$ yields $LT_\varepsilon^{3/2}$. In other words, if the electron temperature be known, one can determine both the concentrations n_1 or n_ε and the thickness L of the prominence.

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[16]) A. UNSÖLD, Physik der Sternatmospharen, p. 415, 419 (SPRINGER 1938).

[17]) Dividing the observed $E_{\lambda c}/E_{Ba_c}$ by its value for 10,000° read from the lower dotted curve of Fig. 3, one obtains directly $\left(\frac{T_\varepsilon}{10^4}\right)^{3/2} \cdot \frac{10^{10}}{\bar{n}_1}$, instead of using (6$b$).

5. *Remark on the continuous spectrum due to the formation of H^-*

CHANDRASEKHAR and Mrs. BREEN [18]) have computed the absorption coefficient of negative hydrogen a_p expressed per neutral hydrogen atom and per unit of electron temperature for various temperatures and wave-lenghts. For temperatures up to about 10,000° and wave-lengths near λ 3646, forced transitions are negligible, as may be seen by comparing their Tables 5 and 6. For a given electron temperature T_ε, electron pressure p_ε and concentration of neutral hydrogen n_0 the number of recombinations and electron switches for a certain velocity interval, and therefore the emission in a certain wave-length interval, is then the same as for thermal equilibrium at a temperature equal to T_ε. Or also: the emission per unit wave-length and unit solid angle per cm³ per second $\varepsilon_{\lambda H^-}$ is the same as the absorption for $T = T_\varepsilon$, and therefore given by

$$\varepsilon_{\lambda H^-} = B(\lambda,\ T_\varepsilon) \cdot a_p(\lambda,\ T_\varepsilon) \cdot k\,T_\varepsilon\, n_\varepsilon\, n_0 \tag{9}$$

where $B(\lambda, T_\varepsilon)$ is the intensity of black body radiation, and $p_\varepsilon = n_\varepsilon\, kT_\varepsilon$ has been substituted.

From their Table 5 for a_p, equation (9) and KRAMERS' theory, viz. equation (4), we obtain, for λ 3647, by a slight linear extrapolation

$$\left\{\begin{array}{ll} T_\varepsilon = 5040^0, & \dfrac{\varepsilon_{3647H^-}}{\varepsilon_{Ba_c}} = 2.2\times 10^{-4}\,\dfrac{n_0}{n_1}, \\ T_\varepsilon = 10{,}080^0, & \dfrac{\varepsilon_{3647H^-}}{\varepsilon_{Ba_c}} = 5.3\times 10^{-3}\,\dfrac{n_0}{n_1}, \end{array}\right. \tag{10}$$

as a general theoretical result, holding therefore for the chromosphere as well as for prominences.

An actual determination of n_0 has not been devised as yet. However, WURM, [19]) by comparing the ionisation of Sr^+ into Sr^{++} with the ionisation of H into H^+, and assuming a plausible abundance ratio of Sr to H estimates for prominences n_0/n_1 between 1 and 0.1, so that the ratio (10) of H^- to Ba_c spectrum would be less than $^1/_{200}$ for an electron temperature of 10,000° or lower. An estimate for the chromosphere has not been carried out, but assuming the same or a smaller $n_0/(n_1\, n_\varepsilon)$ and a ten times larger n_ε, the ratio of the two spectra would be smaller than $^1/_{20}$.

Pending further evidence, it might be well to keep the possibility of the H^- spectrum in the chromosphere in mind. If it be present, this would make the determined T_ε still lower.

The writer wishes to express his gratitude to Professor H. H. PLASKETT for clarifying discussions, in particular regarding the STARK effect as means of determining the chromospheric concentration.

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[18]) S. CHANDRASEKHAR and FRANCES HERMAN BREEN, Ap. J. 104, 430 (1946).

[19]) K. WURM, loc. cit.

SEX CELL FORMATION IN EXPLANTS OF THE FOETAL HUMAN OVARIAN CORTEX. I

BY

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(Communicated at the meeting of September 30, 1950)

In most of the cytological, embryological and endocrinological text-books the *details* of the development of the *human* ovary are described only in rather short terms and often in quite differing ways. However, since the middle of the 19th century a great deal of work has been done on the *cytological* and *histological* development of the mammalian gonad, including some important work on the human gonad, which I should like to consider here.

One of the best descriptions of the cytological development of the female gonad was given by DE WINIWARTER already in 1901.

This author studied a number of serially sectioned human ovaries and compared the results of this work with those obtained by studying a great number of rabbit ovaries. He stated that during the early developmental stages cord-like proliferations were formed, starting from the superficial epithelium of the gonadal region, and he concluded that even the medullary parenchym and the tubules of the rete ovarii originated from the superficial epithelial cells of the celoma. In the ovary of a 7 months old human foetus he was able to find practically the same situation as in a rabbit ovary some 5—10 days after birth, the primitive cortex constituting practically the whole volume of the ovary. In this cortex a great number of ova were found, the youngest always lying just under the superficial epithelium and the more developed ones gradually sinking into the depths of the cortex. The nuclei of the undifferentiated cells of the superficial epithelium mainly had a long shaped form, lying perpendicular to the surface. The nuclear membrane was very thin and the interior of the nuclei showed a fine reticular structure without nucleoli (protobroch A. stage). Going from the periphery into the centre, the nuclei gradually changed their appearance. They became smaller, more ovoid and less stainable, but they still preserved a reticular structure without nucleoli (protobroch B. stage). According to his view, these cells were the future follicle cells. Other nuclei showed a marked growth, became spherical, 1 or

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2 nucleoli developed and the reticular structure became coarser (deutobroch stage). In this way the nucleus of a primary oöcyte was formed and this gradually reached the early prophase of the first meiotic division. Going deeper into the cortex, leptotene, synaptene, pachytene and diplotene stages were found and in the most central parts of the cortex primordial follicles occurred with nuclei in diplotene stages. Apart from these developmental stages, protobroch B. nuclei were found all over the cortex.

De Winiwarter also described many symptoms of karyolysis in the cortex ovarii, while mitotic divisions appeared to be extremely rare, because, as he said, in man mitotic divisions in the ovary stopped a long time before birth.

After the seventh month the peripherial nuclei of the cortex reached further meiotic prophase stages, while the nuclei in the follicles showed the resting type (dictuoid stage) which followed the diplotene stage.

De Winiwarter deduced from his preparations that, in man, all epithelial cells of the primitive cortex originated from the superficial epithelium and were potentially able to form ova and follicle cells.

In 1930 Swezy and Evans also studied the cytological peculiarities of the human ovary.

Summarizing the results of studying the ovaries of a 25 mm human embryo, they wrote: "Here, as in older ovaries, as will be pointed out later, new germ cells arose both by proliferations from the germinal epithelium and by division of already existing germ cells".

According to the ovaries of a 9 cm embryo, they said: "Ova arise from the germinal epithelium by a simple process. Any cell apparently is a potential ovum. The new ovum cannot be distinguished from the surrounding epithelial cells, until both the cell and its nucleus undergo a slight enlargement as the beginning of the process of changing into a germ cell". Moreover a great number of new ova were formed by the division of already existing ones and early maturation of the ova started already in this stage. On this particular point the authors confirmed the findings described by De Winiwarter, but between the deutobroch and the leptotene stages they found a spermatocyt-like stage with condensed prochromosomes.

In the 19 cm embryo no more divisions of preexisting ova were found. "Unlike conditions in the two younger embryos new ova in the embryo of this age arise mainly by proliferations from the germinal epithelium".

Starting from the superficial epithelium, a fine stratification of cells in different stages of development was now found, that is, with protobroch, deutobroch, leptotene, synaptene, pachytene, diplotene and some transitional forms between the diplotene and the dictuoid stages.

Swezy and Evans did not find real follicles in this period. However, many of the larger ova were encapsulated by connective tissue cells, but "other cells, the so-called indifferent cells, probably epithelial derivates, are also found disposed in the same manner".

According to their studies of the *mature* ovary, the authors mentioned that "it has been found that new sex cells arise in a cyclical manner throughout mature life, the embryonic cells having disappeared sometime between birth and the attainment of sexual maturity".

This conclusion brings us to one of the most discussed points in the study of the development of the gonads viz.: the origin of the sex cells.

A number of contradictory opinions can be found in the literature the vital point always being: do the definitive ova belong to a special line of cells "directly descending in an unmodified condition from the segmenting egg", or not.

With reference to this particular point SIMKENS published in 1928 the results of an extensive study of a relatively large number of human ovaries in different stages of development. Unfortunately this author did not describe the structural details of the ova observed, but he put the problem of sex cell genesis as follows:

Do all the sex cells arise from primordial parents migrating from elsewhere into the gonads, or are the definitive sex cells products of the "germinative" epithelium?

Now in a number of young stages (2, 6 and 7 mm embryos) SIMKENS was not able to find any particular primordial germ cells in the celomic epithelium.

In the 10—11 mm embryo he found cord-like proliferations of the epithelium extending from the superficial layers down into the underlying stroma, and he came to the conclusion that the superficial epithelium is the source of all cells of the incipient ovary [1]).

At the 25 mm stage, the superficial epithelium began to take on the character of an insignificant investing layer, more or less completely separated from the core of the gland.

However, after the 90 mm stage had been reached, he says "after which time no more germ cells or any cellular recrutes from the germinative epithelium reach the epithelial nucleus. Hence from the 90 mm stage onward the histogenesis of the ovary is concerned only with the cells already included within the core". Summarising the results of his own work, the author says: "Shortly after the formation of the epithelial nucleus, which I believe to be the result of a continuous delivery of cells from the germinative epithelium and not from an early and late delivery, two well defined parts can be recognized in the gonad, the nucleus itself and the investing or germinative epithelium; the latter continues to deliver cells

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[1]) In a review on the status of the germ cell problem in vertebrates by EVERETT in 1945, this author summarised the most important and contradictory opinions on this particular point. EVERETT suggested another interpretation of the observed proliferation of germ cells from the germinal epithelium. He said: "It seems probable that the cells of the epithelium, which form functional sex elements, are not and never were a part of the mesothelial covering, but are cells which were segregated early and are merely stored in the epithelium".

into the stroma until the tunica albuginea forms and prevents their ingress; this continues as late as the fifth or sixth month. The primordial germ cells, those delivered from the germinative epithelium during the formation of the indifferent gonad lie deep within the stroma, almost to the hilus, where they are found in greater abundance near the peripherial ends of the rete ovarii. Here they remain until the tunica albuginea is formed and the stroma takes on the character of the cortical and medullary zones. The large cells lie yet within the medullary zone, where, because of their large size, thin chromatin and disunion, they give to that zone its characteristic pale and loose structure. In the medullary zone some of them take on the appearance of follicles, lie loosely associated in nests, and disposed toward the cortex, into which zone they eventually move along with the later transformed indifferent cells, where with the second type of genital cells they enlarge apace, some of them degenerating, others attaining the status of mature follicles at birth. The growth of the primary genital cells into follicles is similar to the secondary genital cells which follows this course; the large cell becomes surrounded by a few small spherical cells, outside of which there are a few small fusiform cells.

The small spherical cells arrange themselves in an orderly manner around the peripheral margin of the cytoplasm of the oöcyt and proliferate to form an irregular double layer. As yet there is no definite cytoplasmic membrane formed around the gonocytes, but its margin can be easily determined by the intense staining reaction of the cytoplasm itself. The irregular disposal of the follicle cells soon passes into an orderly one and soon thereafter the vitelline membrane can be distinguished. The follicles increase in size, so that at birth certain ones have acquired the parts of a mature follicle".

Moreover SIMKENS found between the 5th and 7th month of embryonic development a great number of degenerating follicles in the medullary zone of the cortex and consequently he suggested that the definitive egg cells arise from the more peripherial cells that were delivered to the epithelial nucleus from the germinative epithelium.

Comparing the work of SIMKENS with the above mentioned work of SWEZY and EVANS, it will be clear that the latter do not believe that sex cell formation in the female is limited to the early embryonic period only. On the contrary SWEZY and EVANS suggest the following scheme of the development of the ovaries:

a. an early embryonic period with growth and mitotic divisions of germ cells.
b. a mid-embryonic period with early prophase stages of the first maturation division with a prochromosome stage in the pre-leptotene region.
c. a late-embryonic period with early prophase stages of the first maturation division, without a prochromosome stage and ending in a resting stage (dictuoid stage) shortly after having reached the diplotene stage.

d. a period of degeneration of all ova formed in the embryonic period, from birth to sexual maturity.
e. the period of sexual maturity with a cyclical new formation of ova from the superficial epithelium which shows maturation divisions shortly before or after the ovulation.

In 1932 SIMKENS, contributed to our knowledge of the development of the human ovary from birth to sexual maturity. In the first place he came to the conclusion: "Although I am unable to find any evidence that germ cell division takes place in ovaries older than the sixth month of gestation, except the maturation divisions that take place at ovulation".

His second conclusion can best be summarised by saying that he distinguished two types of "primordial" follicles, viz.:

a. The "primordial" follicle which develops only in embryos and young children. These follicles all disappear through degeneration before sexual maturity has been attained.
b. A second type, the "primary follicle" which occurs in a practically constant number between birth and sexual maturity.

These primary follicles are described as larger, they stain more intensively, are surrounded by at least 1 layer of large round or cuboidal cells and originate in the medullary region of the cortex which means that they are formed in the oldest parts of the ovarian parenchym. The author fixed the number of primary follicles at about 30.000 per ovarium at birth, remaining practically constant until sexual maturity.

Finally the author decided on the existence of solid masses of "granulosa" cells without oögonia. Of these cells he says: "In a position of quietude, indistinguishable from other cells, the potential germ cells remain, probably awaiting stimulation to go forth and produce follicles" and again, "....the potential masses of cells are always in the ovary, until in very advanced age".

Apart from the work just mentioned it must be remembered that SCHRÖN in 1863 observed an increase of young ova just under the tunica albuginea during the menstrual period in an adult woman. With regard to the follicle cells he discussed the possibility of a mesenchymal genesis. HIS in 1865 indicated the epithelial origin of the follicle cells. KOSTER in 1868 observed growth and formation of young ova and follicle cells in pregnant women of 32 and 37 years of age, as well as in the ovaries of young women (16—17 years old). SLAVJANSKY in 1874 found "Pflüger tubes" originating from the superficial epithelium in the ovary of a woman of 30. FELIX in 1912 did not believe that the superficial epithelium of the adult was able to regenerate new parenchym, because of "the striking absence of mitosis in the germinal epithelium". ALLEN in 1923 again accepted the formation of new ova during sexual maturity. MOMIGLIANO in 1927 stressed the role of the germinal epithelium in the formation of young ova

during embryogenesis, while the follicle cells should originate from indifferent but epithelial cells.

Furthermore, most of the work on ovarian development which can be compared with the foregoing paragraphs has been done on other mammals, although this work is of great importance for the general problems, it indicates however so many differences between the different species that it was thought valuable to restrict our discussions to man alone, since the potentialities of the human tissues especially must be regarded as of decisive importance for a real insight into human physiology and pathology.

Moreover the foregoing already clearly indicates that there are still many problems to solve. Some of these can be summarised as follows:

1. What is the origin of the sex cells. Are there in man, as in many animals, also special primordial sex cells migrating into the gonadal region (ROBERTS) and, if so, are these cells the real progenitors of all sex cells, or do they perhaps induce (NIEUWKOOP in amphibia) the gonadal parenchym in such a way that this tissue (also?) becomes able to form the future ova?
2. If the primordial sex cells migrate into the superficial epithelium, are they always cells of a morphologically different type, or do they behave at least in this respect like their neighbour cells? (DE ROBERTIS; KINGERY).
3. Accepting the superficial epithelium of the ovary as being or becoming potentially able to form new ova (SWEZY and EVANS; MOORE and WANG HSI) is there any time-limit to this potency?
4. Does the power to form new ova still exist during sexual maturity and if so which part of the ovary and which cells must be regarded as the matrix? Does the superficial epithelium possess the exclusive right to produce new ova? (SCHRÖN; KOSTER; ALLEN; ESSENBERGH; OTTO; SCHWARZ; CLAUDE; YOUNG; CROUSE).
5. Do two types of follicles (primordial and primary ones) exist, originating in different periods of gonadogenesis and do primordial and/or primary follicles remain in a position of quietude until sexual maturity has been reached? (SIMKENS).
6. Why are there so many egg cells in the primary stages of the first meiotic division during the latter part of embryonic development and, if these cells are not used during sexual maturity but degenerate between birth and sexual maturity, what can be the meaning of these cells?
7. Are there any differences in life span between oögonia, oöcytes and oötides in different stages of development?
8. Are the embryonic egg cells (especially those which entered the first meiotic division) potentially able to complete this division and can they eventually go into the second maturation division, or are these processes limited to the sex cells during sexual maturity?

9. What is the origin of the follicle cells? Do they arise from the cortical parenchym (HIS; MOMIGLIANO; EVERETT) and/or from the elements of the connective tissue? (NOVAK; SCHRÖN) etc. etc.

Many of these questions were and still are difficult to solve partly because experiments were not possible in human beings. However, since the method of tissue culture has been developed in such a way that experiments on histogenesis and organogenesis have become possible (FELL; GAILLARD), it seemed to be worth attempting the cultivation of fragments of the human ovary in different stages of development.

The material could easily be obtained from the gynaecological and obstetrical clinics and was prepared as soon after death as possible, but in no case after a period lasting longer than 6 hours. The exact techniques of cultivation will be mentioned separately for each of the experimental series.

PRELIMINARY EXPERIMENTS:

In order to study the technical possibilities, a number of preliminary experiments were done between February 1943 and December 1947. The ovaries were obtained from foetuses between about the 16th and 36th week of development and all series will principally be discussed chronologically, because of the technical procedures developing during that time.

Series 18–3–1943. Sixteen weeks [1]) old human foetus.

The left ovary was cut into small fragments ($\pm \frac{1}{2}$ mm^3). 120 fragments were cultivated during a period of 7 days in embryo containers [2]) (see fig. 1).

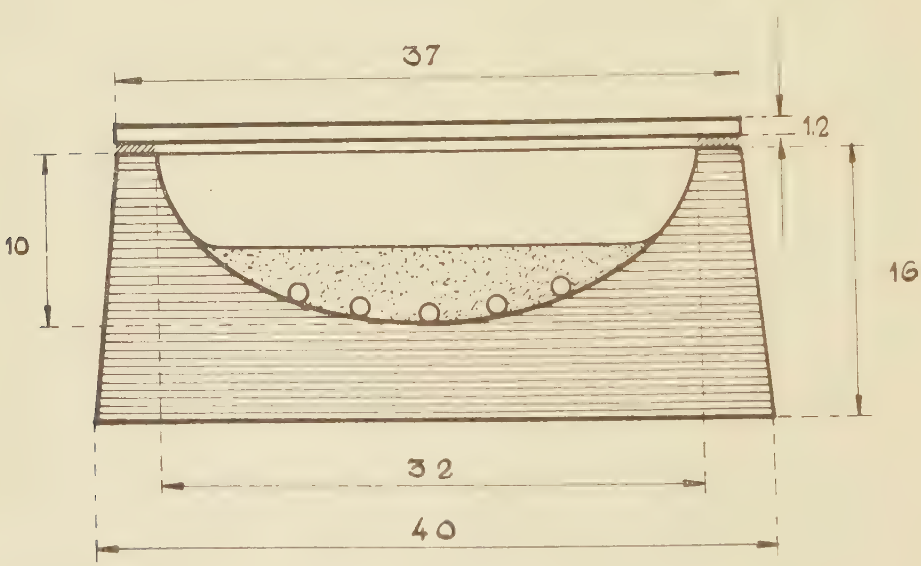

Fig. 1

1) It will be obvious that the age of all foetuses can be given only approximately.

2) In this and all other experiments the cultures were refreshed every three days.

In each of the containers 30 explants were put into 21 drops of a medium composed of:

10 %	Placental vein serum in Gey's saline solution	10
10 %	Serum of a patient suffering from an agenesis ovarii . .	10
	Chick embryo press juice (9 days)	1

20 Explants were cultivated during 7 days on coverglasses each of them in one drop of a medium composed of:

$$\left.\begin{array}{ll} & \left.\begin{array}{l}\text{Human blood plasma}\\ \text{Plac. vein serum}\end{array}\right\}\text{aà} \\ 10\ \% & \\ & \text{Chick embryo press juice (9 days)}\end{array}\right\}\text{aà}$$

Results: In both the media, even after two days of cultivation [1]), most of the fragments became *encapsulated* by a *cubical epithelium.* At the same time the greater part of the inner mass of the tissue degenerated and after four days of cultivation the aspect of the interior mass had completely changed. *The cell detritus had practically disappeared* and a *mesenchym-like structure was observed instead of it.* Moreover cord-like *epithelial structures* were seen, attached to the covering epithelium and there penetrated into the centres of the explants. No oögonia were found. Unfortunately after six days of cultivation all explants appeared to be completely degenerated.

Series 20–4–'43. Thirty-six weeks old human foetus.

The left ovary was cut into 160 small fragments which were cultivated in embryo containers. In each of the containers 20 explants were cultivated in 10 drops of the following medium

10 %	Placental vein serum in Gey's saline solution	5
10 %	Serum of a patient suffering from an agenesis ovarii . .	10
	Chick embryo press juice (9 days)	1

Results: All explants appeared to be degenerated within two days of cultivation. The non-cultivated control sections showed extensive post mortem degenerations.

Series 17–1–'44. Twenty weeks old human foetus.

The left ovary was cut into small fragments, 75 of them being cultivated for 5 days in embryo containers. In each of five containers 10 ovarian explants were cultivated in 16 drops of a medium composed of:

10 %	Placental vein serum in Gey's saline solution	15
	Chick embryo press juice (9 days)	1

In each of another five containers, 5 ovarian explants were cultivated, together with 5 explants of the anterior hypophysis of the same foetus. The cultivation medium in these containers was made up in the same way.

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[1]) These and all further explants were studied in the living state and after fixation in Bouin-Hollande and staining of the sections with hematoxylin-eosin and Heidenhain "Azan" method.

Results: The ovarian explants in *both* conditions behaved practically alike. *Many explants completely degenerated* within the period of five days.

However *some explants were completely encapsulated* by a cuboidal epithelium. In these explants in comparison with the non cultivated controls *the central mass of tissue was practically unchanged.*

No cord-like "proliferations" were observed. The anterior hypophysis explants were also encapsulated by a cuboidal epithelium. The morphological aspect of the interior mass of this tissue showing the typical structure of the human foetal anterior hypophysis.

Series 25–1–'44. Eighteen weeks old human foetus.

10 small ovarian fragments of the right ovary were cultivated per embryo container in 16 drops of a cultivation medium composed as in series 17-1-'44. 8 containers were used.

In each of 8 other containers, 5 ovarian explants were cultivated, together with 10 explants of the anterior hypophysis of the same foetus.

In 8 further containers, 10 anterior hypophysis explants were cultivated separately. The cultivation period lasted 8 days.

Results: As in the previous series, the hypophysis explants did not influence the ovarian explants.

After 2—4 days of cultivation, most of the ovarian explants were *encapsulated* by a cuboidal epithelium. *In the central parts a severe degeneration* occurred and after 4 days of cultivation the degenerated tissue was replaced by a fibrous connective tissue.

However *at the same time, starting from the encapsulating epithelial layer of cells, cord-like proliferations which often anastomosed were seen to extend into the central mass* (see fig. 1, Plate I).

After 6 days of cultivation, no living ovarian explants were observed: the anterior hypophysis explants behaved as described in the previous series.

Series 6–3–'44. Eighteen weeks old human foetus.

The left ovary was cut into 40 small fragments. All explants were cultivated on coverglasses in a medium composed of:

Adult human blood plasma (heparinised with 0.5 ml of a 0.1 % solution per 10 ml of blood)	1
10 % Placental vein serum in Gey's saline solution	1
Chick embryo press juice (9 days)	one tiny drop

I. 10 ovarian explants were cultivated separately.

II. 5 ovarian explants were each cultivated together with 1 fragment of the foetal suprarenal cortex.

III. 5 ovarian explants were each cultivated together with 1 fragment of foetal suprarenal cortex and 1 fragment of the anterior hypophysis.

IV. 10 ovarian explants were each cultivated together with 1 fragment of the anterior hypophysis of an adult rat.

V. 10 ovarian explants were each cultivated together with 1 fragment of the anterior hypophysis of a new born rat.

Results: Apart from only a few ovarian explants out of group II

which appeared to be encapsulated by a cuboidal epithelium after 2—4 days of cultivation, in all other ovarian fragments a complete degeneration occurred. After 6 days of cultivation no living ovarian explants were found in any of the 5 groups.

Series 11–3–'44. Twenty six weeks old human foetus.

The right ovary was cut into small fragments. All explants were again cultivated in coverglasses and the medium was composed as in series 6-3-'44.

5 Explants were cultivated separately and five other explants were each cultivated together with one small fragment of the foetal suprarenal cortex.

Results: *One explant of the first group and two explants of the second group were encapsulated by a cuboidal epithelial layer* after 5 days of cultivation. The central parts of these fragments had completely died.

No proliferation of the encapsulating epithelium. In the suprarenal cortex fragments the glomerular zones appeared to be in good condition.

Series 20–9–'46. Twenty-eight weeks old human foetus.

The right ovary was cut into relatively large ($1-1\frac{1}{2}$ mm^3) fragments.

In each one of twelve embryo containers, 10 explants were cultivated in 16 drops of a medium composed as in series 17-1-'44.

Results: A bacterial contamination occurred. All explants died within 5 days of cultivation.

Series 30–9–'46. Thirty weeks old human foetus.

From the right ovary 150 small fragments were cut and cultivated in fifteen embryo containers with 16 drops of a medium composed as in series 17-1-'44.

Results: Apart from some follicle cells of the pre-existent primordial follicles, all other tissues showed a complete necrosis within 4 days of cultivation.

No epithelial encapsulation.

Series 2–10–'47. Fourteen weeks old human foetus.

The left ovary was divided into three parts (Two poles and a middle region). *The middle region* was cut into slices perpendicular on the axis and from each one of the slices only the most peripheral cortical parts were cut into fragments. Thirty of them *with* a part of the germinal epithelium attached to them and thirty others *without* a part of the germinal epithelium. Embryo containers were used and in each of them 10 explants were cultivated in 15 drops of the following medium:

Gey's saline solution	9
Placental vein serum	1
Streptomycin in Gey's solution	4 (containing 200 U.)
Chick embryo press juice (9 days)	1

Results: With the exception of one explant belonging to the group prepared with germinal epithelium, all others died within 4 days of cultivation. The "intact" one fixed after nine days of cultivation was entirely encapsulated by a cuboidal epithelium.

However the centre of the explant was fairly necrotic.

Series 16–10–'47. Twenty-three weeks old human foetus.

The left ovary was divided into 3 parts. (Two poles and a middle region).

In this series the *poles* were cut into slices perpendicular to the ovarian axis. From the cortical parts of these slices 50 fragments were cut.

In each of the embryo containers five explants were put *on the top* of a coagulum composed of 7 drops of a mixture of:

Human blood plasma (heparinised with 0.5 ml of a 0.1 % solution per 10 ml of blood)	2
Placental vein serum	1
Streptomycin in Gey's saline solution	2 (containing 100 U.)
Chick embryo press juice	2
Gey's saline solution	8

Results: *All explants were surrounded by a flat or cuboidal epithelium* within two days of cultivation. The inner mass of the explants showed at that time an intact cortical structure with a number of deutobroch and leptotene nuclei. Other meiotic prophase stages which were to be seen in the non-cultivated control tissue obviously had disappeared by lysis and other types of degeneration. *After 6 days of cultivation all oöcytes and oögonia had disappeared* and in the cell nest only an indifferent type of epithelial cells remained. After 8 days of cultivation a complete degeneration of the parenchym had occurred.

Series 18–11–'47. Twenty-four weeks old human foetus.

From the poles of both ovaries 140 fragments were cut. In each embryo container 7 explants were cultivated on the top of a coagulum composed of 7 drops of a mixture:

Human blood plasma (heparinised)	2
Placental vein serum	1
Streptomycin in Gey's solution	2 (containing 100 U.)
Chick embryo press juice	4
Chick blood plasma	4

Results: Complete degeneration of all explants after 3 days of cultivation. Chick plasma?

Series 21–11–'47. Twenty-six weeks old human foetus (see fig. 2, Plate I).

From the poles of the ovaries small fragments partly *with* and partly *without* the epithelium were cut. In embryo containers 5 explants were cultivated on the top of a coagulum composed of 5 drops of a mixture of:

Human blood plasma (heparinised)	2
Placental vein serum	1
Streptomycin in Gey's solution	1 (containing 10 U.)
30 % Chick embryo press juice in Gey's solution	2
Gey's saline solution	4

Results: After 4 days of cultivation *all explants with a part of the germinal epithelium attached to them were encapsulated by a columnar*

epithelium with apical (secretory?) vacuoles. The inner mass of tissue showed some localised degeneration phenomena but between a great number of indifferent epithelial cells *some oöcytes with deutobroch nuclei were still present* (see fig. 3, Plate I).

In the explants without a part of the *epithelium* attached to them *extreme degeneration phenomena* occurred (see fig. 4, Plate I), *the follicle cells appearing to be the most resistent ones.*

After 11 days of cultivation a degeneration of the parenchym was clearly visible also in the encapsulated explants, but a part of the detritus had completely disappeared and in this region, starting *from the surrounding epithelium some cord-like or tubule-like regeneration of indifferent epithelial cells* was to be found (see fig. 5, Plate I).

The explants cultivated *without a part of the germinal epithelium* showed *further degeneration* phenomena (see fig. 6, Plate I).

Conclusions based on the results of the preliminary experiments

1. In cultivating explants of the human ovarian cortex from foetuses of different stages of development (16—36 weeks old) and largely independent of the techniques used, the germinal epithelium appeared to be of great importance. Explants cultivated with a part of the germinal epithelium attached to them always showed the best histological structures and remained alive longer than explants cultivated without the adhering epithelium. Moreover, soon after the beginning of the cultivation period, the epithelium tended to encapsulate the explants with one layer of cuboidal or columnar cells, all having the same morphological appearance. Especially in the last series (21–11–'47) large apical vacuoles were observed in all the cells of the covering epithelium.
2. In three series, cord-like or tubule-like "proliferations" were found in connection with the surrounding epithelium and developing after the inner mass of tissue had been degenerated and replaced by a loose connective or mesenchymal tissue.
   Neither in the surrounding epithelium nor in the parenchymal cords could new formed oögonia or oöcytes be diagnosed.
3. In none of the series could the explants be kept alive longer than about 7—11 days. During the degeneration processes the covering epithelial cells and the follicle cells, which may be present in the central parts of the explants, appeared to be the most resistant.
4. Looking after the different technical procedures used some points can be particularly stressed:
   a. The ovarian poles seemed to be more easily to cultivate than the middle region of the organ.
   b. The cultures cultivated in embryo containers and on the top of a coagulum were the most satisfactory.

c. Heterologous medium components (especially chick plasma) became suspect, because of the extreme degeneration phenomena occurring in the explants.
d. The addition of anterior hypophysis and suprarenal cortex fragments either separately or combined apparently did not influence the behaviour of the ovarian tissues.

DEFINITIVE EXPERIMENTS

After the above mentioned results and suggestions had been obtained, a number of new experiments were done with human ovaries obtained from 4—9 months old foetuses. In these experiments only cortical fragments were cultivated. These were prepared from the poles with a part of the germinal epithelium attached to them. Only embryo containers were used and the explants were cultivated on the top of weak coagula containing only homologous components. According to this, the chick embryo press juice was replaced by human foetal brain press juice [1]) or ascitic fluid and the chick plasma was completely omitted.

The results obtained so far indicated some difference in the behaviour of the younger and older ovaries and, for our present discussion, the experimental series are therefore grouped according to the age of the foetal donores.

A. Results of cultivating ovarian cortical fragments from 14—21 weeks old human foetuses

Series 5–5–1948. Sixteen weeks old human foetus (see fig. 1, Plate II).

The ovarian poles were used for cultivation. 160 explants were cut in such a way that most of them had a part of the covering epithelium attached to them. They were cultivated in 20 embryo containers.

In each of the containers 8 explants were put on the top of a weak coagulum composed of 5 drops (± 0.5 ml) of a mixture of:

Human blood plasma (heparinised)	2
Placental vein serum	1
Streptomycin in Gey's saline solution	1 (with 10 U.)
Human foetal brain press juice	2
Gey's saline solution	4

Results: After 4 days of cultivation about half the number of explants showed a flat or cuboidal surrounding epithelium.

In the central parts of these explants most of the original parenchym had disappeared and instead of this a loose connective tissue was present.

[1]) In preparing this press juice the brain tissue was removed from 4—6 months old foetuses aseptically and as soon as possible after death (max. 6 hours). Then the tissue was cut into pieces (diameter ± 1 cm) and placed in a Petri dish at + 4° C. for 24 hours. After that time the tissue was minced with a usual tissue press; an equal quantity of Gey's saline solution was added and the mixture was centrifuged for 15 min. 6—8000 r.m.

The press juice was pipetted in pyrex glass tubes and stored at — 20° C.

Obviously *starting* from the surrounding epithelium a number of *parenchymal cords were observed which often anastomosed* and, *in the cords, numerous oöcytes were observed* (see fig. 2, Plate II).

After 13 days of cultivation all remaining explants were fixed and stained. They appeared to be encapsulated by a *columnar or stratified epithelium from which cord-like structures started* (see fig. 3 and 4, Plate II). *The number of oöcytes had increased considerably*, (see fig. 3 and 5, Plate II) and *leptotene, synaptene* and *pachytene stages of the first maturation division frequently occurred.*

Series 7–6–'48. Twenty weeks old human foetus.

Embryo containers were again used. In each of them 10 explants, cut from the ovarian poles, were cultivated on the top of a coagulum composed as in series 5-5-'48. (Total 200 explants).

Results: Within 4 days of cultivation the explants were *encapsulated* by a cuboidal epithelium grown from the germinal epithelium and in most of the explants the parenchym in the *central parts* had *degenerated* completely (see fig. 6, Plate II).

Some days later the *degenerated mass of tissue* appeared to be partly *replaced* by a very *loose reticular structure* (see fig. 1, Plate III) and at that time a *regeneration* regularly *started from the peripherial epithelium* which could be followed step by step (see fig. 2 and 3, Plate III).

Most of the anastomosing parenchymal cords were *symplasmatic* with numerous nuclei, some of them showing a deutobroch character.

After 10—11 days of cultivation a reticulum of parenchymal cords was obtained and the *number of ova* had *increased* enormously (see fig. 4, Plate III). Deutobroch, leptotene, synaptene, diplotene and early metaphase stages were frequently observed (see fig. 5, 6, 7, 8 and 9, Plate III, and fig. 1, Plate IV).

Moreover, indifferent parenchymal sister cells were seen to envelop the young ova (Epithelial origin of the first layer of follicle cells?) (see fig. 1, Plate IV).

In comparison with the stage of nuclear development, the cytoplasm was generally only poorly developed.

Finally, some of the encapsulating cells showed large apical (secretory?) vacuoles (see fig. 4, Plate III).

Series 2–2–'49. Sixteen weeks old human foetus (170 explants).

In each of the containers 9 explants were cultivated. The medium was composed as in series 5-5-'48, but after 5 days of cultivation the streptomycin was left out and the quantity of blood plasma was halved. Gey's saline solution was added instead of these volumes.

Results: After 5 days of cultivation, a *complete central degeneration* and disappearance of all the remnants of the original structure was observed. From the surrounding epithelium a number of *cord-like pro-*

liferations occurred and in these cords a number of *new oöcytes were found*. After 12 days of cultivation, in two out of five explants *fixed at that time*. a complete filling up of the centres was found by a mass of parenchymal cell cords (see fig. 2, Plate IV). The explants behaved like *miniature ovaries*. In the cords *all stages of the meiotic prophase* were seen (see fig. 2, 3 and 4, Plate IV), *as well as a number of large "primordial" follicles* with their nuclei in the dictuoid stage and with a *well developed cytoplasm* (see fig. 2 and 5, Plate IV). Moreover it became clear that *undifferentiated parenchymal cells tended to envelop the young oöcytes* (see fig. 3 and 4, Plate IV), which again indicates the possibility of a parenchymal origin of the first layer of follicle cells.

Series 8–6–'49. Twenty one weeks old human foetus (150 explants).

The same technical procedures were used as in series 2–2–'49.

Results: The same reactions were observed as usual, viz. the encapsulating epithelium, degeneration of the original tissue and regeneration of cord-like proliferations from the superficial epithelium. It is remarkable that after 6 days of cultivation only very few young ova were seen, but after 12 days of cultivation a number of new oöcytes in early prophase of the first maturation division were observed.

After 15 days of cultivation most of the explants still showed this structure. However, some explants showed another aspect, the centre of the explants being composed of *a mass of collagenous connective tissue*, while *only in the periphery did anastomosing parenchymal cords with oöcytes occur* (see fig. 6, Plate IV). Moreover, these explants were *bordered by an epithelium* of an inverted type, viz. *with the nuclei in the apices of the cells* (see fig. 6, Plate IV). Finally, at some places, a double layered covering epithelium was observed.

Series 25–10–'49. Twenty-one weeks old human foetus.

The explants were made as usual from the polar cortex of the ovaries.

Two media were choosen:

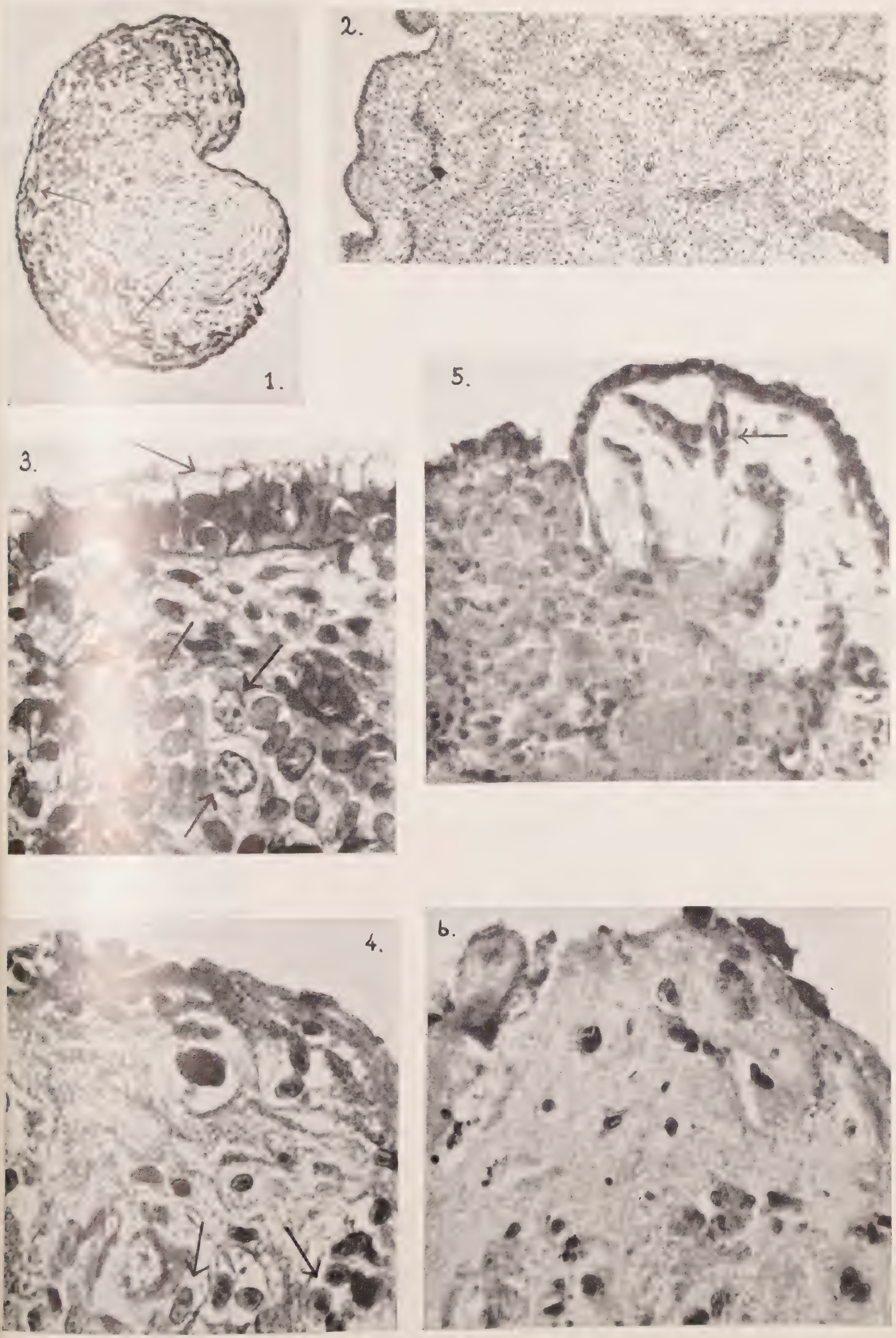
A. The same as the one in the foregoing series.
B. A medium in which the foetal human brain press juice was replaced by ascitic fluid from a patient suffering from a carcinosis peritonei, as indicated by C. M. Pomerat (private discussion, July 1949).

In each of the two media 100 explants were cultivated in embryo containers.

Results: In both media the germinal epithelium encapsulated the fragments as usual. The parenchym in the central parts of the explants degenerated within a few days and a regeneration of cord like proliferations started from the surrounding epithelium. In these epithelial cords, a number of young oöcytes developed, some of them being enveloped by one layer of cuboidal undifferentiated parenchymal cells.

After 17 days of cultivation lumina *in some of the cord-like structures* were seen, which indicated the possibility of tubule formation (see page 1311).

PLATE I



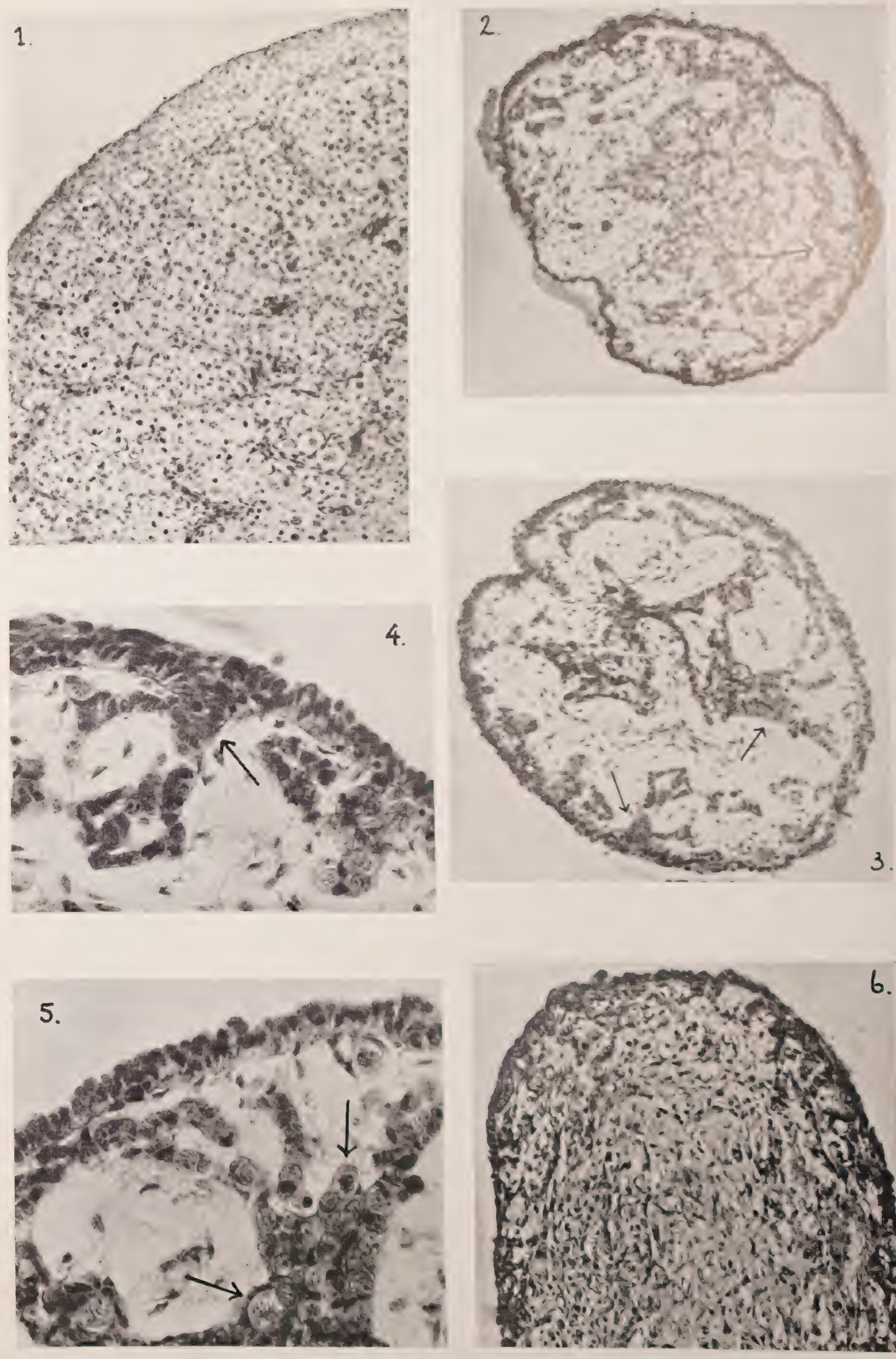
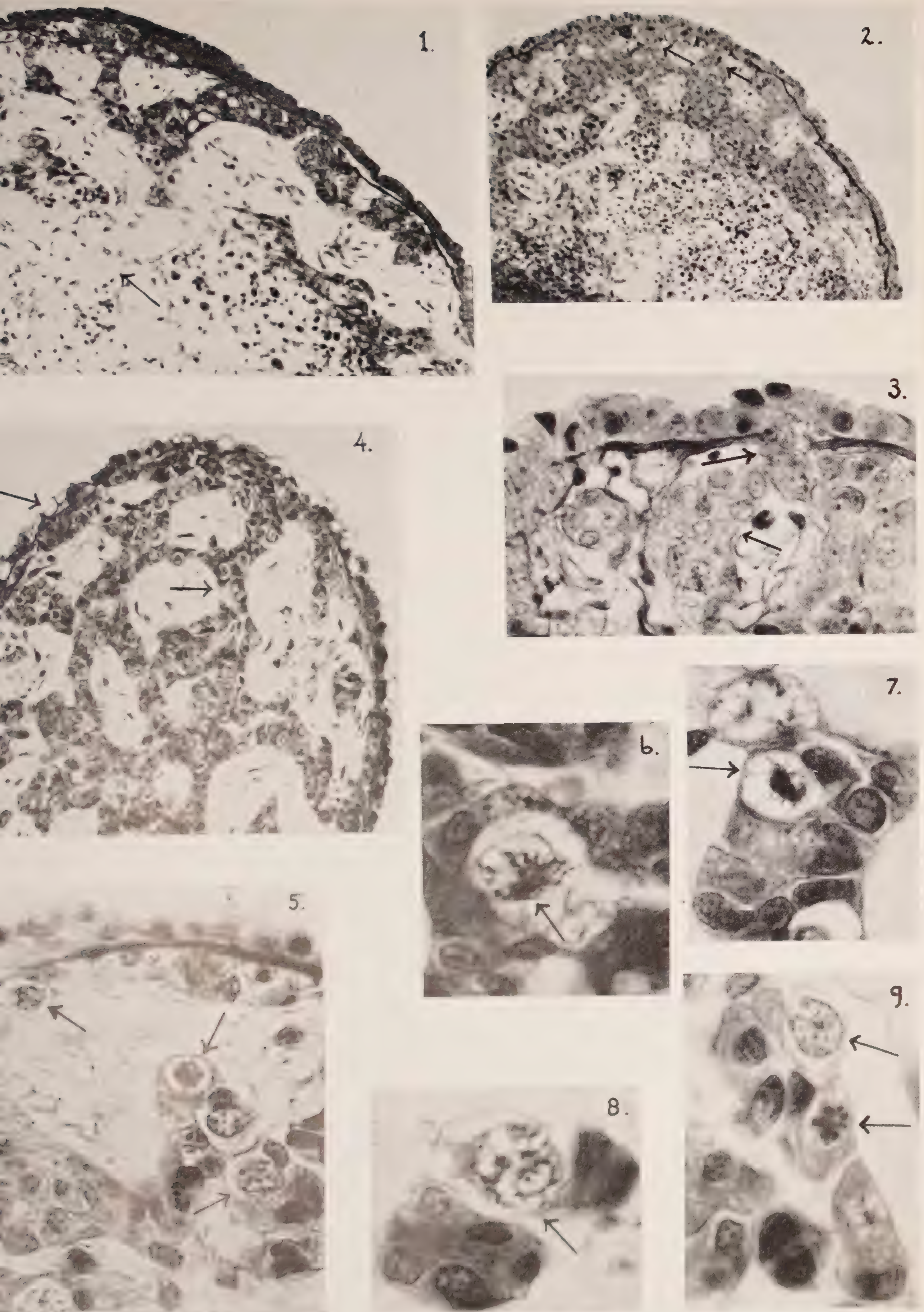
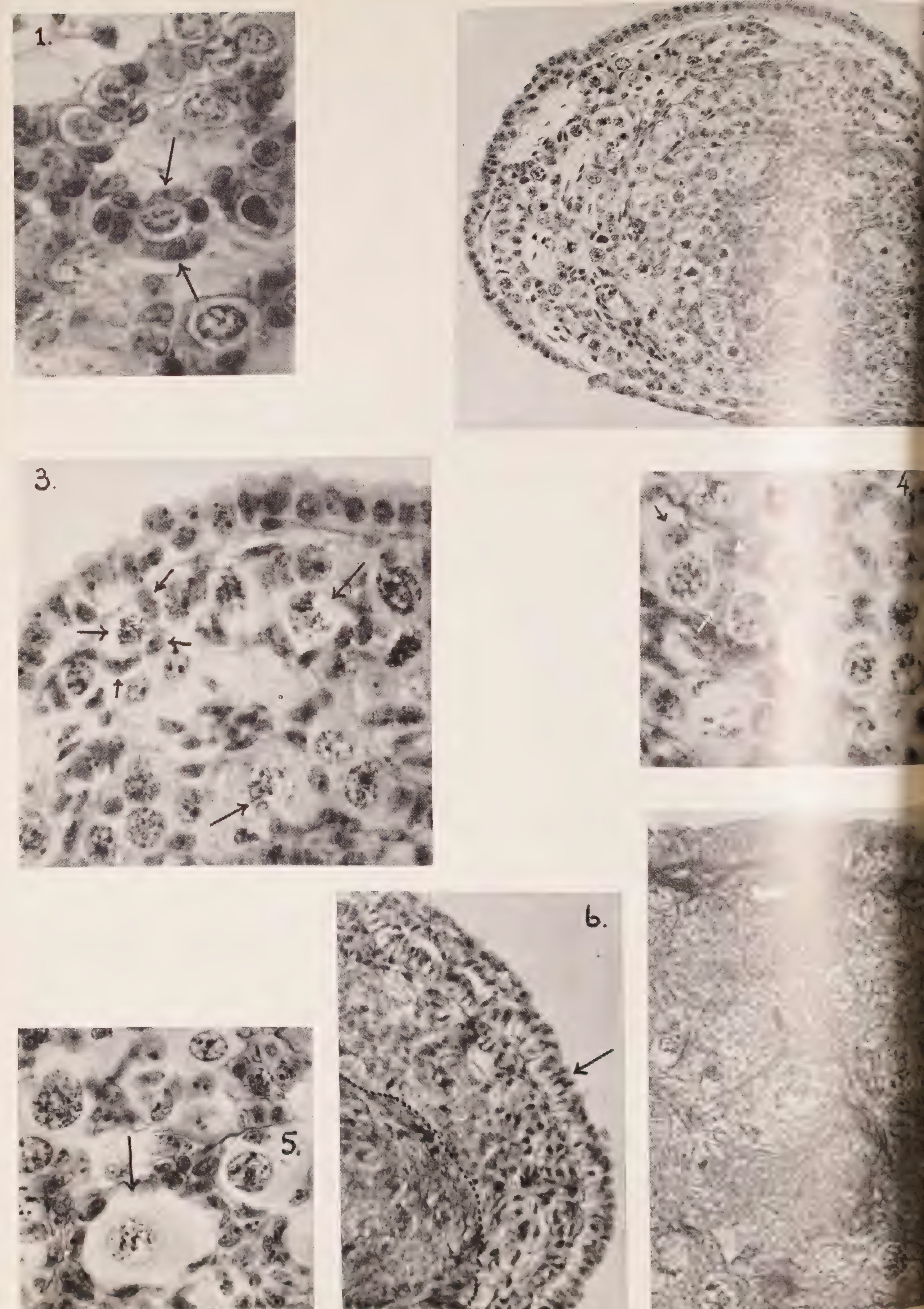


PLATE III



PLATE



There was no difference of any importance between the explants cultivated in the two media A. and B.

Series 25–11–'49. Sixteen weeks old human foetus.

The same technical procedures were used as in series 25–10–'49.

Results: In this series the behaviour of the explants in the media A. and B. was slightly different. In medium A. exactly the same reactions were observed as were described for the series 25-10–'49. However, after cultivation for 10 days in the medium B (ascitic fluid), many explants had died and only a few survived. These exceptional explants did not show anastomosing parenchymal cords but a very massive parenchym, which was divided into *nests* separated by septa of connective tissue. There were many oöcytes in early prophase stages of the first maturation division (see fig. 7, Plate IV).

Conclusions based on the experiments with explants of the ovarian cortex from the 14—21 *weeks old human foetuses*

It would seem desirable to summarise the results of the experiments so far described with explants from ovaries obtained from the youngest foetuses (14—21 weeks old).

a. In comparison with the results of the preliminary series, the *explants in the homologous media behaved far better* and the cultivation period was extended to 15—17 days.

b. *In all cases the germinal epithelium surrounded the cultivated fragments* by a cuboidal or columnar epithelium. In many of the explants the columnar cells possessed *large apical (secretory?) vacuoles*, but in one series (8–6–'49) an inversed type of epithelial cells was seen in some explants with the nuclei in the apical parts of the cells and vacuoles in the basal parts.

c. The degeneration of the parenchym of the original explants was regularly observed and *as a rule cord-like parenchymal proliferations penetrated the centres, starting from the superficial epithelium.*

d. The cord-like structures often appeared to possess a symplasmatic structure and *in all series, after* 4—10 *days of cultivation a great number of new oöcytes developed in the regenerated cords.*

e. *In the oöcytes, the first maturation division* frequently started. All prophase stages (leptotene, synaptene, pachytene, diplotene) occurred as in normal embryonic ovaries of this age, viz. with the nuclei in the centres of the cells.

f. In one series, apart from the meiotic prophase stages, an *early metaphase stage was observed*, wich indicates *the possibility of a further progress of meiotic division in the foetal oöcytes.*

g. In two series in the epithelial cords, primordial follicles with one layer of flat or cuboidal follicle cells developed, indicating the *potential parenchymal origin of these cells.*

h. Obviously *the actual stage of development of the egg cell nucleus was not of decisive importance to follicle development*, as follicles were found with sex cell nuclei in deutobroch, leptotene, synaptene, pachytene and dictuoid stages; but the same stages were also found without follicle cells surrounding the oöcytes.

i. Moreover, *the growth of the cytoplasm of the oöcytes appeared to be independant of the nuclear differentiation phenomena*. In most series the cytoplasm of the oöcytes was only poorly developed. However in the series 2-2-'49 a fine vitello genesis occurred and apparently normal, though young, follicles were found with one layer of *flattened* or *cuboidal* epithelial cells enveloping the oöcytes.

j. In one series (2-2-'49) the cord-like parenchymal proliferations increased in such a way that the parenchym practically filled the whole explant. These explants resembled the appearance of an intact but *miniature foetal ovary* with all stages of sex cell formation up to the "primordial" follicles.

k. Finally there were *slight indications of tubule formation* in the parenchymal cords and in the last series there was a suggestion of some difference between the explants cultivated with human foetal brain press juice as a component of the cultivation medium and those cultivated with ascitic fluid.

AGATHICERAS SUNDAICUM HAN., A LOWER PERMIAN FOSSIL FROM TIMOR

BY

P. KRUIZINGA

ERRATUM

In fascicule 7 of Vol. LIII of the Proceedings of the Kon. Ned. Akad. van Wetenschappen. Sept. 1950. unfortunately a bad typographical error has been left on p. 1056 in the title of the following paper:

Agathiceras sundaicum Han., a Lower Permian fossil from *Timor* by P. KRUIZINGA.

Please read *Billiton* instead of Timor. The correct title is on top of the plate between pp. 1062—1063.

CHEMISTRY

ELASTIC VISCOUS OLEATE SYSTEMS CONTAINING KCl. XIV [1]

a) The problem of the character of the damping in the 0.6 % oleate system.
b) Rotational oscillations characterized by the second, third, - - - - - roots of the equation tg $\zeta = \zeta$.
c) The linear displacement in the equator plane of the sphere as a function of the distance from the centre.

BY

H. G. BUNGENBERG DE JONG, W. W. H. WEYZEN *) AND W. A. LOEVEN *) [2]

(Communicated at the meeting of September 30, 1950)

1. *Introduction*

In the present investigation we will study both with the 0.6 % and the 1.2 % oleate system the elastic deviation in the equator plane of the sphere as a function of the distance from the centre.

In part III of this series we found that the damping of the elastic oscillations is of a different character in the 1.2 % oleate system (A proportional to R) and in the 0.6 % oleate system (A independent of R). As J. M. BURGERS [3]) has shown theoretically, a (small) slipping of the elastic system along the wall of the vessel might give an explanation of the fact that A is independent of R.

Though in part III of this series a number of quantitative consequences of this supposition could be confirmed, later on (see part VIII of this series) doubt arose as to the real existence of slipping in the 0.6 % oleate system. The lack of data which give information on the magnitude of the expected slipping along the wall is a drawback for further experimental investigations. In a letter to the authors, J. M. BURGERS was so kind to send us the following informations:

"Values of $r\Phi$, in which

$$\Phi = \frac{\sin \zeta r}{(\zeta r)^3} - \frac{\cos \zeta r}{(\zeta r)^2}.$$

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*) Aided by grants from the "Netherlands Organisation for Basis Research (Z.W.O.)".

[1]) Part I has appeared in these Proceedings 51, 1197 (1948), Parts II—VI in these Proceedings 52, 15, 99, 363, 377, 465 (1949), Parts VII—XIII in these Proceedings 53, 7, 109, 233, 743, 759, 975, 1122 (1950).

[2]) Publication no. 10 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

[3]) J. M. BURGERS, these Proceedings 51, 1211 (1948) and 52, 113 (1949).

The radius R of the sphere has been taken 1, as a consequence of which $a = \zeta$.

	$\Lambda = 0$ $\zeta = 4.493$	$\Lambda = 0.523$ $\zeta = 4.561 + 0.380\,i$	$\Lambda = 0.770$ $\zeta = 4.658 + 0.570\,i$
r	$100\,r\,\Phi$	$100\,r\,\Phi$	$100\,r\,\Phi$
0	0		
0.1	3.27		
0.2	6.14		
0.3	8.30		
0.4	9.50	9.42	
0.5	9.65	9.52	9.36
0.6	8.77	8.63	
0.7	7.08		
0.8	4.81	4.77	4.68
0.9	2.34	2.67	3.01
0.95		1.97	2.67
1.00	0	1.83	2.75

Φ can not become zero for real values of r if ζ has a complex value."
Compare figure 1, which gives the above values of $r\Phi$ as a function of r.

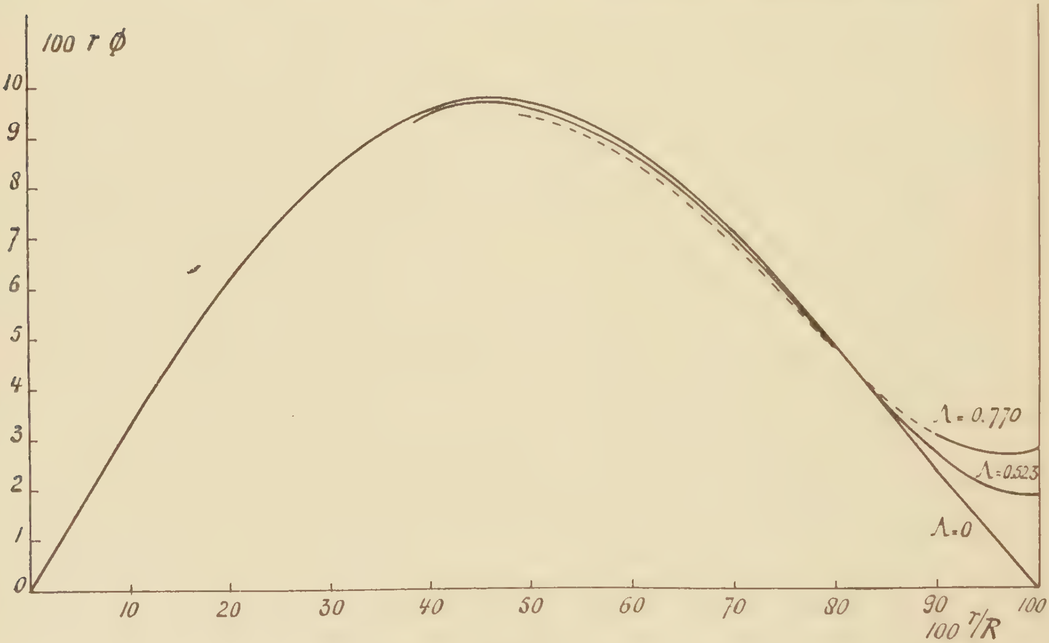

Fig. 1

2. *Methods*

As an oleate preparation we used K-oleate, prepared from chemically pure oleic acid and Na-oleate, neutral powder, from BAKER [4]) (for remarks

[4]) A generous gift of Na-oleate from the ROCKEFELLER Foundation provided the means for the experiments described in this paper.

on these preparations see part X and XIII). The experiments have been performed at room temperature (17—18°), using 0.6 % and 1.2 % oleate systems containing (besides at all times 0.05 N KOH) K_3-citrate or KCl in a concentration which corresponds to the minimum damping of the elastic oscillations (with chemically pure K-oleate: KCl = 0.98 N; with oleate from BAKER: K_3-citrate = 0.52 moles/l, KCl = 1.05 moles/l; compare part XIII of this series).

Our intention was to investigate the elastic deviations in the equator plane of a spherical vessel. Of course it would be ideal to use completely filled vessels, but than we see no way to mark the elastic system in the equator plane satisfactorily. If we use, however, exactly half filled vessels (the period of the rotational oscillation is then practically that of a completely filled spherical vessel; see part I, section 2, part VI, section 1), we have the equator plane automatically and the oleate system can in principle be marked just below this plane with the aid of the electrolytic H_2 mark described in part X. To do this, it will of course be necessary not to use the intact spherical vessels as containers for the elastic system, but to remove beforehand the upperpart of them (cut off at 3—4 mm above the equator plane).

It has appeared that when the cathode is slowly moved in a straight line through the centre we can now obtain a very satisfactory "line" of minute H_2-bubbles just beneath the surface (for further particulars see below). With appropriate illumination this track of H_2-bubbles stands out as a bright white line (of approximately 0.5—1 mm thickness) on a relative dark (citrate containing systems) or greyish (KCl containing systems) background. As the visibility of the line becomes less with time and finally vanishes altogether (gradual coalescence of the minute H_2-bubbles to bigger ones and breaking of the bubbles through the interface oleate system/ air) the photographic registration of the oscillating system must be performed soon after the marking (owing to the much higher viscosity of the 1.2 % oleate system, the line persists longer here than with the 0.6 % oleate system).

The apparatus which is used to bring the oleate system in resonance and to provide for intermittent illumination at the moments of largest elastic deviations, is given in the figs 2 and 3. Fig. 4 gives the device with which the line of H_2-bubbles is drawn.

In fig. 2 the frame, which is constituted of hook-iron, is only given by dotted lines. The cup-like half sphere *a* filled with the oleate system rests on the edge of a wide hollow cylinder *b*, which is attached to disk *c*. Connected to this disk is the axis *d*, supported by the ball-bearings *e* and *f* in the iron plates *g* and *h*. Disk *i* serves to connect the triangular rocker arm *j* (constituted of two bars *k* and *l*, joined together at one end *m*). The connecting bar *n* has ball-bearings *o* and *p* at either end. A pin in ball-bearing *o* is fixed in the hole at the end *m* of the rocker arm *j* by means of screw *q*. A pin fixed in ball-bearing *p* is connected with the middle of bar *r* (of quadratic cross-section). This bar can be moved and fixed in the desired position in the slot (also of quadratic cross-section) of bar *u* by the screws

s and *t*. Bar *u* is fixed to the disk *v*, which latter is connected by a ball-bearing to the iron plate *x*.

An electric gramophone motor (not designed in the figure) fed by direct current and provided with an adjustable friction brake, rests with a rubber coated pulley against the edge of disk *v*. The frequency of rotation of disk *v* can be regulated by

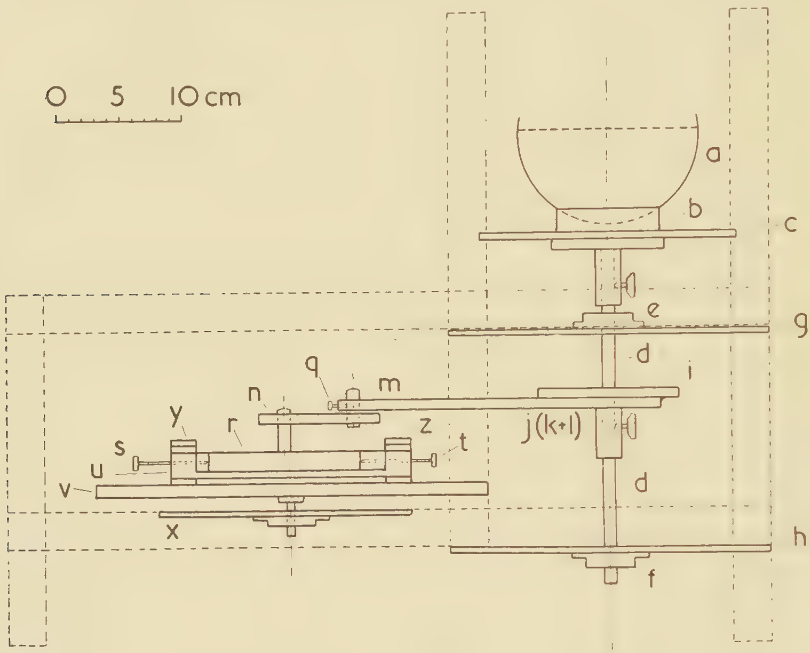

Fig. 2A

the friction brake and, for small changes of the frequency, by adjusting the e.m.f. of the direct current (taken from a stabilized rectifier). The excentricity of the pin on bar *r*, in relation to the axis of rotation of disk *v*, can be regulated by the screws *s* and *t*. As a rule we adjusted it so that cup *a*, containing the oleate system, swings to and fro over an angle of 1° or less. As the connecting rod is about

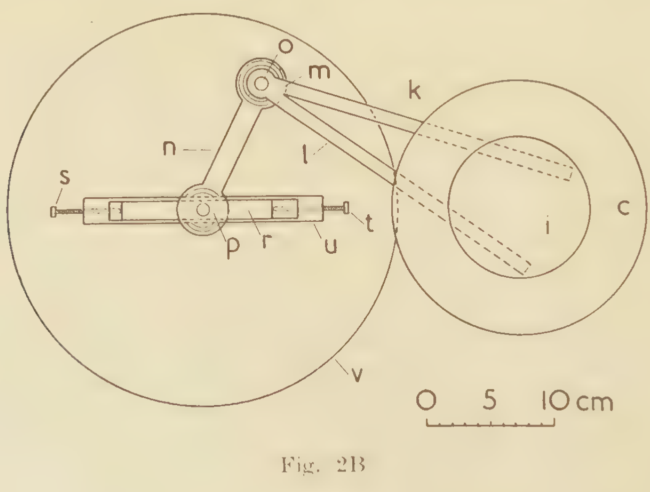

Fig. 2B

30 times (or more) as large as the exentricity which is used, the cup can be considered to swing practically sinussoidal.

The contact strips y and z are fastened to bar u, but they are isolated from it by interposed ebonite plates. They play a role with the repeated intermittent illumination which was used to take photographs of the white track of H_2-bubbles at the moments of largest deviation. Because of contact with two suitably placed contact springs the illumination current is closed at each half rotation of disk v (a 220 volt current was used and 110 volt incandescent lamps). Compare fig. 3, in which many details of fig. 2 have been left out. An iron strip, the cross-section of which is shown in the fig. 3 at a, is attached to the iron-frame above disk v. It carries the bar b, which can be turned round an axis which coincides with the rotation axis of disk v and thus be fixed in the desired position. At one end bar b carries a brass tube c, which serves to adjust the thin brass bar d in the desired

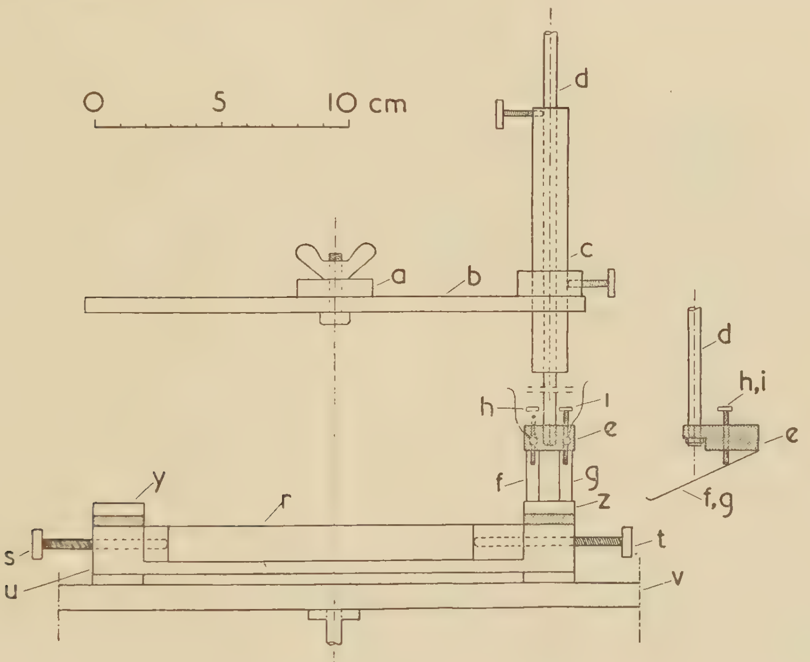

Fig. 3

position. At the lower end of d a hard rubber plate e is fixed. Two contact springs f and g, which are adjustable by the screws h and i are attached to this plate. The springs are connected to wires (not shown in the figure) which are part of the illumination circuit. Each time the contact strip y or z passes underneath the springs, the current is closed.

Because, at resonance of the excited rotational oscillations there will be a phase difference of approximately 90°, bar b is turned and fixed in such a position, that contact is made just at the zero deviation of the exciting oscillation. The photographs will then show the maximum deviations of the oleate system in both directions, whereas the swinging glass vessel is then always at the same position of zero deviation.

For direct observation of the elastic deviations through the telescope of a kathetometer, but also for taking photographs with intermittent illumination, a mirror, under an angle of about 45°, can be placed above vessel a in fig. 2. To support this mirror two of the four vertical hook-iron bars which belong to the frame and

are placed around the axis d, are longer, namely the left one at the foreside and the right one at the backside. Clamps, the details of which are not given in the figure, allow to fasten the axis to which the mirror is fixed and to adjust its inclination.

The mirror can easily be removed and this must be done for the electrolytic marking of the oleate system. Its place must then be taken by a contrivance which

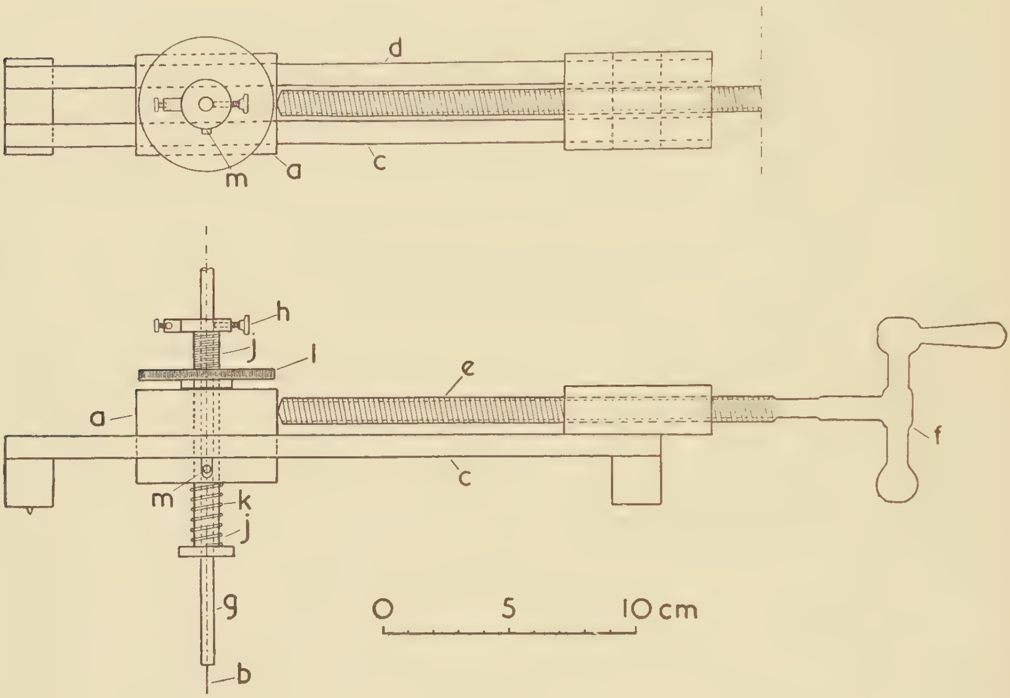

Fig. 4

allows to draw the "white line" through the centre of the equator plane of vessel a. This contrivance (see fig. 4) consists in principle of a suitable shaped massive brassblock a carrying the adjustable Pt-electrode b, two guiding brass bars (of quadratic cross-section) c and d and a long screw e, with handle f.

By turning this handle block a can be slowly pushed from the right to the left, as a result of which the Pt-electrode b moves in a straight line from the glass wall via the centre to the opposite glass wall of the vessel a in fig. 2. The Pt-electrode b is fastened to a thin brass bar g, which by loosening and fastening screw h, can be adjusted to such a position that the Pt-electrode just dips into the surface of the oleate system. This adjusting must be not done close to the glass wall as here the oleate system is curved upwards by capillary forces.

The electrode b, which must be the cathode, receives its current through a wire connected to a clamp opposite to h. As anode we also use a Pt-wire which dips somewhere in the oleate system, but in any case far from the place where the white line must be drawn. It appeared to be of advantage to use a potentiometric device which allows to use the most fávourable e.m.f. (e.g. 2.7 Volt). This is that one which just develops sufficient H_2-bubbles during the relatively long time which the drawing of the line (a few minutes) takes.

A too fast movement of the electrode may lead to breaks and the white line than turns into a number of isolated tracks.

The speed of movement must therefore be such that the relaxation of the elastic

tensions can occur sufficiently. Just before reaching the opposite glass wall the circuit is opened. There are then still enough minute H_2-bubbles around it to finish the last part of the white line.

Having arrived here, we must wait some 20 seconds to allow for the relaxation of the elastic tensions. If one lifted the electrode out of the surface at once, disturbance of the white line would occur.

But also after these 20 seconds the electrode must be removed out of the surface of the oleate system with care, as at a fast movement new elastic tensions are set up. Disk l, tube j, spring k and pin m (attached to j) in a vertical slot in block a, serve this end. By turning disk l the tube j can only move vertically, upwards or downwards, but cannot turn around its axis. So by turning disk l the electrode b can slowly be removed out of the surface of the oleate system.

3. *Preliminary experiments. The existence of a series of rotational oscillations of the elastic oleate system and the ratios of their periods.*

Starting from the experimental disposition described in section 2 and gradually increasing the frequency of the exciting oscillation of the glass vessel, one obtains at first resonance of the simple type of rotational oscillation (the type studied hitherto in the preceding parts of this series) [5]. At the moments of largest deviation the white line of H_2-bubbles has the form as is depicted in fig. 5, case I. At further increase of the frequency

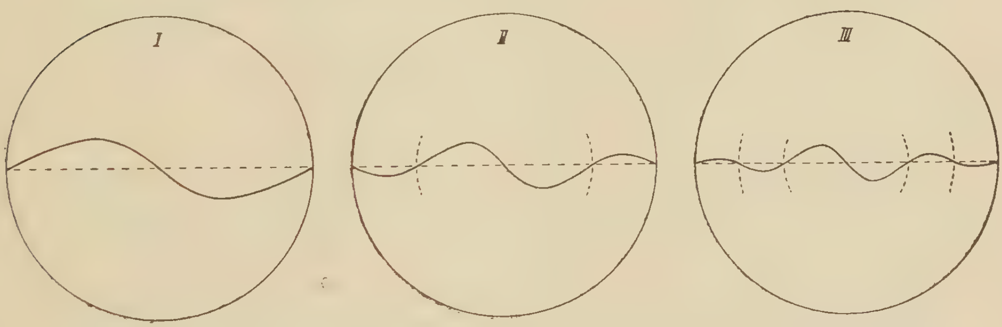

Fig. 5

one obtains resonance of more complicated rotational oscillations, compare in fig. 5 the cases II, III. The oleate system still oscillates in concentric shells, but interposed between glass wall and centre there are now one or two nodal spherical surfaces. The experimental equipment did not allow to increase the frequency further. The aim of this would have been to observe still more members of this series of rotational oscillations.

By direct visual observation we could further observe that in case II of fig. 5, the nodal point does not lie half way centre and glass wall, but somewhat closer to the centre. We observed further that in the cases II

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[5]) For the experiments in this section we used a vessel with $R = 5.0$ cm, the vessel swinging over a total angle of $42'$.

and III the linear elastic deviations are always largest between centre and the nearest nodal point.

By visual observation it is not possible to measure the positions of the nodal points, but we have tried to determine the ratios of the periods of the cases I, II and III. To this end we measured, through the telescope, the deviations as a function of the frequency of the exciting oscillation of the glass vessel and thus obtained resonance curves. From the positions of the resonance frequencies we thus obtained the ratios enlisted in the next survey (in which the suffixes I, II and III written near T indicate, that the period is meant to belong to the types of oscillation I, II and III of fig. 5).

oleate concentr.	salt concentr.	T_{II}/T_{I}	T_{III}/T_{I}
0,6 %	0.52 mol/l K_3-citrate . . .	0.574	0.426
	1.05 ,, KCl	0.564	—
	1.05 ,, KCl	0.572	—
1.2 %	1.05 ,, KCl	0.570	0.440
		mean = 0.570	*mean* = 0.433

The above figures are certainly not very accurate because of the not sufficient constancy of the electromotor driving the swinging device [6]). Still the few measurements which could be performed clearly showed that the rotational oscillations II and III are not harmonics (in the ordinary sense) of I, for the ratios of their resonance frequencies are no integers.

4. *Characteristics of the first, second and third rotational oscillation of an elastic fluid enclosed in a spherical vessel, for the case that slipping along the wall is absent.*

The possibility of an infinite series of rotational oscillations has already been mentioned by Burgers [7]), in a passage which runs as follows: "The rotational oscillation considered in that paper which is determined by the first root $\zeta = 4{,}493$ of the equation tg $\zeta = \zeta$, is the first one of an infinite series of possible rotational oscillations; it is characterized by the absence of a nodal point in the function Φr, whereas the other solutions have 1, 2, - - - nodal points respectively."

It seems indicated to consider if the rotational oscillations of higher order observed in section 3 may correspond to those mentioned by Burgers.

The formula for the first rotational oscillation (case I in fig. 5)

$$T = \frac{2\pi R}{4.493}\sqrt{\frac{\varrho}{G}},$$

[6]) As this error increased more and more the study of resonance curves had to be abandoned for the time present.

[7]) J. M. Burgers, these Proceedings 52, 113 (1949); see footnote 3 on page 113.

will retain its general form for the rotational oscillations of higher order, but for the numerical factor 4.493, which must be replaced by one of the further roots of the equation tg $\zeta = \zeta$. We found the following values for a few further roots: second root = 7.725; third root = 10.904; fourth root = 14.066. With the same elastic system in the same vessel the ratio of the periods of any two rotational oscillations will then be inversely proportional to the ratio of the roots determining these rotational oscillations. Thus:

$$T_{\mathrm{II}}/T_{\mathrm{I}} = 4.493/7.725 = 0.582$$
$$T_{\mathrm{III}}/T_{\mathrm{I}} = 4.493/10.904 = 0.412$$

Comparing the calculated ratios with those found experimentally ($T_{\mathrm{II}}/T_{\mathrm{I}} = 0.570$; $T_{\mathrm{III}}/T_{\mathrm{I}} = 0.433$; see survey in section 3) we get the strong impression, that the series of rotational oscillations observed in the oleate systems is the series determined by the successive roots ζ of the equation tg $\zeta = \zeta$.

Two further properties of the rotational oscillations of the oleate system which were observed visually in section 3, (the node in case II of fig. 5 lies not exactly at 0.5 R, but somewhat closer to the centre; the linear amplitude is largest between centre and adjacent node) also follow from the mathematical equations. Compare table I and fig. 6, in which for the first, second and third rotational oscillations the linear amplitude ($r\Phi$) has been given as a function of the distance r from the centre. The amplitudes have been expressed in percents of that at the maximum (first rotat. oscill.) or at the maximum closest to the centre (second and third rotat. oscill.).

For calculating this table we start from the expression

$$\frac{\sin \zeta r - \zeta r \cos \zeta r}{\zeta\, (\zeta r)^2},$$

which gives — apart from a coefficient — the linear displacement $r\,\Phi$ in the equator plane, if R is taken 1. [8]) As we wish to express the linear displacements in percents of the displacement at the maximum (first rotat. oscill.) or at the maximum closest to the centre (second and third rotat. oscill.), it suffices to calculate the expression

$$\frac{\sin \zeta r - \zeta r \cos \zeta r}{(\zeta r)^2}$$

for a number of values of r (0.1; 0.2; —) and taking for the first rotational oscillation $\zeta = 4.493$, for the second $\zeta = 7.725$ and for the third $\zeta = 10.904$. We must, however, know the value of this expression at the above-mentioned maximum, which involves the knowledge of those values of r where the maxima or minima are situated. By differentiation we obtain the equation $\operatorname{tg} z - \frac{2z}{2 - z^2} = 0$, in which $z = \zeta\, r$. This equation has been solved graphically and we obtained for the positions of the maxima or minima: first rotat. oscill. $r = 0.463$ (max.); second

[8]) J. M. Burgers, these Proceedings **51**, 1211 (1948).

TABLE I

The linear amplitude of the first, second and third rotational oscillation as a function of the distance (r) from the centre. The linear amplitude is expressed in percents of that at the maximum (first rotat. oscill.) or at the maximum closest to the centre (second and third rotat. oscill.)

first rotational oscillation $\zeta = 4.493$			second rotational oscillation $\zeta = 7.725$			third rotational oscillation $\zeta = 10.904$		
	100 r/R	linear amplitude		100 r/R	linear amplitude		100 r/R	linear amplitude
centre	0	0	centre	0	0	centre	0	0
	10	33.7		5	29.1		5	40.4
	20	63.2		10	55.6		10	73.8
	30	85.5		15	77.2		15	94.6
	40	97.7		20	92.2	max.	19.1	100
	45	99.9	max.	26.9	100		25	89.4
max.	46.3	100		30	98.5		30	66.7
	48	99.8		35	90.1		35	37.1
	50	99.3		40	75.3		40	6.7
	60	90.3		45	56.1	node	41.2	0
	70	72.8		50	34.4		45	— 18.4
	80	49.5		55	12.8		50	— 34.0
	90	24.1	node	58.2	0	min.	54.5	— 38.5
	95	11.3		60	— 6.8		60	— 32.5
wall	100	0		65	— 22.5		65	— 19.1
				70	— 33.2		70	— 2.8
			min.	76.9	— 38.5	node	70.8	0
				85	— 32.0		75	12.2
				90	— 22.9		80	22.0
				95	— 11.7	max.	84.4	24.9
			wall	100	0		90	20.7
							95	11.4
						wall	100	0

rotat. oscill. $r = 0.269$ (max.) and $r = 0.769$ (min.); third rotat. oscill. $r = 0.191$ (max.), $r = 0.545$ (min.) and $r = 0.844$ (max.).

For calculating the positions of the nodes, the expression

$$\frac{\sin \zeta r - \zeta r \cos \zeta r}{\zeta (\zeta r)^2}$$

must be zero, therefore $\sin \zeta r - \zeta r \cos \zeta r = 0$ or $\operatorname{tg} \zeta r = \zeta r$, which equations have the roots $\zeta r = 0$, $\zeta r = 4.493$, $\zeta r = 7.725$, $\zeta r = 10.904$, ...

For the *first rotational oscillation* $\zeta = 4.493$, consequently $\zeta r = 0$ and $\zeta r = 4.493$ are the roots of the equation $\operatorname{tg} \zeta r = \zeta r$, in which r is smaller than or equal to 1. Thus here the linear displacement is only zero at the centre ($r = 0$) and at the wall of the vessel ($r = 1$). There is no node between centre and wall.

For the *second rotational oscillation* $\zeta = 7.725$. Therefore here the first three roots of $\operatorname{tg} \zeta r = \zeta r$ must be considered.

The linear displacement is here zero at $r = 0$ (centre), $r = 4.493/7.725 = 0.582$ (node) and $r = 1$ (wall).

For the *third rotational oscillation*, characterized by $\zeta = 10.904$, we find similarly that the linear displacement is zero at $r = 0$ (centre), $r = 4.493/10.904 = 0.412$ (node), $r = 7.725/10.904 = 0.708$ (node) and $r = 1$ (wall).

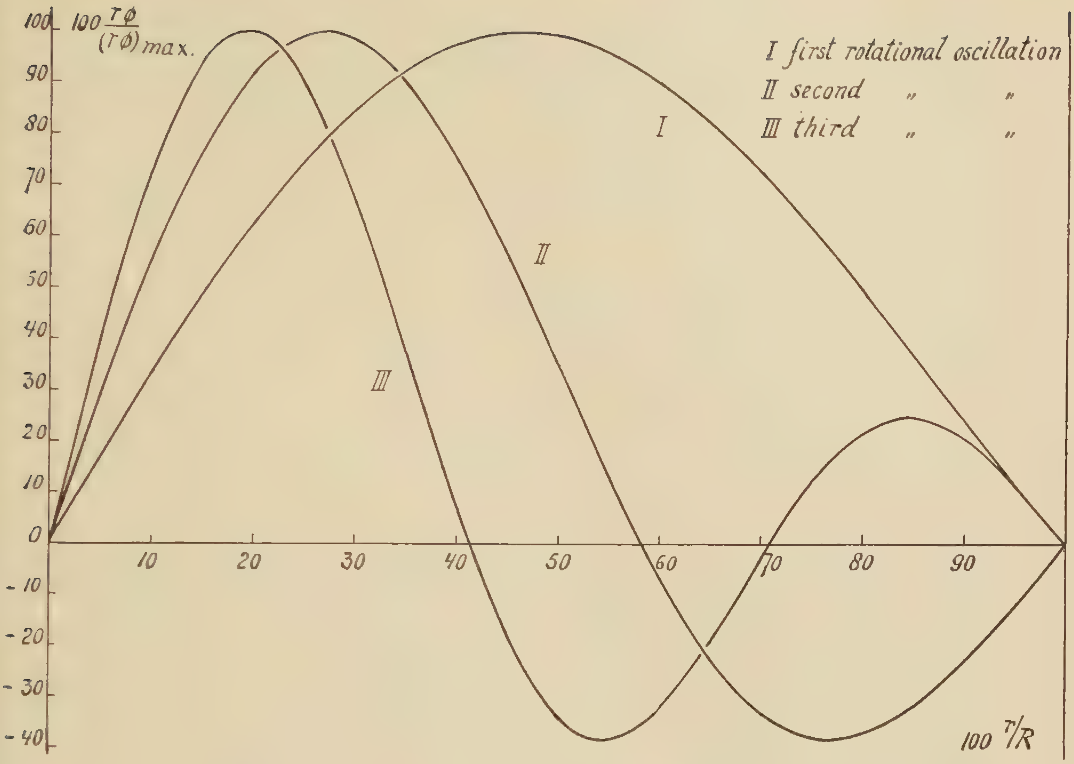

Fig. 6

5. *Measurements of the linear displacement as a function of the distance from the centre, on oscillating* 1.2 % *and* 0.6 % *oleate systems*

As is discussed in section 2) the photographs of the white track of minute H_2-bubbles were made by repeated intermittent illumination at the moments of largest elastic deviations (alternatively in opposite directions). As the frequency of the exciting oscillation was brought as near as possible to the resonance frequency of the rotational oscillation of the oleate system (at which a phase difference of approximately 90° exists between exciting and excited oscillations) the glass vessel is always practically in the same position (approx. zero deviation).

If there is no slip of the oleate system along the glass wall — as in the 1.2 % oleate system, because of $\Lambda \infty R$ — we must therefore expect photographs of the type I in fig. 7. In these photographs the two intersecting (at the centre) S-shaped curves meet at either side in a single point at the glass wall.

If there is a slip of the oleate system (as was supposed by J. M. Burgers in order to give an explanation of Λ being independent of R which was observed in the 0.6 % oleate system; compare fig. 1) we would expect photographs of the type II in fig. 7, in which the two curves do not meet in a single point at the glass wall.

All photographs of the first rotational oscillation actually obtained, as well with 1.2 % as with 0.6 % oleate systems, showed, however, type III of fig. 7, in which the two curves meet in points situated at a short distance before the glass wall (exaggerated in fig. 7; the gap amounting to ± 2 %

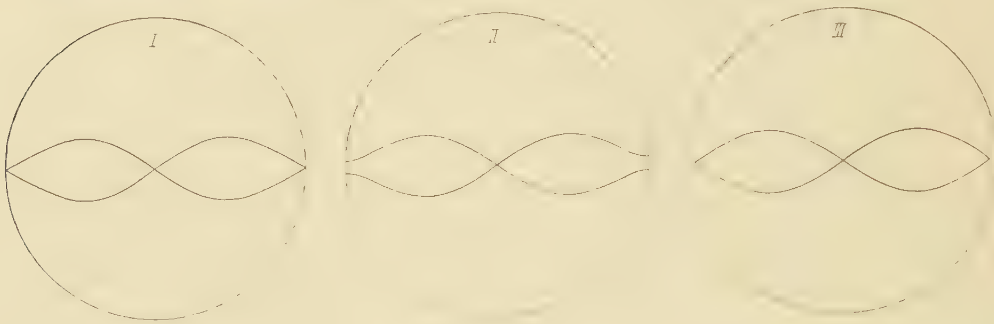

Fig. 7

of R). The same we found for the second rotational oscillation. At first sight one may think that a nodal point lies close to the glass wall, but we never observed the continuation of the apparently intersecting curves between "point of intersection" and glass wall.

The distance between the "point of intersection" of the curves and the glass wall has been shown to be only an optical delusion. Close to the glass wall the oleate system is curved upwards by capillary forces. We made photographs of the glass vessel with a thermometer scale rounded off at both ends resting with its ends on the glass wall (practically in the equatorplane). In the case of the not filled glass vessel (thermometer scale in air) we obtained photographs, in which the image of the ends practically coincides with the image of the inner wall of the glass vessel. If, however, water or the oleate system was poured into the vessel, so that the scale is just beneath the surface, photographs were obtained which show an empty space between the ends of the scale and the wall of the vessel.

Further measurements of the photographs and comparison with the thermometer scale itself resulted in: a) that the used photographic camera gave practically no distortion of the scale if it was photographed in air, b) the same result, except for the regions close to the glass wall, was obtained if it was covered with water or 1.2 % Na-oleate solution. Before evaluating the function $r\Phi = f(r)$ from the photographs of the oscillating systems, the values of r were corrected with the aid of a table based on the above photographs of the scale, when it was covered with the oleate system.

Enlarged images (of approx. 30 cm diameter of the negatives) were projected on millimeter paper with the aid of a projector lantern and with a pencil the curves and the wall of the vessel were drawn over. [9]) On these drawings on large scale

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[9]) We are much indebted to H. L. Booij for making these drawings. Because of the inconstancy of the electromotor during the repeated intermittent exposition the images were often not sharp and therefore difficultly visible.

we measured the distance between the two inner sides of the vessel and the intersection point in the centre. This ought to give two equal values of R, but in some cases gave slightly different values. Now we measured, at both sides of the centre, the vertical distance of the curves at $r/R = 0.1$, 0.2 etc., which gave two sets of the twofold values of the vertical distance at 10 %, 20 % etc. of the radius. From these we obtained the meanvalue l of the single deviation (still as a vertical distance) as a function of r/R [10]), or — if we take $R = 1$ — as a function of r (the latter still as the distance from the centre to the footpoint of the perpendicular). For a comparison with the mathematical theory we must, however, calculate

TABLE II

Measurements on the 1.2 % oleate system of the linear amplitude ($r\,\Phi$) of the first rotational oscillation as a function of the distance (r) from the centre. The linear amplitude is expressed in percents of that at the maximum

no. 1		no. 2		no. 3		no. 4	
$100\,r/R$	$r\,\Phi$	$100\,r/R$	$r\,\Phi$	$100\,r/R$	$r\,\Phi$	$100\,r/R$	$r\,\Phi$
5.9	18.5	5.8	17.1	6.4	19.1	5.3	18.4
11.7	36.3	11.8	35.7	12.8	38.7	10.6	36.6
22.7	64.3	22.9	63.2	25.0	71.7	21.0	66.6
33.4	86.9	33.6	85.9	35.6	90.2	31.2	88.3
43.4	97.6	43.7	96.9	45.4	99.3	41.2	98.4
46.3	100.0	46.6	100.0	48.2	100.0	44.2	100.3
52.8	97.6	53.2	99.0	54.4	97.3	51.0	98.9
62.8	88.6	62.3	89.3	63.2	87.1	60.9	91.5
71.4	71.9	71.5	72.2	71.8	65.6	70.7	75.3
80.8	48.5	80.9	46.7	80.9	40.4	80.6	51.5
90.7	21.0	90.7	20.5	90.7	15.2	90.7	23.6
95.9	6.6	95.9	7.5	95.9	4.4	95.9	9.4
99.4	0	99.4	0	99.0	0	100.5	0

no. 5		no. 6			
$100\,r/R$	$r\,\Phi$	$100\,r/R$	$r\,\Phi$	$100\,r/R$ mean	$r\,\Phi$ mean
5.4	16.9	5.7	18.1	5.8	18.0
10.8	35.7	11.4	35.4	11.5	36.4
21.4	€5.4	22.4	63.9	22.6	65.9
31.7	86.6	33.0	87.3	33.1	87.5
41.7	98.5	43.1	98.8	43.1	98.3
44.7	99.3	46.0	100.0	46.0	99.9
51.5	99.6	52.6	97.8	52.6	98.4
61.2	91.3	62.0	89.8	62.1	89.6
70.9	74.3	71.3	70.9	71.3	71.7
80.6	51.8	80.8	46.8	80.8	47.6
90.6	24.5	90.7	19.4	90.7	20.7
95.9	10.5	96.2	6.4	96.0	7.5
99.9	0	99.4	0	99.5	0

[10]) A correction for the actual value of r/R because of the above discussed distorticn, was only necessary for $r/R = 0.90$ and 0.95.

from this the length of the arc ($= r\,\Phi$) as a function of the corresponding value of r, which we will denote with $r_{corr.}$. From l and r we know tg Φ and therefore Φ in radians. Arc Φ is obtained by multiplication with $r_{corr.}$, which latter is obtained by multiplication of r with sec Φ.

Table II gives the results of six measurements on the first rotational oscillation (no 1—6) with the 1.2 % oleate systems, in which different oleate preparations have been used (no 1, 2, 3, chemically pure K-oleate; no 4, 5, 6 Na-oleate from Baker), also vessels of different diameters (no 1—5 with $R = 5.07$ cm, no 6 with $R = 4.34$ cm) and different salt media (no 1—3 with 0.98 N KCl; no 4—5 with 1.05 N KCl and no 6 with 0.52 moles/l potassium citrate).

Table III (first rotational oscillation) and IV (second rotational oscil-

TABLE III

Measurements on the 0.6 % oleate system of the linear amplitude ($r\,\Phi$) of the first rotational oscillation as a function of the distance (r) from the centre. The linear amplitude is expressed in percents of that at the maximum

no. 7		no. 8			
$100\,r/R$	$r\,\Phi$	$100\,r/R$	$r\,\Phi$	$100\,r/R$ mean	$r\,\Phi$ mean
5.2	19.8	5.1	18.5	5.2	19.2
10.3	33.3	10.2	35.4	10.3	34.4
20.6	66.9	20.3	67.7	20.5	67.3
30.7	89.4	30.3	87.6	30.5	88.5
40.7	99.0	40.4	99.0	40.6	99.0
43.8	100.3	43.4	99.7	43.6	100.0
46.7	100.0	46.4	100.0	46.6	100.0
50.7	100.2	50.3	97.6	50.5	98.9
60.6	91.0	60.4	89.4	60.5	90.2
70.6	73.1	70.4	71.2	70.5	72.2
80.5	46.5	80.4	48.7	80.5	47.6
90.7	16.6	90.6	22.4	90.7	19.5
96.2	3.8	95.9	9.0	96.1	6.4
98.5	0	100.4	0	99.5	0

lation) give the measurements with the 0.6 % oleate systems (at all times Na-oleate from Baker and a salt medium of 0.52 moles/l potassium citrate) with different vessels (no 7, 9 and 10, with $R = 4.34$ cm; no 8 with $R = 5.07$ cm).

The results have been represented in three figures only (fig. 8 A, 8 B and 9) using as "experimental points" the means of the values r and $r\Phi$ on each horizontal row of the tables II, III and IV. These points, the endpoints to the right with $r\Phi = 0$ included, lie close to the theoretical curve in particular in the figs 8 A and 8 B [11]). These curves have been drawn

[11]) The endpoints to the right, which represent $r\,\Phi = 0$, lie at a value of the abscissa which is somewhat lower (0.5 % in fig. 8; 2 % in fig. 9) than $100\,r/R = 100$. This is possibly caused by the fact that we used the correction here following from

TABLE IV

Measurements on the 0.6 % oleate system of the linear amplitude ($r\,\Phi$) of the second rotational oscillation as a function of the distance (r) from the centre. The linear amplitude is expressed in percents of that at the maximum closest to the centre

no. 9		no. 10			
$100\,r/R$	$r\,\Phi$	$100\,r/R$	$r\,\Phi$	$100\,r/R$ mean	$r\,\Phi$ mean
6.1	32.9	5.5	30.4	5.8	31.7
11.8	59.6	10.8	56.0	11.3	57.8
17.3	83.5	16.0	77.4	16.7	80.5
22.5	96.6	21.2	93.9	21.9	95.3
27.2	100.0	26.1	99.3	26.7	99.7
31.8	95.8	30.9	97.6	31.4	96.7
36.4	86.7	35.7	88.0	36.1	87.4
40.9	71.9	40.4	71.2	40.7	71.6
45.6	52.6	45.3	48.1	45.5	50.8
50.3	29.9	50.2	23.9	50.3	26.9
55.2	7.7	55.2	3.2	55.2	5.5
60.3	10.9	60.3	13.0	60.3	−12.0
65.4	27.1	65.3	31.5	65.4	−29.3
70.5	35.8	70.4	40.8	70.5	−38.3
75.5	38.7	75.4	41.4	75.5	−40.1
80.6	37.3	80.6	38.5	80.6	−37.9
85.5	29.7	85.5	29.1	85.5	−29.4
90.7	18.6	90.7	15.8	90.7	−17.2
96.2	4.5	96.2	3.1	96.2	− 3.8
98.8	0	97.2	0	98.0	0

according to the data given in table I, which table represents $r\Phi$ as a function of r in the case that there is no slipping along the wall of the vessel.

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photographs of the thermometer scale (covered with oleate system), which were taken while the vessel was at rest (see above). In the rocking vessel the distortion close to the wall in the direction of the abscissa will presumably be larger because of centrifugal effects.

The latter may even change the plane interface into a slightly hollow one, which will cause an extra optical distortion over the whole diameter of the vessel. This might explain why in fig. 9 the part of the curve from the first maximum up to higher values of r/R is displaced slightly to the left of the theoretical curve.

If one draws the curves for each of the no. 1—6 of table II one observes that at $r/R = 0.96$ the correspondence with the theoretical value of $r\,\Phi = 9$ % (following from table I) is sometimes better, sometimes worse than that of the mean value of $r\,\Phi$ given in table II (= 7.5 %). The same applies for the correspondence of the single abscissa values and the mean abscissa value with the theoretical value $r/R = 1.00$ for the endpoints to the right with $r\Phi = 0$. It appears that both kinds of deviations from the theoretical values are slight (in order of the experimental errors) for those experiments in which the values of the angular displacement Φ at the maximum (at $r/R = 0.463$) were the smallest. These were the series no. 4 and 5 with $\Phi = 0.213$ and 0.256. The differences are larger for the no. 1, 2 and 6, in which Φ at the maximum was respectively 0.375, 0.392 and 0.353. The difference is the largest for no. 3, where Φ was the largest of all ($\Phi = 0.465$).

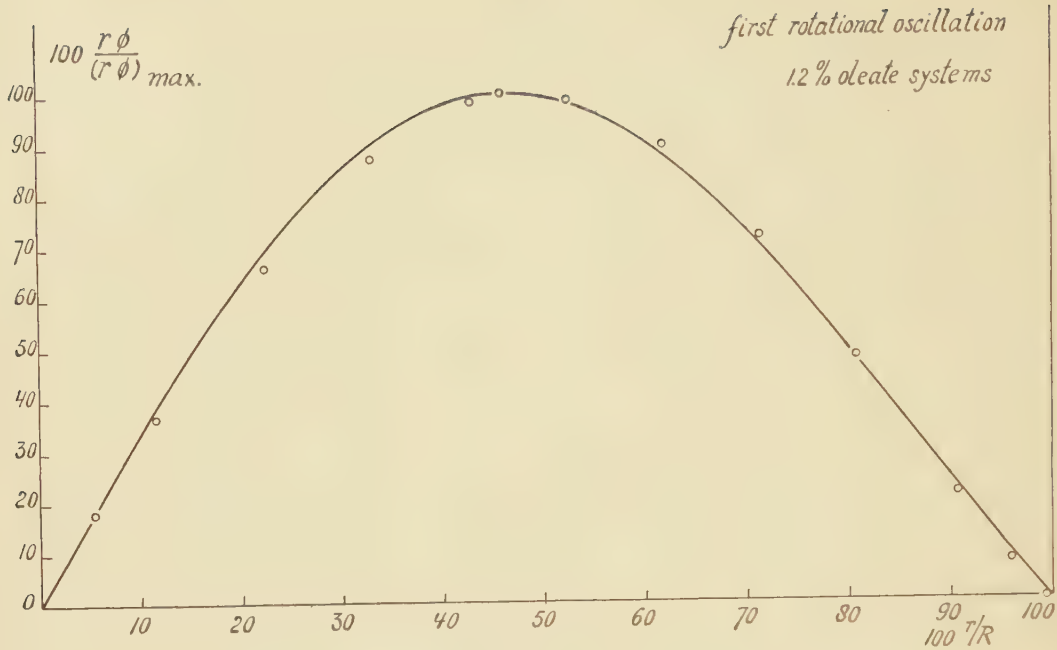
$100 \frac{r\phi}{(r\phi)_{max.}}$
first rotational oscillation
1.2 % oleate systems
0
10
20
30
40
50
60
70
80
90
100
100 r/R

Fig. 8A

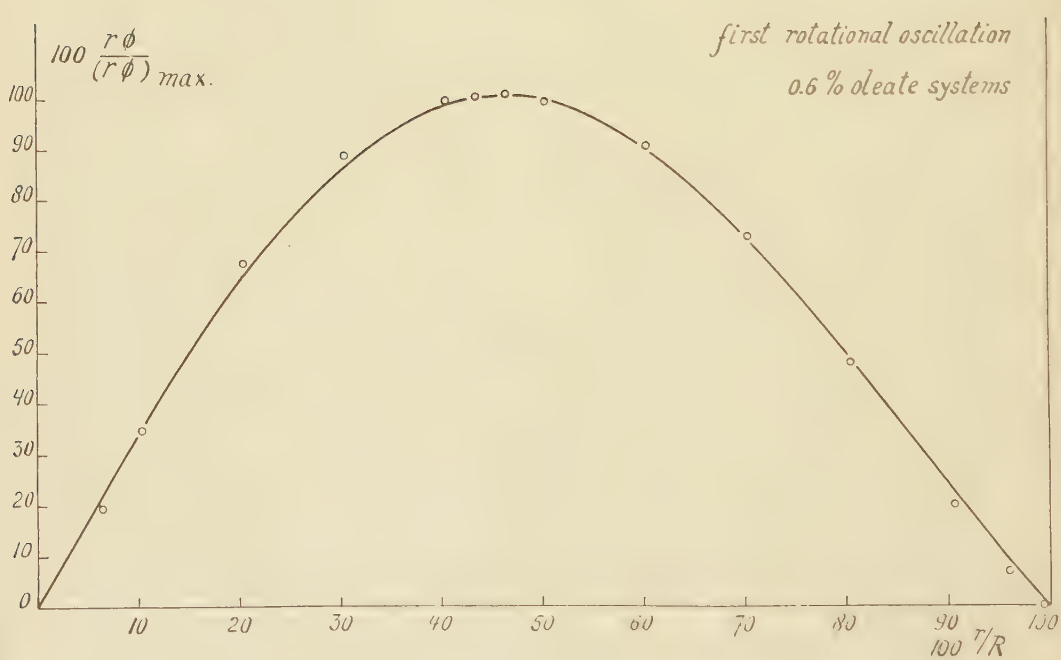
$100 \frac{r\phi}{(r\phi)_{max.}}$
first rotational oscillation
0.6 % oleate systems
0
10
20
30
40
50
60
70
80
90
100
100 r/R

Fig. 8B

The results with the 1.2 % oleate systems are quite as were to be expected (cf part III: $\Lambda \propto R$), but those with the 0.6 % oleate systems enforce our doubt as to the real existence of slipping along the wall of the vessel. Compare in the introduction the different character of the damping in the 1.2 % and in the 0.6 % oleate system.

To be certain that for the 0.6 % oleate systems under the conditions of saltconcentration and temperature as were used in photographing the elastic deviation Λ is still independent of R, Mr H. J. VAN DEN BERG was so kind to determine Λ at 0.48 moles/l potassium citrate [12]) and at 17° in quite the same way as is given in details in the parts II and III of this series. The results (see survey below) leave no doubt as to the reality of Λ being independent of R.

R (cm)	b_1/b_3	Λ	
3.01	1.440 ± 0.008	0.365 ± 0.006	
5.64	1.446 ± 0.010	0.369 ± 0.007	*mean*
6.77	1.436 ± 0.009	0.362 ± 0.006	$\Lambda = 0.366$
9.24	1.444 ± 0.008	0.368 ± 0.006	

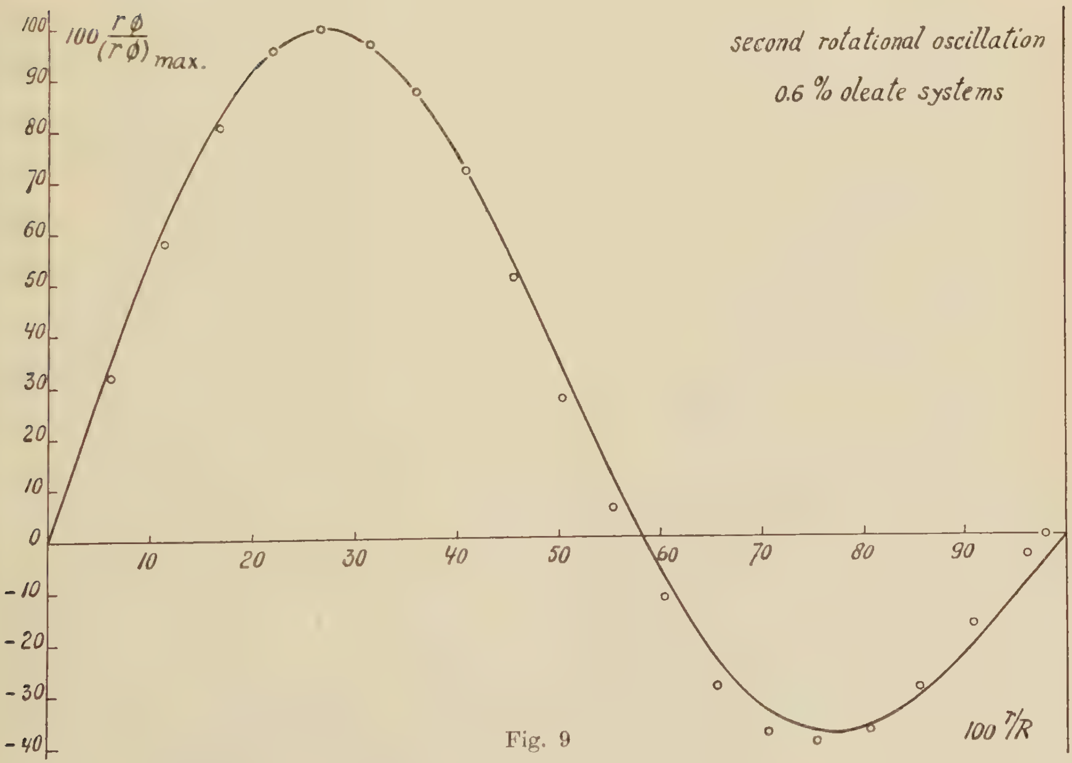

Fig. 9

12) The oleate preparation of BAKER, used in these control experiments showed a slightly different position of the citrate concentration corresponding to minimum damping (0.48 instead of 0.52 moles/l). This explains the choice of the citrate concentration.

Still we have to consider another point before we may safely draw a conclusion. In fig. 1 we perceive that the value of $r\Phi$ at the wall decreases with decreasing Λ. As in our case Λ is lower than in the two examples of fig. 1 with finite values of Λ, one could imagine that $r\Phi$ at the wall might be so small as to disappear in the experimental errors.

We must therefore know which value for $r\Phi$ one might expect in the case of slipping along the wall at $\Lambda = 0.366$. On our request J. M. BURGERS was so kind to provide us with the means to calculate approximately this $r\Phi$ in percents of $r\Phi$ at the maximum and it appeared that we have to expect here a value of 13 %.

Such a great value of $r\Phi$ would easily have been found in our experiments if it had really been present. We therefore come to the conclusion that the problem of the nature of the damping in the 0.6 % oleate systems (Λ independent of R) is still unsolved.

6. *Summary*

1. The linear displacement in the equator plane of the sphere as a function of the distance from the centre has been determined on oscillating 1.2 % and 0.6 % elastic oleate systems. For technical reasons exactly half filled spheres had to be used instead of completely filled ones.

2. The results obtained with the 1.2 % oleate systems (in which Λ is proportional to R) are corresponding reasonably well with the theoretical expectations for an oscillating spherical mass of an elastic fluid, which does not slip along the wall of the vessel.

3. The results obtained with the 0.6 % oleate systems (in which Λ is independent of R) are exactly like the results obtained with the 1.2 % oleate systems, that is a slipping of the elastic system along the wall of the vessel could not be detected.

4. As a consequence we must take back the conclusion made in part III of this series that the damping in the 0.6 % oleate system is described satisfactory by the case treated theoretically by J. M. BURGERS, in which the damping is a consequence of slipping of the elastic fluid along the wall of the vessel (in spite of the fact that this theory gives a Λ which is independent of R and a number of quantitative relationships which were confirmed in part III).

5. By increasing the frequency of the exciting oscillation a number of further rotational oscillations has been observed with the oleate system. They correspond to oscillations characterized by the second, third, - - - roots of the equation tg $\zeta = \zeta$.

6. The linear displacements as a function of the distance to the centre has been determined for the second rotational oscillation and here too a reasonable correspondence was found with theoretical expectations.

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HISTOLOGY

SEX CELL FORMATION IN EXPLANTS OF THE FOETAL HUMAN OVARIAN CORTEX. II

BY

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B. Results of cultivating ovarian cortical fragments from 24—26 weeks old human foetuses

Series 5–3–'49. Twenty-six weeks old human foetus (see fig. 1, Plate V).

The technical procedures were the same as those mentioned on page 1312, series 5–5–'48.

A total of 160 explants was cultivated.

Results: After 3 days of cultivation a deterioration and a cellular degeneration occurred in the central parts of the explants (see fig. 2, Plate V). In some of the explants an increase in collagenous fibres could be observed. In the meantime, in this series also a cord-like proliferation started from the superficial epithelium, in which some oöcytes developed. It is remarkable that between the 12th and 14th day of cultivation the *follicle cells* of the original "primordial" follicles in the centres of many explants *remained intact, while the oöcytes had completely degenerated* (see fig. 3, Plate V). *In some of these explants the intact follicle cells started a proliferation and formed epithelial cords* (see fig. 4, Plate V. *Moreover in these cords too new oöcytes were formed* (see fig. 5, Plate V) which is a higher magnification of the part indicated with an arrow in fig. 4).

Starting on the 12th day (see fig. 6, Plate V) and continuing until the 19th day (see fig. 7, Plate V), some of the *peripherial cords changed into cyst or tubules*, bordered by one layer of cuboidal cells. In the walls of these tubules and in the cords oöcytes in early prophase stages were observed (see fig. 7 and 8, Plate V).

On the 14th day of cultivation *some explants suddenly showed a balloon-shaped extension of the covering epithelium* in which again a strong *regeneration of cords and tubules* occurred, with numerous oöcytes in all stages of the early meiotic prophase. Finally *all cells of the covering epithelium* showed large *apical or basal vacuoles* after the 14th day of cultivation (see fig. 1, Plate VI).

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Series 21–4–'49. Twenty-four weeks old human foetus.

80 Explants from the *most peripheral parts of the cortex* of the ovarian poles were cultivated in a medium made up as for series 5–5–'48 (page 1312).

Results: The cultivation period lasted 14 days, the initial changes in the explants always being the same: encapsulation, degeneration in the centres and regeneration starting from the covering epithelium. Between the ninth and the fourteenth day, numerous new oöcytes were formed in the cords and many stages of the meiotic prophase were found (see fig. 2, Plate VI). *In some explants double nucleated ova were found after 12 days of cultivation* (see fig. 3, Plate VI). In comparison with the prophase nuclei in the same explant and photographed with the same magnification (see fig. 2, Plate VI) *these nuclei were only small and contained less chromatic material which suggested that a complete first maturation division had occurred in a foetal oöcyt.*

Finally, between the 12th and 14th day of cultivation, the formation of tubules was observed.

Series 5–1–'50. Twenty-six weeks old human foetus (see fig. 4, Plate VI).

160 Explants were made from the cortex of the ovarian poles and they were divided over two media:

A. was composed as that for series 5-5-'48 only with less streptomycin. (0.5 U per medium).

B. was composed in a similar way, but the foetal brain press juice was replaced by the ascitic fluid obtained from a patient suffering from a carcinosis peritonei.

Results:

ad A. The initial behaviour of the explants was the same as in the two preceding series. However the formation of tubules had already started after six days of cultivation (see fig. 5, Plate VI). Many new ova were found, some of them in early metaphase stages. After 12 days of cultivation a temporary papillomatous proliferation of the covering epithelium had occurred in some of the explants, many of the cells possessing large apical vacuoles (see fig. 6, Plate VI). *Between the 18th and 21st day of cultivation the cords and tubules became divided into nests and cysts by fine collagenous membranes* (see fig. 8, Plate VI) and at the same time *a number of metaphase and anaphase* stages belonging to *the first maturation division* were observed (see fig. 7 and 8, Plate VI). In the metaphase stages the pairs of chromosomes were easily to be observed and each individual chromosome was split into two chromatids as an introduction to the second maturation division (see fig. 7, Plate VI).

Furthermore, after some 24 days of cultivation, *an important growth of some of the explants suddenly occurred, which resulted in a newly formed regeneration zone with numerous cords and tubules, while in the centres of these explants a number of "primordial" follicles appeared with a normal vitellogenesis in the cytoplasm* (see fig. 9, Plate VI).

ad B. In this series the initial behaviour was the same as in medium *A*. Only, in one of the explants, *some nuclei with prochromosomes* were observed as described by SWEZY and EVANS in very young human embryo's (see fig. 1, Plate VII). Between the 18th and 30th day of cultivation a new phenomenon regularly occurred, viz. *an eccentric balloon-shaped outgrowth* of the explants (see fig. 2 and 4, Plate VII). In these cases the original explants often degenerated, as demonstrated in fig. 2, Plate VII. However from the covering epithelium bordering the "balloons" a number of cord-like (see fig. 4, Plate VII) or tubule-like (see fig. 2 and 3, Plate VII) proliferations always started. In these, numerous new ova were observed (see fig. 4 and the higher magnification of this preparation given in fig. 5 of Plate VII).

The nuclei of these cells showed all stages of the meiotic prophase as well as a number of metaphase and anaphase stages. In some of the "balloons" some highly developed "primordial" follicles were found with an apparently normal vitellogenesis, the occurrence of a Balbiani corpuscle and the formation of a thin zona pellucida (see fig. 6, Plate VII). In one of the explants a triasteric division was formed (see fig. 7 and 8, Plate VII, representing two micrographs of the same cell at different levels).

Finally, after some 21 days of cultivation the cords and tubules in many of the explants were divided by collagenous septa into nests or cysts (see fig. 1, Plate VIII) with a number of indifferent cells and one or two egg cells.

Conclusions based on the experiments with explants of the ovarian cortex from 24—26 weeks old human foetuses

As a rule many of the developmental potencies already observed in the explants of the younger ovaries could be confirmed. However some special points may be particularly stressed:

a. The cultivation period could be extended to about 30 days.

b. From the series 5–3–'49 it became clear that *cord-like proliferations could not only develop from the surrounding epithelium*, but *also from follicle cells remaining intact after the ova to which they belonged had degenerated. In these cords of follicle cells new ova were formed*, which indicates the potency of the follicle cells to form these special elements and *stresses the intimate relationship between the germinal epithelium, the follicle cells and the oöcytes.*

c. In the series 5–1–'50 we observed a number of metaphase and anaphase stages of the first maturation division, indicating that *foetal ova are also potentially able to pass the diplotene stage of the first meiotic division.*

d. In the series 21–4–'49 even some *double nucleated oöcytes* were found, with only a small amount of chromatic substances, which suggests a *completed first meiotic nuclear division.*

Moreover, in the series 5–1–'50, a *triasteric division* was found. *These findings principally indicate that even in embryonic ovarian tissues maturation division processes can proceed to a further stage than usual, depending obviously on the special circumstances of the in vitro cultivation.*

e. After some 21 days of cultivation of the explants of series 5–1–'50, the cords and the tubules which were formed by proliferation of the surrounding epithelium altered their appearance. They became divided into nests and cysts by fine collagenous membranes. In the nests and cysts more than one oöcyte often occurred.

f. In two series (5–3–'49 and 5–1–'50) of cultivation an explosive growth was found in the second or third week, either on all sides of the explants, or only on one side. In the latter case "balloon"-shaped explants were formed.

These extensions were bordered by an indifferent epithelial capsule and were filled by a loose mesenchymal tissue. Shortly afterwards, numerous cords and tubules were formed from the covering epithelium penetrating the mesenchym. In these cords a great number of new oögonia and oöcytes developed and all meiotic prophase stages were seen. Some of the ova developed to primordial follicles with an apparently normal vitellogenesis, a Balbiani corpuscle and a thin zona pellucida. *These findings indicate still more strongly than before the potency of the morphologically indifferent covering epithelium to develop new ova and new follicle cells.*

g. In one series (5–1–'50) a temporary papillomatous proliferation of the covering epithelium occurred, showing a tendency to escape the organo typical organisation, without however any direct results.

C. Results of cultivating cortical fragments from the ovary of a mature new-born human foetus

Only one experiment has been done with explants prepared from the poles of the ovarian cortex of a full-grown foetus (see fig. 2, Plate VIII).

Series 13–1–'50. (150 Explants).

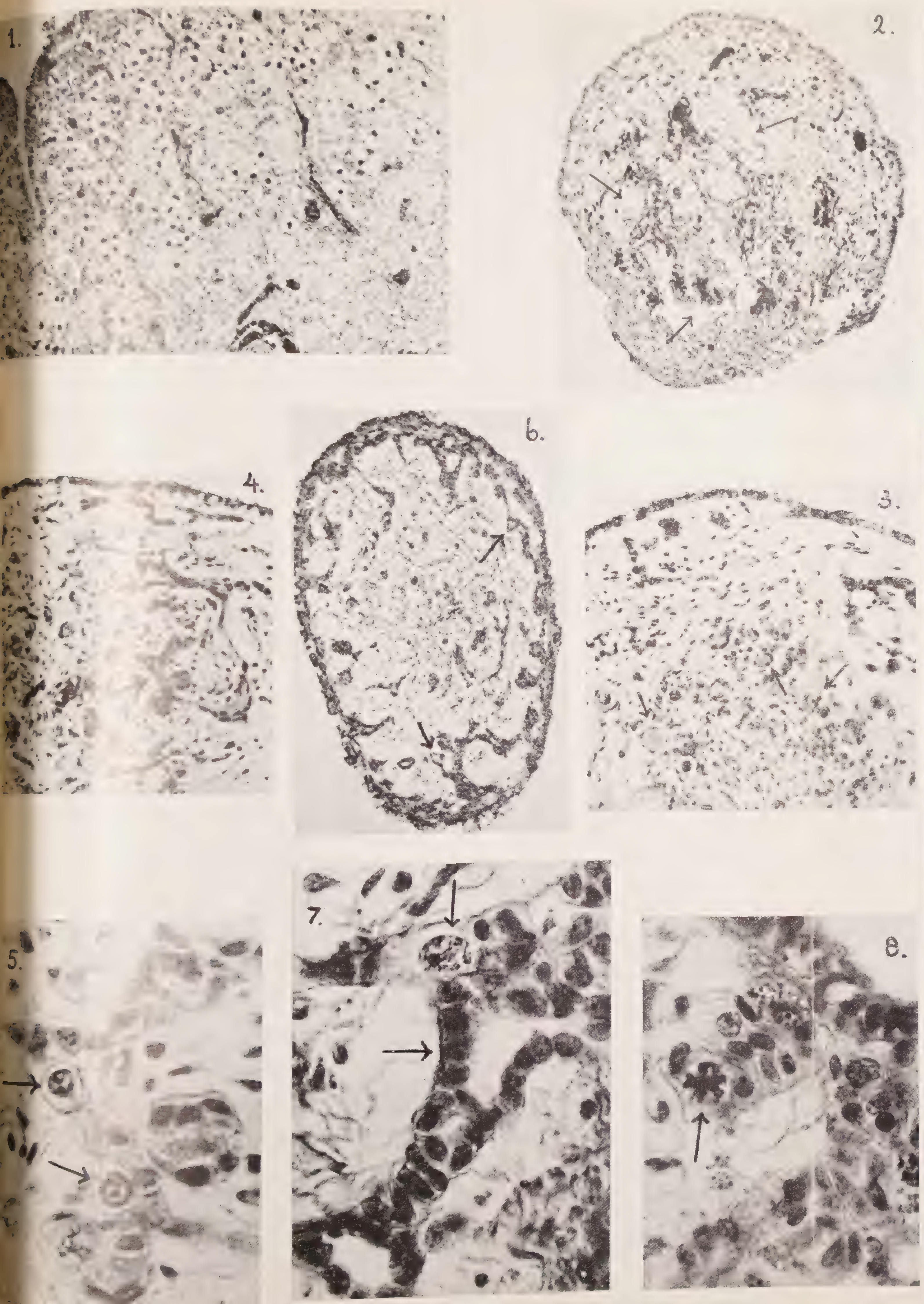
The explants were cultivated in the same way as before, viz. in embryo containers and on the top of a coagulum composed of 5 drops of the following mixtures *A* and *B:*

ad *A*.	Placental vein serum	1	
	Human blood plasma	2	(heparinised)
	Streptomycin in Gey's saline solution	1	(with 1 U.)
	Human foetal brain press juice	2	
	Gey's saline solution	4	

ad *B*. Instead of human foetal brain press juice, human ascitic fluid was taken.

Results:

ad A. Within 6 days of cultivation the explants with a part of the germinal epithelium attached to them were surrounded by a flat epithelial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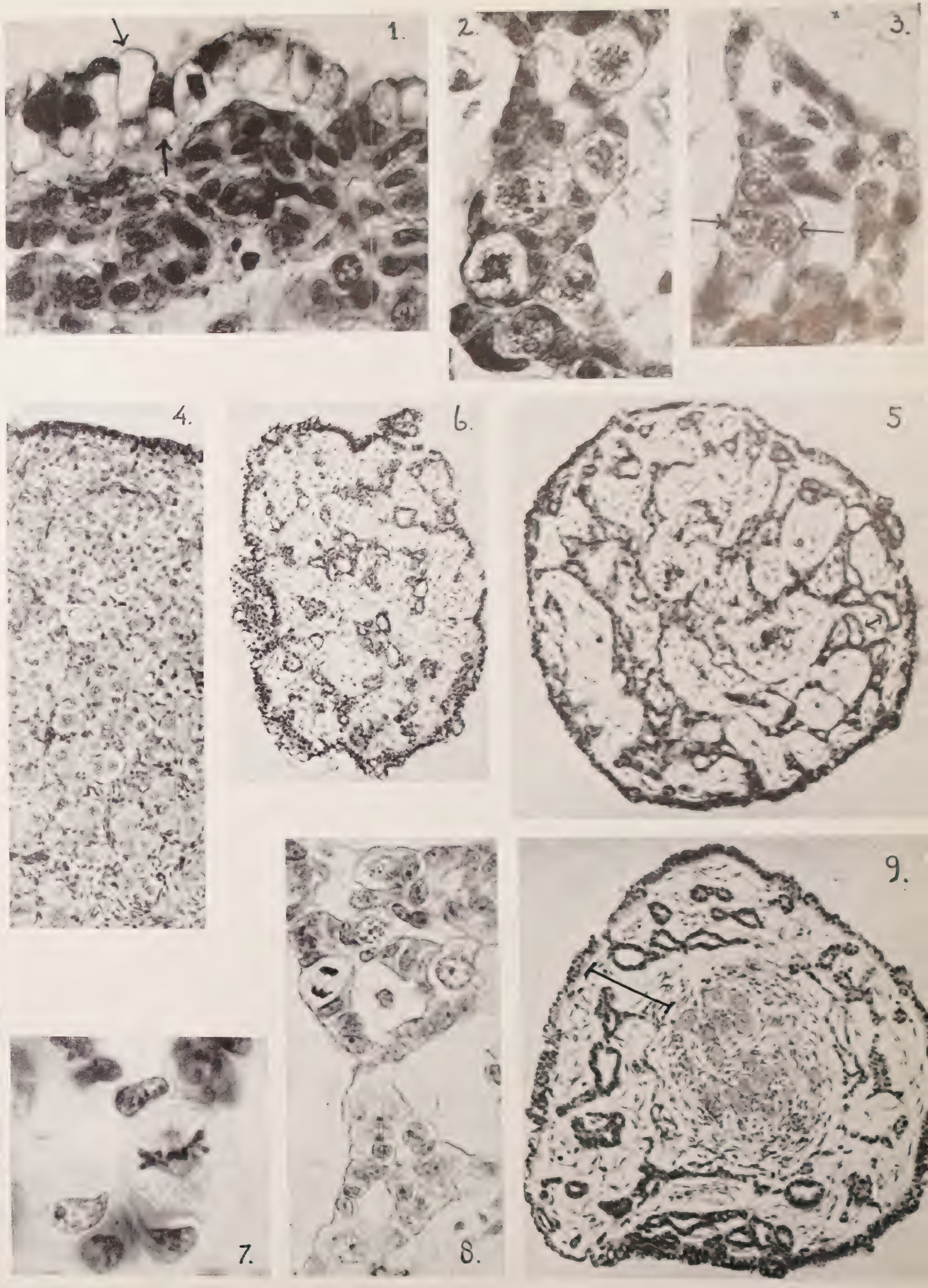
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PLATE VII

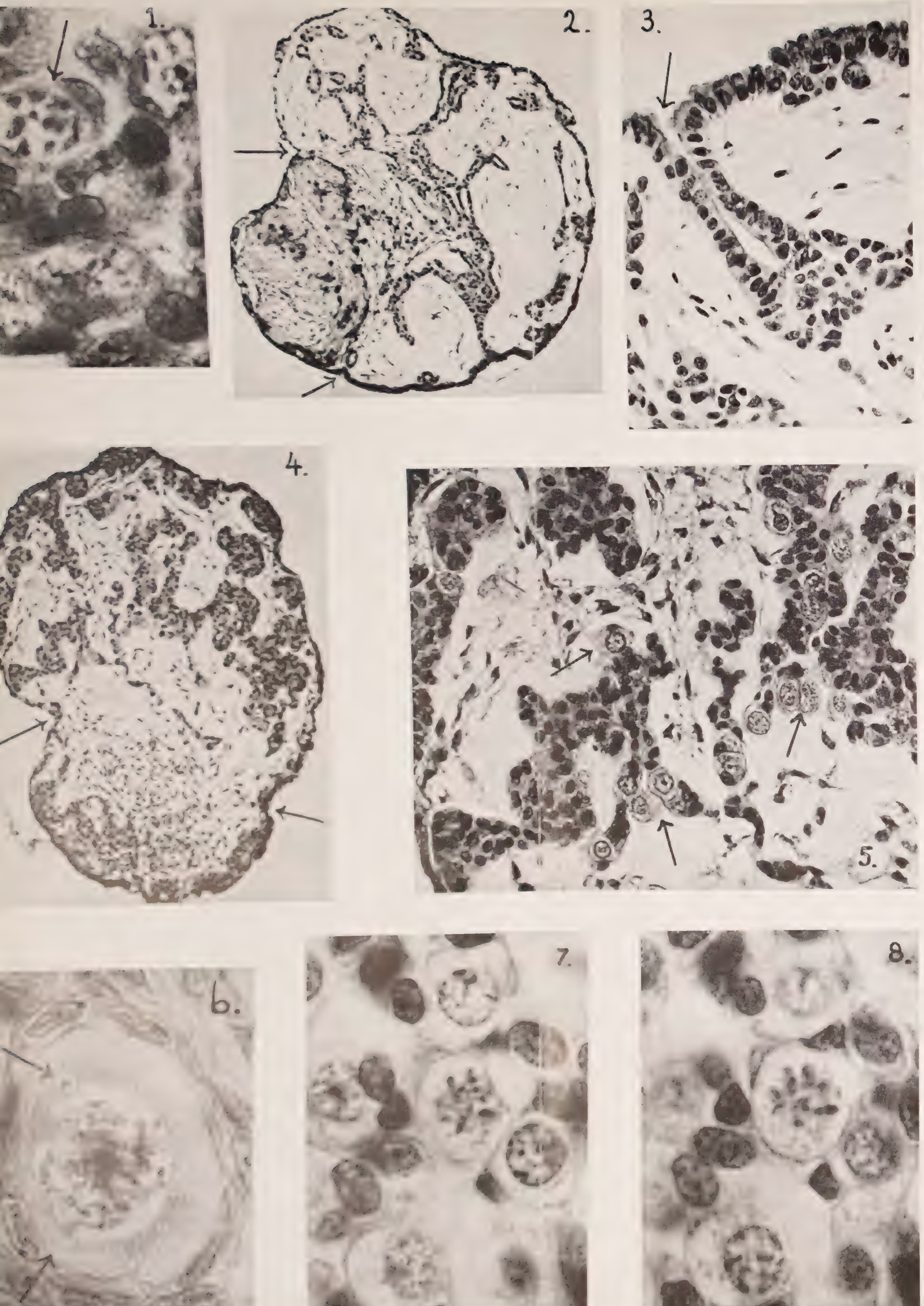
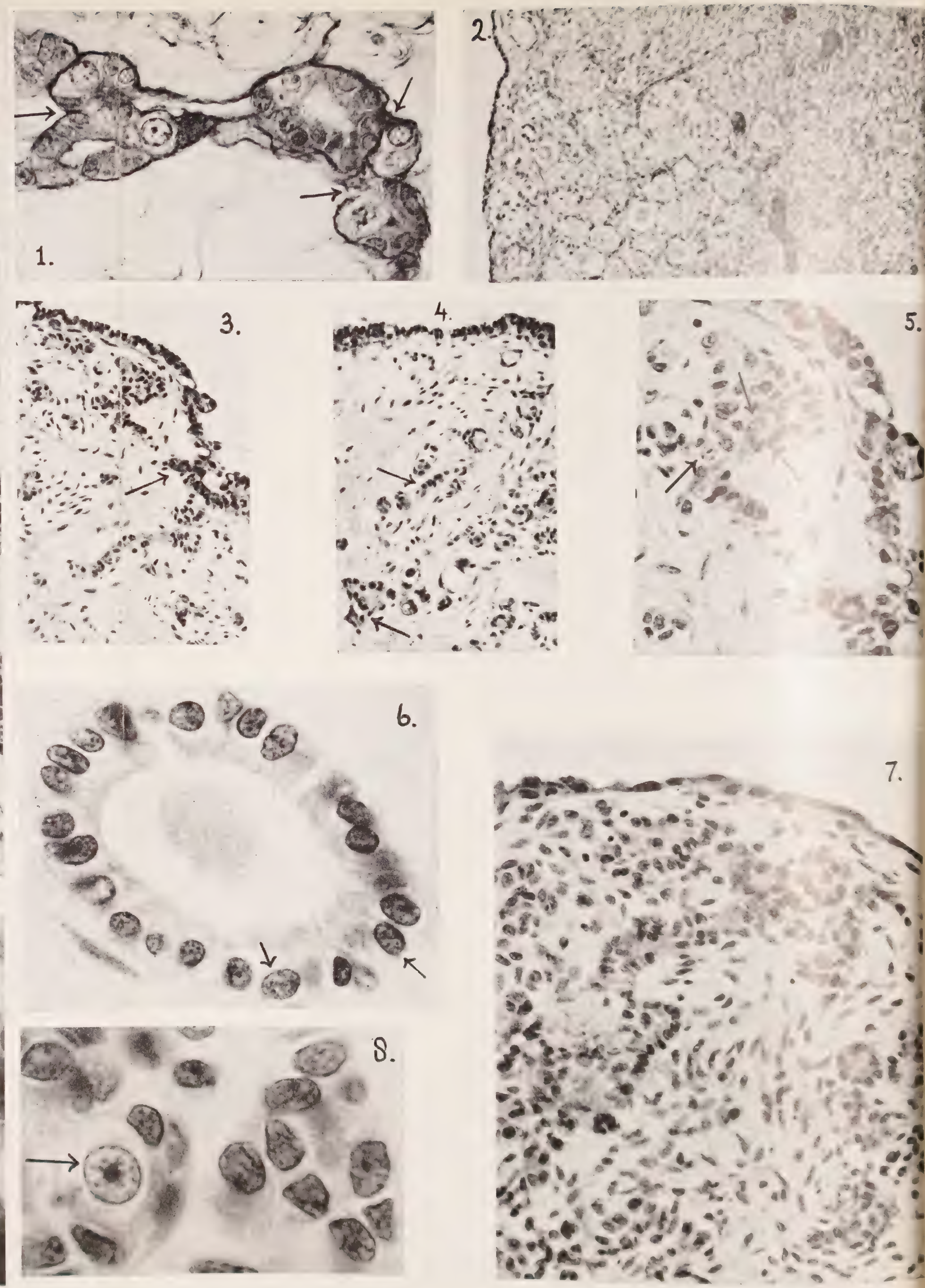


PLATE VIII



layer of cells. The indifferent parenchym and the ova in the centre of the explants appeared to be degenerated and mainly necrotic. However, most of the *follicle cells escaped these degeneration processes*, remained intact *and formed some small cord-like structures*. Instead of the degenerated tissue, large amounts of collagenous fibres developed.

After 9—12 days of cultivation, the surrounding epithelium became cuboidal and many of these cells showed apical vacuoles. Moreover, regenerative cords were present, which were obviously formed either from the covering epithelium (see fig. 3, Plate VIII), or from the central masses of follicle cells (see fig. 4, Plate VIII).

In the peripherial cords a number of new oöcytes were present (see fig. 5, Plate VIII).

After 15—18 days of cultivation many degeneration phenomena again occurred in the centres of the explants and all cords disappeared. However, after 21 days of cultivation, *a new proliferation*, starting from the surrounding epithelium, was observed, but now *only wide tubules, bordered by a high columnar epithelium were formed* (see fig. 6, Plate VIII). These tubules crossed the total diameter of the explants and between them a concentrated mass of collagenous connective tissue was found with only a few fibrocytes in it. In the walls of the tubes the nuclei showed different structures (see fig. 6, Plate VIII) which resembled the protobroch A. and B. stages known from normal ovarian development.

ad B. In this medium practically the same initial events occurred. However, after 18 days of cultivation, the appearance of the explants was quite different. *Numerous parenchymal cords were seen, separated by a cellular connective tissue* (see fig. 7, Plate VIII) and *in the cords some young ova were found* (see fig. 8, Plate VIII).

Conclusions based on the experiments with explants of the ovarian cortex of a full-grown foetus

Principally *the germinal and the follicle epithelium appeared to be able to form cord-like parenchymal structures* after the rest of the parenchym has degenerated. *In a number of cords obviously derived from the germinal epithelium, new ova were formed even after* 18 *days of cultivation.*

The *epithelium* surrounding the explants became *cuboidal just before the regeneration process started.*

In the explants cultivated with human foetal brain press juice at the end of the cultivation period, *wide tubules* were found, bordered by a high columnar epithelium and crossing the total diameter of the explants.

The *nuclei of these cells appeared to be different and resembled the protobroch A. and B. nuclei* described during the normal early stages of egg cell development.

DISCUSSION

In discussing the results obtained by cultivating fragments of the human foetal ovarian cortex in relation to a number of questions summarised on page 1307 and 1308 we can conclude that the presence of the germinal epithelium appeared to be of decisive importance, at any rate with the technical procedures that were used. Only the explants with a part of the covering epithelium attached to them were able to survive, all others dying within a few days of the beginning of the cultivation period. In this respect there was no difference between explants prepared from the ovarian cortex of younger or older foetuses.

A study of the behaviour and aspect of the "germinal epithelium" showed that it appeared to grow and encapsulate the fragments. The shape of the cells was cuboidal or columnar and generally only one layer of cells was found, but in some cases a stratified character occurred. Moreover a flat epithelium was observed in the explants of the ovaries from the full-grown foetus, but only during the first days of cultivation. A peculiarity of all cuboidal or columnar cells of an explant was that they often appeared to possess large apical or basal vacuoles. These vacuoles could indicate a secreting activity, as at the same time the coagulum became liquified in the immediate neighbourhood of the explants. Of the nuclear structures, it can be said that they resembled the protobroch A. and protobroch B. types.

In relation to the question of whether the primordial sex cells are cells of a morphologically different type or not it can be stated that no special morphological type of cell occurred, all the cells being either flat, or cuboidal, or columnar, with or without vacuoles in the apices or in the basal parts of the cytoplasm.

Soon after the epithelium had surrounded the explants, the interior *parenchymal* tissue exhibited an extensive degeneration which led to a complete necrosis. Sometimes a temporary increase of collagenous fibres occurred but generally a loose reticular structure replaced the degenerated tissue after a few days and completely filled the centres of the explants.

However, in the explants of the older foetuses, an exception must be made, for the follicle cells of the "primordial" follicles sometimes occurred in great numbers. Generally these cells appeared to be alive after all parenchymal elements including the egg cells had degenerated and disappeared.

Accordingly, both the follicle cells in the central parts of the fragments and the covering epithelium around the explants, as well as the reticular cells in the interior parts, were the only elements which resisted the factors governing the degeneration processes and were able to survive.

Generally, after about 4—6 days of cultivation, a new phenomenon was observed in all series. Starting from the surrounding epithelium, a proliferation occurred and epithelial or symplasmatic strands or cords penetrated into the interior of the explants. In the preliminary experi-

ments no special elements were found in the cords, but in the definitive experiments which were done in homologous media numerous young ova developed after a few days, many of them entering the first maturation division. Leptotene, synaptene, pachytene and diplotene nuclei frequently occurred. In addition some early metaphase stages were found in the ovarian explants from the younger foetuses, indicating the possibility of a further progress of this type of cell division. In the explants obtained from the ovaries of the 24—26 weeks old foetuses, this possibility was demonstrated still more clearly, some of the oöcytes being in a metaphase or even an anaphase situation. In one of these series double nucleated oöcytes were formed, which indicates with certainty that the new ova in contrast to the behaviour in vivo were potentially able to complete the first maturation division. In this respect it is interesting that in one explant a triasteric division was observed.

In the explants prepared from the ovarian cortex of the new-born baby new genital cells also developed in the cord-like structures, but so far no signs of the maturation divisions could be found.

Apart from the cord-like structures formed by the covering epithelium a second type of cell strands was observed especially in the ovarian explants of the older foetuses (24—26 weeks old), developing from the follicle cells derived from the primordial follicles in which the oöcytes had degenerated. The surviving follicle cells often appeared to form a number of branched epithelial cell cords and it seems to be of particular interest that in a number of these cords new oöcytes also developed.

Consequently egg cells could be observed to develop in the proliferations of the germinal epithelium, as well as in the cords of follicle cells derived from the original primordial follicles.

In discussing the origin of the follicle cells it must be mentioned here that in many series a potential parenchymal origin of the first layer of follicle cells was observed. Hand in hand with the formation of the oöcytes in the new formed parenchymal cords, a number of flat or cuboidal indifferent epithelial cells often ranged themselves around the young ova, thus forming the first layer of follicle cells of young primordial follicles. Moreover the formation of these primordial follicles appeared to be independant of the degree of nuclear differentiation of the egg cells, as follicles were found with nuclei in deutobroch, prophase or dictuoid stages, as well as analogous nuclei occurring without the follicle cells surrounding the ova.

It might be of interest to state that the degree of cytoplasmic differentiation was also not decisive for the formation of the primordial follicles, because sometimes the cytoplasm of the ova in the different primordial follicles was only poorly developed, while in other explants, for reasons unknown, a fine vitellogenesis occurred.

In view of the number of cell layers surrounding the ova, it was a remarkable fact that never more than one cell layer was formed. This

indicates that the formation of a stratified granulosa needs more special "conditions", which were obviously not present in these experiments.

The foregoing paragraphs show the importance of the intimate histogenetic relationships between the germinal epithelium, the first layer of follicle cells and the egg cells.

After 14—21 days of cultivation, the formation of tubules was found in many explants. These tubules were either transformed parenchymal cords, or were directly developed from the surrounding epithelium. In the walls of these tubules, egg cell formation still appeared possible.

In some cases the cords and tubules were divided after some weeks of cultivation into nests or cysts, each containing a variable number of indifferent parenchymal cells and one or more young ova.

Furthermore the explants in some series of the older group suddenly showed an extensive spherical or balloon-shaped outgrowth after some weeks of cultivation. These extensions were covered by a cuboidal morphologically indifferent epithelium, derived from the germinal epithelium, and the interior of the balloons became filled by a mesenchymal tissue. After some days of the covering epithelium of these extensions cords and tubules again regenerated and, in them, numerous new oöcytes in all stages of the first maturation division and primordial follicles were formed, indicating still more strongly than before the special potencies of the germinal epithelium.

Finally, attention must be drawn to the fact that the potency to form new ova was not restricted to the younger ovaries only, but appeared to last certainly until the moment of birth.

SUMMARY

1. The presence of the "germinal epithelium" was of vital importance to the survival of ovarian explants from human foetuses (14—36 weeks old). All fragments cultivated without the epithelium attached to them died within a few days.
   Cultures with an intact part of the epithelium were able to survive (even) after the epithelium had completely surrounded them.
   Consequently the following conclusions are based only on experiments with this type of explant.
2. During cultivation, important changes could be observed in the centres of the explants; there was also some difference between the ovaries from younger and older foetuses in this respect. In the "younger" group, all parenchym in the centres, including the young ova, disappeared through degeneration. In the "older" group, the same processes occurred, but the follicle cells from the "primordial" follicles resisted the degeneration process. In both types, a reticular stroma remained after the remnants of the degeneration had completely disappeared.

3. Then, starting from the covering epithelium, a cord-like or tubule-like regeneration occurred. These cords penetrated into the reticular stroma and often anastomosed. In some explants the new parenchym increased in such a way that the interior of the explants became completely filled, thus suggesting the formation of a miniature ovary.
4. In the cords a great number of newly formed genital cells appeared and all stages between morphologically indifferent epithelial cells and differentiated genital cells could be observed, showing all prophase structures of the first maturation division. In some explants, a completed first maturation division was found and once a triasteric division occurred. In consequence, the "covering" epithelium is not only able to regenerate ovarian parenchym and genital cells, but the foetal genital cells in contrast to the behaviour in vivo, appear potentially able to complete at least the first meiotic division.
5. In the "older" group, a second type of parenchymal cords was found, formed by proliferation of the surviving follicle cells.
   It was remarkable that the formation of new ova was also found in these cords.
6. In many explants the newly formed ova became surrounded by one layer of flat or cuboidal cells and it became clear that this first layer of follicle cells originated from the undifferentiated parenchymal cells, which consequently appeared potentially able to form these special elements.
7. With regard to the preceding points, the histogenetic relationships between covering epithelium, follicle cells and genital cells must be particularly stressed. Moreover, these relationships were demonstrated still more clearly in a number of explants, which form balloon-shaped extensions covered by the germinal epithelium and initially filled by a mesenchymal tissue. Starting from the covering epithelium, a large number of cord-like proliferations, in which numerous genital cells and primordial follicles developed, again penetrated into the mesenchymal tissue, thus repeating the processes already described in the foregoing paragraphs.
8. The above-mentioned results finally indicate that the potency of egg cell formation is not restricted to an early embryonic period, but certainly lasts until the moment of birth.

RÉSUMÉ

1. La survie en culture de fragments d'ovaire d'embryons humains (14 à 36 semaines) dépend largement de la présence de l'epithelium germinatif. Tous les fragments qui en sont dépourvus meurent après quelques jours; au contraire, ceux qui en possèdent une portion intacte sont capables de survie dès le moment où ils sont complètement entourés par l'épithélium.

En conséquence, les conclusions qui suivent sont uniquement basées sur les expériences réalisées avec ce dernier type d'explantats.

2. Pendant la culture, la zone centrale des explantats montre des remaniements importants dont l'allure diffère suivant l'âge du foetus donneur. Dans les fragments d'ovaires provenant des embryons les plus jeunes, tout le parenchyme central, y compris les jeunes oöcytes, disparait par dégénérescence. Dans les fragments provenant d'embryons plus âgés, les cellules folliculeuses des "follicules primordiaux" résistent à la dégénérescence. Dans les deux cas, un stroma réticulaire subsiste après la disparition des structures dégénérées.
3. Ensuite, à partir de l'épithélium de recouvrement, des cordons (ou des tubes) sont régénérés. Ces cordons s'enfoncent dans le stroma réticulaire central et s'anastomosent fréquemment. Dans certains explantats le nouveau parenchyme ainsi formé se développe à tel point qu'il remplit complètement l'intérieur du fragment, donnant l'impression de former un ovaire en miniature.
4. Dans les cordons, apparaissent de nouvelles cellules germinales. On peut observer tous les stades intermédiaires entre les cellules épithéliales morphologiquement indifférentes d'une part, et les cellules germinales complètement différenciées d'autre part. Celles-ci montrent toutes les images prophasiques de la première division de maturation. Dans certains explantats, on peut observer l'achèvement de la première division de maturation, et dans un cas, une division triastrale a été trouvée.
   Il apparait donc que non seulement l'épithélium de recouvrement peut régénérer un parenchyme ovarien, y compris les cellules germinales, mais encore que celle-ci, chez le foetus, sont potentiellement capables de parachever la première mitose de maturation, ce qu'elles ne font pas "in vivo".
5. Dans les fragments provenant d'embryons plus âgés, on observe la formation d'un deuxième type de cordons, régénérés par les cellules folliculeuses qui ont résisté à la dégénérescence. Il est remarquable que dans ces cordons également de nouveaux oöcytes se différencient.
6. Dans beaucoup d'explantats, on voit les néo-oöcytes s'entourer d'une couche de cellule aplaties ou cubiques. Cette première couche de cellules folliculeuses provient indubitablement des cellules parenchymateuses indifférenciées qui se révèlent donc capables de former également ces éléments spéciaux.
7. Les points précédents démontrent l'étroitesse des relations histogénétiques entre l'épithélium de recouvrement, les cellules folliculeuses, et les cellules germinales. Cette parenté est encore mieux soulignée dans les cas où les fragments donnent naissance à des bourgeons sphériques tapissés d'épithélium superficiel, et dont le centre est rempli par un tissu mésenchymateux. Ici encore l'épithélium de

recouvrement donne naissance à des cordons qui s'enfoncent dans le mésenchyme central du bourgeon et dans lesquels se différencient des cellules génitales et des cellules folliculeuses.

8. Les résultats mentionnés ci-dessus indiquent finalement que les potentialités ovo-formatrices ne sont pas limitées aux premiers stades du développement embryonnaire mais persistent jusqu'au moment de la naissance.

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BIOLOGY

DE AFKOMST DER AMPHIBIEËN

DOOR

J. F. VAN BEMMELEN

(Communicated at the meeting of Sept. 30, 1950)

In zijn verhandeling Vertebrate Zoology zegt G. R. DE BEER in het hoofdstuk Origin of Chordates op bl. 431: "That the Amphibia arose from fish there is no doubt". Tegenover een dergelijke absolute uitspraak mag in allen gevalle worden aangevoerd, dat door meerdere autoriteiten op phylogenetisch gebied die twijfel wel degelijk werd en waarschijnlijk nog wordt gekoesterd. Om slechts een enkel voorbeeld aan te halen: In zijn merkwaardig en doorwrocht boek: die Entstehung der Landtiere (Leipzig 1891) schrijft Dr HEINRICH SIMROTH op blz. 352: "Wir finden hier (sc. bij de Amphibia) einen viel continuirlicheren Zusammenhang mit den ältesten Anfängen als bei den erst secundär abgelenkten Fischen".

Wil men trachten om in deze controverse zijn standpunt te bepalen, dan is het in de eerste plaats noodzakelijk zich goed rekenschap te geven, wat men onder Amphibieën en wat onder Visschen te verstaan heeft. Ook daarbij echter stuit men op controversen, waarvan de voornaamste zich laat samenvatten in de vraag: Zijn de Dipnoï Visschen of Amphibieën?

Hun populaire naam Longvisschen wijst hun een plaats aan onder de Visschen waartoe zij dan ook in alle handboeken gerekend worden. De wetenschappelijke naam Dipnoï schijnt eenigszins meer in de richting der Amphibieën te wijzen, in zooverre als hij verband houdt met de samenwerking of afwisseling tusschen kieuw- en longademhaling. Door enkele deskundigen, b.v. door GÖTTE, werden zij dan ook als Amphibieën beschouwd.

Voor de beantwoording van de hoofdvraag: Wat is een Amphibium? behoeft echter aan de quaestie omtrent de Dipnoï niet noodzakelijk de voorrang te worden verleend, daar de voorstelling eener zelfstandige dierklasse tusschen de Visschen en de Reptielen is ontstaan in 't begin der 19e eeuw, dus in een tijd toen de Longvisschen nog onbekend waren.

Ook de Blindwoelers (Gymnophionen of Coecilia) en de uitgestorven Stegocephalen speelden bij het opkomen der voorstelling van een afzonderlijk Amphibieëntype geen rol. Zoodoende zijn het alleen de Kikvorschen en de Salamanders geweest, die door hun afwijkenden lichaamsbouw, maar vooral ook door hun gedaantewisseling en levenswijze aanleiding hebben gegeven, om hen in een afzonderlijke klasse te vereenigen en van de Reptielen waarmede zij te voren saamgevat waren te scheiden.

Beschouwt men nu deze twee orden van recente Amphibieën in hun lichaamsbouw en levenswijze zoowel op zich zelf als vergelijkenderwijze

dan is het duidelijk dat men in beide te doen heeft met landdieren, die zich secundair aan het leven in het zoete water hebben aangepast. Daardoor is hun huid naakt en glibberig geworden, en hebben zij hun schubben en nagels verloren. Dat zij die hoornproducten der opperhuid (typische attributen van het luchtleven, en dan ook verdwijnend bij overgang in het water) oorspronkelijk bezaten, mogen wij besluiten uit het voorkomen van hoornschubben bij de Blindwoelers en van nagels bij den Afrikaanschen en Japanschen genagelden kikvorsch Dactylethra. Veel belangrijker en ingrijpender echter zijn de vervormingen, die toegeschreven mogen worden aan het waterleven der larven, en die gewoonlijk worden saamgevat onder den term metamorphose. Dat de daardoor teweeggebrachte wijzigingen in de organisatie der jonge dieren niet zonder meer mogen opgevat worden als atavismen, dus als terugkeer tot oorspronkelijker toestanden, maar integendeel als latere vervormingen van jeugdstadiën, mag besloten worden uit den aard zelf dier modificaties. De pootloosheid der donderpadden b.v. wordt veroorzaakt door retardatie in de ontwikkeling der ledematen, de ontwikkeling der verhoornde tandjes in de lipranden is een aanpassing aan het (secundaire) plantaardige dieet, evenals de buitengewone verlenging en spiraalsgewijze opwinding van het darmkanaal. Hoe verschillend de larvale aanpassing aan het waterleven zelfs bij na met elkaar verwante dieren kan werken blijkt o.a. uit de geheel afwijkende gedaante van de Dactyletralarve, met haar in een draad uitloopenden staart en haar lange mondvoelers. Zooals de populaire naam Kikkervischjes voor de vorschlarven bewijst, doet hun gedaante oppervlakkig aan visschen denken, maar deze schijnovereenkomst heeft met werkelijke of beter gezegd rechtstreeksche verwantschap even weinig te maken, als b.v. de gelijkenis van Hazelwormen met Slangen of die van Salamanders met Hagedissen.

In zijn bijkomstig karakter is het kikkervisch-stadium vergelijkbaar met de rupsperiode der vlinders, de larven der kevers, de maden der vliegen. Bij de rupsen b.v. zijn de vleugelkiemen in ontwikkeling stil blijven staan en naar binnen gestulpt, zoodat de vleugels schijnen te ontbreken. De monddeelen hebben nog het karakter van kaken, in verband met het secundaire plantendieet. Bij de ontwikkeling van de pop, verborgen in de rupsenhuid, worden de vleugelkiemen naar buiten gestulpt, groeien de kopledematen tot sprieten en zuigtong uit, bereiken de borstpooten hun volle lengte, terwijk de buikpooten en de naschuivers verdwijnen. Al deze vervormingen zijn van secundairen aard, d.w.z., zij hebben niets te maken met het ontstaan der Insecten uit meer oorspronkelijke Geleedpootige dieren.

Bij de Amphibieën moet het kikkervisch-tijdperk een gevolg zijn van de gewoonte hunner voorouders om hun eieren in het zoete water te leggen, wat wederom in verband zal gestaan hebben met de vervanging van een stevige en drooge eischaal door een weeke slijmerige omhulling. Deze veronderstelling vindt steun in het feit dat de Blindwoelers wèl

een eischaal voortbrengen, en dan ook hun eieren op het droge leggen, en met hun gekronkeld slangachtig en pootloos lichaam omslingeren, zoodat men zeggen kan, dat zij hun eieren uitbroeden.

Als volwassen dieren zijn de Blindwoelers althans in sommige opzichten nog meer gespecialiseerd dan de Salamanders en de Kikvorschen, daar zij geheel pootloos zijn. Het verlies der ledematen zal wel in verband staan met hun onderaardsche levenswijze. Hun zeer eigenaardig schubbenkleed is misschien een erfdeel hunner voorouders, daar ook de palaeozoïsche Stegocephalen schubben bezaten. Of de Coecilien geheel vrij van den invloed van het zoete water gebleven zijn, is twijfelachtig, daar sommige waarnemers berichten dat zij zich te water begeven kunnen. Dat hun voorouders van pooten voorzien waren, wordt ook nog waarschijnlijk gemaakt door hun verwantschap met Amphiuma, een Derotreme-Perennibranchiaat, die door bevoegde beoordeelaars wordt opgevat als een neotenie van een Coecilia, op dezelfde wijze als de in bergmeren zwemmende Axolotl' een geslachtsrijp wordende larve is van de landsalamander Amblystoma. De verhouding van de Coecilien tot de landsalamanders is dezelfde als die van de Hazelworm tot de van pooten voorziene Hagedissen.

Dat de watersalamanders (Tritonen) van landsalamanders afstammen, is zonder meer duidelijk. Maar ook de laatstgenoemden zijn op een vochtige omgeving aangewezen en doorloopen een groot gedeelte van hun ontwikkeling in het zoete water, of wanneer dit voor hen moeilijk bereikbaar is, zooals voor de Alpensalamander zijn zij levendbarend geworden.

Ook bij de Kikvorschen vindt men dezelfde afwisseling van land- en waterleven: de waterkikvorsch dankt zijn lange en krachtige achterpooten weliswaar in hoofdzaak aan zijn gewoonte om te springen, maar toch zijn zwemvliezen aan het zwemmen. Zijn aanpassing aan het waterleven is secundair, of eigenlijk tertiair, wanneer men die der Oer-chordaten meetelt. Ook van de uitgestorven Stegocephalen heeft men de zekerheid, dat zij op het land leefden, daar kleine salamandervormige vertegenwoordigers van die merkwaardige diergroep (de Microsauriers) in het inwendige van versteende holle boomstammen uit de Steenkoolperiode zijn aangetroffen. Maar zij moeten hun eieren in het water gelegd hebben, daar larvenvormen met goed ontwikkelde kieuwbogen in de leisteen derzelfde formatie worden gevonden.

Stegocephale Amphibieën komen het eerst voor in het Boven Carboon, en het laatst in de Trias. Dan volgt een lange periode, waarin geen overblijfselen van Amphibieën zijn ontdekt. Eerst in het krijt komen weer versteende skeletstukken voor, die aan Urodelen mogen toegeschreven worden, en in het Eoceen (op één na de oudste periode van het Coenozoïcum) verschijnen voor 't eerst echte Salamanders. De beroemdste daaronder is de "homo-diluvii testis", een reuzensalamander, die in 1726 door den Zwitserschen arts SCHEUCHZER werd ontdekt en voor een menschenskelet aangezien.

Kikvorschen waren tot voor korten tijd eerst uit het Mioceen bekend, thans echter kent men een echte Batrachier uit de Onder Trias van Madagascar. Daarmede is weer een nieuw afdoend bewijs geleverd, hoe weinig het geoorloofd is tot den ouderdom van een diertype te besluiten uit dien der geologische formatie waarin het voor het eerst wordt aangetroffen. In vol. III van ZITTEL's groote Handbuch der Palaeontologie (1887—1890) kan men nog lezen, dat tusschen de jongste Stegocephalen en de oudst bekende Urodelen een geweldige kloof gaapt, en het in allen gevalle niet geoorloofd is, de hedendaagsche Amphibieën als rechtstreeksche afstammelingen der Stegocephalen te beschouwen. De ontdekking van den Madagassischen Trias-kikvorsch bewijst de juistheid dezer bewering, daar er uit blijkt, dat kikvorschen reeds naast de Stegocephalen bestonden en zeker niet minder gespecialiseerd waren dan deze, zoodat zij een even lange voorgeschiedenis als die palaeozoïsche Amphibieën gehad moeten hebben, een periode die tot diep in het Palaeozoicum teruggaat. Dit geldt evengoed voor de Stegocephalen; in het Carboon treden zij als 't ware plotseling op met een rijkdom van vormen, die wel onderling groote overeenkomst bezitten, maar toch volstrekt niet uit elkaar afgeleid kunnen worden. Wel vertoonen sommige duidelijke kenteekenen van aanpassing aan bijzondere levensomstandigheden, zoo is er b.v. een pootlooze vorm (Aistopodus) die met zijn tezelfder tijd levende viervoetige verwanten evenzoo nauw verbonden is, als b.v. de Hazelworm of de Sheltopusik met de gewone van pooten voorziene Hagedissen. Ook in hun verspreiding over het aardoppervlak vertonen deze oudste Stegocephalen een bijna universeele uitgebreidheid, zoodat men met zekerheid mag aannemen, dat reeds lang te voren voorouderlijke leden dezer diergroep moeten bestaan hebben, waarvan mogelijkerwijze nog wel eens overblijfselen kunnen gevonden worden. Daarentegen is het zoo goed als uitgesloten, dat de Stegocephalen de rechtstreeksche voorouders der nog thans voortlevende Salamanders, Kikvorschen en Coecilien zouden zijn. Hiertegen pleit niet zoozeer de boven besproken kloof in de fossiele overlevering tusschen de Trias en het Krijt, maar veel meer het verschil in lichaamsbouw en lichaamsomvang. De Triadische Stegocephalen toch waren veel meer gespecialiseerd dan de Carbonische, getuige de wonderbaarlijke samengesteldheid der emailbekleding hunner tanden (waaraan zij den naam Labyrinthodonten danken), en bereikten daarbij veel grootere lichaamsafmetingen, tot reusachtige toe. ZITTEL aarzelt dan ook niet hun uitsterven aan deze overmatige toename in omvang toe te schrijven. Hetzelfde geldt voor alle overmatig groote dier- en plantvormen, getuige de Dinosauriers, de Reusvogels, de Mammoet, het Reuzenhert en tal van andere. Maar in dat interval tusschen Trias en Krijt moeten kikvorschachtige dieren zijn blijven bestaan, en zonder twijfel ook evengoed Salamanderachtige, de voorouders der hedendaagsche Urodelen. De tegenstelling tusschen deze beide groepen van recente Amphibieën (Batrachia en Urodelen) moet dus reeds vóór de Trias, dus al in het Palaeozoicum

bestaan hebben. Daar nu de eerste veel ingrijpender van het oorspronkelijke type aller Tetrapoden afgeweken zijn dan de laatste, moet hun gemeenschappelijke voorouder dié eigenaardigheden bezeten hebben, die aan Kikvorschen en Salamanders gemeen zijn.

De voornaamste reden nu om aan te nemen, dat reeds deze onbekende voorouders van alle Amphibieën landdieren waren, is de bouw hunner ledematen, die te oordeelen naar de oudst bekende Stegocephalen, reeds vijfstralige pooten en geen vinnen moeten geweest zijn. Hierin zal ook wel de hoofdoorzaak te zoeken zijn, dat zij geen spoor in de fossiel bevattende aardlagen van het Devoon en Siluur hebben achtergelaten; landbewoners verkeren daarbij in zooveel ongunstiger omstandigheden dan zeedieren. Dat desniettegenstaande de steenkoollagen van het Carboon zoo te zeggen plotseling dien grooten rijkdom aan fossiele skeletten hebben opgeleverd, mag toegeschreven worden aan den bij uitzondering gunstigen invloed van de weelderige plantengroei in vochtige omgeving van die periode.

Dat vocht moet echter geen keukenzout hebben bevat; de oudste ons bekende Stegocephalen leefden zeker niet in zee, en hetzelfde mogen wij onderstellen voor de ons onbekende voorouders der Kikvorschen en Salamanders in het Carboon. Wanneer wij ons van die gemeenschappelijke voorouders der Amphibieën een voorstelling trachten te maken op grond van de bovenstaande beschouwingen, dan komen wij wel tot het beeld van een langgestrekt kruipend gewerveld dier met twee paar korte vijfteenige pooten, dat zich in of nabij zoet water bewoog, maar volstrekt niet tot een met breede vinnen toegeruste visch. Wel moeten deze Oer-Amphibieën op hun beurt weer van nog oorspronkelijker gebouwde Chordaten afgestamd zijn, die tevens de voorouders van alle andere Gewervelden, de Visschen incluis, waren, en eveneens van de Tunicaten en Enteropneusten. Die Oerchordaten moeten reeds kieuwspleten bezeten hebben, daar deze bij alle ons bekende vormen worden aangelegd maar bij alle Vertebraten in meerdere of mindere mate worden gereduceerd. Dit wijst er wel op, dat die oudste Chordaten waterbewoners waren, maar daarom behoeven zij nog niet noodzakelijk in de diepe zee geleefd te hebben. Integendeel, het lijkt mij veel waarschijnlijker, dat zij in den ondiepen kustzoom rondkropen, of zelfs aan de oevers van zoetwaterplassen en kommen. In allen gevalle moet er reeds scheiding tusschen land en water bestaan hebben, anders ware de vorming der oudste (z.g. azoïsche slibafzettingen niet mogelijk geweest. Men kan zich dus gemakkelijk voorstellen, dat deze oudste kieuwspleet-dieren tegen vlakke oevers opkropen en daarbij in de bovenachterhoeken van hun kieuwdarm luchtbellen opnamen, waardoor die hoeken in de lichaamsholte ingestulpt werden. Daardoor ontstonden de longen.

De overgang van het water- tot het landleven zal wel meerdere malen, zoowel wat tijd als wat plaats aangaat zijn geschied, en de vervanging der kieuwen door longademhaling meer of minder volledig zijn geweest,

of beter gezegd in een vroeger of later stadium der ontwikkeling hebben plaats gegrepen. Zoodoende ontstonden *naast* niet *uit* de Amphibieën de Reptielen, die in de vochtbehoefte hunner jonge kiemen voorzagen door hun het vermogen te bezorgen zich met een vochtblaas te omgeven (het amnion), en ze tevens met een uitstulping van hun einddarm uitrustten (de allantoïs), die hen in staat stelde adem te halen door een harde of leerachtige, maar poreuze eischaal heen.

Doordien bij sommige dezer oorspronkelijke longademhalers de beschaalde eieren langer in de eileiders bleven vertoeven, en de schaalvorming ten slotte achterwege bleef, ontwikkelde zich de allantoïs tot placenta.

Evengoed als de overgang van het water- tot het landleven op allerlei verschillende wijzen, tijden en plaatsen kan geschied zijn, zal ook de terugkeer tot het water tallooze malen hebben plaats gevonden, en in allerlei phasen van ontwikkeling, zoodat de verschillendste typen van Chordata ontstonden. Slechts enkele daarvan zijn ons in de fossiele nalatenschap behouden gebleven, zooals de wonderlijke gepantserde wezens, die onder den naam Placodermen worden saamgevat. SIMROTH vermoedde, dat deze op hun steltvormige voorste ledematen langs vlakke kusten rondstrompelden, maar tevens in staat waren te zwemmen door wrikbewegingen van hun geschubden staart. Misschien namen zij bij het vertoeven op het land luchtbellen op, waardoor zij in het water niet zonken, niettegenstaande hun zware koppantsering.

Andere Oerchordaten raakten hun schubbekleeding weer kwijt, of bereikten nimmer dezen vorm van differentiatie der opperhuid, maar bleven naakt, zooals Amphioxus en Balanoglossus, of omgaven zich met een mantelplooi, waarin andere stoffen dan hoornstof, b.v. cellulose werden afgescheiden, zooals de Tunicaten. In nog weder andere richting hebben zich de Echinodermen gedifferentieerd; zij zijn zoover van het oorspronkelijke chordatentype afgeweken, dat zelfs van een strengvormige chorda niets meer te onderkennen is.

Aan al deze vervormingen ligt één gemeenschappelijke hoofdoorzaak ten grondslag, de overgang van een vrij bewegelijke tot een meer of minder geïmmobiliseerden toestand, zooals ik dit in mijn opstel over Symmetrie heb trachten te betogen.

De bovenstaande beschouwingen over Cambrische of zelfs Praecambrische Oerchordaten zijn wegens het ontbreken van alle fossiele overblijfselen, geheel hypothetisch. Zij zijn echter in zooverre gerechtvaardigd, als zij er toe kunnen leiden, om de opvatting in twijfel te trekken, dat de Amphibieën uit de Visschen, en de Amnioten uit de Amphibieën ontstaan zouden zijn. Dat de eerste Darwinisten zooals ERNST HAECKEL door die voorstelling beheerscht werden, is begrijpelijk genoeg: zij was een uitvloeisel van de oude lijnrangschikking uit de dagen van LINNAEUS en BUFFON. Maar dat nog heden ten dage die zelfde simplistische verklaring geldig wordt geacht, en de Visschen voor stamouders der Amphibieën

worden verklaard, omdat hun eerste sporen in oudere formaties dan die der Amphibieën worden aangetroffen, lijkt mij even onaannemelijk, als dat men de eierleggende Vogelbekdieren voor den overgang van Reptielen tot Zoogdieren zou aanzien, of de Buideldieren in geologische tijdrekening voor ouder houden dan de niet-buideldragende Zoogdieren.

Als eindbesluit dezer overwegingen moge het volgende gelden: Amphibieën zijn landdieren, die zich aan het zoete water hebben aangepast, Visschen zijn zeedieren, waarvan vele (geheel of voor een deel van hun leven) in het zoete water zijn overgegaan. Beide dierklassen stammen, onafhankelijk van elkaar, van onbekende, primitieve Chordaten af, die evenzeer de voorouders der overige Vertebraten waren.

Summary

The ancestry of the Amphibia

According to the opinion of several authors, e.g. De Beer, Naef, Franck, the Amphibia have derived from Fishes. Older biologists e.g. Simroth and Götte on the contrary suppose that their ancestors were already landliving animals, that got adapted to deposit their eggs in fresh water. As the several types of the oldest-known Stegocephalia start unprovided — for in the same layers of the Uppercarbonic coalmeasures, it is clear that their common Amphibian ancestors must have existed in still older periods, possibly already in the Silurian era. That they did'nt leave any traces in these older formations may be ascribed to the uncompleteness of the palaeontological record, especially with regard to fresh water deposits. As gill slits occur in the embryogenesis of all Chordates without exception, it is evident that the first representatives of this animal type must have lived in water, but need not therefore have been Fishes. On the contrary it is much more probable that they crawled about on the border of land and water, and thereby obtained limbs with five fingers, fit for movement on dry soil. As however their eggs needed liquid surroundings, they deposited them in fresh water, which led to the differentiation of an intermediate larval stage: the tadpole.

The recent Amphibia: Batrachia, Urodeles and Coecilia, must likewise be of much older descent, than was concluded from their first appearance in the upper Cretacean. This supposition has recently been fully affirmed by the discovery of a true frog in the Trias of Madagascar, showing that Batrachia existed at the same time as Stegocephalia and therefore nead not be the direct descendants of the latter. The same conclusion may be assumed for the Urodeles, as these have retained the original commontype in a higher degree than the Batrachia, who are modified by shortening of the spine and loss of the external tail. Their common ancestors, together with those of the Stegocephalia, must have lived in the Palaeozoïcum as terrestrial animals entering fresh water, side by side with the forefathers of the Fishes, who differentiated to marine animals. Some of the latter left the sea by entering rivers and becoming adapted to fresh water.

Résumé

L'origine des Amphibiens

Selon l'opinion de De Beer et plusieurs autres biologistes, les Amphibiens auraient pris leur origine de Poissons, dont les nageoires se modifièrent en membres terrestres, munis de cinq doigts. Beaucoup plus probable me semble la supposition (de Simroth e.a.) qu'ils se sont différentiés de Tetrapodes terrestres qui déposèrent leurs oeux dans l'eau douce et en conséquence subirent une métamorphose plus ou moins complète. En faveur de cette hypothèse on peut citer que les premiers représentants des Stégocéphales apparaissent sans précurseurs dans les couches carbonifères de la formation carbonique supérieure, en forme d'animaux tétrapodes, c'est à dire terrestres habitant les forêts humides, mais sans aucun contacte avec l'eau marin. Les trois différents types des premiers Stégocéphales se montrent en même temps, ce qui prouve que leurs ancêtres communs doivent avoir existé déjà longtemps auparavant. Les derniers représentants de cet ordre d'Amphibiens se trouvent dans le Trias, tandis que les premiers Batraciens n'apparaissent que dans la Craye supérieure, et que les Urodèles sont encore plus récents dans l'Eocène; de sorte qu'ils sont séparés par une espace de temps énorme des derniers Stégocéphales. Mais il y a quelques années cette intervalle a été comblée par la découverte d'une vraie grenouille dans le Trias de Madagascar, ce qui prouve que les Batraciens existaient déjà à côté des Stégocéphales. Puis qu'il va sans dire que les Urodèles sont plus primitifs que les Batraciens, il est fort probable que tous les trois ordres se dérivent d'ancêtres communs beaucoup plus anciens, mais déjà organisés comme animaux terrestres, munis de membres pentadactyles, qui dépoaient leurs oeux dans l'eau douce. Les Poissons au contraire s'adaptèrent à l'eau marin et ne retournèrent qu'en nombre restreint à l'eau douce des fleuves. Dans le cours de leur développement phyloginétique leurs membres se modifiaient en nageoires, en contraste avec ceux des Amphibies, qui ne se modifiaient pas.

Zusammenfassung

Die Herkunft der Amphibien

Nach der Ansicht mehrerer Biologen, wie De Beer, Naef, Franck u.a. stammen die Amphibien von Fischen, die sich auf das Trockne begaben und dabei Lungen bekamen und ihre Flossen zu fünffingerigen Gehfüssen umbildeten. Die entgegengesetzte Meinung, wie sie von älteren Autoren, z.B. Simroth und Götte, behauptet wurde, scheint mir wahrscheinlicher. Zwar sind die ältesten Amphibien, die Stegocephalen, erst aus dem oberen Karbon bekannt, während Fische schon im Silur auftreten, aber die Vorfahren der ersteren müssen schon viel früher als lungenatmende Landtiere gelebt haben, weil schon die allerersten Panzerlurche unvermittelt in grossem Formenreichtum alle zusammen erscheinen

neben verkohlten Stämmen von Morastbäumen, und ausgestattet mit fünffingerigen Gehfüssen. Sie waren also angepasst an eine feuchte Waldumgebung und legten ihre Eier ab im süssen Wasser, worin ihre Larven sich zu Kaulquappen differenzirten. Ihre bisjetzt unbekannten Vorfahren mögen als Landbewohner mit beschupptem Körper und benagelten Fingern während eines grossen Teiles des Paläozoïschen Zeitraums gelebt haben. Sie stammten aus primitiven Chordaten, ebenso wie die übrigen Vertebraten-Klassen aber nicht aus Fischen, die nebenan im Meer zurückblieben oder sich möglicherweise vom Lande darin zurück begaben.

Wie wenig man berechtigt wäre aus dem erstbekannten Vorkommen einer Tiergruppe auf ihr wirkliches Alter zu schliessen, ist neuerdings für die Batrachier wieder bewiesen durch die Entdeckung eines echten Frosches im Trias von Madagascar. Daraus geht hervor, dass Batrachier schon neben den Stegocephalen bestanden, und deshalb nicht aus diesen hervorgegangen zu sein brauchen. Dasselbe gilt in noch höherem Masse für die Urodelen. Die primitiven Amphibien waren vermutlich Landtiere, die sich mehr weniger an das Süsswasser anpassten, während die Fische Meerestiere wurden, und erst nachträglich aus dem Salzwasser in die Flüsse zurückkehrten.

A RHIZOCEPHALAN PARASITE OF THE CRAB PTYCHOGNATHUS BARBATUS (A. M. E.) FROM TERNATE

BY

H. BOSCHMA

(Communicated at the meeting of October 28, 1950)

During one of the visits of the Snellius Expedition to Ternate in the Moluccas, on the beach of the island numerous small Grapsid crabs (afterwards proving to belong to *Ptychognathus barbatus* (A. M. E.)) were found on the coarse sand and between and under pebbles. A great number of these crabs were infested by a parasite described here as a new species.

Sacculina ternatensis nov. spec.

Snellius Expedition, Ternate, September 27—29, 1929, 86 specimens.

Specific characters. Male genital organs in the posterior part of the body, outside the visceral mass. Testes of about equal size, globular, passing with a narrow canal with internal chitinous covering into the wide oval vasa deferentia. Colleteric glands with a very small number of branched canals. External cuticle without excrescences. Retinacula unknown.

The parasites are of small size, their greater diameter varying from 1.7 to 5 mm; they are, however, large in comparison to the size of their hosts, and almost invariably they are but partly covered by the abdomen. The smaller specimens (fig. 3 *u*, *v*, *w*) are of an oval shape: during further development the larger diameter increases more pronouncedly than the antero-posterior diameter, so that they become largely oblong, whilst often the central parts of the lateral surfaces remain narrower than the dorsal and ventral regions, so that then the shape of the animals may become distinctly panduriform (fig. 1*a*, *b*, *c*, *d*). The mantle opening lies at the top of a short tube in the central part of the anterior margin: it is slightly turned towards the left side (the surface of the parasite that is facing the thorax of the host).

Though as a rule the parasites of the larger crabs are larger than those of the smaller, there is not a direct proportion between the size of the parasite and that of the host. The following table refers to parasites on crabs which show a shape of the abdomen not noticeably differing from that of normal female crabs. In this table "No." means the specimen as indicated in the figures 1 and 2; "S." is the length (in mm) of the proximal border of the penultimate segment of the abdomen of the host: "P." is the greater diameter (in mm) of the parasite.

Fig. 1. *Sacculina ternatensis* nov. spec. $a-d$, left side of four specimens detached from their hosts; $e-z$, right side of various specimens, each partly covered with the abdomen of its host. × 6.

No.	S.	P.	No.	S.	P.	No.	S.	P.
1 *e*	2.5	4.7	1 *f*	2.8	4.7	2 *a*	2.9	4.2
1 *w*	2.6	3.6	1 *i*	2.8	5.0	2 *d*	2.9	4.7
1 *k*	2.7	4.7	1 *r*	2.8	3.7	2 *i*	3.0	4.3
1 *l*	2.7	4.4	1 *s*	2.8	3.4	1 *g*	3.0	4.3
1 *o*	2.7	3.9	2 *j*	2.8	4.2	1 *m*	3.0	4.7
2 *n*	2.7	4.2	1 *n*	2.9	4.4	1 *t*	3.3	3.8
1 *h*	2.7	4.6	1 *p*	2.9	4.0	1 *j*	3.3	5.0

Here the crabs are arranged according to size; it is evident that there is no direct correlation between the size of the hosts and that of the parasites. The same obtains when the parasites are measured that occur on crabs possessing an abdomen not differing in shape of that of normal male crabs. In the following table "S." and "P." have the same meaning as in the table above, whilst "No." refers to the specimen as indicated in fig. 3.

No.	S.	P.	No.	S.	P.	No.	S.	P.
3 *t*	0.9	2.6	3 *x*	1.1	2.8	3 *w*	1.3	2.1
3 *v*	0.9	2.2	3 *q*	1.2	3.7	3 *z*	1.3	3.0
3 *s*	1.0	3.3	3 *y*	1.2	2.7	3 *r*	1.3	3.6
3 *u*	1.1	1.7	3 *p*	1.2	3.2			

Fig. 2. *Sacculina ternatensis* nov. spec. Right side of various specimens, each partly covered with the abdomen of its host. × 6.

Here again there is no direct correlation between the size of the hosts and that of the parasites.

In the majority of the specimens there is no manifestation of parasitic castration in so far that the shape of the abdomen differs from that in normal female or male crabs. In some specimens, however, in which the abdomen still has the appearance of that of the female crab, it has become narrower in proportion to its length, thereby slightly changing towards that in the male sex (fig. 1*q*; fig. 2*c*, *e*, *o*, *p*, *r*; fig. 3*f*, *h*). In other specimens this process has gone further, so that now the abdomen, besides being narrower than that of the normal female, has an ultimate segment of a more or less equilateral triangular shape, thereby changing still more towards that of the male abdomen (fig. 2*t*; fig. 3*d*, *l*, *m*, *n*). Finally in one specimen (fig. 3 *o*) the ultimate segment of the abdomen is very similar to that of the normal male, whilst the rest of the abdomen resembles that of female crabs though being much narrower.

The parasite has not a fixed spot for penetrating the abdomen of its host. In the crabs possessing an abdomen showing the characters of the female sex the parasite may be attached to the proximal half of the

Fig. 3. *Sacculina ternatensis* nov. spec. Right side of various specimens, each (except *i*) partly covered with the abdomen of its host. × 6.

abdomen (e.g., fig. 1 *v*, *x*, *y*; fig. 2 *s*, *t*, *z*; fig. 3 *j*) or to the distal half (e.g., fig. 1 *g*, *k*, *l*, *m*, *n*, *o*, *q*). On the other hand the parasites of crabs that have an abdomen showing the characters of the male sex almost invariably are attached to the distal half of the abdomen (fig. 3 *p*—*z*).

Occasionally a crab is infested by two parasites. These may be attached to the proximal half of the abdomen (fig. 2 *p*; fig. 3 *b*) or to the distal half (fig. 2 *b*; fig. 3 *g*, *i*). When two parasites occur on one crab they do not seem to suffer from want of space, for as a rule their shape does not differ from that of solitary specimens.

The internal anatomy could be studied in longitudinal sections of four specimens and in transverse sections of one. Especially the transverse sections distinctly show the structure of the male organs. These are found in the posterior part of the body, outside the visceral mass. Fig. 4 *f* represents a transverse section from the posterior region. Here the male genital opening of the right side is to be seen and the left vas deferens closely adhering to the body wall; the section moreover shows the posterior part of the right testis. In the next section (fig. 4 *g*) the posterior part of the left testis and the cavity of the right testis are visible besides the two vasa deferentia. The following section (fig. 4 *h*) in both male organs shows the narrow tube with its chitinous wall that connects the testis with its vas deferens; in the left male organ this tube leaves the vas deferens, in the right male organ it enters the testis. In the last section (fig. 4 *i*) the chitinous

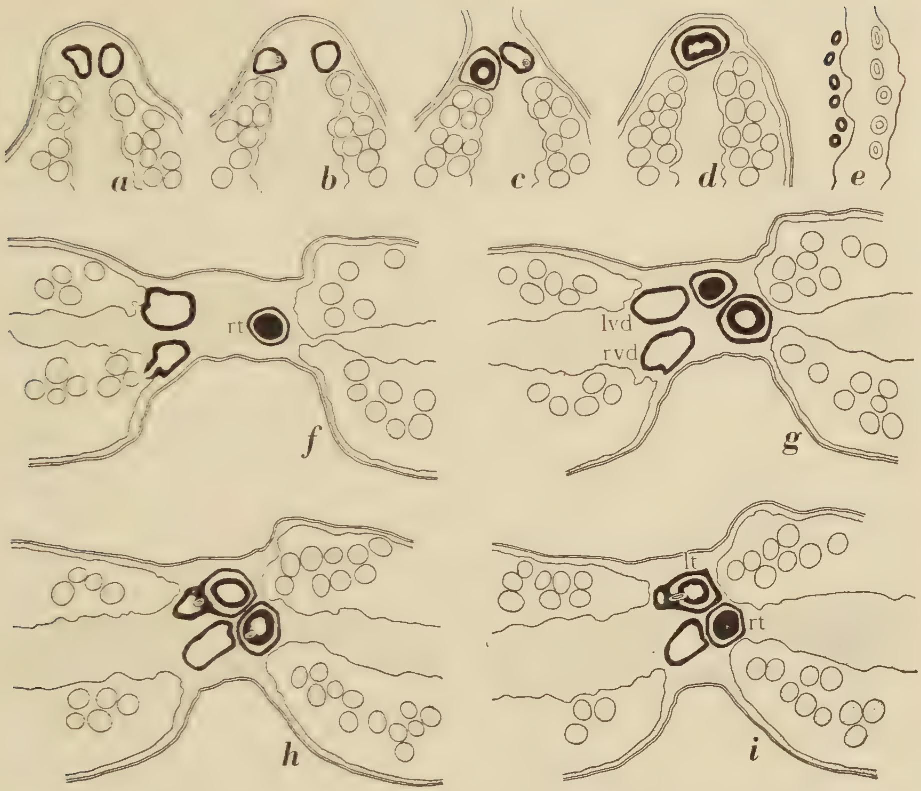

Fig. 4. *Sacculina ternatensis* nov. spec. *a—d*, posterior parts of longitudinal sections of one specimen, *a* in the region of the vasa deferentia, each following section from a slightly more dorsal region than the preceding; *e*, longitudinal sections of the colleteric glands of two specimens; *f—i*, central parts of transverse sections of one specimen, *f* from the posterior region, each following section from a slightly more anterior region than the preceding. *lt*, left testis; *lvd*, left vas deferens; *rt*, right testis; *rvd*, right vas deferens. *a—d*, *f—i*, × 30; *e*, × 63.

tube joins the left vas deferens to its testis, of the right male organ the anterior parts of the vas deferens and the testis are shown. The two testes are more or less globular; the two vasa deferentia are wide, slightly oval, their inner wall does not possess any ridges.

Of the longitudinal sections fig. 4 *a* represents the central parts of the vasa deferentia. In fig. 4 *b* the chitinous tube is seen on the inner wall of the left vas deferens. Fig. 4 *c* shows the central part of the left testis and the chitinous tube on the inner wall of the right vas deferens. In fig. 4 *d* the central part of the right testis is visible. The longitudinal sections show that the male organs are contained in the posterior part of the body, outside the visceral mass.

Longitudinal sections of the colleteric glands of two specimens are represented in fig. 4 *e*. In one of these the number of canals amounts to 6, these are drawn in black as the canals have no internal covering of chitin. In the other specimen there are 5 canals possessing distinct layers of

chitin. The sections are from regions in which the colleteric glands show their most pronounced division of the canal system.

In nearly all specimens examined the external cuticle of the mantle is entirely devoid of excrescences. Often this cuticle is completely smooth, in many cases it shows an outer layer that is slightly rough on account of minute rugosities as represented in fig. 5 *c*. In one specimen a part of the external cuticle is covered with small papillae of a circular or oval shape (fig. 5 *a*). These have a diameter of 2 to 6 μ, their height is about 2 μ. In

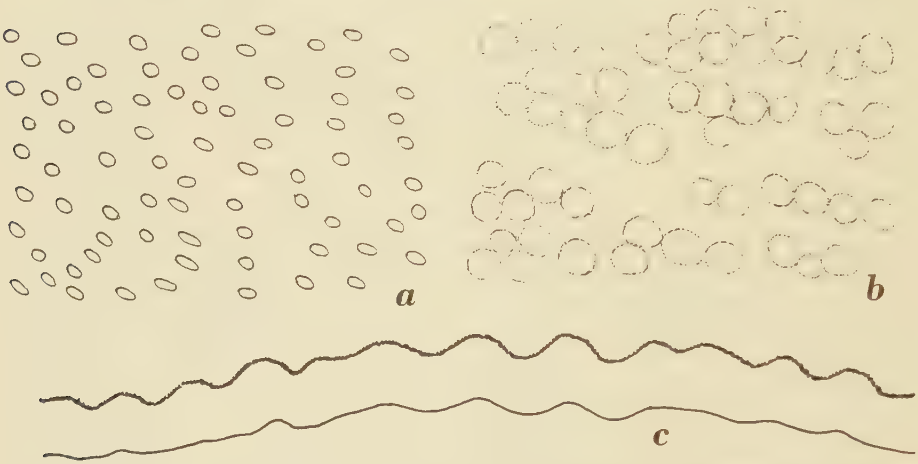

Fig. 5. *Sacculina ternatensis* nov. spec. *a*, upper surface of the external cuticle of a specimen showing small papillae; *b*, upper surface of the external cuticle of a specimen showing small tubercles; *c*, section of the external cuticle of a specimen having the usual structure. × 530.

another specimen a part of the external cuticle shows small tubercles, very little extending above the surface, having a diameter of 6 to 11 μ (fig. 5 *b*). These tubercles are arranged in rows.

As these peculiarities of the external cuticle were found in two specimens only, and in these in a small part of the cuticle only, we may safely regard these as abnormal features, and state that a smooth cuticle is characteristic of the species.

The external cuticle of the mantle is comparatively thin (thickness about 15 μ), it has a tendency to appear in sections as an undulating layer (fig. 5 *c*), which may, however, be a result of contraction during preservation.

No retinacula were found on the parts of the internal cuticle examined for this purpose.

A fairly large number of species of the genus *Sacculina* have, as the specimens dealt with above, an external cuticle that is devoid of excrescences, or has a more or less uneven or roughened surface, or shows small excrescences of a very indefinite kind. Among these there are two species described by KOSSMANN (1872), *Sacculina captiva* of which no distinctive

characters are known, and *S. pomum* of which it is stated that the two testes are united, proving that at least the latter is specifically distinct from the species described above.

In a previous paper (BOSCHMA, 1937) the characters are noted of twenty species of the genus with an external cuticle without excrescences or with very insignificant excrescences. In seven of these, *Sacculina caelata*, *S. calva*, *S. confragosa*, *S. glabra*, *S. pertenuis*, *S. rathbunae*, and *S. scabra*, the male organs are situated in the visceral mass. The other species correspond with *S. ternatensis* by having the male organs in the posterior part of the body, outside the visceral mass. In contradistinction to *S. ternatensis* eight of these, *S. anceps*, *S. curvata*, *S. flexuosa*, *S. gregaria*, *S. irrorata*, *S. plana*, *S. punctata*, and *S. rugosa*, have colleteric glands with a well developed system of canals, a longitudinal section showing twenty or more of these canals. The remaining species, *S. bicuspidata*, *S. gibba*, *S. pustulata*, *S. schmitti*, and *S. sulcata*, possess a smaller number of these canals. But in all of these the vasa deferentia are narrow canals, so that on account of this character they are distinct from *S. ternatensis*.

Among the species described by SHIINO (1943) there are six of the genus *Sacculina* that have an external cuticle without or with indistinct excrescences. In four of these, *S. nigra*, *S. fabacea*, *S. pugettiae*, and *S. upogebiae*, a longitudinal section of the most strongly branched region of the colleteric glands shows more than 20 canals. In two species, *S. imberbis* and *S. pinnotherae*, this number is smaller. These two species, however, differ from *S. ternatensis* by having comparatively narrow vasa deferentia.

In *Sacculina robusta*, another species without excrescences of the external cuticle, the number of canals in a longitudinal section of the colleteric glands is up to 23 or 24 (BOSCHMA, 1948). Moreover, there are other characters separating the species from *S. ternatensis*.

Notwithstanding the lack of distinctive characters in the structure of its external cuticle the species described above, therefore, proves to be distinct from all other well known species of the genus by definite characters.

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THEORY ON CENTRAL RECTILINEAR RECESSION OF SLOPES IV

BY

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(Communicated by Prof. F. A. VENING MEINESZ at the meeting of Sept. 30, 1950)

6. *The ratio between the flat part and the curved part of the rocky nucleus*

The simplified construction of the isoclines by the drawing of some special points, however, is not our principal aim. The integral-curve has a flat part and a curved part. The unprotected flat part of the rocky nucleus being subject to further softening, we should like to know something about the ratio of these parts and about the influence of the data a, b and c upon this ratio.

This is one of the central problems of the interpretation of slopes in Nature. As the reader will be aware W. PENCK [18, p. 127—132] attempted to demonstrate that the slopes of our figures 18 and 19 [W. PENCK's figures 7 (p. 129) and 8 (p. 131)] are the results of accelerated vertical erosive action of streams.

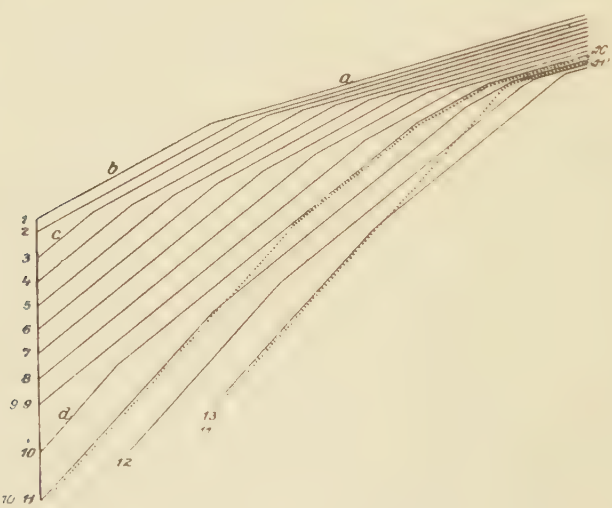

Fig. 18 (after W. PENCK)

"Das veranschaulicht fig. 7 (our figure 18) in der 13 aufeinanderfolgende Hanglagen konstruiert worden sind, die sich unter der Annahme *wachsender Erosionsintensität* ergeben" [W. PENCK, Morpholog. Analyse p. 128].

"Es ist ersichtlich (fig. 8, our figure 19), dasz sich die Hänge umso schärfer

konvex gegen die Talkerben krümmen, je gröszer die Erosionsbeschleunigung ist, je rascher die Erosionsintensität wächst" [W. PENCK, Morphol. Analyse p. 131].

In this part of his theory W. PENCK jumped to the conclusion, that the convex "Gefällsbruch" between the steep part at the foot of his slopes and their flat upper parts could be formed exclusively by accelerated vertical fluvial erosion:

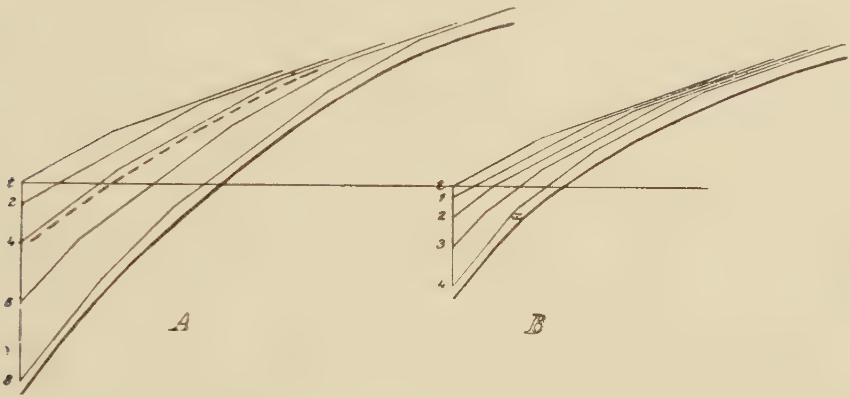

Fig. 19 (after W. PENCK)

"Der Gefällsbruch ist eine Diskontinuität der Abdachung; er wird aber verursacht durch eine kontinuierliche Zunahme der Erosionsintensität. Zum erstenmal begegnen wir dem Fall, dasz eine stetige Ursache morphologisch eine Unstetigkeit bewirkt" (Morphol. Analyse p. 131).

It is clear that especially the undermost curves of W. PENCK's figures 8*A* and 8*B* (our figures 19*A* and 19*B*) have approximately the same shapes as the curves of central rectilinear recession for $\alpha = 25°$ and $22°$, $\beta = 52°$ and c about 1/3. If it is assumed, that the screes were removed afterwards it is possible to explain W. PENCK's slopes in quite an other way. These slopes may be the results of central rectilinear recession by weathering removal of a wall with an initial slope angle of 52°, without any direct effect on the rocky nucleus of accelerated vertical fluvial erosion. Moreover we demonstrated in the second part of our theory (16, p. 1158—1159) that in special conditions knickpoints in convex slopes can be formed exclusively by the free play of weathering removal recession, without having any direct effect of vertical fluvial erosion.

It is our conviction, that W. PENCK's theory is not justified from a quantitatively exact standpoint, a fact which has been pointed out before by JOH. SÖLCH (19) and J. P. BAKKER (6*a*, 20 and 21). In our opinion the theoretical knowledge of slope development based upon a mathematical treatment of the phenomena in Nature has not yet proceeded sufficiently to jump to W. PENCK's morphotectonic conclusions. It should not be overlooked, that without a rigorous mathematical treatment such series

of two-dimensial drawings have not any deductive value. In the most favourable case series of two-dimensial figures or blockdiagrams etc., drawn with the purpose to demonstrate a morphogenetic succession — for instance of cyclic landscape development in arid or humid climates — are nothing but series of co-existing types of present-day landscapes. Reasoning inductively — induction is methodical guesswork — scientists such as DAVIS and W. PENCK assume, that there must be a logical morphogenetic relation between the landscapetypes of their series, but a deductive evidence for the exactness of this assumption is completely lacking. It is clear that from a logical standpoint there is a great danger in these pseudo-deductive treatments of physionomic geomorphologic problems. There are two lacunae in their logical construction: 1. an external one, 2. an internal one:

1. The conditions on which the pseudo-deductions are based, are not established at all or quite insufficiently. For instance in W. PENCK's treatise on slope development over an unchanging baselevel (18, p. 109—115), the author started from one or a small number of constant a-values and a high negative c-value, but overlooked, that halfway he changed his constant a-values into completely unknown variable ones.
2. As we already mentioned the internal lacuna in the logical construction of several present day theories on slope development is the most dangerous one. It is the lacuna of exactly justified evidence of the reality of a special landscape succession. Is a landscape B really the successor of A and C of B etc.?

If a scientist is not able to establish the physico-mathematical relation between C and B, B and A etc. his opinion has only the value of a supposition or an apriorism — we shall call it in future the special morphogenetic apriorism — and the demonstration in his pseudo-deductions has only the value of a parable. From most textbooks and other publications on geomorphology the impression might be gathered, that we know already fairly much of cyclic development of mountain forms. In our opinion, however, our actual knowledge of several parts of physionomic geomorphology is extremely poor, especially as regards such regions, where it is impossible to find datable sediments and fossil soiltypes on the different plateau-like forms, terraces etc. For our knowledge of slope development and several other physionomic geomorphologic problems a decision pro or con mathematical methods is at the same time a choice between science and parable, a question of to be or not to be.

Apart from W. PENCK's ideas the problem of knickpoints in slopes is a very important one for the bicyclic or monocyclic interpretation of rounded summits, ridges and plateau-like forms. Already 30 years ago SÖLCH discussed this problem in his very interesting publication: "Eine Frage der Talbildung" (22, p. 67—68). For the special case of "Eckflurs" the

author's conclusion is, that a bicyclic interpretation as well as a monocyclic one is possible, but he believes that the former interpretation is more probable. "Allein es wird doch wohl nur in Ausnahmen aus einem Grat oder einem gewölbten Rücken ebenflurige, scharfrandige Platte mit steilen Gehängen hervorgehen".

Indeed, if we follow the current opinions a priori a bicyclic interpretation seems to be the most probable one, but we must not forget, that up to this day the knickpointproblem and his various possibilities have never been investigated from a completely exact standpoint; not even for the simplest conditions (see our theory p. 1158—1159 and some aspects of this problem in BAKKER, 23, p. 10—29).

In the following lines we shall give a mathematical method, which is necessary as introduction for an exact treatment of the knickpointproblem.

In the first place, a mathematical definition of the expression "flat part" is wanted. We will call "flat part" of the integral-curve the part limited by the isoclines $p = \tan\alpha$ and $p = \tan\alpha'$, if the difference $\tan\alpha' - \tan\alpha$ is 0,1. Of course, another small value may be taken. So, $\alpha' - \alpha$ is the angle between the directions of the tangents to the integral-curve in the extremities of the nearly straight part. In fig. 14 and 15, the flat part lies between the isoclines 0,4 and 0,5; $\alpha' - \alpha$ is nearly 5°.

The ratio between the flat part and the curved part depends upon the intersection-point of isocline $\tan\alpha - 0{,}1$ with the integral-curve. Observing the course of the isoclines in fig. 14 and 15 (part III of our theory), we see that the scale of their meeting-points with the curve shows some resemblance to the scale of their meeting-points with the slope.

Therefore, our following examination will be based on the definition:

> *The ratio between the flat part and the curved part of the integral-curve depends upon the intersection-point S of isocline $\tan\alpha + 0{,}1$ with the slope in the case of a plateau as well as in the case of a crest.*

Construction of S from given values of a, b and c

As we have seen sub 5 (part III of our theory) the simplified construction of S requires the construction of Q (fig. 20). Now Q is the meeting-point of HA and NB. Choosing OK as unit, $KH = h = \tan\beta = 1/b$. So, the coordinates of H are $(1, 1/b)$. Since $AD = KG = \tan\alpha = 1/a$ and $OA = AD \cot\beta = b/a$, the coordinates of A are $(b/a, o)$. Thus, the equation of HA is

$$b\,y = \frac{ax - b}{a - b}.$$

Since $BE = KF = \tan\gamma = p$ and $OB = BE \cot\beta = pb$, the coordinates of B are (pb, o). The coordinates of C being $(m, 1/b)$, the equation of NB is

$$b\,y = \frac{x - pb}{m - pb}.$$

From $\frac{ax-b}{a-b} = \frac{x-pb}{m-pb}$ we derive $x = \frac{b\,(m-ap)}{a\,(m-1)-b\,(ap-1)}$.

Now again $m = 2(1-c)\,(ap-1)+1$ so $m-1 = 2\,(1-c)\,(ap-1)$ and $m-ap = (1-2c)\,(ap-1)$. Therefore

$$x = \frac{b\,(1-2c)\,(ap-1)}{(ap-1)\,\{2a\,(1-c)-b\}} = \frac{b\,(1-2c)}{2a\,(1-c)-b}$$

and

$$y = \frac{1}{b}\cdot\frac{ax-b}{a-b} = \frac{-1}{2a\,(1-c)-b}.$$

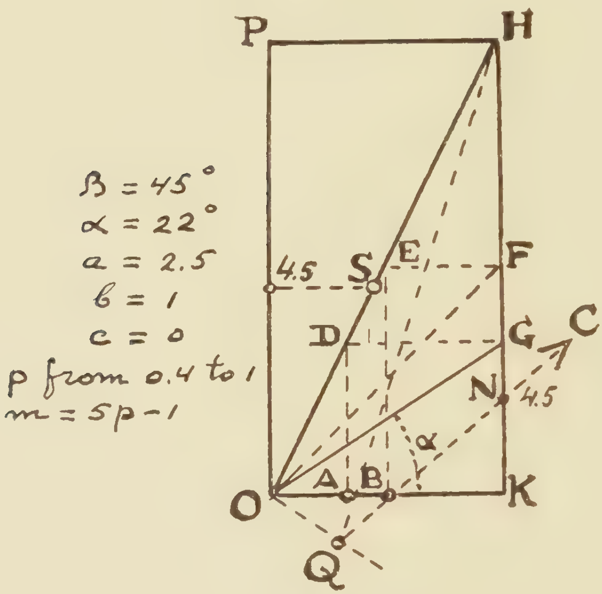

Fig. 20

Consequently: Q lies on the straight line

(20) $$x = -\,b(1-2c)y$$

To get some idea of the ratio between the flat part and the curved part of the rocky nucleus, without drawing any curve, we now proceed as follows.

In the right-angled triangle of fig. 20, OH is the slope and $\angle\, HOK = \beta$ the initial slope-angle of the wall of a plateau or crest.

Draw OP parallel to KH with a regular scale from 0 to 10. KH gets a quadratic scale from 0 to 10. Draw OG so that $\angle\, GOK = \alpha$, the slope-angle of the screes. Cot $\alpha = a$. Draw OF to define the "flat part". $\angle\, FOG$ is the angle between the directions of the tangents in the extremities of the nearly straight part of the integral-curve. FG may be taken 0,1. Transpose the points G and F to OK by drawing GDA and FEB. Draw HA and OQ [cot $KOQ = -\,b(1-2c)$] with the point of intersection Q. Finally, join Q to B and produce this line till the quadratic scale is intersected

in N. The number of N gives the value of the ordinate of S. Construct S by means of the regular scale upon OP, then $HS : SO$ is approximately the ratio between the flat part and the curved part (in horizontal sense for a plateau and in vertical sense for a crest).

7. *Discussion of the influence of a, b and c upon the ratio*

I. *Influence of a*

To discuss the influence of a upon the place of the point N (fig. 20), we will calculate the ordinate KN of this point.

For the coordinate of Q we have found

$$x = \frac{b\,(1-2c)}{2a\,(1-c)-b} \quad , \quad y = \frac{-1}{2a\,(1-c)-b}.$$

The coordinates of B are $OB = x = BE$ cot $\beta = KF$. $b =$ tan α'. $b = b/a'$ and $y = o$ (tan α' = tan $\alpha + 0.1 = (10+a)/10a$).

The equation of QB is:

$$\frac{x - \dfrac{b}{a'}}{\dfrac{b\,(1-2c)}{2a\,(1-c)-b} - \dfrac{b}{a'}} = \frac{y}{\dfrac{-1}{2a\,(1-c)-b}}.$$

For $x = 1$, $y = KN = \dfrac{b-a'}{b\,\{(a-a')\,(2c-1)-(a-b)\}}$

y becomes zero, if $b = a'$ or tan $\beta -$ tan $\alpha = 0.1$.

Conclusion:

In the case of a small difference between β and α, the rocky nucleus may be regarded as a "flat part" over its whole length.

Introducing the value of $a' = \dfrac{10a}{10+a}$ we find

$$y = \frac{1}{b}\,\frac{10\,(a-b)-ab}{2a^2\,(1-c)+10\,(a-b)-ab}.$$

The ratio: $\dfrac{\text{flat part}}{\text{curved part}}$ becomes

$$\frac{1-y}{y} = \frac{2a^2}{10\,(a-b)-ab}\cdot(1-c) = (1-c)\,f\,(a).$$

The function $f(a)$ becomes minimum for $a = \dfrac{20\,b}{10-b}$, independent of c and then, the ratio has the value

$$\frac{80\,b}{(10-b)^2}\cdot(1-c) \text{ dependent upon } c.$$

Since $a = \dfrac{1}{\tan\alpha}$ and $b = \dfrac{1}{\tan\beta}$, the condition $a = \dfrac{20\,b}{10-b}$ becomes tan $\beta - 0.1 = 2$ tan α.

Conclusion:

If there exists the relation tan β $- 0.1 = 2$ *tan a between the initial slope-angle of the wall of a plateau or crest and the slope-angle of the screes the ratio between flat part and curved part of the rocky nucleus has a minimum value, dependent upon the ratio between rock-volume and screes-volume.*

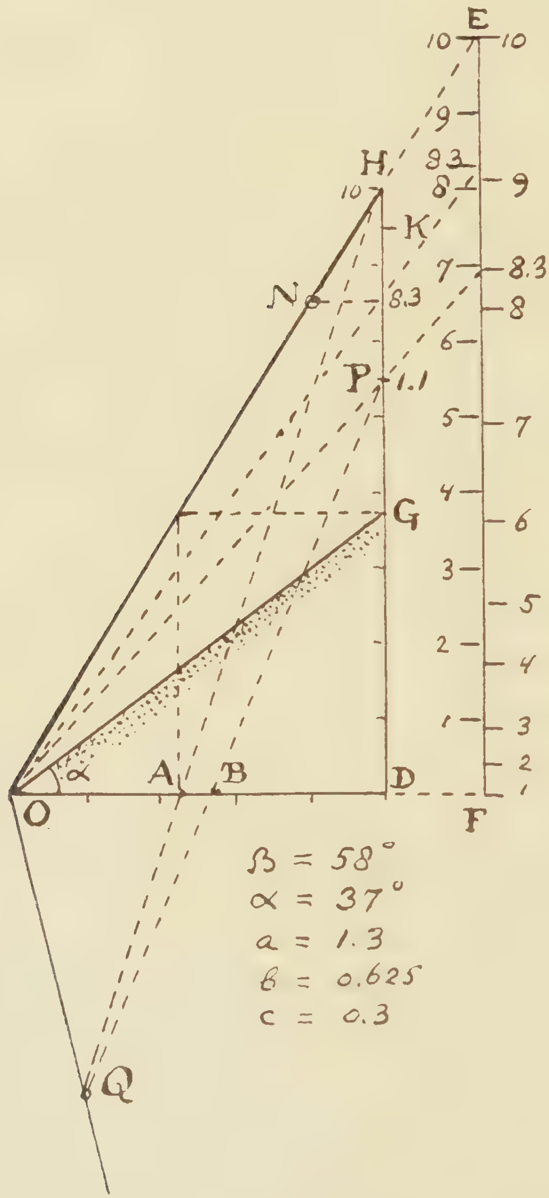

Fig. 21

Example:

In fig. 21 we have chosen $\angle\ HOD = \beta = 58°, OD = 1, DH = \tan\beta = 1.6$ The angle a can easily be constructed by taking $HK = 0.1$ and bisecting KD in G. It follows, that $\tan a = \frac{3}{4}$.

G transposed to OD gives A and H joined to A meets OQ in Q ($\tan DOQ = = \frac{-1}{b(1-2c)} = -4$, if we choose $c = 0.3$). Q joined to B ($AB = 0.1$) meets DH in P. The value of $PD = y$ is obtained by putting $a = \frac{20b}{10-b}$ in the formula $y = \frac{1}{b} \frac{10(a-b) - ab}{2a^2(1-c) + 10(a-b) - ab}$ which gives

$$y = \frac{1}{b} \frac{(10-b)^2}{(10+b)^2 + 40b(1-2c)}$$

and then putting $b = \frac{5}{8}$, $c = 0.3$. We find $y = 1.1$ measured with OD as unit. The number of P on the (auxiliary) quadratic scale is 8.3 (right side of EF). Transposed to the regular scale (left side of EF) and brought back to HD, we find the point N on the slope.

We can now discuss the influence of a on the shape of the integral-curve. In fig. 21, for given values of b and c, the flat part of this curve cannot be smaller than HN, in horizontal parallel projection for a plateau and in vertical parallel projection for a crest. The flat part increases, if $\tan a$ decreases from GD to zero and if $\tan a$ increases from DG to DK (maximum flat part for $\tan \beta - a = 0.1$).

In general terms:

If a decreases from ∞ *to* $\frac{20b}{10-b}$, *the flat part decreases.*

If a decreases from $\frac{20b}{10-b}$ *to b, the flat part increases.*

II. *Influence of b.*

It is a remarkable fact, that the value b does not appear in the equation of the isoclines

$$px = y\frac{h^2 - my^2}{h^2 - y^2} \text{ and } px = y\frac{k^2 - mx^2}{k^2 - x^2} \qquad [m = 2(1-c)(ap-1) + 1].$$

That means: shape and position of the isoclines are independent of the initial slope-angle of the wall of a plateau or a crest. Calculating the point of intersection of an isocline and the slope with equation $x = by$, we find:

$$pb = \frac{h^2 - my^2}{h^2 - y^2} \text{ (plateau) and } pb = \frac{k^2 - mx^2}{k^2 - x^2} \text{ (crest)}$$

($x = 0$, $y = 0$ are the coordinates of the second point of intersection in the origin).

If we seek the locus of the meeting-point, when b is variable, the value b must be eliminated between the two last equations and $x = by$. The result is:

$$px = y\frac{h^2 - my^2}{h^2 - y^2} \text{ and } px = y\frac{k^2 - mx^2}{k^2 - x^2}$$

the equations of the isoclines themselves.

We have seen that the angle α of the screes being given, the ratio between the flat part and the curved part of the integral curve is determined by the meeting-point of the isocline $p = \tan \alpha + 0.1$ with the slope. *Thus, the locus of this meeting-point, when β varies from 90° to α, is the isocline $p = \tan \alpha + 0.1$ in the case of a plateau as well as in the case of a crest.*

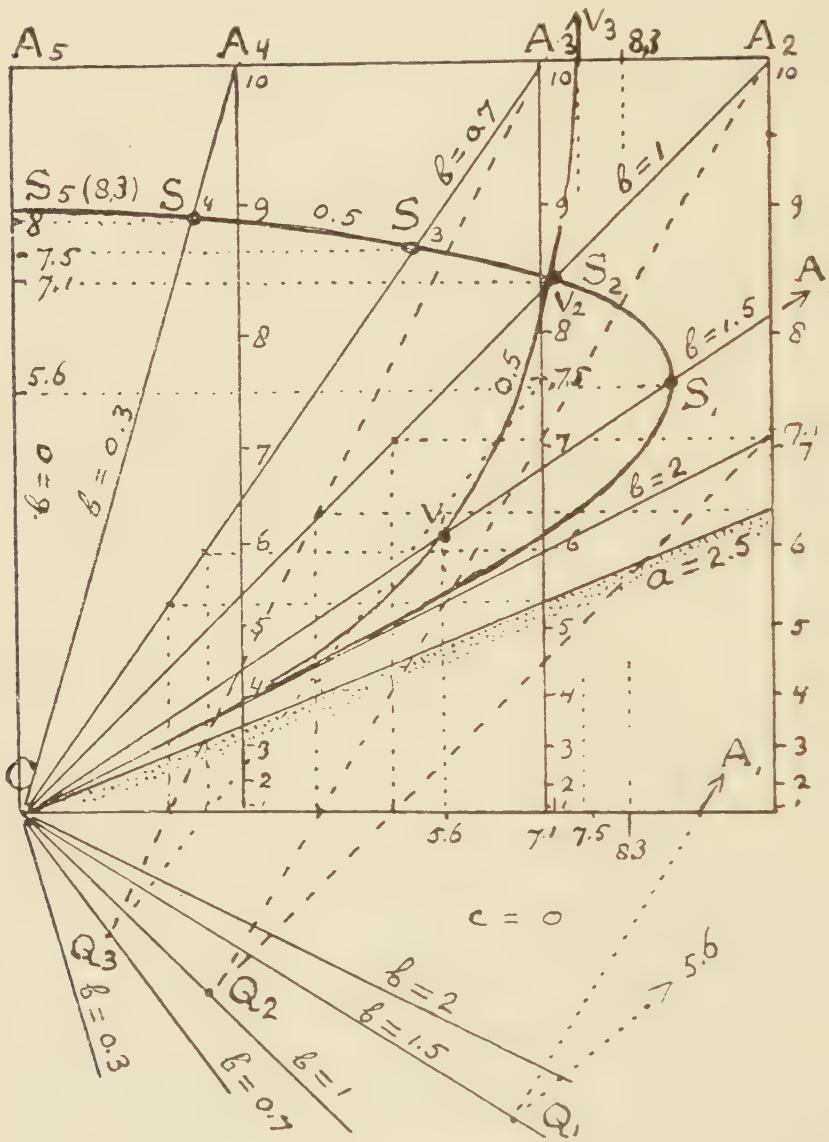

Fig. 22

Example: In fig. 22, $\alpha = 2.5$; $c = 0$. Since $\tan \alpha = 0.4$, we have constructed isocline 0.5 for a plateau and for a crest. Both isoclines meet on the slope OA_2 in S_2, because $k = h$. The ordinate of S_2, constructed by means of Q_2 ($b = 1$) is 7.1. Now the ratio between the flat part and the

curved part of the integral curve in S_2A_2: S_2O (horizontally measured for a plateau, vertically measured for a crest). In this manner, the ratio can easily be read off if b varies from 0 to 2. For a plateau the points of division are S_5—S_1 for a crest V_5—V_1. For $b = 2$, the slope is tangent to isocline 0.5, the curved part disappears and we find again: *In the case of a small difference between* β *and* α ($\tan \beta - \tan \alpha = 0.1$), *the integral curve has no curved part.*

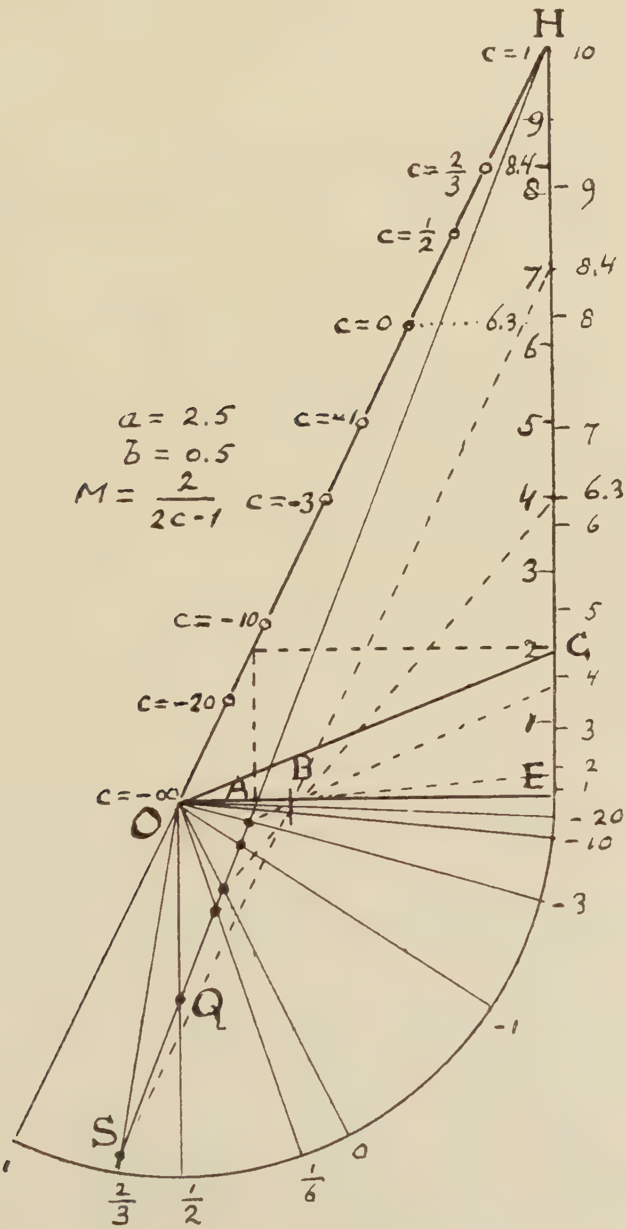

Fig. 23

Reversely, the isoclines may be constructed by seeking their meetingpoints upon the lines OA_1, OA_2 etc.

In fig. 22, the flat part in both cases cannot be smaller than 1.7 $(OS_5 = \frac{h}{\sqrt{m}} = \frac{10}{\sqrt{1,5}} = 8.3$ and the asymptote of isocline 0.5, $x = \frac{k}{\sqrt{m}})$.

Assuming that the lines AS are measured up to their points of intersection with $A_5\,A_2$, we can say about the influence of b upon the shape of the integral curve: *If, for given values of a and c, the value of b increases, the flat part of the integral curve increases also, till the maximum is reached for* $b = \frac{10a}{10+a}$

III. *Influence of c*

In fig. 23, $a = 2.5$; $b = 0.5$. The fragment $AB = 0.1$ remains invariable in position when c varies from 1 to $-\infty$ (for the c-values in Nature, see part II of our theory).

We have drawn the direction of OQ $(M = \frac{2}{2c-1})$ for $c = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{6}, 0,$

$-1, -3, -10$ and -20.

The invariable line HA meets these directions in the points Q, which are joined to B. The meetingpoints with the quadratic scale are transposed to the slope OH by means of the double-scale EH. About the influence of c upon the shape of the integral curve, we observe: *Changes in the positive values of c between 0 and $\frac{1}{2}$ have little influence: the flat part increases if c decreases. The curved part disappears for high negative values of c* (practically beginning with $c = -25$. $M = -\frac{1}{51}\tan\beta$).

Finally, we wish to express our gratitude to Mrs A. v. d. Zee-de Bruijne for her help in the correction of the text and to Miss C. Vervoort and Mr W. Koldewijn, assistants for physical geography at the University of Amsterdam for their kindness in preparing text and figures for the press.

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MATHEMATICS

FORMALISTISCHE BETRACHTUNGEN ÜBER INTUITIONISTISCHE UND VERWANDTE LOGISCHE SYSTEME. IV

VON

J. RIDDER

(Communicated by Prof. L. E. J. BROUWER at the meeting of Sept. 30, 1950)

Nähere Betrachtung des "logistischen Kalküls" LG_1

§ 32. Das System von Regel und Schemata für LG_1 läszt folgende Abänderungen zu (ohne dasz dabei die Klasse der (ableitbaren) Sequenzen oder Sätze sich ändert).

1° *Die Einsetzungsregel kann auf Einsetzungen in P oder Q beschränkt bleiben.*

Denn jede von einer Einsetzung in P oder Q verschiedene Einsetzung (etwa von $\mathfrak{M}$ für X) findet in einer Untersequenz U eines Ableitungsschemas A_1—A_4, UES, UEA, FES oder FEA statt (U soll dabei in U^* übergehen). Sie ist zu ersetzen durch eine Einsetzung in der Obersequenz O oder den Obersequenzen O_1, O_2 des Schemas (O, O_1, O_2 mögen dabei bzw. in $O^*, O_1^*. O_2^*$ übergehen), wobei im Falle von A_1 oder UEA auszerdem in $\mathfrak{D}$ bzw. $\mathfrak{B}$ ($\mathfrak{A}$) vorkommende X durch $\mathfrak{M}$ ersetzt werden sollen ($\mathfrak{D}$, $\mathfrak{B}$, $\mathfrak{A}$ mögen dabei dann bzw. in $\mathfrak{D}^*$, $\mathfrak{B}^*$, $\mathfrak{A}^*$ übergehen); nun geht U^* mittels desselben Schemas, ev. unter Ersetzung von $\mathfrak{D}$, $\mathfrak{B}$, $\mathfrak{A}$ durch $\mathfrak{D}^*$, $\mathfrak{B}^*$, $\mathfrak{A}^*$, aus der abgeänderten Obersequenz O^* oder den abgeänderten Obersequenzen O_1^*, O_2^* hervor.

In einer Ableitungsreihe können wir also die erste Anwendung einer von einer Einsetzung in P oder Q verschiedenen Einsetzung nach oben verschieben; dies braucht nur endlich viele Male wiederholt zu werden, damit nur Einsetzungen in P oder Q übrig bleiben. Man wiederholt das Verfahren für eine ev. vorkommende zweite Anwendung der Einsetzungsregel, u.s.w. Schlieszlich sind dann alle Einsetzungen der Reihe in Einsetzungen in P oder Q verwandelt, während doch die Endsequenz dieselbe geblieben ist.

2° Ist etwa $\mathfrak{D}$ eine Abkürzung von $A \cdot (B \cdot C)$, $\mathfrak{A}$ von $G \cdot (H \subset J)$, so soll $\mathfrak{D} \cdot \mathfrak{D} \cdot \mathfrak{A}$, nach Definition, eine Abkürzung von

$$[\{A \cdot (B \cdot C)\} \cdot \{A \cdot (B \cdot C)\}] \cdot \{G \cdot (H \subset J)\}$$

sein. Mit P, UES, A_1, A_3 [62]) leitet man ab:

$$\mathfrak{D} \cdot \mathfrak{D} \cdot \mathfrak{A} \to \{[\{[\{A \cdot (B \cdot C)\} \cdot A] \cdot (B \cdot C)\} \cdot G] \cdot (H \subset J)\}$$

[62]) Oder mit den Sätzen 1—5.

und ebenso, umgekehrt,

$$\{[\{[\{A \cdot (B \cdot C)\} A] \cdot (B \cdot C)\} \cdot G] \cdot (H \subset J)\} \rightarrow (\mathfrak{D} \cdot \mathfrak{D} \cdot \mathfrak{A}).$$

Allgemein folgt aus P, UES, A_1—A_4:

Ist $\mathfrak{D}_1$ ein Produkt (eine Konjunktion) von n_1 Faktoren (Kalkülformeln) $\mathfrak{C}_1^{(p_1)}$ $(p_1 = 1, 2, \ldots, n_1)$, $\mathfrak{D}_2$ von n_2 Faktoren $\mathfrak{C}_2^{(p_2)}$ $(p_2 = 1, 2, \ldots, n_2), \ldots, \mathfrak{D}_m$ von n_m Faktoren $\mathfrak{C}_m^{(p_m)}$ $(p_m = 1, 2, \ldots, n_m)$, so läszt sich ableiten:

$(\alpha) \ldots \mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_m \rightarrow (\mathfrak{C}_1^{(1)} \cdot \mathfrak{C}_1^{(2)} \ldots \mathfrak{C}_1^{(n_1)} \cdot \mathfrak{C}_2^{(1)} \cdot \mathfrak{C}_2^{(2)} \ldots \mathfrak{C}_2^{(n_2)} \ldots \mathfrak{C}_m^{(1)} \cdot \mathfrak{C}_m^{(2)} \ldots \mathfrak{C}_m^{(n_m)})$,

und

$(\beta) \ldots \mathfrak{C}_1^{(1)} \cdot \mathfrak{C}_1^{(2)} \ldots \mathfrak{C}_1^{(n_1)} \ldots \mathfrak{C}_m^{(1)} \cdot \mathfrak{C}_m^{(2)} \ldots \mathfrak{C}_m^{(n_m)} \rightarrow (\mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_m)$. [63]

Auszerdem behalten die Sequenzen (α) und (β) ihre Gültigkeit, wenn man (eins oder mehrere Male) ohne die Ordnung zu ändern zwei oder mehrere aufeinander folgende Faktoren $\mathfrak{C}_j^{(k)}$ zu einer Formel zusammenfaszt; ebenso wenn man die Ordnung der Faktoren $\mathfrak{C}_j^{(k)}$ willkürlich ändert, oder von Formeln $\mathfrak{C}_j^{(k)}$ von gleicher Gestalt eine oder mehrere, doch nicht alle, fortläszt.

Daraus folgt u.a., dasz der Umfang des Kalküls LG_1 ungeändert bleibt, *wenn wir von nun an das Schluszschema A_3 wie folgt lesen*:

Schema A_3. Vertauschung im Antezedens:

$$\frac{\mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_{k_1} \cdot \mathfrak{D} \cdot \mathfrak{E} \cdot \mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_{k_2} \rightarrow \mathfrak{B}}{\mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_{k_1} \cdot \mathfrak{E} \cdot \mathfrak{D} \cdot \mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_{k_2} \rightarrow \mathfrak{B}}$$

(die $\mathfrak{A}_j$ und $\mathfrak{C}_j$ dürfen teilweise oder alle fehlen).

Mit Schema A_3 allein folgt nun aus Schema P und Einsetzungsregel, dasz in jedem Produkt die Ordnung der Faktoren willkürlich geändert werden darf, d.h. ist $j_1, j_2, \ldots, j_n$ eine Permutation von $1, 2, 3, \ldots, n$, so gilt

$$\mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n \rightarrow (\mathfrak{D}_{j_1} \cdot \mathfrak{D}_{j_2} \ldots \mathfrak{D}_{j_n}) \text{ und } \mathfrak{D}_{j_1} \cdot \mathfrak{D}_{j_2} \ldots \mathfrak{D}_{j_n} \rightarrow (\mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n),$$

oder

$$\mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n = \mathfrak{D}_{j_1} \cdot \mathfrak{D}_{j_2} \ldots \mathfrak{D}_{j_n}.$$

Die in Fuszn. 63 *angegebene abgekürzte Schreibweise lassen wir von nun an nur für Antezedentia von Sequenzen zu* (siehe 4°).

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[63]) $\mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_m$ ist dabei eine Abkürzung von $(\ldots(((\mathfrak{D}_1 \cdot \mathfrak{D}_2) \cdot \mathfrak{D}_3) \cdot \mathfrak{D}_4) \ldots) \cdot \mathfrak{D}_m$; analog das Produkt der $\mathfrak{C}_j^{(p_j)}$ $(j = 1, \ldots, m;\ p_j = 1, 2, \ldots, n_j)$. Ein Produkt enthält mindestens einen Faktor. — Fur die in § 32[bis] folgenden Anwendungen ist es wichtig zu bemerken, dasz, während $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_n$ eine Abkürzung für *eine* bestimmte Formel ist (bei gegebenen $\mathfrak{A}_j$), jedoch umgekehrt eine gegebene Formel sich oft auf *mehrere* Weisen abkürzen läszt; Beispiel: $[\{(A_1 \cdot A_2) \cdot A_3\} \cdot A_4]$ hat die Abkürzungen: $A_1 \cdot A_2 \cdot A_3 \cdot A_4$; $(A_1 \cdot A_2) \cdot A_3 \cdot A_4$; $\{(A_1 \cdot A_2) \cdot A_3\} \cdot A_4$ oder $(A_1 \cdot A_2 \cdot A_3) \cdot A_4$, und sich selbst. Siehe die Verabredungen unter 4°, welche dazu dienen die hierdurch entstehenden Schwierigkeiten zu entgehen.

Mit dem neuen Schema A_3 beweist man leicht, dasz A_1 sich in folgender Weise erweitern läszt:

Schema A_1 Verdünnung im Antezedens: $\dfrac{\mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_n \rightarrow \mathfrak{B}}{\mathfrak{D} \cdot \mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_n \rightarrow \mathfrak{B}}$,

und dasz daneben A_2 die Erweiterung:

Schema A_2. Zusammenziehung im Antezedens: $\dfrac{\mathfrak{D} \cdot \mathfrak{D} \cdot \mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_n \rightarrow \mathfrak{B}}{\mathfrak{D} \cdot \mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_n \rightarrow \mathfrak{B}}$

hat. *Von nun an wollen wir die Schemata A_1 und A_2 in der eben gegebenen Form lesen.*

Das Schema A_3 führt ebenso zu folgenden Erweiterungen:

Schema UEA. $\dfrac{\mathfrak{A} \cdot \mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n \rightarrow \mathfrak{C}}{(\mathfrak{A} \cdot \mathfrak{B}) \cdot \mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n \rightarrow \mathfrak{C}}$ und $\dfrac{\mathfrak{B} \cdot \mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n \rightarrow \mathfrak{C}}{(\mathfrak{A} \cdot \mathfrak{B}) \cdot \mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n \rightarrow \mathfrak{C}}$;

(die $\mathfrak{D}_j$ dürfen fehlen);

und:

Schema FES. $\dfrac{\mathfrak{A} \cdot \mathfrak{B}_1 \cdot \mathfrak{B}_2 \ldots \mathfrak{B}_n \rightarrow \mathfrak{C}}{\mathfrak{B}_1 \cdot \mathfrak{B}_2 \ldots \mathfrak{B}_n \rightarrow (\mathfrak{A} \subset \mathfrak{C})}$:

auch:

Schema FEA. $\dfrac{\mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n \rightarrow \mathfrak{A} \quad \mathfrak{B} \cdot \mathfrak{E}_1 \cdot \mathfrak{E}_2 \ldots \mathfrak{E}_m \rightarrow \mathfrak{C}}{(\mathfrak{A} \subset \mathfrak{B}) \cdot \mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n \cdot \mathfrak{E}_1 \cdot \mathfrak{E}_2 \ldots \mathfrak{E}_m \rightarrow \mathfrak{C}}$

(die $\mathfrak{E}_j$ dürfen fehlen).

Von nun an wollen wir UEA, FES und FEA in dieser Form lesen.

In UES lassen wir abgekürzte Formen für das Antezedens zu. In P und Q sollen $\mathfrak{D}$ und $\mathfrak{A}$ immer in nicht-abgekürzter Form gemeint sein. Aus 4° geht hervor, dasz nachherige Überführung der Antezedentia in abgekürzte Formen möglich ist.

3° *Das Schluszschema A_4 läszt sich durch folgendes Mischungsschema M ersetzen:*

Schema M. Mischung: $\dfrac{\overline{\mathfrak{A}} \rightarrow \mathfrak{B} \quad \overline{\mathfrak{C}} \rightarrow \mathfrak{D}}{\overline{\mathfrak{A}} \cdot \overline{\mathfrak{C}}^* \rightarrow \mathfrak{D}}$

[$\overline{\mathfrak{C}}$ ein Produkt $\mathfrak{D}_1 \cdot \mathfrak{D}_2 \ldots \mathfrak{D}_n$ ($n \geqq 1$), in welchem mindestens einer der Faktoren $\mathfrak{D}_j$ ein $\mathfrak{B}$ ist; $\overline{\mathfrak{A}} \cdot \overline{\mathfrak{C}}^*$ entsteht aus $\overline{\mathfrak{C}}$ durch Fortlassung aller $\mathfrak{D}_j$ mit der Gestalt $\mathfrak{B}$, und Voranstellung der Faktoren von $\overline{\mathfrak{A}}$ in der Ordnung, in welcher sie in $\overline{\mathfrak{A}}$ auftreten]. [64])

$\mathfrak{B}$ wollen wir *die Mischformel* nennen.

M folgt aus A_4. Denn nach A_3 (wie unter 2°) und A_2 ist:

$$\mathfrak{B} \cdot (\overline{\mathfrak{C}}^*) \rightarrow \mathfrak{D}$$

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[64]) Eine überstrichene deutsche Buchstabe wird als Produkt von Kalkülformeln aufgefaszt, deren Anzahl auch 1 sein darf; sie deutet somit eine Abkürzung an.

($\overline{\mathfrak{C}}^*$ aus $\overline{\mathfrak{C}}$ hervorgehend durch Fortlassung aller $\mathfrak{D}_j$ mit der Gestalt $\mathfrak{B}$). Anwendung von A_4 liefert:

$$(\overline{\mathfrak{A}}) \cdot (\overline{\mathfrak{C}}^*) \rightarrow \mathfrak{D},$$

woraus mit A_3 (unter 2°) die Untersequenz von M folgt.

Umgekehrt läszt sich jede Anwendung von A_4 auf Anwendung von M zurückführen.

4° Dasz eine Obersequenz $\mathfrak{P} \rightarrow \mathfrak{B}$ eines Schluszschemas, wobei das Antezedens eine abgekürzte Formel (eine Konjunktion) mit $n\ (\geqq 2)$ Faktoren, sich auf mehrere Weisen auffassen läszt (siehe die Definitionen in § 17 u. Fuszn. 63), verursacht dasz die Untersequenz oft mehrere Formen haben kann. Um dies von nun an zu vermeiden denken wir uns *die Faktorzerlegung eines derartigen Antezedens immer unzweideutig bestimmt; Übergang zu weiteren Faktorzerlegungen soll nur mittels der Schemata* (in der unter 2° u. 3° angegebenen Form) *zugelassen sein.* Diese genügen dazu; denn mit ihnen sind folgende Schemata ableitbar: [64bis])

$$\text{FaA. (1)}\ \frac{\mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_k \ldots \mathfrak{A}_n \rightarrow \mathfrak{B}}{(\ldots(((\mathfrak{A}_1 \cdot \mathfrak{A}_2) \cdot \mathfrak{A}_3) \cdot \mathfrak{A}_4) \ldots \mathfrak{A}_k) \ldots \mathfrak{A}_n \rightarrow \mathfrak{B}};$$

$$(2)\ \frac{(\ldots((\mathfrak{A}_1 \cdot \mathfrak{A}_2) \cdot \mathfrak{A}_3) \ldots \mathfrak{A}_k) \ldots \mathfrak{A}_n \rightarrow \mathfrak{B}}{\mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_k \ldots \mathfrak{A}_n \rightarrow \mathfrak{B}}$$

$(n \geqq 2;\ 1 < k \leqq n)$

(„Faktoränderung" im Antezedens).

§ 32[bis]. *Jede (affirmative) Sequenz $v \rightarrow \mathfrak{A}$, bei der $\mathfrak{A}$ v nicht enthält, läszt sich (in LG_1) ohne Benutzung von Schema Q ableiten; dazu genügt eine Sequenzenreihe, bei der:*

I. *am Anfang jeder auftretenden Sequenz $v \rightarrow$ steht;*

II. *in jeder solchen Sequenz $v \rightarrow \mathfrak{B}$ die Formel $\mathfrak{B}$ v nicht enthält;*

III. *die ausschlieszliche Anwendung der Schluszschemata $\overline{P}$, $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$, $\overline{M}$, $\overline{UES}$, $\overline{UEA}$, $\overline{FES}$, $\overline{FEA}$, die bzw. aus $P, A_1, \ldots, FEA$ dadurch hervorgehen [65]), dasz in jeder Ober- und Untersequenz $\rightarrow$ in $\subset$ geändert, und dabei $v \rightarrow$ vor der Sequenz gesetzt wird, zu $v \rightarrow [(X_n \subset X_n) \subset \mathfrak{A}]$ führt (mit X_n nicht in $\mathfrak{A}$ vorkommend); Einsetzungen bleiben dabei auf solche in $\overline{P}$ beschränkt;* [66])

IV. *der letzte Schlusz von $v \rightarrow [(X_n \subset X_n) \subset \mathfrak{A}]$ zu $v \rightarrow \mathfrak{A}$ führt; hier wird somit das Abtrennungsschema:*

$$\frac{v \rightarrow \mathfrak{A} \quad v \rightarrow (\mathfrak{A} \subset \mathfrak{B})}{v \rightarrow \mathfrak{B}}$$

angewandt.

Nun können wir den Hauptsatz für LG_1 formulieren.

[64bis]) *FaA* (1) mit *UEA*, A_2, A_3; *FaA* (2) mit *P*, *UES*, $A_1 - A_3$ und *M*.

[65]) A_1,– A_3, *UEA*, *FES*, *FEA* in erweiterter Form, *UES* mit-, *P* ohne Abkürzungen als Antezedens.

[66]) Siehe Satz 12; vergl. Fuszn. 16.

Hauptsatz. *Zu jeder Sequenz $\nu \rightarrow \mathfrak{A}$, bei der $\mathfrak{A}$ ν nicht enthält*[67]), *gibt es*, wie wir sahen, *eine Herleitungsreihe mit den Eigenschaften I—IV. Diese Reihe läszt sich derart umwandeln, dasz die Eigenschaften I, II, IV erhalten bleiben, während unter III $\overline{M}$ gestrichen werden kann.*

Neben Schema $\overline{Q}$ (Q) sind somit Schnittschema $\overline{A}_4 (A_4)$ und Mischungsschema $\overline{M} (M)$ überflüssig [68]).

Beweis [69]). Es genügt zu zeigen, dasz eine beliebige Herleitungsreihe mit den Eigenschaften I—IV, deren letztes Schluszschema eine Mischung $\overline{M}$ ist, während sie keine weiteren Mischungen enthält, sich so umwandeln läszt, dasz die Untersequenz der Mischung ohne $\overline{M}$ hergeleitet wird, und der übrige Teil der Herleitung ungeändert bleibt.

Grad einer Kalkülformel ist die Anzahl der in ihr vorkommenden logischen Zeichen. Als *Grad der eben genannten Herleitungsreihe* bezeichnen wir den Grad der Mischformel $\mathfrak{B}$ in $\overline{M}$.

Rang ϱ der Herleitungsreihe ist die Summe der rechten und linken Rangzahl; dabei ist die linke [rechte] Rangzahl die gröszte Anzahl von in einem Faden aneinander anschlieszenden Sequenzen, deren unterste die linke [rechte] Obersequenz der Mischung ist, und von denen jede im Sukzedens [Antezedens] die Mischformel allein [als Faktor einer abgekürzten Formel oder allein] enthält.

Der mindest mögliche Rang ist 2.

Der Beweis geschieht mittels vollständiger Induktion.

Man beweist den Satz für den Grad γ unter Voraussetzung, dasz sie für Grade $< \gamma$ (falls es solche gibt) schon gültig ist. Ferner wird zunächst der Fall $\varrho = 2$ behandelt, darauf der Fall $\varrho > 2$ unter Voraussetzung, dasz der Satz für Herleitungen desselben Grades, jedoch kleineren Ranges schon bekannt ist.

Es sei $\varrho = 2$. Wir unterscheiden eine Anzahl von Einzelfällen:

a) Die linke Obersequenz der Mischung sei durch Einsetzung in $\overline{P}$ entstanden. Dann lautet die Mischung:

$$\frac{\nu \rightarrow (\mathfrak{B} \subset \mathfrak{B}) \qquad \nu \rightarrow (\overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \rightarrow (\mathfrak{B} \cdot \overline{\mathfrak{C}^*} \subset \mathfrak{D})}$$ ($\mathfrak{B} \cdot \overline{\mathfrak{C}^*}$ ist die mit $\mathfrak{B}$ als Anfangsfaktor erweiterte Konjunktion $\overline{\mathfrak{C}^*}$).

Sie läszt sich umwandeln zu:

$$\frac{\nu \rightarrow (\overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \rightarrow (\mathfrak{B} \cdot \overline{\mathfrak{C}^*} \subset \mathfrak{D})} \quad (\overline{A}_3,\ \overline{A}_2)$$ (ev. mehrmalig anzuwenden).

[67]) Ihre Klasse ist dieselbe wie die der HEYTINGschen Theoreme von § 1, Bemerkung III.

[68]) Dadurch ist $X_n \subset X_n$ die einzige Kalkülformel, welche beim Verfahren des Hauptsatzes wieder völlig eliminiert werden kann; von Konjunktionen in einer Obersequenz einer unter III zugelassen gebliebenen Schluszfigur treten alle durch deutsche Buchstaben angegebenen Kalkülformeln als Teilformeln der in der Untersequenz $\nu \rightarrow \boldsymbol{R}$ auftretenden Formel $\boldsymbol{R}$ auf.

[69]) Vergl. GENTZEN, loc. cit. 8), S. 197—210.

Das ist bereits eine Herleitung ohne Mischung: $\nu \to (\mathfrak{B} \subset \mathfrak{B})$ ist überflüssig geworden.

β) Die rechte Obersequenz der Mischung sei durch Einsetzung in $\overline{P}$ entstanden:

$$\frac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \qquad \nu \to (\mathfrak{B} \subset \mathfrak{B})}{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B})}.$$

Die Mischung ist überflüssig.

γ) Keine Obersequenz der Mischung sei aus $\overline{P}$ durch Einsetzung entstanden. Dann sind beide Untersequenz von Schluszschemata, und wegen $\varrho = 2$ kommt die Mischformel $\mathfrak{B}$ nicht im Sukzedens der Obersequenz(en) des linken Schluszschemas und nicht im Antezedens der Obersequenz(en) des rechten Schluszschemas vor.

Wir haben nun die folgenden Möglichkeiten:

γ_1) Die rechte Obersequenz der Mischung ist Untersequenz einer Verdünnung $\overline{A}_1$:

$$\frac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \qquad \dfrac{\nu \to (\overline{\mathfrak{P}} \subset \mathfrak{D})}{\nu \to (\mathfrak{B} \cdot \overline{\mathfrak{P}} \subset \mathfrak{D})}\;(\overline{A}_1)}{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\mathfrak{P}} \subset \mathfrak{D})}\;(\overline{M})$$

($\overline{\mathfrak{A}} \cdot \overline{\mathfrak{P}}$ ist die Konjunktion $\overline{\mathfrak{P}}$ erweitert durch Voranstellung der Faktoren von $\overline{\mathfrak{A}}$ in gleicher Anordnung wie in $\overline{\mathfrak{A}}$).

Dies ist zu ersetzen durch:

$$\frac{\nu \to (\overline{\mathfrak{P}} \subset \mathfrak{D})}{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\mathfrak{P}} \subset \mathfrak{D})}\;(\overline{A}_1):$$

$\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B})$ ist nun überflüssig.

γ_2) Die Mischformel $\mathfrak{B}$ habe die Form $(\mathfrak{B}_1 \cdot \mathfrak{B}_2)$; sie sei nicht eingeführt wie unter γ_1. Die rechte Obersequenz der Mischung kann nur aus $\overline{UEA}$ hervorgegangen sein:

$$\frac{\dfrac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}_1) \qquad \nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}_2)}{\nu \to [\overline{\mathfrak{A}} \subset (\mathfrak{B}_1 \cdot \mathfrak{B}_2)]}\;(\overline{UES}) \qquad \dfrac{\nu \to (\mathfrak{B}_1 \cdot \overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \to [(\mathfrak{B}_1 \cdot \mathfrak{B}_2) \cdot \overline{\mathfrak{C}} \subset \mathfrak{D}]}\;(\overline{UEA})}{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\mathfrak{C}} \subset \mathfrak{D})}\;(\overline{M})$$

($\overline{\mathfrak{C}}$ darf fehlen).

Dies läszt sich ersetzen durch:

$$\frac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}_1) \qquad \nu \to (\mathfrak{B}_1 \cdot \overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\mathfrak{C}} \subset \mathfrak{D})}\;(\overline{M}).$$

Da der Grad der neuen Mischung niedriger als der der ursprünglichen ist, läszt sie sich, nach Annahme, entfernen [70]).

γ_3) Die Mischformel $\mathfrak{B}$ kann auch die Gestalt $\mathfrak{B}_1 \subset \mathfrak{B}_2$ haben; sie sei

[70]) Angenommen wurde, dasz $\overline{\mathfrak{C}}$ nicht den Faktor $\mathfrak{B}_1$ enthält; sonst ist Anwendung von $\overline{M}$ „und" $\overline{A}_1$, $\overline{A}_3$ notwendig. — Die Umwandlung ist analog falls $\nu \to (\mathfrak{B}_2 \cdot \overline{\mathfrak{C}} \subset \mathfrak{D})$ gegeben ist anstatt $\nu \to (\mathfrak{B}_1 \cdot \overline{\mathfrak{C}} \subset \mathfrak{D})$.

wieder nicht eingeführt wie unter γ_1. Dann ist das Ende der Herleitung:

$$\cfrac{\cfrac{\nu \to [\mathfrak{B}_1 \cdot \bar{\mathfrak{A}} \subset \mathfrak{B}_2]}{\nu \to [\bar{\mathfrak{A}} \subset (\mathfrak{B}_1 \subset \mathfrak{B}_2)]}\,(\overline{FES}) \qquad \cfrac{\nu \to (\bar{\mathfrak{D}} \subset \mathfrak{B}_1) \quad \nu \to (\mathfrak{B}_2 \cdot \bar{\mathfrak{E}} \subset \mathfrak{C})}{\nu \to [(\mathfrak{B}_1 \subset \mathfrak{B}_2) \cdot \bar{\mathfrak{D}} \cdot \bar{\mathfrak{E}} \subset \mathfrak{C}]}\,(\overline{FEA})}{\nu \to [\bar{\mathfrak{A}} \cdot \bar{\mathfrak{D}} \cdot \bar{\mathfrak{E}} \subset \mathfrak{C}] \qquad (\bar{\mathfrak{E}} \text{ darf fehlen}).}\,(\bar{M})$$

Es läszt sich umwandeln zu:

$$\frac{\nu \to [\mathfrak{B}_1 \cdot \bar{\mathfrak{A}} \subset \mathfrak{B}_2] \quad \nu \to (\mathfrak{B}_2 \cdot \bar{\mathfrak{E}} \subset \mathfrak{C})}{\nu \to [\mathfrak{B}_1 \cdot \bar{\mathfrak{A}} \cdot \bar{\mathfrak{E}} \subset \mathfrak{C}]^{71)}}\,(\bar{M})$$

und

$$\cfrac{\cfrac{\nu \to (\bar{\mathfrak{D}} \subset \mathfrak{B}_1) \quad \nu \to [\mathfrak{B}_1 \cdot \bar{\mathfrak{A}} \cdot \bar{\mathfrak{E}} \subset \mathfrak{C}]}{\nu \to [\bar{\mathfrak{D}} \cdot \bar{\mathfrak{A}} \cdot \bar{\mathfrak{E}} \subset \mathfrak{C}]^{72)}}\,(\bar{M})}{\nu \to [\bar{\mathfrak{A}} \cdot \bar{\mathfrak{D}} \cdot \bar{\mathfrak{E}} \subset \mathfrak{C}]}\,(\bar{A}_3).$$

Die Mischformeln $\mathfrak{B}_1$ und $\mathfrak{B}_2$ der beiden letzten Mischungen sind von kleinerem Grade als $\mathfrak{B}_1 \subset \mathfrak{B}_2$. Nach der Induktionsannahme sind sie somit zu beseitigen.

Es sei $\varrho > 2$.

I. *Die rechte Rangzahl sei gröszer als* 1 (d.h. die rechte Obersequenz der Mischung ist Untersequenz einer Schluszfigur $\mathfrak{Sf}$, welche die Mischformel $\mathfrak{B}$ im Antezedens von mindestens einer Obersequenz enthält).

Der Grundgedanke des Umwandlungsverfahrens ist, wie bei Gentzen, loc. cit. S. 203, folgender:

Falls die Mischung sich nicht sofort wegschaffen läszt, so wird sie zurückgeführt auf die Erledigung von Herleitungen des *gleichen Grades*, aber *kleineren Ranges*. Meistens genügt dazu dasz man die Mischung sozusagen um eine Stufe nach oben verschiebt, über die Schluszfigur $\mathfrak{Sf}$ hinweg, genau gesagt: Die linke Obersequenz der Mischung $\nu \to [\bar{\mathfrak{A}} \subset \mathfrak{B}]$, die zunächst neben der Untersequenz von $\mathfrak{Sf}$ steht, wird statt dessen neben die Obersequenzen von $\mathfrak{Sf}$ geschrieben, welche nun Obersequenzen von neuen Mischungen werden; und die Untersequenzen dieser Mischungen dienen dann als Obersequenzen einer neuen, an Stelle von $\mathfrak{Sf}$ tretenden Schluszfigur, durch die wir sofort oder nach Zufügung weiterer Schluszfiguren wieder zu der alten Endsequenz gelangen. Zu den neuen Mischungen gehört nun offenbar jeweils ein Rang, der kleiner als ϱ ist, denn die linke Rangzahl bleibt ungeändert, und die rechte vermindert sich um mindestens 1.

Wir unterscheiden die beiden Fälle:

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71) Hier wurde angenommen, dasz die Faktoren von $\bar{\mathfrak{E}}$ nicht die Gestalt $\mathfrak{B}_2$ haben; im entgegengesetzten Fall wird diese Sequenz erst mittels $\bar{M}$, $\bar{A}_1$ und $\bar{A}_3$ erreicht.

72) Enthält $\bar{\mathfrak{A}}$ oder (und) $\bar{\mathfrak{E}}$ Faktoren mit der Gestalt $\mathfrak{B}_1$, so fordert Ableitung dieser Sequenz auch noch Anwendung von $\bar{A}_1$ und $\bar{A}_3$.

a. Die Mischformel $\mathfrak{B}$ komme im Antezedens $\overline{\mathfrak{A}}$ der linken Obersequenz der Mischung vor. Das Ende der Herleitung lautet:

$$\frac{\nu \rightarrow [\overline{\mathfrak{A}} \subset \mathfrak{B}] \quad \nu \rightarrow [\overline{\mathfrak{C}} \subset \mathfrak{D}]}{\nu \rightarrow [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{C}}^* \subset \mathfrak{D}]}.$$

Diese Mischung läszt sich wegschaffen:

$$\frac{\nu \rightarrow [\overline{\mathfrak{C}} \subset \mathfrak{D}]}{\nu \rightarrow [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{C}}^* \subset \mathfrak{D}]} \; (\overline{A}_1,\ \overline{A}_2,\ \overline{A}_3) \text{ (ev. mehrmalig anzuwenden).}$$

b. $\mathfrak{B}$ komme im Antezedens $\overline{\mathfrak{A}}$ der linken Obersequenz der Mischung nicht vor. Wir haben nun die folgende Unterteilung dieses Falles:

b₁. $\mathfrak{S}\mathfrak{f}$ sei ein $\overline{A}_1$ oder $\overline{A}_2$ oder $\overline{A}_3$. Das Ende der Herleitung hat die Form:

$$\cfrac{\nu \rightarrow (\overline{\mathfrak{A}} \subset \mathfrak{B}) \quad \cfrac{\nu \rightarrow (\overline{\mathfrak{E}} \subset \mathfrak{D})}{\nu \rightarrow (\overline{\mathfrak{C}} \subset \mathfrak{D})}\,(\mathfrak{S}\mathfrak{f})\ (\overline{\mathfrak{E}} \text{ und } \overline{\mathfrak{C}} \text{ enthalten den Faktor } \mathfrak{B})}{\nu \rightarrow [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{C}}^* \subset \mathfrak{D}]}\ (\overline{M}).$$

Die Umwandlung fängt dann an mit:

$$\frac{\nu \rightarrow (\overline{\mathfrak{A}} \subset \mathfrak{B}) \quad \nu \rightarrow (\overline{\mathfrak{E}} \subset \mathfrak{D})}{\nu \rightarrow [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{E}}^* \subset \mathfrak{D}]}\ (\overline{M}).$$

Die Herleitung für die Untersequenz der neuen Mischung hat dieselbe linke Rangzahl wie die alte Herleitung, während ihre rechte Rangzahl um 1 kleiner ist. Nach der Induktionsannahme ist nun auch die neue Mischung wegzuschaffen.

Auf die neue Mischung kann noch Anwendung von $\overline{A}_1$, $\overline{A}_2$ und (oder) $\overline{A}_3$ folgen.

b₂. $\mathfrak{S}\mathfrak{f}$ sei ein von $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$ verschiedenes Schluszschema mit **einer** Obersequenz, also $\overline{UEA}$ oder $\overline{FES}$. Das Ende der Herleitung lautet dann:

$$\cfrac{\nu \rightarrow (\overline{\mathfrak{A}} \subset \mathfrak{B}) \quad \cfrac{\nu \rightarrow (\mathfrak{D} \cdot \overline{\mathfrak{E}} \subset \mathfrak{C})}{\nu \rightarrow [(\mathfrak{D} \cdot \mathfrak{F}) \cdot \overline{\mathfrak{E}} \subset \mathfrak{C}]}\,(\overline{UEA})}{\nu \rightarrow [\overline{\mathfrak{A}} \cdot (\mathfrak{D} \cdot \mathfrak{F}) \cdot \overline{\mathfrak{E}}^* \subset \mathfrak{C}] \text{ oder } \nu \rightarrow [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{E}}^* \subset \mathfrak{C}],}\ (\overline{M})$$

je nachdem $(\mathfrak{D} \cdot \mathfrak{F})$ von $\mathfrak{B}$ verschieden ist oder nicht;

oder:

$$\cfrac{\nu \rightarrow (\overline{\mathfrak{A}} \subset \mathfrak{B}) \quad \cfrac{\nu \rightarrow (\mathfrak{F} \cdot \overline{\mathfrak{E}} \subset \mathfrak{C})}{\nu \rightarrow [(\mathfrak{D} \cdot \mathfrak{F}) \cdot \overline{\mathfrak{E}} \subset \mathfrak{C}]}\,(\overline{UEA})}{\nu \rightarrow [\overline{\mathfrak{A}} \cdot (\mathfrak{D} \cdot \mathfrak{F}) \cdot \overline{\mathfrak{E}}^* \subset \mathfrak{C}] \text{ oder } \nu \rightarrow [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{E}}^* \subset \mathfrak{C}],}\ (\overline{M})$$

je nachdem $(\mathfrak{D} \cdot \mathfrak{F})$ von $\mathfrak{B}$ verschieden ist oder nicht;

oder:

$$\cfrac{\nu \rightarrow (\overline{\mathfrak{A}} \subset \mathfrak{B}) \quad \cfrac{\nu \rightarrow (\mathfrak{C} \cdot \overline{\mathfrak{E}} \subset \mathfrak{D})}{\nu \rightarrow [\overline{\mathfrak{E}} \subset (\mathfrak{C} \subset \mathfrak{D})]}\,(\overline{FES})}{\nu \rightarrow [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{E}}^* \subset (\mathfrak{C} \subset \mathfrak{D})]}\ (\overline{M}).$$

In den drei Fällen enthält $\overline{\mathfrak{E}}$ $\mathfrak{B}$ als Faktor.

Beschränken wir uns für die zugehörigen Umwandlungen auf den ersten Fall. Für $(\mathfrak{D}\cdot\mathfrak{F})$ von $\mathfrak{B}$ verschieden sind sie:

$$\dfrac{\dfrac{\nu\to(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad\qquad \nu\to(\mathfrak{D}\cdot\overline{\mathfrak{E}}\subset\mathfrak{C})}{\begin{array}{c}\nu\to(\overline{\mathfrak{A}}\cdot\mathfrak{D}\cdot\overline{\mathfrak{E}^*}\subset\mathfrak{C})\ \text{ oder }\ \nu\to(\overline{\mathfrak{A}}\cdot\overline{\mathfrak{E}^*}\subset\mathfrak{C}),\\ \text{je nachdem } \mathfrak{D} \text{ von } \mathfrak{B} \text{ verschieden ist oder nicht}\end{array}}\,(\overline{M})}{\nu\to[\overline{\mathfrak{A}}\cdot(\mathfrak{D}\cdot\mathfrak{F})\cdot\overline{\mathfrak{E}^*}\subset\mathfrak{C}]}\,(\overline{A}_1,\ \overline{A}_3,\ \overline{UEA}),$$

und für $(\mathfrak{D}\cdot\mathfrak{F})$ und $\mathfrak{B}$ von gleicher Gestalt:

$$\dfrac{\nu\to(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad \dfrac{\dfrac{\dfrac{\nu\to(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad\qquad \nu\to(\mathfrak{D}\cdot\overline{\mathfrak{E}}\subset\mathfrak{C})}{\nu\to[\overline{\mathfrak{A}}\cdot\mathfrak{D}\cdot\overline{\mathfrak{E}^*}\subset\mathfrak{C}]}\,(\overline{M})}{\nu\to[\mathfrak{D}\cdot\overline{\mathfrak{A}}\cdot\overline{\mathfrak{E}^*}\subset\mathfrak{C}]}\,(\overline{A}_3)}{\nu\to[\mathfrak{B}\cdot\overline{\mathfrak{A}}\cdot\overline{\mathfrak{E}^*}\subset\mathfrak{C}]}\,(\overline{UEA})}{\dfrac{\nu\to[\overline{\mathfrak{A}}\cdot\overline{\mathfrak{A}}\cdot\overline{\mathfrak{E}^*}\subset\mathfrak{C}]}{\nu\to[\overline{\mathfrak{A}}\cdot\overline{\mathfrak{E}^*}\subset\mathfrak{C}]}\,(\overline{A}_3,\ \overline{A}_2).}\,(\overline{M})$$

In den neuen Mischungen ist die rechte Rangzahl kleiner als ursprünglich; nach der Induktionsannahme sind diese neuen Mischungen fortzuschaffen.

b_3. $\mathfrak{S}\mathfrak{f}$ sei ein Schluszschema mit zwei Obersequenzen, also $\overline{UES}$ oder $\overline{FEA}$. Das Ende der Herleitung lautet für $\overline{UES}$:

$$\dfrac{\nu\to(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad \dfrac{\nu\to(\overline{\mathfrak{G}}\subset\mathfrak{D})\qquad \nu\to(\overline{\mathfrak{G}}\subset\mathfrak{C})}{\nu\to[\overline{\mathfrak{G}}\subset(\mathfrak{D}\cdot\mathfrak{C})]}\,(\overline{UES})}{\nu\to[\overline{\mathfrak{A}}\cdot\overline{\mathfrak{G}^*}\subset(\mathfrak{D}\cdot\mathfrak{C})]}\,(\overline{M}),$$

und für $\overline{FEA}$:
entweder

$$\dfrac{\nu\to(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad \dfrac{\nu\to(\overline{\mathfrak{G}}\subset\mathfrak{F})\qquad \nu\to(\mathfrak{E}\cdot\overline{\mathfrak{D}}\subset\mathfrak{C})}{\nu\to[(\mathfrak{F}\subset\mathfrak{E})\cdot\overline{\mathfrak{G}}\cdot\overline{\mathfrak{D}}\subset\mathfrak{C}]}\,(\overline{FEA})}{\begin{array}{l}\nu\to[\overline{\mathfrak{A}}\cdot(\mathfrak{F}\subset\mathfrak{E})\cdot\overline{\mathfrak{G}^*}\cdot\overline{\mathfrak{D}^*}\subset\mathfrak{C}]\ \text{ oder }\ \nu\to[\overline{\mathfrak{A}}\cdot\overline{\mathfrak{G}^*}\cdot\overline{\mathfrak{D}^*}\subset\mathfrak{C}],\quad (\overline{\mathfrak{D}}\text{ darf fehlen})\\ \text{je nachdem } \mathfrak{F}\subset\mathfrak{E} \text{ und } \mathfrak{B} \text{ verschiedene Gestalt haben oder nicht;}\end{array}}\,(\overline{M})$$

oder

$$\dfrac{\nu\to(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad \dfrac{\nu\to(\overline{\mathfrak{D}}\subset\mathfrak{F})\qquad \nu\to(\mathfrak{E}\cdot\overline{\mathfrak{G}}\subset\mathfrak{C})}{\nu\to[(\mathfrak{F}\subset\mathfrak{E})\cdot\overline{\mathfrak{D}}\cdot\overline{\mathfrak{G}}\subset\mathfrak{C}]}\,(\overline{FEA})}{\begin{array}{r}\nu\to[\overline{\mathfrak{A}}\cdot(\mathfrak{F}\subset\mathfrak{E})\cdot\overline{\mathfrak{D}^*}\cdot\overline{\mathfrak{G}^*}\subset\mathfrak{C}]\ \text{ oder }\ \nu\to[\overline{\mathfrak{A}}\cdot\overline{\mathfrak{D}^*}\cdot\overline{\mathfrak{G}^*}\subset\mathfrak{C}],\ \text{ je nachdem}\\ \mathfrak{F}\subset\mathfrak{E} \text{ und } \mathfrak{B} \text{ verschiedene Gestalt haben oder nicht.}\end{array}}\,(\overline{M})$$

In diesen Fällen enthält $\overline{\mathfrak{G}}$ $\mathfrak{B}$ als Faktor.

Die zugehörigen Umwandlungen sind für den ersten Fall:

$$\dfrac{\dfrac{\nu\to(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad \nu\to(\overline{\mathfrak{G}}\subset\mathfrak{D})}{\nu\to(\overline{\mathfrak{A}}\cdot\overline{\mathfrak{G}^*}\subset\mathfrak{D})}\,(\overline{M})\qquad \dfrac{\nu\to(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad \nu\to(\overline{\mathfrak{G}}\subset\mathfrak{C})}{\nu\to(\overline{\mathfrak{A}}\cdot\overline{\mathfrak{G}^*}\subset\mathfrak{C})}\,(\overline{M})}{\nu\to[\overline{\mathfrak{A}}\cdot\overline{\mathfrak{G}^*}\subset(\mathfrak{D}\cdot\mathfrak{C})]}\,(\overline{UES}).$$

Im zweiten Fall gibt es verschiedene Möglichkeiten: 1. $\overline{\mathfrak{D}}$ enthält $\mathfrak{B}$ nicht, $\mathfrak{F} \subset \mathfrak{E}$ ist von $\mathfrak{B}$ verschieden; hier genügt eine Mischung kleineren Ranges, 2. $\overline{\mathfrak{D}}$ enthält den Faktor $\mathfrak{B}$, $\mathfrak{B}$ und $\mathfrak{F} \subset \mathfrak{E}$ haben verschiedene Gestalt; nun genügen zwei Mischungen kleineren Ranges, 3. $\mathfrak{F} \subset \mathfrak{E}$ und $\mathfrak{B}$ haben dieselbe Gestalt, während $\overline{\mathfrak{D}}$ keinen Faktor $\mathfrak{B}$ enthält; die Umwandlungen enthalten zwei Mischungen kleineren Ranges:

$$\dfrac{\dfrac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \quad \nu \to (\overline{\mathfrak{G}} \subset \mathfrak{F})}{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\mathfrak{G}^*} \subset \mathfrak{F})}\,(\overline{M}) \qquad \nu \to (\mathfrak{E} \cdot \overline{\mathfrak{D}} \subset \mathfrak{C})}{\nu \to [(\mathfrak{F} \subset \mathfrak{E}) \cdot \overline{\mathfrak{A}} \cdot \overline{\mathfrak{G}^*} \cdot \overline{\mathfrak{D}} \subset \mathfrak{C}]}\,(\overline{FEA})$$

oder

$$\dfrac{\dfrac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \quad \nu \to [\mathfrak{B} \cdot \overline{\mathfrak{A}} \cdot \overline{\mathfrak{G}^*} \cdot \overline{\mathfrak{D}} \subset \mathfrak{C}]}{\nu \to [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{A}} \cdot \overline{\mathfrak{G}^*} \cdot \overline{\mathfrak{D}} \subset \mathfrak{C}]}\,(\overline{M})}{\nu \to [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{G}^*} \cdot \overline{\mathfrak{D}} \subset \mathfrak{C}]}\,(\overline{A}_3,\ \overline{A}_2).$$

4. $\overline{\mathfrak{D}}$ enthält $\mathfrak{B}$ als Faktor, während $\mathfrak{F} \subset \mathfrak{E}$ und $\mathfrak{B}$ dieselbe Gestalt haben; Umwandlungen mit drei Mischungen kleineren Ranges. Auch im dritten Fall hat man dieselben Möglichkeiten zu berücksichtigen wie im zweiten.

II. *Die rechte Rangzahl sei* 1, *also die linke gröszer als* 1.

c. Die Mischformel $\mathfrak{B}$ komme im Antezedens $\overline{\mathfrak{A}}$ der linken Obersequenz der Mischung vor; die Mischung läszt sich wegschaffen wie unter *a*.

d. Der zu *c* entgegengesetzten Fall führt zu folgender Unterteilung:

d_1. $\mathfrak{Sf}$ sei ein $\overline{A}_1$ oder $\overline{A}_2$ oder $\overline{A}_3$. Das Ende der Herleitung ist nun:

$$\dfrac{\dfrac{\nu \to (\overline{\mathfrak{E}} \subset \mathfrak{B})}{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B})}\,(\mathfrak{Sf}) \qquad \nu \to (\overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\mathfrak{C}^*} \subset \mathfrak{D})}\,(\overline{M}).$$

Es ist umzuwandeln zu:

$$\dfrac{\nu \to (\overline{\mathfrak{E}} \subset \mathfrak{B}) \quad \nu \to (\overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \to (\overline{\mathfrak{E}} \cdot \overline{\mathfrak{C}^*} \subset \mathfrak{D})}\,(\overline{M})$$

mit darauf folgender Anwendung des zugehörigen Schemas $\mathfrak{Sf}$.

Die neue Mischung ist kleineren Ranges als die ursprüngliche, somit wegzuschaffen.

d_2. Ist $\mathfrak{Sf}$ ein von $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$ verschiedenes Schluszschema mit einer Obersequenz, so kann dies nur $\overline{UEA}$ sein; der eine von beiden Fällen erhält die Form:

$$\dfrac{\dfrac{\nu \to [\mathfrak{A} \cdot \overline{\mathfrak{E}} \subset \mathfrak{B}]}{\nu \to [(\mathfrak{A} \cdot \mathfrak{F}) \cdot \overline{\mathfrak{E}} \subset \mathfrak{B}]}\,(\overline{UEA}) \qquad \nu \to (\overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \to [(\mathfrak{A} \cdot \mathfrak{F}) \cdot \overline{\mathfrak{E}} \cdot \overline{\mathfrak{C}^*} \subset \mathfrak{D}]}\,(\overline{M}).$$

Umwandlung liefert:

$$\cfrac{\cfrac{\nu \rightarrow [\mathfrak{A} \cdot \overline{\mathfrak{E}} \subset \mathfrak{B}] \quad \nu \rightarrow (\overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \rightarrow [\mathfrak{A} \cdot \overline{\mathfrak{E}} \cdot \overline{\mathfrak{C}^*} \subset \mathfrak{D}]}\ (\overline{M})}{\nu \rightarrow [(\mathfrak{A} \cdot \mathfrak{F}) \cdot \overline{\mathfrak{E}} \cdot \overline{\mathfrak{C}^*} \subset \mathfrak{D}]}\ (\overline{UEA}).$$

Der Rang ist erniedrigt.

d_3. Als Schluszfigur $\mathfrak{S}\mathfrak{f}$ mit zwei Obersequenzen ist nur $\overline{FEA}$ möglich:

$$\cfrac{\cfrac{\nu \rightarrow (\overline{\mathfrak{D}} \subset \mathfrak{F}) \quad \nu \rightarrow (\mathfrak{E} \cdot \overline{\mathfrak{C}} \subset \mathfrak{B})}{\nu \rightarrow [(\mathfrak{F} \subset \mathfrak{E}) \cdot \overline{\mathfrak{D}} \cdot \overline{\mathfrak{C}} \subset \mathfrak{B}]}\ (\overline{FEA}) \qquad \nu \rightarrow (\overline{\mathfrak{A}} \subset \mathfrak{G})}{\nu \rightarrow [(\mathfrak{F} \subset \mathfrak{E}) \cdot \overline{\mathfrak{D}} \cdot \overline{\mathfrak{C}} \cdot \overline{\mathfrak{A}^*} \subset \mathfrak{G}]}\ (\overline{M}).$$

Hier ist die Umwandlung:

$$\cfrac{\nu \rightarrow (\overline{\mathfrak{D}} \subset \mathfrak{F}) \qquad \cfrac{\nu \rightarrow (\mathfrak{E} \cdot \overline{\mathfrak{C}} \subset \mathfrak{B}) \quad \nu \rightarrow (\overline{\mathfrak{A}} \subset \mathfrak{G})}{\nu \rightarrow (\mathfrak{E} \cdot \overline{\mathfrak{C}} \cdot \overline{\mathfrak{A}^*} \subset \mathfrak{G})}\ (\overline{M})}{\nu \rightarrow [(\mathfrak{F} \subset \mathfrak{E}) \cdot \overline{\mathfrak{D}} \cdot \overline{\mathfrak{C}} \cdot \overline{\mathfrak{A}^*} \subset \mathfrak{G}]}\ (\overline{FEA}).$$

Wieder ist der Rang erniedrigt.

Nähere Betrachtung der logistischen Kalküle LG_2, LM, LI und LK

§ 33. Für die Kalküle LG_2, LM, LI und LK gelten gleichartige Bemerkungen wie für LG_1 (§ 32):

1° *Die Einsetzungsregel kann für LG_2 und LM auf Einsetzungen in P und Q, für LI auf solche in P, Q, R, für LK auf solche in P, Q, R, S beschränkt bleiben.*

2° *Die in § 32, 2° getroffenen Verabredungen sollen auch für LG_2, LM, LI und LK erhalten bleiben.*

3° *Das Schema A_4 ersetzen wir auch in diesen Kalkülen durch das Mischungsschema M*, was wieder erlaubt ist (siehe § 32, 3°).

4° *Auch § 32, 4° wollen wir gelten lassen.*

Hinzu kommt:

5° *In den Schemata R und S soll $\mathfrak{A}$ in nicht-abgekürzter Form gemeint sein.*

6° Mit Schema A_3 beweist man leicht, *dasz wir in LG_2, LM, LI und LK fortan lesen dürfen*:

Schema OEA. $$\frac{\mathfrak{A} \cdot \mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_n \rightarrow \mathfrak{D} \qquad \mathfrak{B} \cdot \mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_n \rightarrow \mathfrak{D}}{(\mathfrak{A} + \mathfrak{B}) \cdot \mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_n \rightarrow \mathfrak{D} \qquad \text{(die } \mathfrak{C}_i \text{ dürfen fehlen)}};$$

auch wollen wir lesen (vergl. § 32, 4°):

Schema OES. $$\frac{\mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_n \rightarrow \mathfrak{A}}{\mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_n \rightarrow (\mathfrak{A} + \mathfrak{B})} \text{ und } \frac{\mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_n \rightarrow \mathfrak{B}}{\mathfrak{C}_1 \cdot \mathfrak{C}_2 \ldots \mathfrak{C}_n \rightarrow (\mathfrak{A} + \mathfrak{B})}.$$

Ebenso folgt mit Schema A_3, *dasz wir in LM, LI und LK fortan lesen dürfen:*

Schema NES. $\dfrac{\mathfrak{A}\cdot\mathfrak{B}_1\ldots\mathfrak{B}_n\rightarrow\lambda}{\mathfrak{B}_1\cdot\mathfrak{B}_2\ldots\mathfrak{B}_n\rightarrow\mathfrak{A}'}$; Schema NEA. $\dfrac{\mathfrak{B}_1\cdot\mathfrak{B}_2\ldots\mathfrak{B}_n\rightarrow\mathfrak{A}}{\mathfrak{A}'\cdot\mathfrak{B}_1\ldots\mathfrak{B}_n\rightarrow\lambda}$.

§ 33[1]. *In LG_2 läszt sich jede (affirmative) Sequenz $v\rightarrow\mathfrak{A}$, bei der $\mathfrak{A}$ v nicht enthält, ohne Benutzung von Schema Q ableiten: dazu genügt eine Sequenzenreihe, bei der:*

I. *am Anfang jeder auftretenden Sequenz $v\rightarrow$ steht;*

II. *in jeder solchen Sequenz $v\rightarrow\mathfrak{B}$ die Formel $\mathfrak{B}$ v nicht enthält;*

III. *die ausschlieszliche Anwendung der Schluszschemata $\overline{P}$, $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$, $\overline{M}$, $\overline{UES}$, $\overline{UEA}$, $\overline{FES}$, $\overline{FEA}$, $\overline{OES}$, $\overline{OEA}$, die bzw. aus $P, A_1, \ldots, OEA$ dadurch hervorgehen, dasz in jeder Ober- und Untersequenz $\rightarrow$ in $\subset$ geändert, und dabei $v\rightarrow$ vor der Sequenz gesetzt wird, zu $v\rightarrow[(X_n\subset X_n)\subset\mathfrak{A}]$ führt (mit X_n nicht in $\mathfrak{A}$ vorkommend); Einsetzungen bleiben dabei auf solche in $\overline{P}$ beschränkt;* [73])

IV. *der letzte Schlusz von $v\rightarrow[(X_n\subset X_n)\subset\mathfrak{A}]$ und $v\rightarrow(X_n\subset X_n)$ zu $v\rightarrow\mathfrak{A}$ führt; nur hier wird das Abtrennungsschema $\overline{A}$ angewandt.*

Der Hauptsatz für LG_2 lauter nun:

Zu jeder Sequenz $v\rightarrow\mathfrak{A}$, bei der $\mathfrak{A}$ v nicht enthält [74]), *gibt es eine Herleitungsreihe mit den obigen Eigenschaften I–IV, wobei jedoch unter III die Mischungsregel $\overline{M}$ fortfällt.*

Der Beweis ist eine Erweiterung des Beweises für den Hauptsatz in LG_1.

Zu den Fällen γ_1—γ_3 kommt:

Fall γ_4. Die Mischformel $\mathfrak{B}$ hat die Form $\mathfrak{B}_1+\mathfrak{B}_2$; sie sei nicht eingeführt wie unter γ_1. Das Ende der Herleitung ist in dem einen der zwei möglichen Fälle (beide mit $\overline{OES}$ und $\overline{OEA}$):

$$\dfrac{\dfrac{v\rightarrow(\overline{\mathfrak{C}}\subset\mathfrak{B}_1)}{v\rightarrow[\overline{\mathfrak{C}}\subset(\mathfrak{B}_1+\mathfrak{B}_2)]}\,(\overline{OES})\qquad\dfrac{v\rightarrow(\mathfrak{B}_1\cdot\overline{\mathfrak{D}}\subset\mathfrak{E})\quad v\rightarrow(\mathfrak{B}_2\cdot\overline{\mathfrak{D}}\subset\mathfrak{E})}{v\rightarrow[(\mathfrak{B}_1+\mathfrak{B}_2)\cdot\overline{\mathfrak{D}}\subset\mathfrak{E}]}\,(\overline{OEA})}{v\rightarrow[\overline{\mathfrak{C}}\cdot\overline{\mathfrak{D}}\subset\mathfrak{E}]\qquad(\overline{\mathfrak{D}}\text{ darf fehlen})}\,(\overline{M})$$

Die Umwandlung liefert:

$$\frac{v\rightarrow(\overline{\mathfrak{C}}\subset\mathfrak{B}_1)\quad v\rightarrow(\mathfrak{B}_1\cdot\overline{\mathfrak{D}}\subset\mathfrak{E})}{v\rightarrow[\overline{\mathfrak{C}}\cdot\overline{\mathfrak{D}}\subset\mathfrak{E}]}\ (\overline{M}),$$

falls $\overline{\mathfrak{D}}$ keinen Faktor $\mathfrak{B}_1$ enthält; kommt $\mathfrak{B}_1$ als Faktor in $\overline{\mathfrak{D}}$ vor, so musz nach $\overline{M}$ noch $\overline{A}_1$ und $\overline{A}_2$ folgen. Der Grad der Mischung ist erniedrigt.

Im Falle b_2 gibt es nun auch noch zwei gleichartige Möglichkeiten mit $\overline{OES}$, deren eine ist:

$$\dfrac{v\rightarrow(\overline{\mathfrak{A}}\subset\mathfrak{B})\qquad\dfrac{v\rightarrow(\overline{\mathfrak{C}}\subset\mathfrak{D})}{v\rightarrow[\overline{\mathfrak{C}}\subset(\mathfrak{D}+\mathfrak{E})]}\,(\overline{OES})}{v\rightarrow[\overline{\mathfrak{A}}\cdot\overline{\mathfrak{C}}^*\subset(\mathfrak{D}+\mathfrak{E})]}\,(\overline{M}).$$

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73) Siehe Satz 12; vergl. die Fuszn. 16 u. 26.

74) Ihre Klasse ist dieselbe wie die der Heytingschen Theoreme von § 3, Bemerkung A.

Die Umwandlung ist:

$$\dfrac{\dfrac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \qquad \nu \to (\overline{\mathfrak{C}} \subset \mathfrak{D})}{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{D})}\;(\overline{M})}{\nu \to [\overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset (\mathfrak{D} + \mathfrak{E})]}\;(\overline{OES})$$

Im Falle b_3 kommt folgende Möglichkeit hinzu:

$$\dfrac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \qquad \dfrac{\nu \to (\mathfrak{D} \cdot \overline{\mathfrak{C}} \subset \mathfrak{F}) \qquad \nu \to (\mathfrak{E} \cdot \overline{\mathfrak{C}} \subset \mathfrak{F})}{\nu \to [(\mathfrak{D} + \mathfrak{E}) \cdot \overline{\mathfrak{C}} \subset \mathfrak{F}]}\;(\overline{OEA})}{\nu \to [\overline{\mathfrak{A}} \cdot (\mathfrak{D} + \mathfrak{E}) \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F}] \text{ oder } \nu \to [\overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F}], \text{ je nachdem } \mathfrak{D} + \mathfrak{E} \text{ und } \mathfrak{B} \text{ verschiedene Gestalt haben oder nicht.}}\;(\overline{M})$$

Umwandlung, mit Rangerniedrigung, für den Fall dasz $\mathfrak{D} + \mathfrak{E}$ und $\mathfrak{B}$ verschieden sind:

$$\dfrac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \qquad \nu \to (\mathfrak{D} \cdot \overline{\mathfrak{C}} \subset \mathfrak{F})}{\nu \to (\overline{\mathfrak{A}} \cdot \mathfrak{D} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F});\ \text{oder:}\ \dfrac{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F})}{\nu \to (\overline{\mathfrak{A}} \cdot \mathfrak{D} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F})}\;(\overline{A}_1,\ \overline{A}_3)}\;(\overline{M})$$

$$\dfrac{\dfrac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \qquad \nu \to (\mathfrak{E} \cdot \overline{\mathfrak{C}} \subset \mathfrak{F})}{\nu \to (\overline{\mathfrak{A}} \cdot \mathfrak{E} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F});\ \text{oder:}\ \dfrac{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F})}{\nu \to (\overline{\mathfrak{A}} \cdot \mathfrak{E} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F})}\;(\overline{A}_1,\ \overline{A}_3)}\;(\overline{M})}{\nu \to [\overline{\mathfrak{A}} \cdot (\mathfrak{D} + \mathfrak{E}) \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F}]}\;(\overline{A}_3, \overline{OEA}).$$

Für $\mathfrak{D} + \mathfrak{E}$ und $\mathfrak{B}$ von derselben Gestalt fängt man bei der Umwandlung wieder mit obigen Mischungen an und erreicht die letzte Sequenz $\nu \to [(\mathfrak{D} + \mathfrak{E}) \cdot \overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F}]$ oder $\nu \to [\mathfrak{B} \cdot \overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F}]$. Dann folgt die Mischung, deren rechte Rangzahl 1 ist:

$$\dfrac{\dfrac{\nu \to (\overline{\mathfrak{A}} \subset \mathfrak{B}) \qquad \nu \to [\mathfrak{B} \cdot \overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F}]}{\nu \to [\overline{\mathfrak{A}} \cdot \overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \cdot \subset \mathfrak{F}]}\;(\overline{M})}{\nu \to (\overline{\mathfrak{A}} \cdot \overline{\overline{\mathfrak{C}}^*} \subset \mathfrak{F})}\;(\overline{A}_3,\ \overline{A}_2).$$

Im Fall d_3 kommt hinzu:

$$\dfrac{\dfrac{\nu \to (\mathfrak{A} \cdot \overline{\mathfrak{D}} \subset \mathfrak{B}) \qquad \nu \to (\mathfrak{C} \cdot \overline{\mathfrak{D}} \subset \mathfrak{B})}{\nu \to [(\mathfrak{A} + \mathfrak{C}) \cdot \overline{\mathfrak{D}} \subset \mathfrak{B}]}\;(\overline{OEA}) \qquad \nu \to (\overline{\mathfrak{G}} \subset \mathfrak{H})}{\nu \to [(\mathfrak{A} + \mathfrak{C}) \cdot \overline{\mathfrak{D}} \cdot \overline{\overline{\mathfrak{G}}^*} \subset \mathfrak{H}] \quad (\overline{\mathfrak{D}} \text{ darf fehlen})}\;(\overline{M}).$$

Umwandlung mit Rangerniedrigung:

$$\dfrac{\dfrac{\nu \to (\mathfrak{A} \cdot \overline{\mathfrak{D}} \subset \mathfrak{B}) \qquad \nu \to (\overline{\mathfrak{G}} \subset \mathfrak{H})}{\nu \to (\mathfrak{A} \cdot \overline{\mathfrak{D}} \cdot \overline{\overline{\mathfrak{G}}^*} \subset \mathfrak{H})}\;(\overline{M}) \qquad \dfrac{\nu \to (\mathfrak{C} \cdot \overline{\mathfrak{D}} \subset \mathfrak{B} \qquad \nu \to (\overline{\mathfrak{G}} \subset \mathfrak{H})}{\nu \to (\mathfrak{C} \cdot \overline{\mathfrak{D}} \cdot \overline{\overline{\mathfrak{G}}^*} \subset \mathfrak{H})}\;(\overline{M})}{\nu \to [(\mathfrak{A} + \mathfrak{C}) \cdot \overline{\mathfrak{D}} \cdot \overline{\overline{\mathfrak{G}}^*} \subset \mathfrak{H}]}\;(\overline{OEA}).$$

§ 33[2]. *In LM läszt sich jede (affirmative) Sequenz $\nu \to \mathfrak{A}$, bei der $\mathfrak{A}$ λ*

und ν nicht enthält, ohne Benutzung von Schema Q ableiten; dazu genügt eine Sequenzenreihe, bei der:

I. *am Anfang jeder auftretenden Sequenz $\nu \rightarrow$ steht;*

II. *in jeder solchen Sequenz $\nu \rightarrow \mathfrak{B}$ die Formel $\mathfrak{B}$ ν nicht enthält (λ darf vorkommen);*

III. *die ausschlieszliche Anwendung der Schluszschemata $\overline{P}$, $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$, $\overline{M}$, $\overline{UES}$, $\overline{UEA}$, $\overline{FES}$, $\overline{FEA}$, $\overline{OES}$, $\overline{OEA}$, $\overline{NES}$, $\overline{NEA}$, die bzw. aus $P, A_1, \ldots, NEA$ dadurch hervorgehen, dasz in jeder Ober- und Untersequenz $\rightarrow$ in $\subset$ geändert, und dabei $\nu \rightarrow$ vor der Sequenz gesetzt wird, zu $\nu \rightarrow [(X_n \subset X_n) \subset \mathfrak{A}]$ führt (mit X_n nicht in $\mathfrak{A}$ vorkommend); Einsetzungen bleiben auf solche in $\overline{P}$ beschränkt* [75];

IV. *der letzte Schlusz von $\nu \rightarrow [(X_n \subset X_n) \subset \mathfrak{A}]$ und $\nu \rightarrow (X_n \subset X_n)$ zu $\nu \rightarrow \mathfrak{A}$ führt; nur an dieser Stelle wird das Abtrennungsschema $\overline{A}$ angewandt.*

Hauptsatz fur *LM: Zu jeder Sequenz $\nu \rightarrow \mathfrak{A}$, bei der $\mathfrak{A}$ λ und ν nicht enthält* [76], *gibt es eine Herleitungsreihe mit den eben genannten Eigenschaften I—IV. Diese Reihe läszt sich derart umwandeln, dasz I, II, IV ungeändert bleiben, dagegen unter III $\overline{M}$ gestrichen werden kann* [77].

Der Beweis ist eine Erweiterung des Beweises für den Hauptsatz in LG_2.

Zu den Fällen γ_1—γ_4 kommt:

Fall γ_5. Die Mischformel $\mathfrak{B}$ hat die Gestalt $\mathfrak{A}'$, und sei nicht eingeführt wie unter γ_1. Das Ende der Herleitung lautet:

$$\dfrac{\dfrac{\nu \rightarrow [\mathfrak{A} \cdot \overline{\mathfrak{C}} \subset \lambda]}{\nu \rightarrow (\overline{\mathfrak{C}} \subset \mathfrak{A}')}\,(\overline{NES}) \qquad \dfrac{\nu \rightarrow (\overline{\mathfrak{D}} \subset \mathfrak{A})}{\nu \rightarrow (\mathfrak{A}' \cdot \overline{\mathfrak{D}} \subset \lambda)}\,(\overline{NEA})}{\nu \rightarrow (\overline{\mathfrak{C}} \cdot \overline{\mathfrak{D}} \subset \lambda)}\,(\overline{M}).$$

Umwandlung (mit Graderniedrigung):

$$\dfrac{\nu \rightarrow (\overline{\mathfrak{D}} \subset \mathfrak{A}) \qquad \qquad \nu \rightarrow [\mathfrak{A} \cdot \overline{\mathfrak{C}} \subset \lambda]}{\dfrac{\nu \rightarrow (\overline{\mathfrak{D}} \cdot \overline{\mathfrak{C}^*} \subset \lambda)}{\nu \rightarrow (\overline{\mathfrak{C}} \cdot \overline{\mathfrak{D}} \subset \lambda)}\,(\overline{A}_1,\, \overline{A}_3) \quad \text{oder:} \quad \dfrac{\nu \rightarrow (\overline{\mathfrak{D}} \cdot \overline{\mathfrak{C}} \subset \lambda)}{\nu \rightarrow (\overline{\mathfrak{C}} \cdot \overline{\mathfrak{D}} \subset \lambda)}\,(\overline{A}_3)}\,(\overline{M})$$

Im Falle b_2 kommen hinzu:

$$\dfrac{\nu \rightarrow (\overline{\mathfrak{D}} \subset \mathfrak{B}) \qquad \dfrac{\nu \rightarrow (\overline{\mathfrak{C}} \subset \mathfrak{A})}{\nu \rightarrow (\mathfrak{A}' \cdot \overline{\mathfrak{C}} \subset \lambda)}\,(\overline{NEA})}{\nu \rightarrow (\overline{\mathfrak{D}} \cdot \mathfrak{A}' \cdot \overline{\mathfrak{C}^*} \subset \lambda) \text{ oder } \nu \rightarrow (\overline{\mathfrak{D}} \cdot \overline{\mathfrak{C}^*} \subset \lambda),}\,(\overline{M})$$

je nachdem $\mathfrak{A}'$ von $\mathfrak{B}$ verschieden ist oder nicht;

$$\text{und:} \quad \dfrac{\nu \rightarrow (\overline{\mathfrak{D}} \subset \mathfrak{B}) \qquad \dfrac{\nu \rightarrow (\mathfrak{A} \cdot \overline{\mathfrak{C}} \subset \lambda)}{\nu \rightarrow (\overline{\mathfrak{C}} \subset \mathfrak{A}')}\,(\overline{NES})}{\nu \rightarrow (\overline{\mathfrak{D}} \cdot \overline{\mathfrak{C}^*} \subset \mathfrak{A}')}\,(\overline{M}).$$

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[75]) Siehe Satz 12; vergl. die Fusznoten 16, 26 u. 33.

[76]) Ihre Klasse ist dieselbe wie die der Theoreme im JOHANSSONschen Minimalkalkül. Vergl. § 6^{bis}.

[77]) Die Formel λ kann mit $\overline{NES}$ eliminiert werden. Vergl. Fuszn. 68.

Die zugehörigen Umwandlungen (mit Rangerniedrigungen) sind bzw.:

1°

$$\frac{\nu\to(\overline{\mathfrak D}\subset\mathfrak B)}{\nu\to(\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\mathfrak A)}\,;\qquad\text{oder:}\qquad\frac{\nu\to(\overline{\mathfrak C}\subset\mathfrak A)}{\nu\to(\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\mathfrak A)}\ (\overline{M})$$

$$\cfrac{\cfrac{(\mathfrak A'\text{ u. }\mathfrak B\text{ verschieden})}{\nu\to[\mathfrak A'\cdot\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\lambda]}\ (\overline{NEA})}{\nu\to[\overline{\mathfrak D}\cdot\mathfrak A'\cdot\overline{\mathfrak C^*}\subset\lambda]}\ (\overline{A}_3)\qquad\qquad\cfrac{\cfrac{\nu\to(\overline{\mathfrak D}\subset\mathfrak A')\qquad\cfrac{(\mathfrak A'\text{ u. }\mathfrak B\text{ gleich})}{\nu\to[\mathfrak A'\cdot\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\lambda]}\ (\overline{NEA})}{\nu\to(\overline{\mathfrak D}\cdot\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\lambda)}\ (\overline{M})}{\nu\to(\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\lambda)}\ (\overline{A}_3,\ \overline{A}_2);$$

2°

$$\cfrac{\cfrac{\nu\to(\overline{\mathfrak D}\subset\mathfrak B)}{\nu\to(\overline{\mathfrak D}\cdot\mathfrak A\cdot\overline{\mathfrak C^*}\subset\lambda)}}{\nu\to(\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\mathfrak A')}\ (\overline{A}_3,\ \overline{NES})\qquad\text{oder:}\qquad\cfrac{\cfrac{\cfrac{\nu\to(\mathfrak A\cdot\overline{\mathfrak C}\subset\lambda)}{\nu\to(\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\lambda)}\ (\overline{M})}{\nu\to(\mathfrak A\cdot\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\lambda)}\ (\overline{A}_1)}{\nu\to(\overline{\mathfrak D}\cdot\overline{\mathfrak C^*}\subset\mathfrak A')}\ (\overline{NES})$$

je nachdem $\mathfrak B$ und $\mathfrak A$ verschieden sind, oder nicht.

Bemerkung. Nach Fuszn. 37 läszt sich in jeder Sequenzenreihe, welche sich gemäsz dem Hauptsatz dieses Par. bilden läszt, λ durch $Y'_m\cdot Y''_m$ ersetzen, wobei es immer möglich ist m so zu wählen, *dasz* Y_m *nicht zu den in der Sequenzenreihe vorkommenden elementaren Kalkülformeln gehört.* Das kommt darauf hinaus, dasz man $\overline{NES}$ und $\overline{NEA}$ unter III durch folgende Schemata ersetzt:

$$\overline{\overline{NES}}.\ \frac{\nu\to[\mathfrak A\cdot\overline{\mathfrak B}\subset(Y'_m\cdot Y''_m)]}{\nu\to(\overline{\mathfrak B}\subset\mathfrak A')};\qquad\overline{\overline{NEA}}.\ \frac{\nu\to(\overline{\mathfrak B}\subset\mathfrak A)}{\nu\to[\mathfrak A'\cdot\overline{\mathfrak B}\subset(Y'_m\cdot Y''_m)]}.$$

Im Endresultat (in der letzten Sequenz) kommen Y'_m und Y''_m nicht vor.

MATHEMATICS

DISTRIBUTION MODULO 1 OF SOME CONTINUOUS FUNCTIONS

BY

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This paper consists of two parts. In part I we study an invariant additive set function of sets of real numbers. As an application of the main theorem we obtain the distribution modulo 1 of functions of a certain class. In part II we prove the non-existence of a distribution modulo 1 of functions of another class.

I. *The invariant finitely additive set function ν.*

The space R of the real numbers is group space of the group G of translations $(x \to x + b)$. A set $\mathfrak{S}$ of subsets of R is called *an invariant class of sets,* if:

(1) $$S_1, S_2 \in \mathfrak{S},\ S_1 \cap S_2 = 0 \text{ implies } S_1 + S_2 \in \mathfrak{S};$$

(2) $$S \in \mathfrak{S},\ g \in G \text{ implies } g \cdot S \in \mathfrak{S}.$$

A set function defined on $\mathfrak{S}$ is called an *invariant finitely additive set-function* (ν), if

(3) $$S_1 \cap S_2 = 0,\ S_1, S_2 \in \mathfrak{S},\ \text{implies } \nu(S_1 + S_2) = \nu(S_1) + \nu(S_2);$$

(4) $$S \in \mathfrak{S},\ g \in G,\ \text{implies } \nu(g \cdot S) = \nu(S).$$

We call ν an invariant semi-distribution on $\mathfrak{S}$, if moreover:

(5) $$S \in \mathfrak{S},\ \text{implies } \nu(S) \geqq 0;$$

(6) $$R \in \mathfrak{S},\ \text{and } \nu(R) = 1.$$

Example 1. Throughout the paper $S(x)$ $(x > 0)$ will denote the Lebesgue measure (if existing) of the intersection of a set of real numbers S and the segment $0 - x$. Let $\mathfrak{S}$ be the class of the following sets: *a*) the sets that are bounded by a greater number (bounded above); *b*) the sets for which $\lim_{x\to\infty} S(x)/x$ exists; *c*) any set which is a sum of a set of *a*) and a set of *b*).

Throughout the paper $\nu(S)$ will denote the invariant semi distribution on the class of sets $\mathfrak{S}$, defined by: (3); $\nu(S) = 0$ if S is bounded above; $\nu(S) = \lim_{x\to\infty} S(x)/x$ if this exists. We observe that (3) is not contradicted, (5) and (6) are obviously fullfilled, and so we only need to prove (4): If g is

a translation over a distance $d < 0$, and S is a set mentioned under b), then

$$\nu(g\cdot S) = \lim_{x\to\infty} \frac{(g\cdot S)(x)}{x} = \lim_{x\to\infty} \frac{S(x+d)-S(d)}{x} =$$

$$\lim_{x\to\infty} \frac{S(x+d)}{x} = \lim_{x\to\infty} \frac{S(x)}{x} = \nu(S) \text{ q.e.d.}$$

Here, and in the sequel, we restrict to proofs concerning sets mentioned under b); the proofs for the other sets are simple consequences.

Example 2. Let $a > 1$, and let $\mathfrak{S}^a$ be the class of sets for which

$$(\nu^a \overset{\text{def}}{=}) \lim_{x\to\infty} \frac{S(ax)-S(x)}{ax-x} \text{ exists.}$$

$S(ax) - S(x)$ denotes the Lebesgue measure of the intersection of S and the interval $(x - ax)$.

The best way to show that also ν^a is a semi distribution on $\mathfrak{S}^a$, is to prove the

Theorem 1. $\mathfrak{S}^a = \mathfrak{S}$, $\nu^a = \nu$, for any $a > 1$.

First we prove a lemma which we will also need later on.

Lemma. Let S be a subset of R, and $a > 1$. *If* for any $\varepsilon > 0$, $N(\varepsilon)$ exists such that $x > N$ implies

$$\nu' - \varepsilon < \frac{S(ax)-S(x)}{ax-x} < \nu'' + \varepsilon, \tag{7}$$

then $M(\varepsilon, \nu', \nu'')$ exists, such that $y > M$ implies

$$\nu' - 2\varepsilon < \frac{S(y)}{y} < \nu'' + 2\varepsilon. \tag{8}$$

Proof: For $x > N$, we conclude from (7)

$$\nu' - \varepsilon < \frac{S(a^{k+1}.x)-S(a^k.x)}{a^{k+1}.x - a^k.x} < \nu'' + \varepsilon, \qquad k = 0, 1, 2, \ldots$$

or

$$(\nu' - \varepsilon)\,(a^{k+1}\cdot x - a^k\cdot x) < S(a^{k+1}\cdot x) - S(a^k\cdot x) < (\nu'' + \varepsilon)\,(a^{k+1}\cdot x - a^k\cdot x).$$

Summation over $k = 0, 1, 2, \ldots, n-1$, yields

$$(\nu' - \varepsilon)\,(a^n\cdot x - x) < S(a^n\cdot x) - S(x) < (\nu'' + \varepsilon)\,(a^n\cdot x - x)$$

or if $y = a^n\cdot x$:

$$\nu' - \varepsilon < \frac{S(y)-S(a^{-n}.y)}{y - a^{-n}.y} < \nu'' + \varepsilon.$$

This holds in particular for any $n > 0$ and $N < a^{-n}\cdot y < a\cdot N^2$. It is now possible to change y continuously from N to infinity, while leaving these conditions fullfilled (by shifting the integer n). Hence it follows because $a^{-n}\cdot y$ and $S(a^{-n}\cdot y)$ are bounded, that for y sufficiently large, say $y > M\,(\varepsilon, \nu', \nu'')$ (8) holds.

Proof of theorem 1:

a. Suppose $S \in \mathfrak{S}^a$. From the definition in example 2, and the lemma, it follows immediately, that also $S \in \mathfrak{S}$, and $\nu^a(S) = \nu(S)$.

b. Suppose $S \in \mathfrak{S}$, S is a set as mentioned in example 1 under b) $\nu(S) = \nu$.

The function $N(\eta)$ defined for $\eta > 0$ exists, such that if $x_2 > x_1 > N(\eta)$, $\eta > 0$ being fixed, then

$$(\nu-\eta)\, x_i < S(x_i) < (\nu+\eta)\, x_i, \qquad i = 1, 2.$$

or

$$(\nu-\eta)\, x_2-(\nu+\eta)\, x_1 < S(x_2)-S(x_1) < (\nu+\eta)\, x_2-(\nu-\eta)\, x_1$$

$$\nu-\frac{x_2+x_1}{x_2-x_1}\,\eta < \frac{S(x_2)-S(x_1)}{x_2-x_1} < \nu+\frac{x_2+x_1}{x_2-x_1}\,\eta.$$

Let $x_2 = a \cdot x_1 = a \cdot x$, $a = 1+\delta > 1$, and $\varepsilon > 0$. Choose

$$\eta < \frac{\delta}{2+\delta}\,\varepsilon.$$

Then

$$\nu-\varepsilon < \frac{S(ax)-S(x)}{ax-x} < \nu+\varepsilon \qquad (x > N(\eta) = N(\eta(\varepsilon))).$$

This holds for any $\varepsilon > 0$ and $x > N(\eta(\varepsilon))$, hence:

$$\nu^a(S) \overset{\text{def}}{=} \lim_{x\to\infty} \frac{S(ax)-S(x)}{ax-x} = \nu$$

and $S \in \mathfrak{S}^a$. The theorem follows.

Example 3. Let the function $y = f(x)$ be defined and be monotonously increasing for $x > x_0$, and $f(x) < f(x_0)$ or not defined for $x < x_0$. Let $\mathfrak{S}^* = f^{-1}(\mathfrak{S})$ consist of the subsets of R that are the sum of a bounded (above) set and the image under f^{-1} of a set of $\mathfrak{S}$ (example 1) (it is clear that choice of a larger number then x_0, instead of x_0, has no influence on the result $\mathfrak{S}^*$. The function $f(x)$ is only of interest for values $x >$ the arbitrarily large value x_0(!)).

An additive set function ν^* on $\mathfrak{S}^*$ is defined by (3) and: $\nu^*(S^*) = 0$ if $S^*(\in \mathfrak{S}^*)$ is bounded; if $S^* = f^{-1}(S)$ then

$$\nu^*(S^*) = \nu^*(f^{-1}(S)) = \nu(S) = \nu(f(S^*)).$$

Notation: $\mathfrak{S}^* = f^{-1}(\mathfrak{S})$, $\nu^* = \nu \cdot f$.

The following theorem gives a condition under which the setfunction just defined is an invariant semi distribution, and it is even the same as ν in example 1.

Main Theorem 2:

If $\alpha > 0$, x_0 is a constant, $f(x)$ is bounded above or not defined for $x < x_0$, $f(x)$ is differentiable for $x \geqq x_0$, and

(9) $$\lim_{x\to\infty} \frac{f'(x)}{\alpha \cdot x^{\alpha-1}} = K > 0,$$

then $\mathfrak{S}^* = f^{-1}(\mathfrak{S}) = \mathfrak{S}$ *and* $\nu^* = \nu \cdot f = \nu \cdot$; *or in words : then the semi distribution* ν *is invariant under the "transformation"* f.

Proof: Let $K = 1$ (a restriction not essential for the proof). From (9) follows the existence of $N'(\varepsilon) > x_0$, defined for $\varepsilon > 0$, such that $x > N'(\varepsilon)$ implies:

$$\alpha(1-\varepsilon)\, x^{\alpha-1} < f'(x) < \alpha(1+\varepsilon)\, x^{\alpha-1}. \tag{10}$$

We integrate from N' to z and replace z by x:

$$(1-\varepsilon)\, x^\alpha + C_1 < f(x) - f(N') < (1+\varepsilon)\, x^\alpha + C_2.$$

Therefore $N''(\varepsilon) > N'(\varepsilon)$ exists, such that $x > N''(\varepsilon)$ implies (10) and

$$(1-2\varepsilon)\, x^\alpha < f(x) < (1+2\varepsilon)\, x^\alpha. \tag{11}$$

Now let $\nu^* = \nu^*(S^*)$ be the value of the setfunction $\nu^* = \nu \cdot f$ at the set $S^* = f^{-1}(S)$ $(S = f(S^*))$. Then because $\nu^* = \lim_{x\to\infty} S(f(x))/f(x)$, $N^0(\eta)$ defined for $\eta > 0$ exists, such that $x > N^0(\eta)$, $a > 1$, implies

$$(\nu^* - \eta)\, f(ax) < S(f(ax)) < (\nu^* + \eta)\, f(ax),$$

$$(\nu^* + \eta)\, f(x) > S(f(x)) > (\nu^* - \eta)\, f(x),$$

$$(\nu^* - \eta)\, f(ax) - (\nu^* + \eta)\, f(x) < S(f(ax)) - S(f(x)) < (\nu^* + \eta)\, f(ax) - (\nu^* - \eta)\, f(x),$$

$$\nu^* - \frac{f(ax)-f(x)}{f(ax)-f(x)}\,\eta < \frac{S(f(ax))-S(f(x))}{f(ax)-f(x)} < \nu^* + \frac{f(ax)+f(x)}{f(ax)-f(x)}\,\eta. \tag{12}$$

From now on we suppose $\alpha \geqq 1$. The proof of the other case $0 < \alpha < 1$ is obtained by obvious alterations.

The derivative of the function f in the interval $x - ax$ ($x > N''(\varepsilon)$ and $x > N^0(\eta)$) is bounded by $\alpha(1-\varepsilon)\, x^{\alpha-1}$ and $\alpha(1+\varepsilon)\, (ax)^{\alpha-1}$ (compare (10)). Therefore $f(ax) - f(x)$ is bounded by (!):

$$\alpha(1-\varepsilon)\, x^{\alpha-1}\,(ax-x) \text{ and } \alpha(1+\varepsilon)\,(ax)^{\alpha-1}\,(ax-x). \tag{13}$$

The Lebesgue measure $S(f(ax)) - S(f(x))$ is bounded by (!):

$$\alpha(1-\varepsilon)\, x^{\alpha-1}\,(S^*(ax)-S^*(x)) \text{ and } \alpha(1+\varepsilon)\,(ax)^{\alpha-1}\,(S^*(ax)-S^*(x)). \tag{14}$$

From (13) and (14) we get:

$$\frac{1-\varepsilon}{1+\varepsilon}\, a^{1-\alpha}\, \frac{S^*(ax)-S^*(x)}{ax-x} < \frac{S(f(ax))-S(f(x))}{f(ax)-f(x)} < \frac{1+\varepsilon}{1-\varepsilon}\, a^{\alpha-1}\, \frac{S^*(ax)-S^*(x)}{ax-x} \tag{15}$$

From (11) and (13) we get:

$$\frac{f(ax)+f(x)}{f(ax)-f(x)}\,\eta < \frac{(1+2\varepsilon)\,[(ax)^\alpha + x^\alpha]}{\alpha(1-\varepsilon)\, x^{\alpha-1}\,(ax-x)}\,\eta = \frac{1+2\varepsilon}{\alpha(1-\varepsilon)}\,\frac{a^\alpha+1}{a-1}\,\eta. \tag{16}$$

We now choose for any $a > 1$, $\varepsilon > 0$, $\eta = \eta\,(a, \varepsilon)$ so small that the right hand side of (16) is less then ε. This inequality is used in (12) and combining the result with (15) we obtain:

$$\frac{1-\varepsilon}{1+\varepsilon}\, a^{1-\alpha}\,(\nu^* - \varepsilon) < \frac{S^*(ax)-S^*(x)}{ax-x} < \frac{1+\varepsilon}{1-\varepsilon}\, a^{\alpha-1}\,(\nu^* + \varepsilon). \tag{17}$$

Applying the lemma we conclude to the existence of $M(a, \varepsilon')$, defined for all $a > 1$, $\varepsilon' > 0$, such that $x > M(a, \varepsilon')$, a, ε' being fixed, implies

$$\nu^* a^{1-\alpha} - \varepsilon' < \frac{S^*(x)}{x} < a^{\alpha-1} \nu^* + \varepsilon'$$

hence

$$\lim_{x\to\infty} \frac{S^*(x)}{x} = \nu^*.$$

By definition this limit is $\nu(S^*)$, and therefore $S^* \in \mathfrak{S}$. It follows that $\mathfrak{S}^* \subset \mathfrak{S}$. Because $f(x)$ has an inverse for $x > x_0$, which obeys conditions (9) (with other constants), also $\mathfrak{S} \subset \mathfrak{S}^*$. Therefore $\mathfrak{S}^* = \mathfrak{S}$, and for any $S \in \mathfrak{S} : \nu^*(S) = \nu(S)$. q.e.d.

Application.

Let s be a Lebesgue measurable (L.m.) subset of the interval $0 \leqq x < 1$, with measure $\mu(s)$. Let $S = S(s)$ be the set of all numbers x that differ an integer from a number in the set s. Obviously $\nu(S) = \mu(s)$.

Let $f(x)$ be a function and let $S^*(s) = f^{-1}(S)$ be the set of all x for which $f(x)$ is a number in the set S. If $\nu(S^*) = \nu(S^*(s), f) = \nu(s, f)$ exists for any L.m. set s, and $\nu(s, f)$ is an infinitely additive setfunction, then $f(x)$ is said to possess the C^{III}*-distribution mod* 1: $\nu(s, f)$. If $\nu(s, f)$ exists for any s that is a finite sum of intervals, and $\nu(s, f)$ is finitely additive, then $f(x)$ is said to possess the C^{I}*-distribution mod* 1: $\nu(s, f)$. If the distribution mod 1 $\nu(s, f)$ coincides with the Lebesgue measure $\mu(s)$, then the distribution is called *uniform*.

As a *corrollary of theorem* 2 we now have:

Theorem 3 (KUIPERS–MEULENBELD). *If $f(x)$ obeys the conditions of theorem* 2 (*in particular* (9)), *then $f(x)$ is C^{III} uniformly distributed mod* 1. [1])

Proof: If s is L.m. then $\nu(s, f) = \nu(S^*) = \nu(S) = \mu(s)$.

II. *Some functions which do not possess a C^{I}-distribution mod* 1.

In theorem 2 we proved the invariance of the finitely additive setfunction $\nu(S)$ under a transformation which is in a certain sense not much different from $x \to x^\alpha$, $\alpha > 0$. It is easily seen that $\nu(S)$ is *not* invariant under $x \to e^x$ or the inverse $x \to \ln x$. For these functions the conclusion of theorem 3 is not a corrollary of theorem 2. e^x happens to be C^{III}-uniformly distributed mod 1; $\ln x$ not. We prove:

Theorem 4. *If $M, L > 0$ are constants, $f(x)$ is continuous,* $\lim_{x\to\infty} f(x) = \infty$ [2]), *and if $\gamma > \beta > M$, $f(\gamma) - f(\beta) > 1/4$ implies*

$$\beta \frac{f(\gamma) - f(\beta)}{\gamma - \beta} < L, \tag{18}$$

then $f(x)$ does not possess a C^{I}-distribution mod 1.

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[1]) For $\alpha \geqq 1$, this is, but for a condition of monotony which I do not need, a theorem of KUIPERS ([2] Ch. III, th. 6). For $0 < \alpha < 1$, KUIPERS and MEULENBELD recently gave a proof of an n-dimensional generalisation of th. 3 ([4] th. IV).

[2]) This condition is superfluous, but it is convenient in the proof.

Proof: Suppose the finitely additive setfunction $\nu(s) = \nu(s, f)$ is the C^I-distribution mod 1 of the given function $f(x)$. We will show that the assumption of its existence leads to a contradiction.

Let s be the interval of numbers x which obey $0 \leq p < x < q < 1$, $q - p = b > 1/4$. $S = S(s)$ and $S^* = f^{-1}(S)$ are defined as before.

By definition $N(\varepsilon)$ exists, such that $x > N(\varepsilon)$ implies

$$\nu(s) - \varepsilon < \frac{S^*(x)}{x} < \nu(s) + \varepsilon. \tag{19}$$

Let also $N(\varepsilon) > M$.

Next we choose two numbers β and $\gamma > \beta > N(\varepsilon)$, such that: γ is the smallest number greater than β, for which $f(\gamma) \equiv q$ mod 1; β is the greatest number smaller than γ, for which $f(\beta) \equiv p$ mod 1; $f(\gamma) > f(\beta)$. Because $f(x)$ is continuous:

$$S^*(\gamma) = S^*(\beta) + (\gamma - \beta).$$

In view of (19):

$$S^*(\gamma) > (\nu(s) - \varepsilon)\,\beta + (\gamma - \beta) = \gamma - (1 - \nu(s) + \varepsilon)\,\beta. \tag{20}$$

Application of (18) yields:

$$\frac{\beta\, b}{\gamma - \beta} < L \text{ or } \beta < \frac{L}{L+b}\,\gamma.$$

Substitute in (20), divide by γ, and rearrange:

$$\frac{S^*(\gamma)}{\gamma} > \nu(s) + \frac{(1 - \nu(s))\, b}{L+b} - \varepsilon\, \frac{L}{L+b}. \tag{21}$$

If $\nu(s) \neq 1$, then ε can be choosen so small, and $\gamma > N(\varepsilon)$ exists, such that

$$\frac{S^*(\gamma)}{\gamma} > \nu(s) + \varepsilon$$

in contradiction with (19).

Hence for all intervals like s we get the same value $\nu(s) = 1$ so that $\gamma(s)$ cannot be an additive setfunction · q.e.d.

Examples:

1. If $f(x)$ is differentiable and $x \cdot f'(x)$ $(x > 0)$ is bounded, then $f(x)$ does not possess a C^I-distribution mod 1. In particular $f(x)$ is not uniformly distributed, which was also proved by KUIPERS and MEULENBELD ([4] th. II). E.g. $\ln x$, $\ln(x + \sin x + 1)$, have no C^I-distribution mod 1.

2. A step function with discontinuities at the integervalues of x, can be approximated by a continuous function in such a way, that the measure of the set of all x for which the values of the two functions differ, is bounded. We are therefore able to prove: If the sequence $n(f(n+1) -$ $- f(n))$ $(n > 0)$ is bounded, then the function $f(x)$, defined by $f(x) = f(n)$

if $n \leqq x < n+1$, does not possess a C^{I}-distribution mod 1; in other words: $f(n)$ does not possess a C^{I}-distribution mod 1.

Compare [1] E.g. $f(n) = \ln n$.

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A RANK-INVARIANT METHOD OF LINEAR AND POLYNOMIAL REGRESSION ANALYSIS. III [1]

BY

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3. CONFIDENCE REGIONS FOR THE PARAMETERS OF POLYNOMIAL REGRESSION EQUATIONS

The probability set

3. 0. The probability set Γ underlying the probability statements of this section is the $n(\nu+2)$-dimensional Cartesian space $R_{n(\nu+2)}$ with coordinates

$$u_{11}, \ldots, u_{1n}, \ldots, u_{\nu 1}, \ldots, u_{\nu n}, v_1, \ldots, v_n, w_1, \ldots, w_n.$$

Every random variable mentioned is supposed to be defined on this probability set.

We suppose $n(\nu+2)$ random variables $\boldsymbol{u}_{\lambda i}$, $\mathbf{v}_i$, $\mathbf{w}_i$ $(\lambda = 1, \ldots, \nu;$ $i = 1, \ldots, n)$ to have a simultaneous probability distribution on Γ. Furthermore we consider $n\nu$ parameters $\xi_{\lambda i}$ and N parameters $a_{p_1 \ldots p_\nu}$ for all sets of non-negative integers $p_1, \ldots, p_\nu$ satisfying

$$0 \leqq \sum_{\lambda-1}^{\nu} p_\lambda \leqq h.$$

Now we put [2]

$$\left.\begin{array}{lrl} (10) & \theta_i & = \sum a_{p_1 \ldots p_\nu} \xi_{1i}^{p_1} \cdots \xi_{\nu i}^{p_\nu} \\ (11) & \eta_i & = \theta_i + \mathbf{w}_i \\ (12) & \mathbf{x}_{\lambda i} & = \xi_{\lambda i} + \boldsymbol{u}_{\lambda i} \\ (13) & \mathbf{y}_i & = \eta_i + \mathbf{v}_i . \end{array}\right\} \begin{cases} i = 1, \ldots, n \\ \lambda = 1, \ldots, \nu \end{cases}$$

So, for any set of values of the $(N + n\nu)$ parameters $a_{p_1, \ldots, p_\nu}$, $\xi_{\lambda i}$, the variables $\mathbf{x}_{\lambda i}$ and $\mathbf{y}_i$ have a simultaneous distribution on Γ, and are there fore random variables.

The parameters $\xi_{\lambda i}$ $(i = 1, \ldots, n)$ are interpreted as values assumed by the variable ξ_λ. The equation (10) is the polynomial regression equation. The random variables $\mathbf{w}_i$ are called "the true deviations" from the poly-

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[1]) This paper is the third of a series of papers, the first of which appeared in these Proceedings, 53, 386—392 (1950); the second appeared in these Proceedings, 53, 521—525 (1950).

[2]) Σ in equation (10) denotes summation over all sets $p_1, \ldots, p_\nu$.

nomial of degree h; the random variables $\mathbf{u}_{\lambda i}$ and $\mathbf{v}_i$ are called "the errors of observation" of the "true" values $\xi_{\lambda i}$ and η_i respectively. [3])

Conditions; approximation

3. 1. In order to give confidence regions for the parameters $a_{p_1,\dots,p_\nu}$ we consider the following conditions:

Condition I: All n $(\nu+2)$-uples $(\mathbf{u}_{\lambda i}, \mathbf{v}_i, \mathbf{w}_i)$ are stochastically independent.

Condition IIa: 1. Each of the errors $\mathbf{u}_{\lambda i}$ vanishes outside a finite interval $|\mathbf{u}_{\lambda i}| < g_{\lambda i}$.

2. For each $i \neq j$ we have $|\xi_{\lambda i} - \xi_{\lambda j}| > g_{\lambda i} + g_{\lambda j}$.

Condition IIb: 1. Each of the errors $\mathbf{u}_{\lambda i}$ vanishes outside a finite interval $|\mathbf{u}_{\lambda i}| < g_{\lambda i}$.

2. For each $i \neq j$, for each set $p_1, \dots, p_\nu$ and for any real $h_{\lambda i}$ such that $|h_{\lambda i}| \leqq g_{\lambda i}$ we have

$$\operatorname{sgn}\left\{\prod_{\lambda=1}^{\nu}(\xi_{\lambda i}+h_{\lambda i})^{p_\lambda}-\prod_{\lambda=1}^{\nu}(\xi_{\lambda j}+h_{\lambda j})^{p_\lambda}\right\}=\operatorname{sgn}\left\{\prod_{\lambda=1}^{\nu}\xi_{\lambda i}^{p_\lambda}-\prod_{\lambda=1}^{\nu}\xi_{\lambda j}^{p_\lambda}\right\}.$$

Condition III: For all fixed values of the constants $\varrho_{\lambda i}$ the n random variables $\sum_{\lambda=1}^{\nu} \varrho_{\lambda i}\, \mathbf{u}_{\lambda i} + \mathbf{v}_i + \mathbf{w}_i = \mathbf{z}_i$ have continuous distribution functions, which are symmetrical with the median med $(\mathbf{z})$.

Finally we mention that the solution will be given subject to the following

Approximation: For any positive s the quantities

$$\mathbf{u}_{\lambda i}^{s}\, \mathbf{u}_{\lambda' i}, \quad \mathbf{u}_{\lambda i}^{s}\, \mathbf{v}_i, \quad \mathbf{u}_{\lambda i}^{s}\, \mathbf{w}_i \qquad (\lambda, \lambda' = 1, \dots \nu;\ i = 1, \dots, n)$$

are neglected. [4])

Confidence regions

3. 2. We consider the case $\nu = 1$, so that equation (10) can be written as

$$\eta_i = \sum_{p=0}^{h} a_p\, \xi_i^p .$$

Let us arrange the n observed points (x_i, y_i) according to increasing values of x:

$$x_1 < \dots < x_n.$$

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[3]) It is clear that the random variables $\mathbf{v}_i$ and $\mathbf{w}_i$ cannot be separated in one sample of observations; if, however, the experiment is repeated for the same "true" values $\xi_{\lambda i}$, η_i (e.g. if — when the relation between income and consumption is investigated — for the same families and the same period the amounts of their incomes and outlays are repeatedly calculated), then the errors $\mathbf{v}_i$ can be mitigated by averaging, whereas the deviations $\mathbf{w}_i$ cannot.

[4]) The approximation implies that the errors $\mathbf{u}_{\lambda i}$ are sufficiently small. This restriction is not very serious, because, unless the number of points n is very large, large values of $\mathbf{u}_{\lambda i}$ will cause the confidence region for the parameters of the polynomial to be so large as to render the method useless.

We leave 0, 1,... or h points out of consideration until the remaining number n' is such that $n'/(h+1)$ is an integer, and write $n_h = n'/(h+1)$. (It seems advisable with respect to the power of the method to omit the points with rank $n_h + 1$, $2n_h + 1, \ldots$ and / or $hn_h + 1$). From now on we write n for the remaining number n', so that $(h+1)\, n_h = n$.

We define the following quantities:

$$\Delta(i, n_h + i) = \frac{\mathbf{y}_i - \mathbf{y}_{n_h+i}}{\mathbf{x}_i - \mathbf{x}_{n_h+i}}$$

$$\Delta^{(2)}(i, n_h + i, 2n_h + i) = \frac{\Delta(i, n_h + i) - \Delta(n_h + i, 2n_h + i)}{\mathbf{x}_i - \mathbf{x}_{2n_h+i}}$$

$$\vdots$$

$$\Delta^{(h)}(i, n_h + i, \ldots, hn_h + i) =$$

$$= \frac{\Delta^{(h-1)}(i, n_h + i, \ldots, \overline{h-1}\, n_h + i) - \Delta^{(h-1)}(n_h + i, 2n_h + i, \ldots, hn_h + i)}{\mathbf{x}_i - \mathbf{x}_{hn_h+i}}$$

We arrange the observed quantities $\Delta^{(h)}(i, \ldots, hn_h + i)$ according to increasing magnitude:

$$\Delta_1^{(h)} < \ldots < \Delta_{n_h}^{(h)},$$

in which

$$\Delta_j^{(h)} = \Delta^{(h)}(i_j, \ldots, hn_h + i_j).$$

3. 3. Then we have the following theorem:

Theorem 6: Under conditions I, IIa, and III the interval $(\Delta_{r_h}^{(h)}, \Delta_{n_h - r_h + 1}^{(h)})$ is a confidence interval for a_h to the approximate level of significance $2I_{\frac{1}{2}}(n_h - r_h + 1, r_h)$. [5])

In order to prove this theorem we shall use the following *Lemma*. Define for all non-negative integers s and for all positive integers c and i

$$P^s_{i, n_h+i, \ldots, (c-1)\, n_h+i} = \sum_{s_i} \cdots \sum_{\substack{s_{(c-1)n_h+i} \\ s_j \geq 0,\, \Sigma s_j = s}} x_i^{s_i} \ldots x_{(c-1)n_h+i}^{s_{(c-1)n_h+i}}.$$

Then we have

$$\frac{P^s_{i, \ldots, (c-1)n_h+i} - P^s_{n_h+i, \ldots, cn_h+i}}{x_i - x_{cn_h+i}} = P^{s-1}_{i, \ldots, cn_h+i}.$$

Proof of the lemma: We have

$$P^s_{i, \ldots, (c-1)\, n_h+i} - P^s_{n_h+i, \ldots, cn_h+i} =$$

$$= \sum_{s_i} \cdots \sum_{s_{(c-1)\, n_h+i}} x_{n_h+i}^{s_{n_h+i}} \ldots x_{(c-1)\, n_h+i}^{s_{(c-1)\, n_h+i}} (x_i^{s_i} - x_{cn_h+i}^{s_i}),$$

[5]) In the first and second part of this paper the arguments of the incomplete Beta-function must be reversed.

in which $\Sigma s_j = s$. It follows that

$$\frac{P^s_{i,\ldots,(c-1)n_h+i} - P^s_{n_h+i,\ldots,cn_h+i}}{x_i - x_{cn_h+i}} =$$

$$= \sum_{s_i} \cdots \sum_{s_{(c-1)n_h+i}} x^{s_{n_h+i}}_{n_h+i} \cdots x^{s_{(c-1)n_h+i}}_{(c-1)n_h+i} \left(x_i^{s_i-1} + x_i^{s_i-2}\, x_{cn_h+i} + \ldots + x^{s_i-1}_{cn_h+i}\right) =$$

$$= P^{s-1}_{i,\ldots,cn_h+i}.$$

Proof of Theorem 6: The relation between $\mathbf{x}_i$ and $\mathbf{y}_i$ is given by

$$\mathbf{y}_i = \sum_{p=0}^{h} a_p\, (\mathbf{x}_i - \mathbf{u}_i)^p + \mathbf{v}_i + \mathbf{w}_i$$

$$\approx \sum_{p=0}^{h} a_p\, \mathbf{x}_i^p - \mathbf{u}_i\, (a_1 + 2\, a_2\, \xi_i + \ldots + h a_h\, \xi_i^{h-1}) + \mathbf{v}_i + \mathbf{w}_i,$$

in which we neglected (in accordance with the Approximation) $\mathbf{u}_i^s$ for $s > 1$. Putting $\mathbf{z}_i = \varrho_i\, \mathbf{u}_i + \mathbf{v}_i + \mathbf{w}_i$, in which

$$-\varrho_i = a_1 + 2\, a_2\, \xi_i + \ldots + h\, a_h\, \xi_i^{h-1},$$

we get

$$\mathbf{y}_i \approx \sum_{p=0}^{h} a_p\, \mathbf{x}_i^p + \mathbf{z}_i.$$

Now we have according to the lemma:

$$\Delta\,(i, n_h + i) \approx a_1 + a_2 \frac{\mathbf{x}_i^2 - \mathbf{x}^2_{n_h+i}}{\mathbf{x}_i - \mathbf{x}_{n_h+i}} + \ldots + a_h \frac{\mathbf{x}_i^h - \mathbf{x}^h_{n_h+i}}{\mathbf{x}_i - \mathbf{x}_{n_h+i}} + \frac{\mathbf{z}_i - \mathbf{z}_{n_h+i}}{\mathbf{x}_i - \mathbf{x}_{n_h+i}}$$

$$= a_1 + a_2\, \mathbf{P}^1_{i,n_h+i} + \ldots + a_h\, \mathbf{P}^{h-1}_{i,n_h+i} + \frac{\mathbf{z}_i - \mathbf{z}_{n_h+i}}{\mathbf{x}_i - \mathbf{x}_{n_h+i}}.$$

$$\Delta^{(2)}\,(i, n_h + i, 2\, n_h + i) \approx a_2 + a_3\, \mathbf{P}^1_{i,n_h+i,2n_h+i} + \ldots + a_h\, \mathbf{P}^{h-2}_{i,n_h+i,2n_h+i} + \frac{\dfrac{\mathbf{z}_i - \mathbf{z}_{n_h+i}}{\mathbf{x}_i - \mathbf{x}_{n_h+i}} - \dfrac{\mathbf{z}_{n_h+i} - \mathbf{z}_{2n_h+i}}{\mathbf{x}_{n_h+i} - \mathbf{x}_{2n_h+i}}}{\mathbf{x}_i - \mathbf{x}_{2n_h+i}}.$$

$$\Delta^{(h)}\,(i, n_h + i, \ldots, hn_h + i) \approx a_h + \mathbf{Z}_i,$$

in which $\mathbf{Z}_i$ is a random variable depending on

$$\mathbf{x}_i, \ldots, \mathbf{x}_{hn_h+i} \quad , \quad \mathbf{z}_i, \ldots, \mathbf{z}_{hn_h+i}.$$

$\mathbf{Z}_i$ can be written as a fraction, the denominator being a product of terms $(\mathbf{x}_{cn_h+i} - \mathbf{x}_{c'n_h+i})$ $(c, c' = 0, \ldots, h;\ c \neq c')$; according to condition II*a* this denominator has a definite sign. The numerator consists of a sum of terms

$$(\mathbf{z}_{cn_h+i} - \mathbf{z}_{c'n_h+i}) \prod_{c'', c'''} (\mathbf{x}_{c''n_h+i} - \mathbf{x}_{c'''n_h+i}).$$

But to our order of approximation this is equal to

$$(\mathbf{z}_{cn_h+i} - \mathbf{z}_{c'n_h+i}) \prod_{c'',c'''} (\xi_{c''n_h+i} - \xi_{c'''n_h+i}),$$

so that the numerator can be written as

$$\sum_{c=0}^{h} \tau_c \, \mathbf{z}_{cn_h+i} \text{ with } \sum_c \tau_c = 0.$$

It follows from condition III that this quantity has zero median. From this and from the above-mentioned property of the denominator it follows that

$$P\,[\Delta^{(h)}(i, \ldots, hn_h + i) < a_h \,|\, a_h] = P\,[\Delta^{(h)}(i, \ldots, hn_h + i) > a_h \,|\, a_h] = \tfrac{1}{2}.$$

From this and from condition I the theorem immediately follows.

Additional remarks

3. 5. If a_h is known, a confidence interval for a_{h-1} can be found. Consider the equation

$$\mathbf{y}_i - a_h \, \mathbf{x}_i^h \approx \sum_0^{h-1} a_p \, \mathbf{x}_i^p + \mathbf{z}_i,$$

which shows that the problem is reduced to the case of a polynomial of degree $(h-1)$. So, if a confidence interval for a_h is given, a confidence region for a_h and a_{h-1} can be found. This can be generalized to an $(h + 1)$-dimensional confidence region for the parameters $a_0, \ldots, a_h$ in a way analogous to the one described in 2. 2. and 2. 3.

3. 6. If $\nu > 1$, an N-dimensional confidence region for the parameters $a_{p_1 \ldots p_\nu}$ can be found in the following way:

1. Given the other parameters, a confidence region for the parameters $a_{p_1 0 \ldots 0}$ $(p_1 = 0, \ldots, h)$ in the N-dimensional parameter space can be constructed, the level of significance being ε_1 (cf. 3. 5.).
2. In the same way one can proceed with the parameters

$$a_{0p_2\ldots 0}, \ldots, a_{00\ldots p_\nu} \quad (p_\lambda = 1, \ldots, h),$$

the levels of significance being $\varepsilon_2, \ldots, \varepsilon_\nu$.

3. Finally the parameters $a_{p_1^1 \ldots p_\nu^1}$ which have at least two indices $p_\lambda^1 \neq 0$. We suppose that (apart from the conditions I and III) condition II*b* is valid, which is a more stringent condition than condition II*a*. Consider the equation

$$\mathbf{R}_i = \mathbf{y}_i - \Sigma\, a_{p_1 \ldots p_\nu} \mathbf{x}_{1i}^{p_1} \ldots \mathbf{x}_{\nu i}^{p_\nu} \approx a_{p_1^1 \ldots p_\nu^1} \, \mathbf{x}_{1i}^{p_1^1} \ldots \mathbf{x}_{\nu i}^{p_\nu^1} + \mathbf{z}_i,$$

in which Σ denotes summation over all sets $p_1, \ldots, p_\nu$ except the set $p_1^1, \ldots, p_\nu^1$. We can then state regarding the quantities

$$\mathbf{S}_{ij} = \frac{\mathbf{R}_i - \mathbf{R}_j}{\mathbf{x}_{1i}^{p_1^1} \ldots \mathbf{x}_{\nu i}^{p_\nu^1} - \mathbf{x}_{1j}^{p_1^1} \ldots \mathbf{x}_{\nu j}^{p_\nu^1}}$$

that
$$P[\mathbf{S}_{ij} < a_{p_1^1 \ldots p_\nu^1} \mid a_{p_1^1 \ldots p_\nu^1}] = P[\mathbf{S}_{ij} > a_{p_1^1 \ldots p_\nu^1} \mid a_{p_1^1 \ldots p_\nu^1}] = \tfrac{1}{2},$$

so that in a well-known way confidence regions for each of the parameters $a_{p_1^1 \ldots p_\nu^1}$ can be found with levels of significance

$$\varepsilon_{\nu+1}, \ldots, \varepsilon_{N-(h-1)\nu-1}.$$

The common part of the $N-(h-1)\nu-1$ regions is a confidence region for the "true parameter point" in the N-dimensional parameter space, the level of significance being

$$\leqq \sum_{q=1}^{N-(h-1)\nu-1} \varepsilon_q.$$

4. CONFIDENCE REGIONS FOR THE PARAMETERS OF SYSTEMS OF REGRESSION EQUATIONS.

4. 0. In recent years considerable work has been done on the subject of systems of regression equations (see e.g. T. HAAVELMO (1943, 1944), T. KOOPMANS (1945, 1950), R. BENTZEL and H. WOLD (1946), M. A. GIRSHICK and T. HAAVELMO (1947)). In this section we shall give a brief investigation into the application of the methods considered on this subject.

The probability set

4. 1. Our probability set Γ will be the $n(\nu+2\tau)$-dimensional Cartesian space $R_{n(\nu+2\tau)}$ with coordinates

$$u_{11}, \ldots, u_{1n}, \ldots, u_{\nu 1}, \ldots, u_{\nu n}$$
$$v_{11}, \ldots, v_{1n}, \ldots, v_{\tau 1}, \ldots, v_{\tau n}$$
$$w_{11}, \ldots, w_{1n}, \ldots, w_{\tau 1}, \ldots, w_{\tau n}.$$

We suppose $n(\nu+2\tau)$ random variables $\mathbf{u}_{\lambda i}$, $\mathbf{v}_{\varkappa i}$, $\mathbf{w}_{ti}$ $(i = 1, \ldots, n;$ $\lambda = 1, \ldots, \nu;\ \varkappa, t = 1, \ldots, \tau)$ to have a simultaneous probability distribution on Γ. Furthermore we consider $n\nu + m\tau$ parameters $\xi_{\lambda i}$, a_{tj} $(i = 1, \ldots, n;$ $j = 1, \ldots, m;$ $\lambda = 1, \ldots, \nu;$ $t = 1, \ldots, \tau)$. Finally we consider the following equations

$$f_t(\eta_{1i}, \ldots, \eta_{\tau i}, \xi_{1i}, \ldots, \xi_{\nu i}, a_{t1}, \ldots, a_{tm}) = \mathbf{w}_{ti} \tag{14}$$
$$\mathbf{x}_{\lambda i} = \xi_{\lambda i} + \mathbf{u}_{\lambda i} \tag{15}$$
$$\mathbf{y}_{\varkappa i} = \eta_{\varkappa i} + \mathbf{v}_{\varkappa i}. \tag{16}$$
$$\left\{\begin{array}{l} i = 1, \ldots, n \\ \lambda = 1, \ldots, \nu \\ \varkappa, t = 1, \ldots, \tau \end{array}\right.$$

The equations (14) are supposed to have a unique solution for $\eta_{\varkappa i}$ $(\varkappa = 1, \ldots, \tau;\ i = 1, \ldots, n)$ on every element of Γ, except possibly on a set of elements with zero probability.

The equations (14) are called the "stochastic regression equations". The parameters $\xi_{\lambda i}$ $(i = 1, \ldots, n)$ are interpreted as the values which the variable ξ_λ assumes $(\lambda = 1, \ldots, \nu)$. The random variables $\mathbf{w}_{ti}$ are called

the "true deviations" in the stochastic regression equations. Finally, the random variables $\mathbf{u}_{\lambda i}$ and $\mathbf{v}_{\varkappa i}$ are called the "errors of observation" of the "true" values $\xi_{\lambda i}$ and $\eta_{\varkappa i}$ respectively.

The problem is again, to determine confidence regions for the parameters α_{tj}.

Confidence regions

4. 2. We reduce the equations (14), (15) and (16) to the forms

$$\mathbf{y}_{\varkappa i} = g_\varkappa (\mathbf{x}_{1i}, \ldots, \mathbf{x}_{\nu i}, \mathbf{u}_{1i}, \ldots, \mathbf{u}_{\nu i}, \mathbf{v}_{1i}, \ldots, \mathbf{v}_{\tau i}, \mathbf{w}_{1i}, \ldots, \mathbf{w}_{\tau i}, \alpha_{11}, \ldots, \alpha_{1m}, \ldots, \alpha_{\iota 1}, \ldots, \alpha_{\tau m}). \qquad (\varkappa = 1, \ldots, \tau)$$

Consider e.g. the case

$$f_t = H_t (\xi_{1i}, \ldots, \xi_{\nu i}) + \sum_{\varkappa=1}^{\tau} \beta_{t\varkappa} \eta_{\varkappa i}, \qquad (t = 1, \ldots, \tau)$$

in which $\beta_{t\varkappa}$ are real numbers and H_t are polynomials of degree h in the ξ's. Suppose that the errors $\mathbf{u}_{\lambda i}$ are sufficiently small in order that terms containing $\mathbf{u}_{\lambda i}\, \mathbf{u}_{\lambda' i}$ $(\lambda, \lambda' = 1, \ldots, \nu)$ can be neglected (cf. the Approximation of section 3. 1.); then we have

$$H_t (\mathbf{x}_{1i}, \ldots, \mathbf{x}_{\nu i}) + \sum_{\varkappa=1}^{\tau} \beta_{t\varkappa} \mathbf{y}_{\varkappa i} \approx \mathbf{z}_{ti}, \qquad (t = 1, \ldots, \tau)$$

in which $\mathbf{z}_{ti}$ are linear functions of $\mathbf{u}_{\lambda i}, \mathbf{v}_{\varkappa i}, \mathbf{w}_{ti}$ $(\lambda = 1, \ldots, \nu; \varkappa, t = 1, \ldots, \tau)$. The random variables $(\mathbf{z}_{1i}, \ldots, \mathbf{z}_{\tau i})$ $(i = 1, \ldots, n)$ have a simultaneous probability distribution, while the n τ-uples $(\mathbf{z}_{1i}, \ldots, \mathbf{z}_{\tau i})$ are supposed to be stochastically independent. So we have

$$\mathbf{y}_{\varkappa i} \approx \sum_{t=1}^{\tau} \frac{B_{t\varkappa}}{B} \{-H_t (\mathbf{x}_{1i}, \ldots, \mathbf{x}_{\nu i}) + \mathbf{z}_{ti}\}, \tag{17}$$

in which

$$B = \begin{vmatrix} \beta_{11} \cdots \beta_{1\tau} \\ \vdots \qquad \vdots \\ \beta_{\tau 1} \cdots \beta_{\tau\tau} \end{vmatrix}$$

and $B_{t\varkappa}$ is the cofactor of the element $\beta_{t\varkappa}$.

Then the problem is reduced to the case considered in section 3. Call N the number of parameters of a polynomial of degree h. Then, in a way and under conditions which are analogous to those stated in section 3, a confidence region for τN parameters of the equations (17) can be given. But the original equations contain $\tau(N + \tau - 1)$ parameters. This means that, if $\tau(\tau-1)$ parameters of the original equations are given, a confidence region for the remaining τN parameters can be constructed. If the level of significance of the confidence regions for the parameters of the equations (17) are $\varepsilon_\varkappa (\varkappa = 1, \ldots, \tau)$, this level of the confidence region for the τN parameters of the original equations is $\leq \sum_1^\tau \varepsilon_\varkappa$.

4. 3. We shall now elaborate a simple example, which is due to T. HAAVELMO (1944), p. 99 seq. Suppose we have the following equations:

$$\left.\begin{aligned} \eta_{1i} - \beta\,\eta_{2i} &= \mathbf{w}_{1i} \\ \alpha\,\xi_{1i} + \eta_{1i} - \alpha\,\eta_{2i} &= \mathbf{w}_{2i} \\ x_{1i} &= \xi_{1i} \\ \mathbf{y}_{1i} &= \eta_{1i} + \mathbf{v}_{1i} \\ \mathbf{y}_{2i} &= \eta_{2i}, \end{aligned}\right\} \quad i = 1, \ldots, n$$

in which α and $-\beta$ are positive.

We obtain:

$$\mathbf{y}_{1i} = \frac{\alpha\beta}{\alpha-\beta}\, x_{1i} + \frac{\alpha\,\mathbf{w}_{1i} - \beta\,\mathbf{w}_{2i}}{\alpha-\beta} + \mathbf{v}_{1i}$$

$$\mathbf{y}_{2i} = \frac{\alpha}{\alpha-\beta}\, x_{1i} + \frac{\mathbf{w}_{1i} - \mathbf{w}_{2i}}{\alpha-\beta}.$$

Suppose that the complete or the incomplete method gives two confidence intervals

$$\boldsymbol{a}_1 \leqq \frac{\alpha\beta}{\alpha-\beta} \leqq \boldsymbol{a}_2$$

$$\boldsymbol{b}_1 \leqq \frac{\alpha}{\alpha-\beta} \leqq \boldsymbol{b}_2$$

with levels of significance ε_1 and ε_2 respectively. Then we obtain two confidence regions in the α, β-plane, bounded by hyperbolas and by straight lines respectively. The probability that the common part contains the "true" point (α, β) is $\geqq 1 - \varepsilon_1 - \varepsilon_2$. (See fig. 2).

On multicollinearity

4. 4. As a final application we consider the following case. The following equations are given (cf. section 2. 0.):

$$\left.\begin{aligned} \theta_i &= \alpha_0 + \alpha_1\,\xi_{1i} + \alpha_2\,\xi_{2i} \\ \eta_i &= \theta_i + \mathbf{w}_i \\ \mathbf{x}_{\lambda i} &= \xi_{\lambda i} + \mathbf{u}_{\lambda i} \\ \mathbf{y}_i &= \eta_i + \mathbf{v}_i. \end{aligned}\right\} \quad \begin{aligned} i &= 1, \ldots, n \\ \lambda &= 1, 2. \end{aligned}$$

Hence

$$\mathbf{y}_i = \alpha_0 + \alpha_1\,\mathbf{x}_{1i} + \alpha_2\,\mathbf{x}_{2i} + \mathbf{z}_i$$

with

$$\mathbf{z}_i = \mathbf{v}_i + \mathbf{w}_i - \alpha_1\,\mathbf{u}_{1i} - \alpha_2\,\mathbf{u}_{2i}.$$

Suppose that the observed values x_{1i}, x_{2i} $(i = 1, \ldots, n)$ are such that the following *condition* is satisfied:

For each pair i, j $(i, j = 1, \ldots, n)$ the quotient

$$\frac{x_{1i} - x_{1j}}{x_{2i} - x_{2j}}\,(i \neq j)$$

has the same sign.

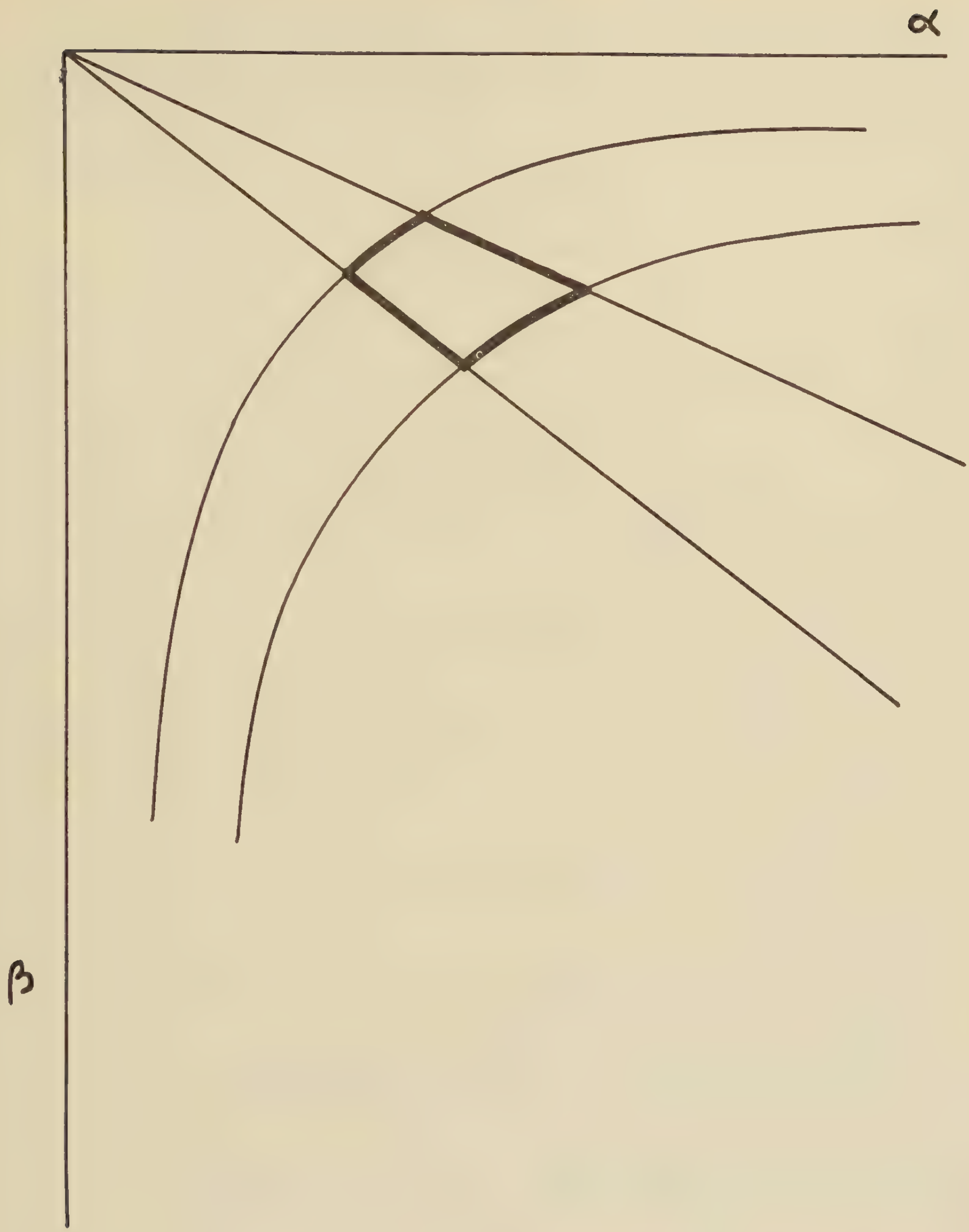

Fig. 2

This condition implies that, apart from the above-mentioned linear relation between $\mathbf{x}_{1i}$, $\mathbf{x}_{2i}$ and $\mathbf{y}_i$, we have an additional monotonic relation between the observed values x_{1i} and x_{2i} (if this relation also is — approximately — linear, we have a case of "multicollinearity").

We now have the following

Theorem 7. Under the above-mentioned condition the regions A_1 and A_2 (cf. section 2. 2.) are identical, and their common part A is unbounded.

Proof. If the condition is satisfied the arrangement of the observed points (x_{1i}, x_{2i}, y_i) according to increasing values of x_1 is the same as (or just the reverse of) the arrangement according to increasing values of x_2. Moreover (cf. section 2. 2.) the quantities $K^{(1)}(ij)$ and $K^{(2)}(ij)$ which are estimates of a_1, given a_2, and of a_2, given a_1, respectively are represented by the same set of straight lines in the a_1, a_2-plane:

$$(x_{1i}-x_{1j})\,K^{(1)}(ij)+(x_{2i}-x_{2j})\,a_2=y_i-y_j$$

$$(x_{1i}-x_{1j})\,a_1+(x_{2i}-x_{2j})\,K^{(2)}(ij)=y_i-y_j.$$

As the slopes of these straight lines $-(x_{1i}-x_{1j})/(x_{2i}-x_{2j})$ have the same sign, the regions A_1 and A_2 are identical, from which the theorem follows.

If the incomplete method instead of the complete method is used, the same theorem holds with respect to the regions A_1' and A_2', whereas the condition that all quantities $(x_{1i}-x_{1j})/(x_{2i}-x_{2j})$ have the same sign is weakened to the condition that all quantities $(x_{1i}-x_{1.n_1+i})/(x_{2i}-x_{2.n_1+i})$ have the same sign $(i=1,\ldots,n_1)$.

5. PROBLEMS OF PREDICTION

The probability set

5. 0. For the probability set and the random variables defined on it we refer to section 4. 1. We assume, however, that all errors $\boldsymbol{u}_{\lambda i}$, $\boldsymbol{v}_{\varkappa i}$ are identically equal to zero $(\lambda=1,\ldots,\nu;\ \varkappa=1,\ldots,\tau;\ i=1,\ldots,n)$.

Conditions

5. 1. We impose the following conditions:

Condition I: All n τ-uples $(\boldsymbol{w}_{1i},\ldots,\boldsymbol{w}_{\tau i})$ are distributed independently of each other.

Condition IIIa: All n τ-uples $(\boldsymbol{w}_{1i},\ldots,\boldsymbol{w}_{\tau i})$ have the same continuous simultaneous distribution function.

Apart from these conditions we shall use the additional conditions, which are necessary for the determination of a confidence region for the parameters of the regression equations.

The problem

5. 2. Suppose that the following n points are observed:

$$(\xi_{1i},\ldots,\xi_{\nu i},\eta_{1i},\ldots,\eta_{\tau i}).$$

Suppose further that the following ν parameters are given:

$$\xi_{1,n+1},\ldots,\xi_{\nu,n+1}.$$

These parameters are interpreted as the ξ-coordinates of an $(n+1)$-th point, which is not observed. The problem is to determine a confidence region for the η-coordinates of this point, i.e. for

$$\eta_{1,n+1},\ldots,\eta_{\tau,n+1}.$$

Confidence regions

5. 3. Consider again the case (cf. section 4. 2.):

$$f_t \equiv H_t(\xi_{1i}, \dots, \xi_{\nu i}) + \sum_{t=1}^{\tau} \beta_{t\varkappa} \eta_{\varkappa i}, \qquad (t = 1, \dots, \tau)$$

so that we have

$$\eta_{\varkappa i} = \sum_{t=1}^{\tau} \frac{B_{t\varkappa}}{B} \{-H_t(\xi_{1i}, \dots, \xi_{\nu i}) + \mathbf{w}_{ti}\}. \qquad (\varkappa = 1, \dots, \tau;\ i = 1, \dots, n)$$

Putting $i = n + 1$ we can write

$$\eta_{\varkappa, n+1} = g_\varkappa(\xi_{1,n+1}, \dots, \xi_{\nu,n+1}, \beta) + h_\varkappa(\mathbf{w}_{1,n+1}, \dots, \mathbf{w}_{\tau,n+1}, \beta),$$

in which

$$g_\varkappa(\xi_{1,n+1}, \dots, \xi_{\nu,n+1}, \beta) = -\sum_{t=1}^{\tau} \frac{B_{t\varkappa}}{B} H_t(\xi_{1,n+1}, \dots, \xi_{\nu,n+1})$$

$$h_\varkappa(\mathbf{w}_{1,n+1}, \dots, \mathbf{w}_{\tau,n+1}, \beta) = \sum_{t=1}^{\tau} \frac{B_{t\varkappa}}{B} \mathbf{w}_{ti}$$

and in which β is the "true parameter point"; β may be considered as a vector, the components of which are $\beta_{t\varkappa}(t, \varkappa = 1, \dots, \tau)$ and all parameters determining the polynomials $H_t\,(t = 1, \dots, \tau)$.

Suppose β is known. Then we can arrange the n quantities $h_\varkappa(w_{1i}, \dots, w_{\tau i}, \beta)$ according to increasing magnitude:

$$h_{\varkappa 1} < \dots < h_{\varkappa n}, \qquad (\varkappa = 1, \dots, \tau)$$

in which

$$h_{\varkappa j} = h_\varkappa(w_{1i_j}, \dots, w_{\tau i_j}, \beta).$$

We have the following

Theorem 8: Under conditions I and III*a* a confidence interval for $\eta_{\varkappa . n+1}$ is given by

$$\left(g_\varkappa(\xi_{1,n+1}, \dots, \xi_{\nu,n+1}, \beta) + \mathbf{h}_{\varkappa s},\ g_\varkappa(\xi_{1,n+1}, \dots, \xi_{\nu,n+1}, \beta) + \mathbf{h}_{\varkappa, n-s+1}\right)$$

if β is the known "true parameter point"; the level of significance is $2s\,(n + 1)^{-1}$.

In order to prove this theorem, we shall use the following lemma (see W. R. Thompson (1936)):

Lemma: If a random sample of size n is drawn from a universe with continuous distribution function; if the sample values are arranged in ascending order; if an $(n + 1)$-th draw from the same universe is to be effected; then the probability that the stochastic interval bounded by the s-th and the $(n - s + 1)$-th of these values will contain the $(n + 1)$-th is equal to $1 - 2s/(n + 1)$.

Proof of Theorem 8: As $g_\varkappa(\xi_{1,n+1}, \dots, \xi_{\nu,n+1}, \beta)$ is ex hypothesi a known quantity, the problem is to determine a confidence interval for

$h_\varkappa(\mathbf{w}_{1,n+1}, \ldots, \mathbf{w}_{\tau,n+1}, \beta)$. But n sample values $h_\varkappa(\mathbf{w}_{1i}, \ldots, \mathbf{w}_{\tau i}, \beta)$ from the same universe (cf. condition IIIa) are obtained; hence the lemma is sufficient in order to show the validity of the theorem.

5. 4. Generally, however, β is unknown, and we can only calculate a confidence region $\boldsymbol{R}$ for β. Let now β vary through $\boldsymbol{R}$, and denote by $\boldsymbol{J}_\varkappa$ the interval bounded by the lowest of all lower limits of the interval considered in Theorem 8 and by the highest of all upper limits ($\varkappa = 1, \ldots, \tau$). If the level of significance of $\boldsymbol{R}$ is ε, the following theorem immediately follows:

Theorem 9:

$$P\left[\eta_{1,n+1} \in \boldsymbol{J}_1, \ldots, \eta_{\tau,n+1} \in \boldsymbol{J}_\tau, \beta \in \boldsymbol{R}\right] \geqq (1-\varepsilon)\left(1-\frac{2\tau s}{n+1}\right).$$

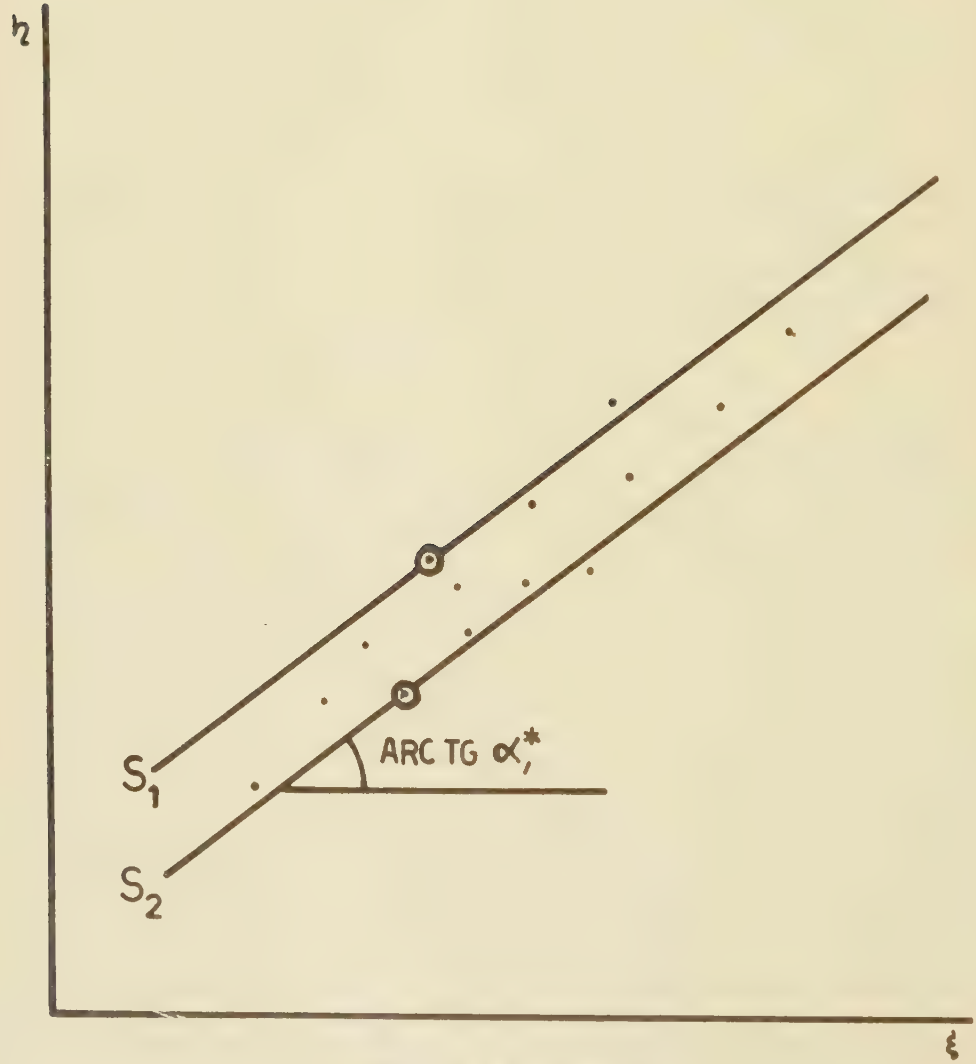

Fig. 3. $n = 14$ $s = 2$

The linear case in two variables

5. 5. For the linear case of two variables

$$\eta_i = a_0 + a_1 \xi_i + \mathbf{w}_i$$

a simple graphical representation can be given. Suppose that a_1^* is the "true" a_1; after arranging the sample values

$$w_i (a_1^*) = \eta_i - a_0 - a_1^* \xi_i$$

in order we find two straight lines:

S_1: $$\eta = a_0 + a_1^* \xi + w_s (a^*)$$

and S_2: $$\eta = a_0 + a_1^* \xi + w_{n-s+1} (a^*).$$

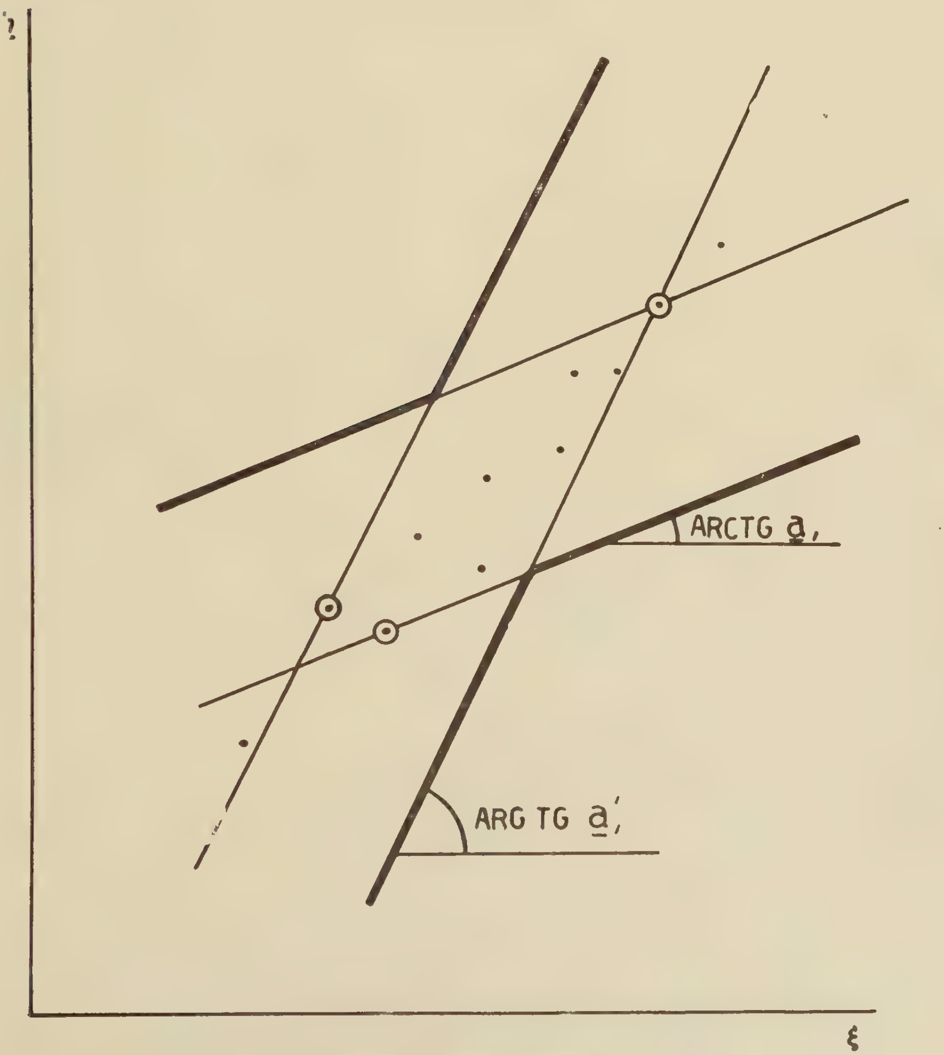

Fig. 4. $n = 11$ $s = 2$

The probability that the region bounded above by S_1 and below by S_2 will contain an $(n+1)$-th sample point D is, under the condition that $a_1^* = a_1$, equal to $1 - 2s/(n+1)$. (See fig. 3).

Suppose that the confidence interval for a_1 is $(\mathbf{a}_1, \mathbf{a}_1')$. When a_1^* varies through this interval the lines S_1 and S_2 revolve around the observed points (ξ_i, η_i) with ranks s and $(n - s + 1)$ respectively with respect to increasing values of w. As long as the observed points having these properties remain the same, S_1 and S_2 revolve around one point; but as soon as variation of a_1^* causes another point to have this property, the revolution takes place around this point. The figures 4 and 5 elucidate the fact that sometimes the region is bounded by the straight lines S_1 and S_2 for $a^* = \mathbf{a}_1$ and $\mathbf{a}_1'$ only, whereas it is sometimes necessary to consider values between $\mathbf{a}_1$ and $\mathbf{a}_1'$ as well.

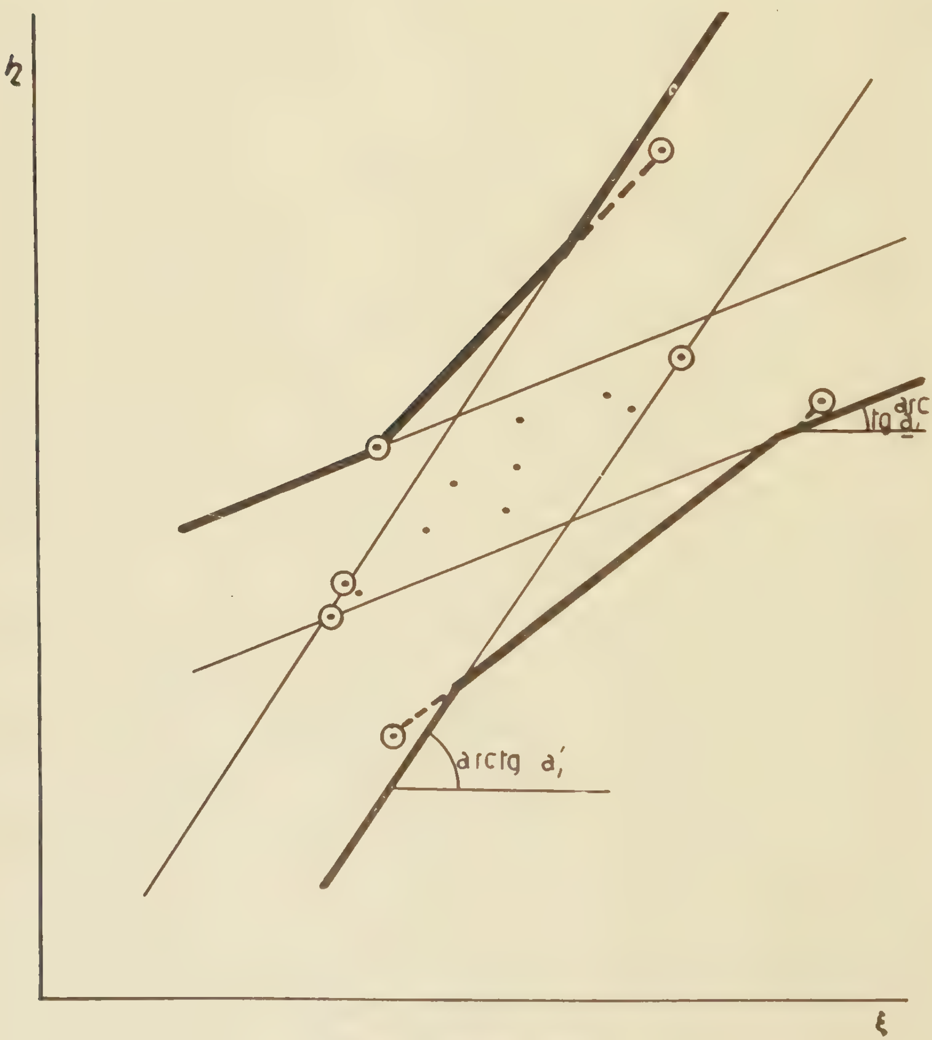

Fig. 5. $n = 15$ $s = 2$

6. CONCLUDING REMARKS

6. 0. The methods of determining confidence regions which may be derived from this kind of analysis have not been exhaustively treated. In order to elucidate this statement we shall give a confidence interval for a in the stochastic regression equation

$$\eta_i = A\,\xi_i^a + \mathbf{w}_i, \qquad (i = 1, \ldots, n)$$

in which all ξ_i are positive, and in which

$$P\,[A\,\eta_i \leqq 0] = 0$$

holds for $i = 1, \ldots, n$. $\xi_1, \ldots, \xi_n$ are known, A and a are unknown parameters, and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are random variables, which are supposed (1) to be distributed stochastically independent, (2) to have continuous symmetrical distribution functions with zero median.

We arrange the observed points (ξ_i, η_i) according to increasing magnitude of ξ and define

$$\mathbf{d}_i = \frac{\log|\eta_{n_1+i}| - \log|\eta_i|}{\log \xi_{n_1+i} - \log \xi_i}, \qquad (i = 1, \ldots, n_1)$$

in which $n_1 = \frac{1}{2}n$ (if n is odd the point $(\xi_{\frac{1}{2}(n+1)}, \eta_{\frac{1}{2}(n+1)})$ is neglected). After arranging the observed quantities d_i according to increasing magnitude:

$$d_{(1)} < \ldots < d_{(n_1)}$$

we have the following

Theorem 10. Under conditions (1) and (2) the interval $(\mathbf{d}_{(r_1)}, \mathbf{d}_{(n_1-r_1+1)})$ is a confidence interval for a to the level of significance $2I_{\frac{1}{2}}\,(n_1 - r_1 + 1, r_1)$.

Proof. We have

$$\frac{\eta_{n_1+i} - \mathbf{w}_{n_1+i}}{\eta_i - \mathbf{w}_i} = \left(\frac{\xi_{n_1+i}}{\xi_i}\right)^a$$

or:

$$\eta_{n_1+i} - \left(\frac{\xi_{n_1+i}}{\xi_i}\right)^a \eta_i = \mathbf{w}_{n_1+i} - \left(\frac{\xi_{n_1+i}}{\xi_i}\right)^a \mathbf{w}_i.$$

It follows from condition (2), that

$$\mathbf{w}_{n_1+i} - \left(\frac{\xi_{n_1+i}}{\xi_i}\right)^a \mathbf{w}_i$$

has a continuous distribution function with zero median. Hence:

$$P\left[\eta_{n_1+i} - \left(\frac{\xi_{n_1+i}}{\xi_i}\right)^a \eta_i < 0\right] =$$

$$= P\,[\log \eta_{n_1+i} - \log \eta_i < a\,(\log \xi_{n_1+i} - \log \xi_i)] =$$

$$= P\left[\frac{\log \eta_{n_1+i} - \log \eta_i}{\log \xi_{n_1+i} - \log \xi_i} < a\right] = P\left[\frac{\log \eta_{n_1+i} - \log \eta_i}{\log \xi_{n_1+i} - \log \xi_i} > a\right] = \tfrac{1}{2},$$

if $\eta_1, \ldots, \eta_n$ are positive; if they are negative we have to replace η_i and η_{n_1+i} by $-\eta_i$ and $-\eta_{n_1+i}$ respectively. From this and from condition (1) the theorem follows.

The theorem shows that this method of determining a confidence interval for α is identical with the incomplete method for α in the linear equation

$$\log \eta_i = \log A + \alpha \log \xi_i + \mathbf{w}'_i$$

(which can be written as

$$\eta_i = A\, \xi_i^\alpha\, e^{\mathbf{w}'_i}),$$

if $\mathbf{w}'_1, \ldots, \mathbf{w}'_n$ satisfy the same conditions (1) and (2).

6. 1. Finally we mention that it is possible to find estimates instead of confidence intervals. Consider e.g. the statistics $\Delta(ij)$; each of these $\binom{n}{2}$ statistics has the property that its sampling median is equal to α_1 (cf. section 1. 3.). Hence one can use the sample median of the observed quantities $\Delta(ij)$ as an estimate of α_1.

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CHEMISTRY

INFLUENCE OF ORGANIC COMPOUNDS ON SOAP AND PHOSPHATIDE COACERVATES — XV [1])

THE ACTION OF HALOGEN DERIVATIVES OF PARAFFINS ON AN OLEATE COACERVATE

BY

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(Communicated by Prof. H. G. BUNGENBERG DE JONG at the meeting of Sept. 30, '50)

1. *Introduction*

The historical development of the investigations on the influence of organic compounds on oleate coacervates has taken the following course. The action could be studied qualitatively by bringing the organic compound into contact with the coacervate and observing the result under the microscope ("contact method": BUNGENBERG DE JONG and SAUBERT, 1937). Especially substances which are sparingly soluble in water may be investigated in this manner. With the aid of this method it was possible to see if a substance has a "salt-sparing" or "salt-demanding" action. The former is demonstrated by the appearance of a great number of small vacuoles in the system.

Thereafter a number of publications (BUNGENBERG DE JONG et al., 1937, 1938) were devoted to the study of compounds soluble in water. The action of the shorter alcohols, ketones, esters of carbamic acid, derivates of urea, amines and some miscellaneous compounds was investigated. Here a quantitative study enabled us to compare the action of the various compounds and to study the influence of alterations in the molecules (increase of the number of carbon atoms, branching and ring closure of the chain, introduction of another polar group, introduction of halogen atoms or double bonds etc.). This kind of experiments may prove of great value for biology as many substances are found in the cell, which, generally speaking, resemble soap molecules in having an amphipatic character (phosphatides etc.).

However, the method then used could only be applied to the study of substances soluble in water. It was felt that another method (giving the

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[2]) Publication no. 11 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

opportunity to investigate hydrophobic substances) was wanted. KOETS and BUNGENBERG DE JONG (1938) first dissolved the organic compounds in oleic acid, from which oleate containing the added substance was prepared. This method proved to be too troublesome to become a general method for measuring the influence of a great number of organic compounds.

The next step was that the compound was dissolved in a small quantity of propylalcohol, to which the oleate solution was added afterwards. Then coacervation was produced by the addition of KCl in the usual manner (BOOIJ et al., 1950, 2). This new method opened a large field of investigations. We have already described some experiments on the action of hydrocarbons (BOOIJ et al., 1950, 2 and 3). The object of the present publication is to describe the effect of the introduction of one or more halogen atoms into aliphatic chains as regards their action on the oleate coacervate.

2. *The influence of alkylhalides on the coacervation of Na-oleate*

The method has been described by BOOIJ et al. (1950, 2). We started with an investigation of some mono-chlorine derivatives of alkanes (fig. 1). It is obvious that here the activity increases with increasing length of the

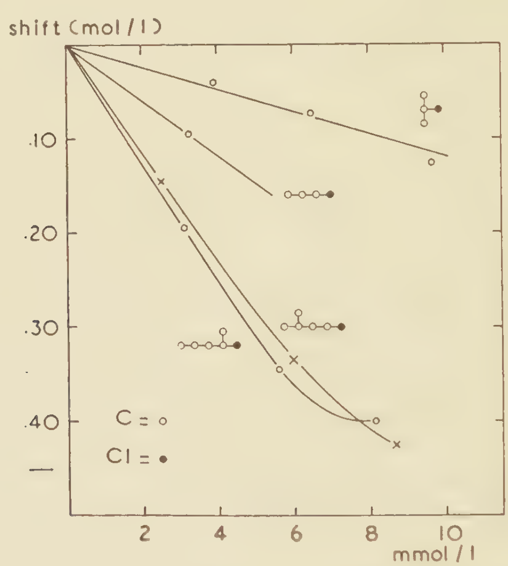

Fig. 1. Action of some chlorine derivatives of alkanes. The activity is given as the shift of the KCl concentration needed for a coacervate volume of 50 % (— = salt sparing activity). Abscissa = concentration of the added compound.

carbon chain. Branching of the chain (compare propyl- and isopropylchloride) results in a decrease of the effect (in this case a "salt-sparing" effect). These effects are comparable to those described earlier for alcohols and other organic molecules. When we introduce a chlorine atom into an

alcohol (Bungenberg de Jong et al., 1938) the salt sparing action is enforced. In this respect the introduction of a halogen atom resembled that of a methyl group more or less. In many substances with biological activity too the introduction of a chlorine atom has approximately the same influence as the introduction of a methyl group.

In our case, however, the introduction of a halogen atom into an aliphatic alkane results in a compound with totally different properties. Thus iso-amylchloride does not resemble 2-methylpentane, etc. The introduction of a chlorine atom gives an aliphatic alkane a strong salt-sparing activity.

This is still better demonstrated when more chlorine atoms are introduced. These substances have a pronounded salt-sparing activity (fig. 2).

We had at our disposal a larger group of bromine derivatives. Fig. 3

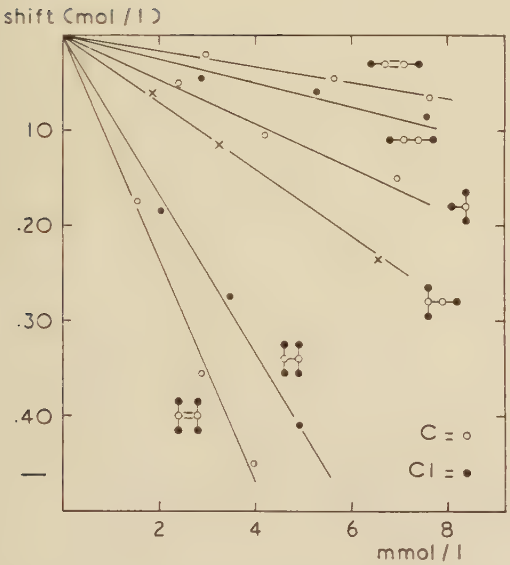

Fig. 2. Influence of alkylhalides with two or more chlorine atoms.

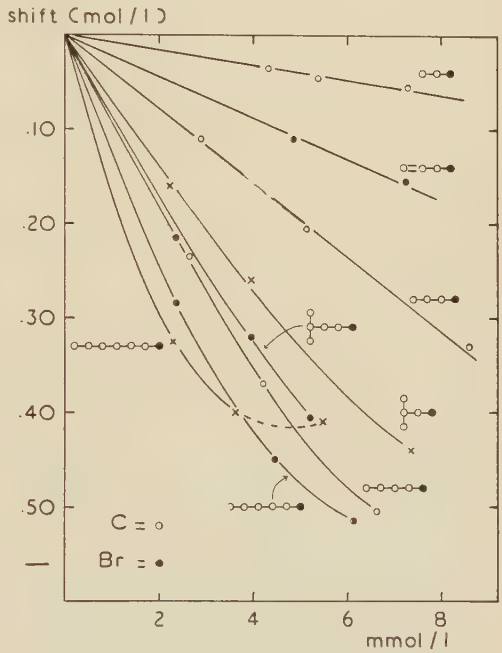

Fig. 3. Action of bromine derivatives of alkanes.

shows the influence of the increase of the carbon chain. The result of branching of the chain and of introduction of a double bond is also quite distinct. These effects are the same as those described earlier with hydrophilic substances. Hexylbromide begins to show a phenomenon which we have already met with the aliphatic alkanes. The action tends to become reverse when the concentration becomes high enough. Long alkanes even showed a salt-demanding in stead of a salt-sparing activity (Booij et al., 1950, 2).

This phenomenon is still better shown by some di-bromine derivatives (fig. 4, see di-Br-hexane and di-Br-decane). Here too it was clear that the introduction of a bromine atom results in quite different effects than the introduction of methyl groups.

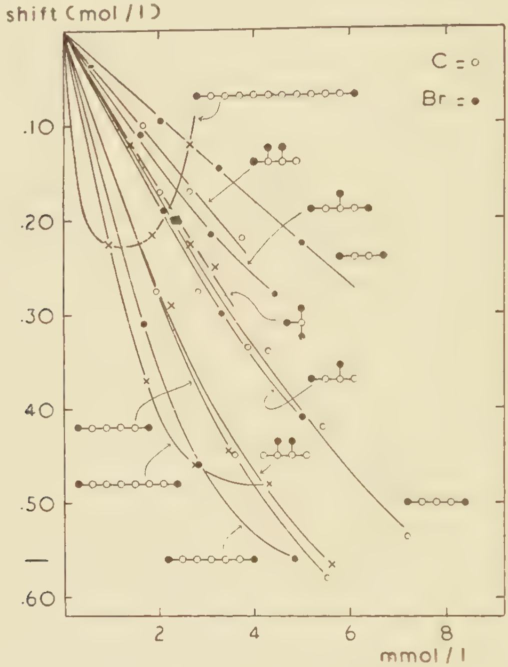

Fig. 4. Influence of alkylhalides with two of more bromine atoms.

3. *Discussion*

It seems worth while to compare some of the bromine derivatives with their parent substances. When this is done (fig. 5) the influence of the bromine atoms is seen to give a change toward salt-sparing activity in each case. In a preceding publication we assumed that the action of alkanes was the result of two opposing tendencies, viz. a salt-sparing factor produced by the molecules present between the parallel carbon chains of the soap ions and a salt-demanding factor (as a result of the presence of some molecules between the planes of the CH_3 end groups of the soap ions in the micelle (Booij et al. 1950, 2)). The resulting action would be a reflection of the distribution of the added molecules over the different places available in the soap micelle. The difference between the action of alcohols and alkanes we tried to explain with the hypothesis that alcohols would be found preferently in the position between the parallel carbon chains of the soap micelle by virtue of their hydrophilic OH-group, which would "anchor" the molecule in this position. Now we see that introduction of a halogen atom in an alkane tends to intensify the salt-sparing factor in the molecule. In the light of our hypothesis this would

mean that the bromine atom is more hydrophilic than the CH_3 group. Then the distribution of the molecules over the soap micelle would shift in favour of the position parallel to the carbon chains of the soap ions forming the micelle. When the carbon chain of the added bromine deriva-

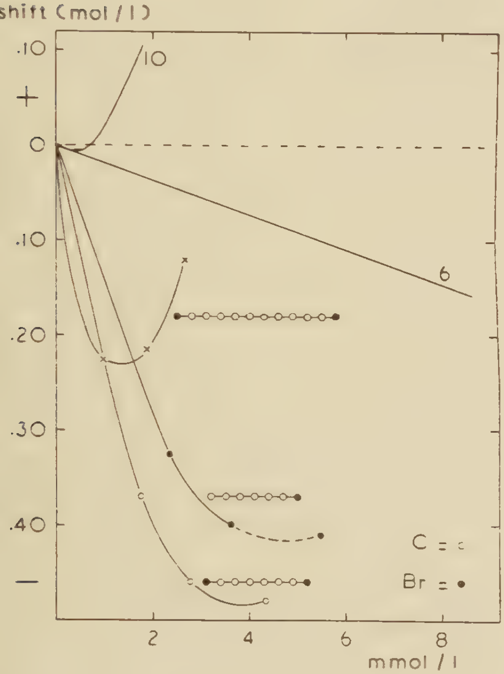

Fig. 5. A comparison of the action of alkanes (the numbers denote the number of C atoms in the paraffin, thus 10 = decane etc.) with bromine derivatives of alkanes.

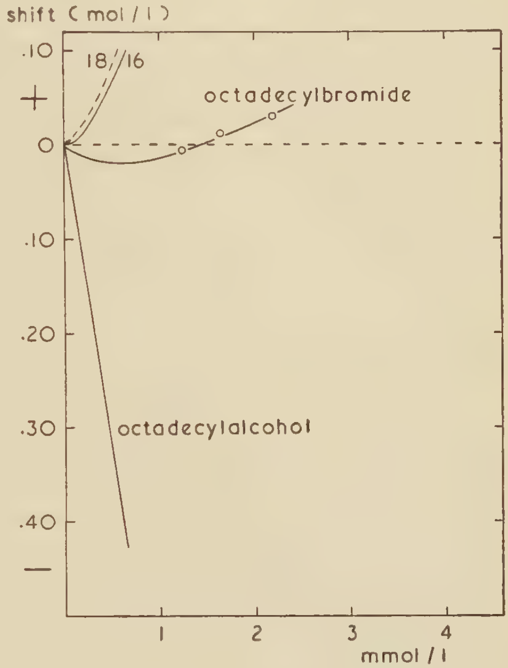

Fig. 6. Influence of the introduction of polar groups into a long alkane. 18 = activity of octadecane (deduced from BOOIJ et al., 1950, 2, who measured the action of alkanes up to hexadecane), 16 = hexadecane.

tive becomes very long (e.g. in octadecylbromide) it is expected that it will resemble an alkane very much (fig. 6). Our experiments showed that this is the case indeed. However, the salt-sparing factor is still of some importance as octadecylbromide has not at all the same salt-demanding activity as e.g. hexadecane. In 1,10-di-Br-decane the "aliphatic factor" begins to show itself (fig. 5), but the difference in activity between this compound and decane itself is still very important.

So we arrive at the conclusion that the introduction of a bromine atom changes the distribution equilibrium of the added molecules within the soap micelle. This must be ascribed to the slightly hydrophilic character of the Br-atom. The difference between this atom and a really hydrophilic group (e.g. an OH-group) is demonstrated strikingly if we compare octadecylbromide and octadecylalcohol (fig. 6). In the former case the long aliphatic chain gives the molecule practically the character of a hydrocarbon.

Now the difference between the introduction of a chlorine atom into a paraffin and into an alcohol has become clear. The halogen atom is much more hydrophobic than the OH-group of the alcohol. Introduction of the halogen will in this case influence the partition of the substance between the medium and the soap micelles in favour of the latter. Thus the halogen derivative will have a stronger salt-sparing activity than the alcohol itself. Introduction of a methyl group will act in the same direction.

In contrast, the halogen is more hydrophilic than a methyl group. Introduction into an aliphatic alkane will influence the partition within the micelle. The substance will concentrate between the parallel soap molecules in stead of between the CH_3-planes in the micelle. A strong salt-sparing effect will be the result. Here, however, the introduction of a methyl group will shift the equilibrium in the other direction.

4. *On the spreading of molecules with long aliphatic chains and weakly polar groups*

The introduction of a benzene nucleus into a aliphatic alkane has qualitatively the same influence on its activity (as regards the oleate coacervate) as that of a halogen atom (BOOIJ et al. 1950, 3). There too the action can be explained by the result of two opposing factors (a salt-demanding factor of the long alkyl group and a salt-sparing action of the aromatic nucleus).

If this hypothesis is right, it must be possible to get additional evidence by experiments on the spreading of molecules of this character. It is a well known fact, however, that it is not possible to measure the force area diagram of long apolar molecules with weakly polar groups (e.g. octadecylbromide or octadecylbenzene). No film is formed or the film collapses at very low pressures. We have tried to imitate the conditions in the soap micelle by spreading these compounds together with an excess of stearic acid.

The same two possibilities as regards the position of the added substance in the stearic acid film are present which we had supposed in the soap micelle (fig. 7). If the molecule is situated in position A one will observe a change in the force/area diagram if this is computed for the stearic acid molecules alone. This could be demonstrated by spreading stearic acid and octadecylalcohol together (fig. 8). The alcohol is "anchored" to the water by its OH-group and thus held in position A. If our hypothesis is correct the addition of hexadecane will not change the force/area diagram of stearic acid. This proved to be the case indeed (fig. 8). A small addition of hexadecane does not change the picture (if the amount is too large, however, the stearic acid film collapses soon and measurements are well nigh impossible).

We had at our disposal a pure sample of octadecylbenzene [3]) which

[3]) We are much indebted to the Koninklijke/Shell-Laboratorium, Amsterdam for the gift of this compound.

substance we used to see whether our second hypothesis might be right. Here we expected that the stearic acid diagram would be influenced as the benzene nucleus might act as a — be it weak — anchor to the water. This was found indeed (see fig. 9). The difference with octadecylalcohol was

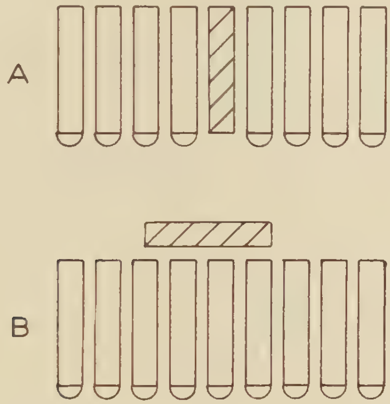

Fig. 7. When an apolar substance with a weakly polar group is spread in an excess of stearic acid, two possibilities arise.

A. The substance is situated between the carbon chains of the stearic acid.

B. The substance is pressed out of the film.

In case *A* the computed area per stearic acid molecule will be apparently too large. In case *B* the area per stearic acid molecule will be the same as if the stearic acid had been spread alone.

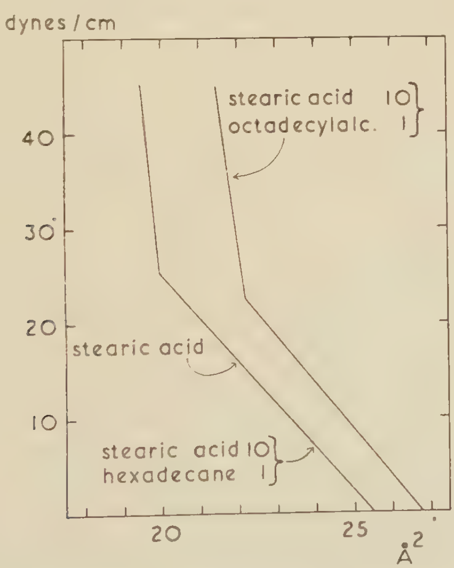

Fig. 8. Force/area diagrams of *a*) stearic acid, *b*) stearic acid and hexadecane in a proportion 10 to 1 molecules and *c*) stearic acid and octadecylalcohol in a proportion 10 to 1. The area is computed for the stearic acid molecules alone. We must conclude that octadecylalcohol takes its place parallel to the stearic acid molecules, while hexadecane is pressed out of the film.

that the film collapses when the proportion stearic acidoctadecylbenzene is below 5. With octadecylalcohol a film is formed at any proportion of the two compounds.

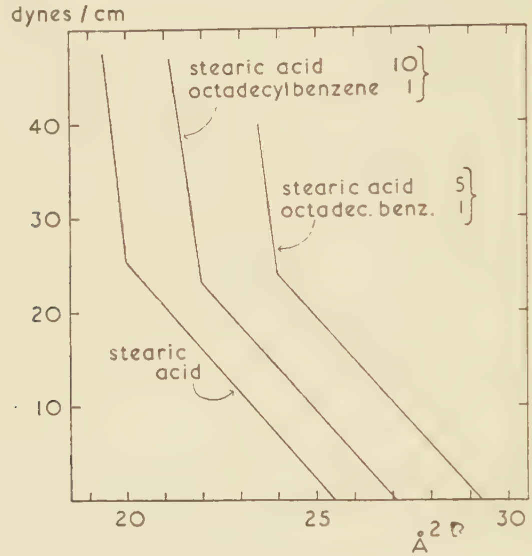

Fig. 9. Force/area diagrams of mixtures of stearic acid and octadecylbenzene as compared with that of stearic acid.

So we may conclude that the idea of a distribution of organic compounds over different places in the soap micelle has gained some support from these experiments.

Summary

1. We investigated the action of several halogen derivatives of aliphatic hydrocarbons on the oleate coacervate with the aid of the "propylalcohol method".

2. The introduction of a halogen atom produces a great change in the action of these hydrocarbons.

3. This change was explained by the hypothesis that the halogen atom is slightly hydrophilic and thus "anchors" the molecule in a position parallel to the soap ions in the micelle.

4. Experiments on the spreading of stearic acid with octadecyl-alcohol, hexadecane and octadecylbenzene indicate that there is a striking difference between the two substances first mentioned. The first is present between the parallel stearic acid molecules, the second will be pressed out of the film very soon. In low concentrations octadecylbenzene is also found between the parallel carbon chains of stearic acid, presumably by virtue of its slightly hydrophilic benzene group.

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University of Leiden

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X-RAY INVESTIGATION OF THE NUCLEAR SPOT OF CRYSTALS OBTAINED BY RECRYSTALLIZATION

BY

T. J. TIEDEMA *)

(Communicated by Prof. J. M. BIJVOET at the meeting of Sept. 30, 1950)

1. *Introduction*

In an effort to understand some recrystallization phenomena, as for instance stimulation and the orientation of crystals obtained by recrystallization of a matrix with a given texture, it seems worth while to try to get more information about nucleation. The present paper describes some experiments carried out in this field. Theoretical conclusions from these experiments will be discussed in a paper by BURGERS and TIEDEMA (1950). In the course of the investigation of recrystallization phenomena, many theories about nucleation are brought forward. Until now, however, no theory is confirmed by experiments. Most likely this is due to the fact that it is rather difficult to detect the nuclear spot of a crystal. Yet there is a type of crystal of which the nuclear spot can be determined easily, namely the roundish crystals obtained by primary recrystallization of a critically-stretched fine-grained quasi-isotropic material (see fig. 1). Such crystals were used in the experiments described hereafter. The material used in these experiments was cold rolled aluminium sheet, either of a purity of 99.5 % and a thickness of 1 mm, or of a purity of 99.998 % and a thickness of 2 mm.

2. *Experimental part*

Preparation of the test-pieces. A slab of 2×3 cm was sawed from the aluminium sheets. In the case of aluminium of purity 99.5 %, this slab was either annealed during 5 minutes at a temperature of 630° C, then cold rolled to a thickness of 0.2 mm and again annealed during 3 minutes at the same temperature; or immediately cold rolled to a thickness of 0.4 mm and annealed during 3 minutes at a temperature of 630° C. The length of the so obtained test-pieces varied from 10 to 15 cm.

In the case of aluminium of purity 99.998 %, the slab was cold rolled to a thickness of 0.4 mm and annealed during $1\frac{1}{2}$ minute at a temperature of 630° C. The length of these test-pieces was about 10 cm. The so obtained test-pieces were subjected to a stretching of 2 % and recrystallized at a temperature of 640° C.

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*) Associate worker of the Foundation for Fundamental Research in Holland.

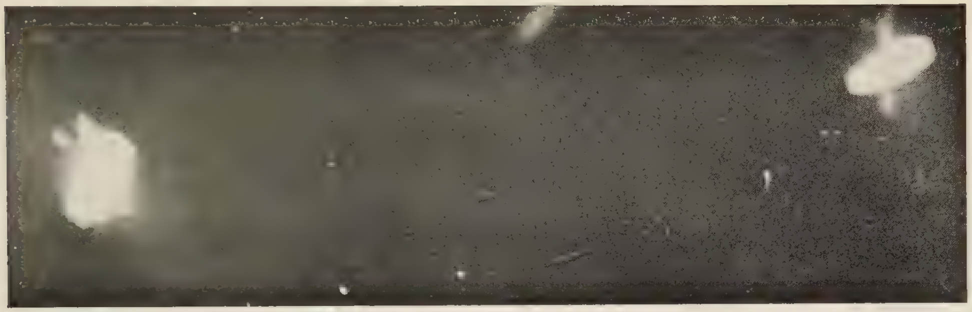

Fig. 4*a*

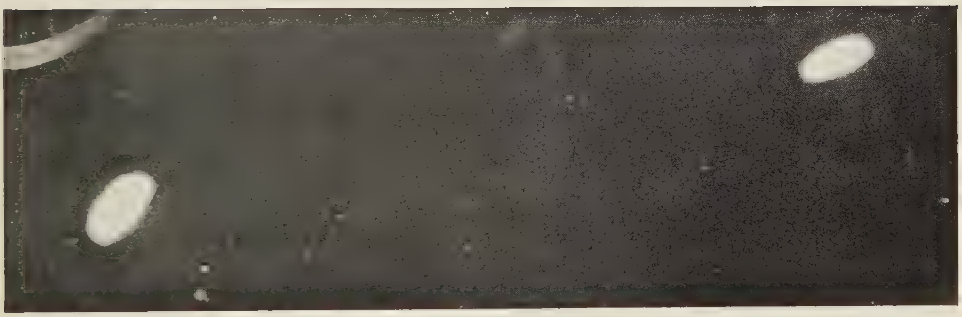

Fig. 4*b*

Fig. 12

Fig. 6

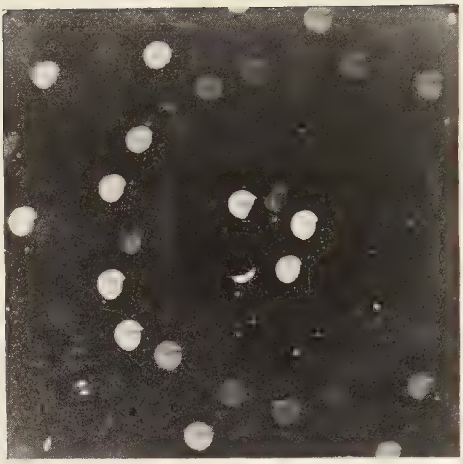

Fig. 7

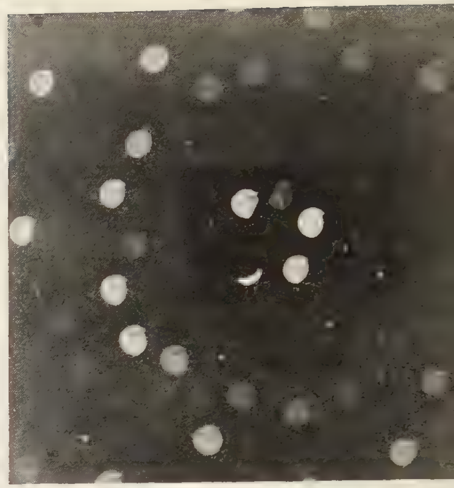

Fig. 8

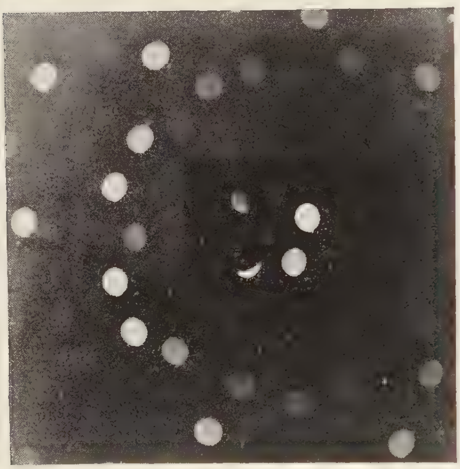

Fig. 9

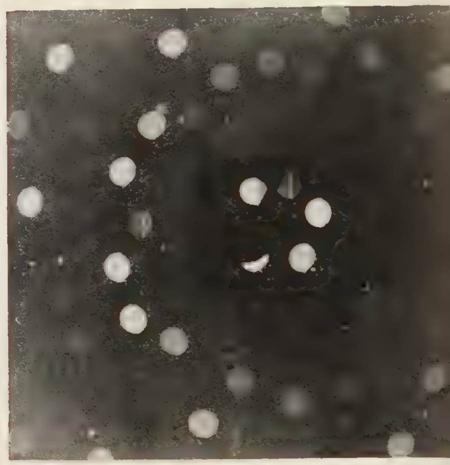

Fig. 10

Fig. 11

Recrystallization procedure. First the test-pieces were annealed during 8 minutes. Then the pieces were taken out of the furnace and, after cooling, etched with a mixture of 1 part of HNO_3 (s.w. 1.4), 3 parts of HCl (s.w. 1.19) and 3 parts of water. Of course the test-pieces were handled with the utmost care to avoid deformation of the new crystals. The roundish crystals so obtained had diameters varying from 0.5 to 6 mm. Then the test-pieces were annealed during 48 hrs. to complete recrystallization. After this annealing no further etching was applied. In this way it was possible to locate the nuclear spot easily, especially for the small crystals.

X-ray investigation of the nuclear spots. This investigation was carried out in two ways:

1. by means of normal Laue exposures and
2. by means of a variant of the Laue method as described in a paper of GUINIER and TENNEVIN (1949).

Normal Laue photographs were made of a number of roundish crystals which were obtained in the way as described before. The time of exposure was 1 hour, the distance between crystal and film 5 cm; the diaphragm used had a diameter of 0.5 mm and a length of 50 mm. It turned out that, if the irradiated part of the crystal was the centre, every Laue spot had a peculiar appearance, as if a dash passed through it: see figures 2*a* and 3*a* which give Laue photographs of the central spots of two different crystals. Enlargement of the photographs showed clearly that the dashes were striated: see figure 4*a*, which is an enlargement of the enclosed part of figure 2*a*. Figures 2*b* and 3*b* give Laue photographs of the same two crystals of figures 2*a* and 3*a*, but now irradiated 0.5 mm beside the centre, whereas figure 4*b* is an enlargement of the enclosed part of figure 2*b*. It appears from these experiments that the centre (nuclear spot) of a crystal formed by recrystallization contains some unconsumed parts of the original matrix. The difference in orientation between the crystal and the unconsumed parts in the centre of the crystal amount to about 1° as can be determined from the photographs.

As discussed in the paper by BURGERS and TIEDEMA, the result of these experiments is compatible with the conception of the nucleation process described by CAHN (1950).

In order to get more information about the structure of the nuclear spot, the second X-ray method was used, namely the Laue variant of GUINIER and TENNEVIN. First this method will be described briefly (see figure 5). If a polychromatic divergent beam of X-rays, originating from the very small focal spot S, falls on a crystal slab PP' and if the distance between focus and specimen is large, the reflected X-rays will be focussed in a point M, which lies at approximately the same distance from the crystal as the focus. For details we must refer to the paper of GUINIER and TENNEVIN. If now the film is placed close to the specimen, one will obtain

a kind of image of the irradiated part of the crystal. In this way the roundish crystals were investigated. The focus used had a diameter of 0.3 mm, the distance between focus and specimen was 100 cm whereas the distance between specimen and film was $4\frac{1}{2}$ cm. The surface of the crystal irradiated by the beam was a circle with a diameter of 4 mm.

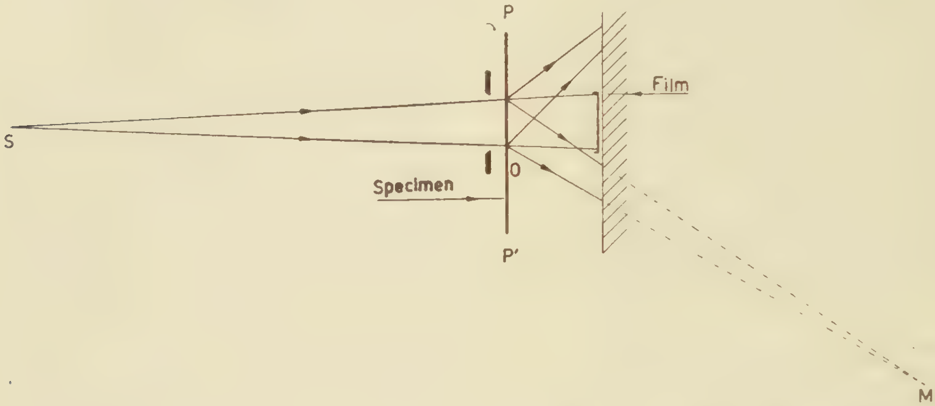

Fig. 5. Schematic representation of the Laue variant of GUINIER and TENNEVIN, taken from their paper (1949).

A striking result was obtained in this way. If the central part of the crystal was irradiated, the image was broken up into two or even more parts: see figure 6. In order to be sure that this effect was not due to the surface of the crystal, the crystal was etched from 0.4 mm to 0.2 mm thickness. As figure 7 shows, no difference in the image is caused by this etching.

From these photographs one gets the impression that a crystal is built up of two or even more parts with orientations, which lie close to each other. It was therefore tried to follow the "boundary" between two of those parts. Indeed this was possible by mounting the specimen upon a slide and making photographs every 3 millimeters. The so obtained photographs are given in figures 7—10 [1]). It is of interest that the "boundary" between the two parts is straight, as could be expected if both parts started their growth at the same moment and at equal rate. Finally figure 11 gives an image of the crystal outside the nuclear spot and lying wholly inside one of the parts of which the crystal is composed. As is shown clearly, no lines at all are visible on this image. An estimate of the angle between two crystal parts can be obtained by means of the mounting of figure 5 if the film is placed approximately in the point M. Figure 12 shows an example of the lines obtained in this way from two adjacent parts of one single crystal. The rather poor appearance of these lines is due to the bad focus and to the fact that the film was not placed exactly in the point

[1]) In figure 10 the irradiated region lies partly outside the investigated crystal as follows both from the shape of its Laue spots and from the presence of "complementary shaped" spots of the adjacent crystal.

M. This, however, is of no importance for the calculation of the difference in orientation between the two adjacent parts of the crystal.

The distance between film and specimen was 60 cm. It turned out that the orientation difference varied from 7′ to 18′. In agreement herewith, there was no difference between the appearance of normal Laue photographs taken on a "boundary" or beside a "boundary". This is clear since a normal Laue photograph only shows deviations in orientation in the order of 1°.

Finally we want to point out that also in the case of the very pure aluminium the same phenomena were observed, so that it seems improbable that the described "boundaries" originate from impurities in the original material.

3. *Summary*

It appears from these experiments that the centre (nuclear spot) of a crystal formed by recrystallization contains some unconsumed lattice regions with an orientation deviating about 1° from that of the main body of the crystal.

Moreover the main body itself consists generally of two or even more parts with nearly the same orientation (differences up to about 20′ of arc).

The meaning of these results as to nucleation theory are discussed by BURGERS and TIEDEMA (1950).

This work is part of the research programme of the "Stichting voor Fundamenteel Onderzoek der Materie" and was made possible by a finantial support from the "Nederlandse Organisatie voor Zuiver wetenschappelijk Onderzoek".

Delft, August 1950.

Laboratory for Physical Chemistry of the Technical University.

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DETERMINATION OF VITAMIN A IN MARGARINE

BY

J. BOLDINGH AND J. R. DROST

(Communicated by Prof. J. H. DE BOER at the meeting of Oct. 25, 1950)

Various difficulties are experienced in the quantitative determination of vitamin A. The Carr-Price reaction, in which the blue colour given with antimony trichloride is measured, is unreliable not only because of the transient nature of the colour but also because of the fact that other substances in addition to vitamin A give similar colours with the reagent. The measurement of the extinction in the ultra-violet at 325 mμ is far and away the most preferable, provided that no other substances are present which interfere with the absorption curve in the neighbourhood of this wavelength.

GRIDGEMAN and co-workers [1]) have considerably improved the determination of vitamin A in liver oil concentrates using this method by purifying the unsaponifiable part of such concentrates chromatographically (with Al_2O_3 as adsorbent) and by eliminating just those substances which interfere to a greater or lesser degree with the absorption curve of vitamin A in the ultra-violet.

However, for the determination of vitamin A in margarine, this method cannot be applied. Interfering substances, originating from the unsaponifiable part of the fats which are included in the margarine, cause an extremely strong distortion of the absorption curve of the vitamin A and they cannot be removed by chromatographic adsorption on Al_2O_3. It was therefore necessary in this case to carry out the determination using the older Carr-Price reaction, in which, however, widely varying results and often seemingly big "losses" occurred.

WILKIE and collaborators [2]) have suggested measuring the vitamin A in the chromatographed extract not at the maximum (325 mμ) but at 340 mμ, using an appropriate conversion factor. It was assumed that at this wavelength no irrelevant absorption would occur. Although this assumption has been found to be essentially correct, the method has the objection that the measurement takes place in a wavelength area in which the absorption curve rises very steeply. Slight deviations manifest themselves as big differences in the calculation of the vitamin A content so

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[1]) N. T. GRIDGEMAN, G. P. GIBSON and J. P. SAVAGE, Analyst 73, 662 (1948).

[2]) J. B. WILKIE, J. Assoc. Off. Agric. Chem. 30, 382 (1947); J. B. WILKIE and J. B. DE WITT, J. Assoc. Off. Agric. Chem. 32, 455 (1949).

that this method of determination must be considered as being in principle wrong.

In this laboratory, success has recently been attained in eliminating by a modified chromatography the substances causing the irrelevant absorption from the saponification extract, without loss of vitamin A. The measurement of $E^{1\%}_{1\,cm}$ 325 mμ and the use of the conversion factor 1900 gives the number of international units of vitamin A in the margarine. By this, the determination is adapted to the internationally established requirements of February 1950 [3]). The chromatography is carried out with two adsorbents connected in series. Al_2O_3 functions as the first adsorbent in the same way as it is used for concentrates. With this, substances which percolate more quickly that the vitamin A and which have a rather small absorption at 325 mμ are eliminated. The still contaminated vitamin A fraction which comes out of the column is then passed through a column of alkaline Al_2O_3 (10 % by weight of NaOH), the substances with irrelevant absorption being then completely retained (tocopherols and related compounds, as well as kitol). The form of the absorption curve of the vitamin A fractions from margarine thus obtained is pretty well identical with that of pure vitamin A and no correction is necessary. The procedure is carried out on a semi-micro scale with a total of only 200 I.U. vitamin A (so that only about 10 g margarine need be taken for the determination). The reproducibility of the method is very high (2—4 %) deviation), the recovery is better than 95 %, and the time taken is only about 2½ hours.

Full details of the procedure, reproducibility, and recovery will be published shortly elsewhere.

Unilever Research Laboratory
Zwijndrecht (The Netherlands), October 1950.

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[3]) World Health Org. Techn. Rep. Ser. 1950, 3.

BOTANY

DEVELOPMENTAL PROCESSES OF THE RICE-PLANT IN RELATION TO PHOTOPERIODISM. I [1])

BY

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(*Landbouwhogeschool Wageningen, The Netherlands*)

(Communicated by Prof. S. J. WELLENSIEK at the meeting of Sept. 30, 1950)

1. Introduction. There is an extensive literature dealing with the influence of environmental conditions on the development of the rice plant. The influence of nutrients, temperature and light on the germination, the tillering, the time of earing and on the processes of ripening has been studied. Nevertheless the developmental processes succeeding each other or occurring simultaneously in a normal freely developing rice plant, are not yet fully understood. In most cases the plants have only been observed externally, without the systematic dissection necessary to determine their condition. With a better knowledge of the developmental processes occurring in control series, the results of many experiments on rice could have been interpreted more succesfully. The physiology of wheat and rye is far better known.

This knowledge may be the key to the solution of practical problems: the time of application of fertilizers depends on the requirements during the different phases of growth; abnormal development in case of diseases or physiological disturbances can only be recognized if the normal course of development is known. The mèntèk-disease in Java has been considered for a long time as a "yellowing", until KUILMAN [19] observed a shortening of the leaf-sheaths, a phenomenon which he could reproduce by cultivating plants on modified solutions. Moreover, knowledge of varietal differences in development will make it possible to choose varieties with factors adapted to prevailing conditions as daylength, altitude, water supply, etc. By breeding new varieties with a combination of favourable factors, earliness and a high yield may be promoted.

The scope of this paper is confined to the study of the development of two rice varieties under known conditions and to the effect of different daylengths on the developmental processes.

2. Literature. After the morphology of the rice plant had been studied (VAN BREDA DE HAAN [8]; BALSAC [3]; BHALERAO [5]; JULIANO and ALDAMA [16], compiled by COPELAND [11]), more attention was

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1) The experiments have been carried out at the "Algemeen Proefstation voor de Landbouw", Buitenzorg, Java. They were ended by war circumstances.

directed to the physiological processes. Germination has been studied frequently (VALETON [41]; JONES [15]; CHIEN LIAN PAN [10]), as well as the process of tillering. Exact data on this part of the developmental cycle have been given by RAMIAH and NARASIMHAM [36], KUILMAN [21, 22] and others, cited by the latter. Both authors pay more attention to the individual plant than had been done before. KUILMAN studied the behaviour of plants on nutrient solutions under controlled conditions. In the Untung-variety the process of tillering started about 18 or 19 days after germination, with an interval after the appearance of the first lateral shoots and a second pause from about the 45th to the 65th day. In the Baok-variety an interval in the process of tillering occurred between the 75th and the 85th day. Similar intervals have been observed by SUMMERS [40], ADAIR [2] and RAMIAH and NARASIMHAM [36]. Afterwards tillering increases again, though many buds die off or remain dormant. Attention has also been paid to stem elongation as well as to the development of the inflorescence (NOGUCHI [29]). RAMIAH [34] gave a growth-phase concept, similar to the schemes of PURVIS and GREGORY [32] for rye and of KcKINNEY [25] for wheat. In rice the photophase is far more important than the thermophase, which may even be lacking.

The effect of daylength on the time of flowering has been studied in many rice growing countries (NOGUCHI [30]; FUKE [13]; KONDO [18]; EGUCHI [12]; KUILMAN [20]; BEACHELL [4]; and others). The awned varieties are either indifferent to daylength or ear initiation is delayed to a small extent by short days. In the unawned varieties, endemic in northern regions, short days hasten earing. Japanese varieties introduced into Java flowered at such an early stage, that the yield was reduced. The reverse was observed by BEACHELL [4], who states, that tropical varieties introduced into the Southern States of the U.S.A. "fail to head, or head too late". The long days of summer prevent heading and the low temperature in autumn retards maturity.

Even in the same country the growing period may vary, depending on the date of sowing. The effect of this date on the length of the growing period has been described by MITRA [28]; by JENKINS [14]; by ADAIR [1] and by BEACHELL [4]. Two groups of varieties may be distinguished: 1. those whose growing period is shortened if the date of seeding is delayed. These "timely fixed" (MITRA) or "sensitive" (JENKINS) varieties flower at a particular time, irrespective of the date of sowing. They head when the days are shortening. 2. The second group of varieties flowers at a definite length of time more or less irrespective of their date of sowing. They are "periodically fixed" or "indifferent". Some sensitive varieties, if sown late, give a reduction in growth period as well as in yield.

SCRIPCHINSKY [38] interpreted the experiments of RAMIAH [35], LORD and DE SILVA [24] and HAIGH [17]. He comes to the conclusion that the seasonal variation of daylength, though small in the tropics, may be the cause of the seasonal growth habit.

At Buitenzorg (Java), VAN DER MEULEN [27] controlled the growth period of some awned or bulu-varieties and some unawned or tjereh-varieties by monthly sowings throughout the year. The awned varieties endemic in the tropics, showed only slight differences throughout the seasons. Of the unawned varieties the growth period of Bayang and Skrivimankoti was shortest if grown between May and October, longest if grown between November and April. A difference of 60 days or more occurred (fig. 1). Daylength at Buitenzorg (7° Lat. S.) differs between

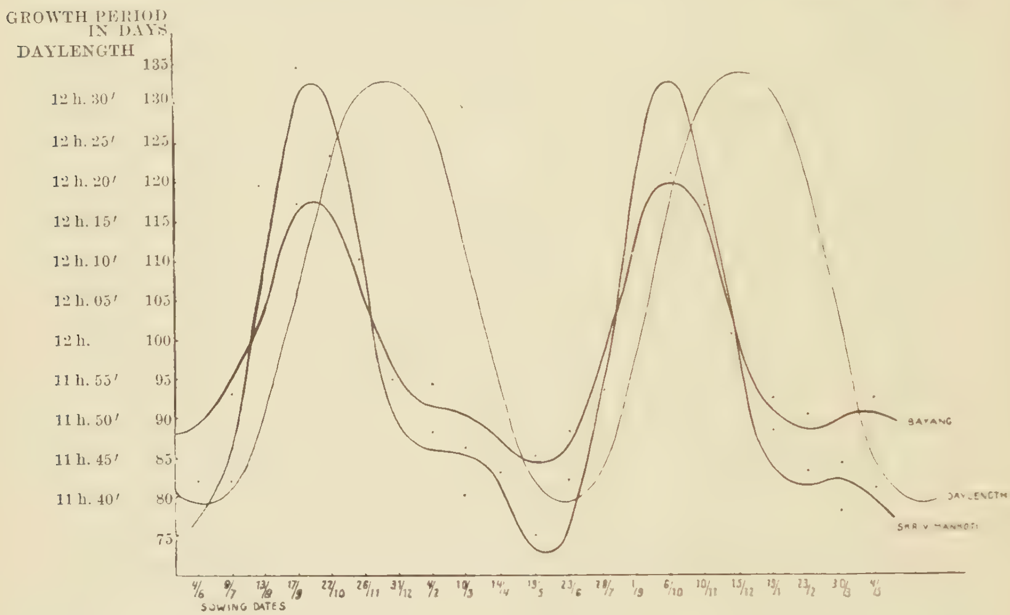

Fig. 1. Daylength at Buitenzorg and lengths of the periods from sowing to ripeness of the varieties "Bayang" and "Skrivimankoti" throughout the year (after VAN DER MEULEN, 27).

12 h + 30′ and 12 h — 18′. That a difference by only 48′ can induce such a difference in growth period is made clear by the same experiment North of the equator: the varieties reacted in the same way, but here the longest growth period occurred between May and October, the shortest between November and April (VAN DER MEULEN, unpublished).

Inducing earliness by short-day treatment has been tried frequently with more or less effect. A review has been given by KUILMAN [23]. One of the most striking reports is that of SIRCAR and PARIJA [38], who obtained an earlier flowering by short-day treatment of a Bengal variety. They could reduce the period before flowering from 133 to 47 days.

The way in which the developmental processes are influenced, in all experiments mentioned, has not yet been elucidated, a fact affirmed by WHITE ([43], p. 328).

3. Material and methods. In examining the developmental processes by the dissecting method used by BLAAUW and his coworkers

(Wageningen), care must be taken that all individuals are of the same genetic composition, of the same sowing date and that environmental conditions are similar. Three series of plants were grown on the nutrient solution used by KUILMAN and modifed by VAN RAALTE [33] [1]):

1. Two series of the variety "Untung", one sown January 16th, the other July 21st. Untung is an unawned or tjereh-variety, a pure line, selected by VAN DER MEULEN from an Indian variety named "Tilak-kacheri". The growth period is dependent on the date of sowing;
2. One series of the variety "Baok", sown in January, an awned or bulu-variety, a pure line with a growth period of 130 days, independent of the date of sowing;
3. Plants of the same varieties and of the same date of sowing from the rice fields were examined as well.

The plants were grown in the garden of the laboratory at Buitenzorg, Java. The temperature of the salt solution and of the air varied between 21° and 32° C; the number of hours of sunshine was determined; the daily light period was obtained from the Meteorological Institute. The grains were soaked in water for 24 hours; for another 24 hours they were kept in closed jars on moist paper in darkness, as germination is more regular under these conditions. Germination was continued on gauze covering the top of cylinders filled with the nutrient solution. In the early stages each seedling was kept on a 1 litre container, in the later stages a 4 litre container was used for every plant. In the seedling stage 10 plants were dissected every day; after the 16th day only 2 or 3 plants could be examined daily. In the later stages dissection of a plant with its tillers took so much time, that only 2 or 3 plants a week could be examined. The total number of dissected plants was about 300. In order to determine the stage of the growing-point and of the primordia a binocular microscope was used. Axillary buds and tips of culms were cleared by chloralhydrate in order to facilitate examination.

4. Nomenclature. As the conceptions concerning the morphologic status of the parts of the Gramineae-embryo differ greatly, the nomenclature of the laterals gave some difficulties. The opinion of BOYD and AVERY [87], who regard the meristematic epiblast as the ligule of the scutellum seems most satisfactory. The scutellum is thus the first leaf, the coleoptile the second, the first green leaf that becomes visible after germination must then be considered as the third leaf. This system is difficult to use in practice, just as all other systems, in which the first green

[1])

distilled water	1 litre	$Fe_2(SO_4)_3$	25 mg
NH_4NO_3	340 mg	ferric citrate	10 mg
KNO_3	170 mg	$MnSO_4 \cdot 4H_2O$	20 mg
$MgSO_4 \cdot 7H_2O$	250 mg	H_3BO_3	0,5 mg
$CaSO_4 \cdot 2H_2O$	100 mg	$Fe_3(PO_4)_2$	400 mg
KCl	300 mg	$Ca_3(PO_4)_2$	400 mg

leaf of the main stem is not indicated by number one. For convenience therefore I labelled the first green leaf number one, its axillary bud was called 1/1, the lateral in the second leaf 1/2 etc. Every lateral has a closed prophyll at the base, which can be compared with the coleoptile. A secondary lateral originating in the axil of a prophyll is indicated by 0. For example: after tillering a bud 1/4/0 may be found, i.e. the bud in the prophyll of the tiller in the axil of the 4th leaf of the main stem.

Identification of the leaves and laterals of young plants is not difficult, if the coleoptile or the bladeless first leaf is still present. During development, these leaves die off. After some experience it was possible to recognize groups of tillers belonging to a lateral of the first order, which could be torn away from the main stem, leaving only a small scar. The main stem deprived of all laterals of the first order with their tillers shows the scutellum at the pointed, somewhat curved base, even when maturity is reached.

5. Results.

5.1. *The fruit.* Much attention has been paid to the different parts of the grain except to the embryo. The short stem not only bears the

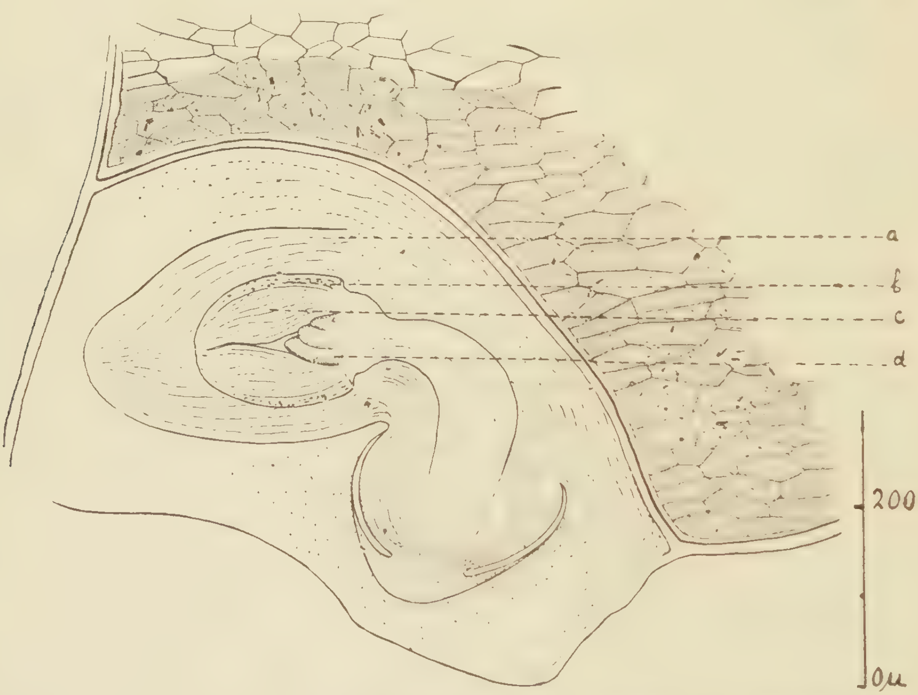

Fig. 2. Embryo of the Untung-variety after 24 h. soaking. *a*: coleoptile; *b–d*: 1st, 2nd and 3rd leaf.

scutellum and the epiblast, but the closed coleoptile covers three leaf primordia, a fact observed by CHI TUNG YUNG [9]. The second primordium is so tightly enclosed by the first that the two can hardly be distinguished

(fig. 2). The third primordium is of the same height as the growing point: 0,080 mm. The three leaves can be detected a fortnight after flowering, at which time the grain is still immature.

5.2. *Germination.* After one day of exposure on wet filter paper, the swollen margins of the scutellum and the epiblast penetrate through the wall of the third glume, forming a cuplike cavity in which the tip of the plumule can be seen (fig. 3). The function of the hair-like epidermis cells

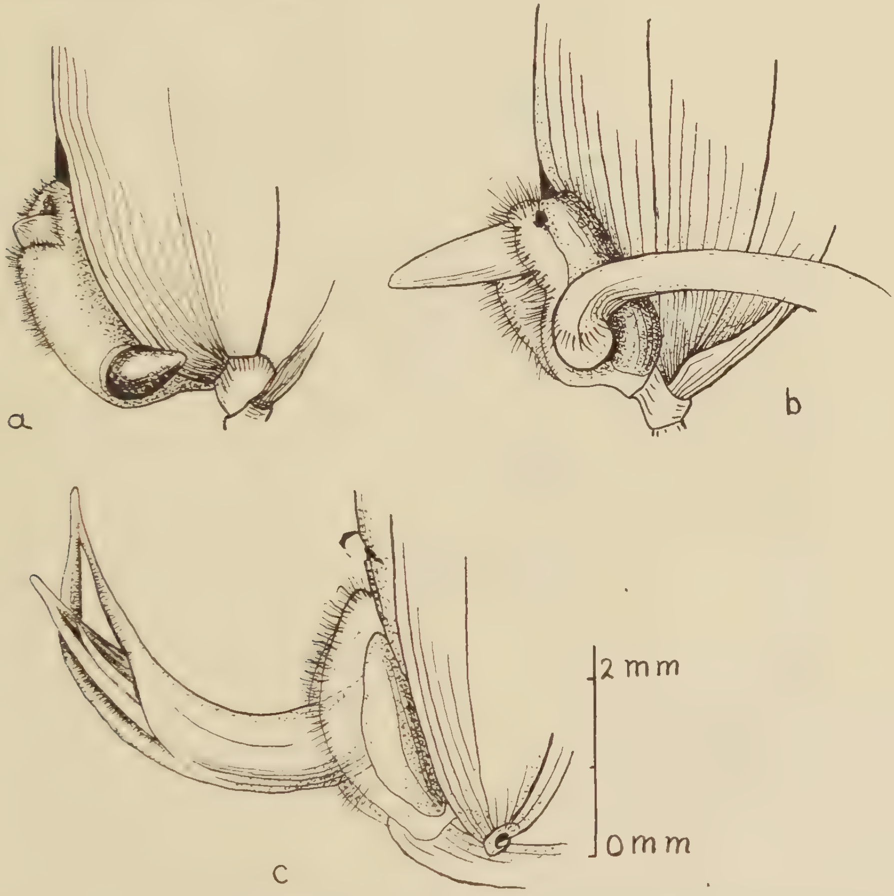

Fig. 3. *a–c:* development of the root and the plumule after 24 h. soaking and respectively 1, 2 and 3 days germinating. In *c* the coleoptile is split open by the 1st and 2nd leaf.

of the margins may be to retain water and to prevent dessication of the germinating grain, (JULIANO and ALDAMA [16]). On the other hand the hairs, retaining air if submerged, may protect the embryo against oxygen deficiency, which causes abnormal germination (VALETON [41]; JONES [15]).

5.3. *The development of the leaves in the first vegetative period.* At the second day after germination the coleoptile, the first and the second

primordia elongate rapidly, the third primordium only to a small extent (fig. 4). A 4th primordium, which forms a wall with a higher and a lower side is developed at the growing point. The highest part, opposite the midrib of the preceding leaf elongates till it reaches the height of the growing point. By lateral growth the margins overlap each other, enclosing the growing point. CHI TUNG YUNG [9], who probably employed longitudinal sections, described this process, but overlooked the lateral growth.

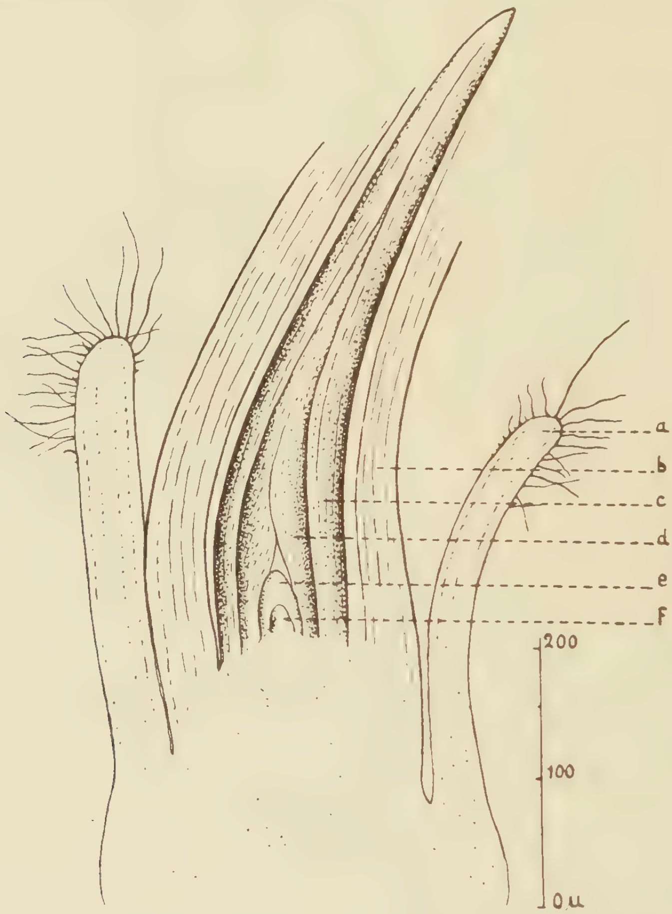

Fig. 4. Longitudinal section through the plumule of a 2 days-old plant of the Untung-variety. *a:* epiblast; *b:* coleoptile; *c–e:* 1st, 2nd and 3rd leaf; *f:* 4th primordium, covering the growing point.

At the 3rd day the coleoptile is split open by the horny point of the first leaf. This leaf and the second one elongate simultaneously. The sheaths of all leaves reach maturity after the blades, all parts arising from meristematic cells at the base of the primordium.

At the 6th day the coleoptile and the first leaf have reached maturity, the second leaf has nearly stopped elongating, the 3rd leaf just enters the grand period of growth, the 4th leaf has hardly reached 1 mm, and a 5th

primordium appears (fig. 5). Every growing point in the vegetative period is always surrounded by the following sequence of leaves:

one or more mature leaves; one which has nearly reached maturity; one in the grand period of growth; one primordium from 0,080 to 0,200 mm in length; one primordium of 0,080 mm or less, just developed.

Fig. 5. Longitudinal section of a 6 days old seedling of the Untung-variety. The growing point is surrounded by the 5th leaf, axillary buds are developed. Adventitious roots piercing through the coleoptile.

The blade and the sheath of a preceding leaf has to reach maturity before the next leaf primordium can enter the grand growing period. At that moment the growing point is surrounded by two younger primordia not yet elongating. The blades of the highest leaves, however, start elongating at the time the sheath of the precedent leaf still shows immature cells at the base.

Every 3 or 5 days another leaf has reached maturity and a new primordium is initiated (fig. 6). This applies in the Untung- as well as in the Baok-variety for the earlier stage of development. The leaf production rate seems to be rather independent of variety and of environmental

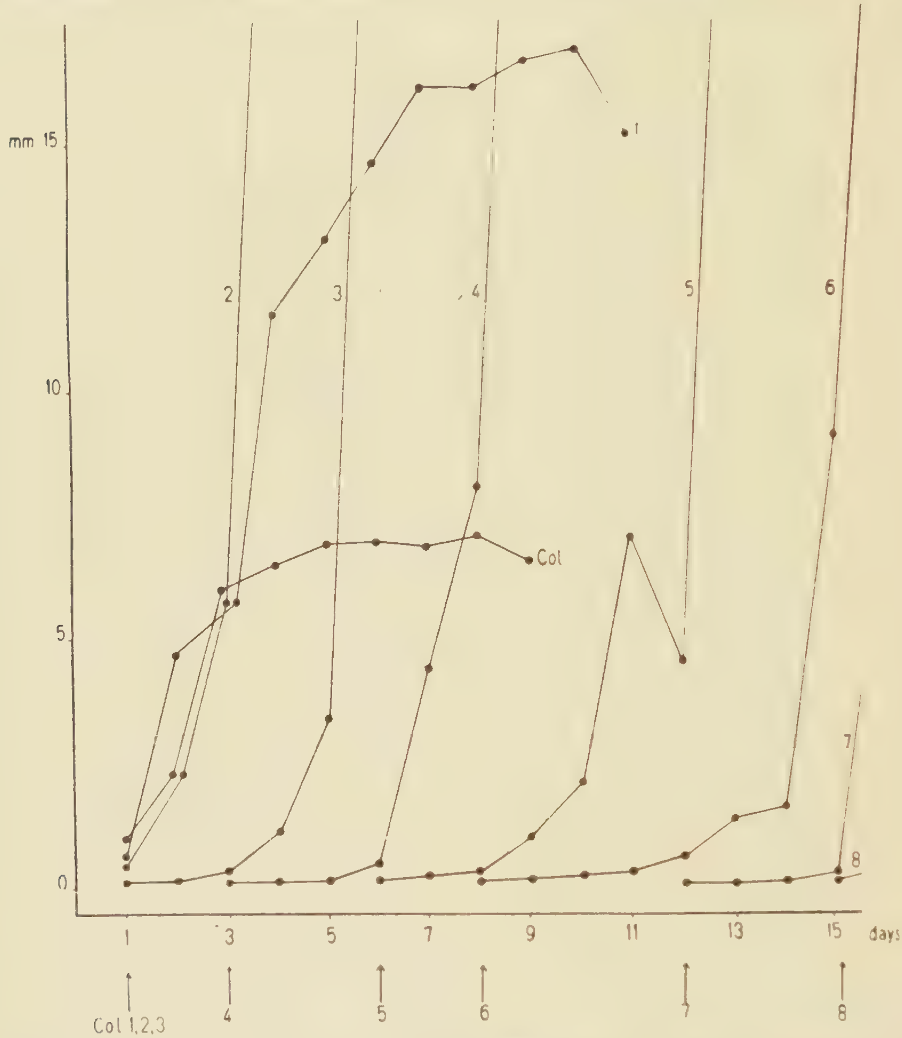

Fig. 6. Parts of the growing curves of the first 7 leaves. ↑ time of initiation of a leaf with its number; col: coleoptile. The coleoptile and the leaves 1, 2 and 3 are present in the embryo.

conditions up to the 4th or 6th week. The same phenomenon has been observed by PURVIS and GREGORY in rye [32].

In the later stages the successive leaves reach greater lengths; a younger one has to wait for 7, 8 or even more days before the preceding one has reached maturity and elongation can start. Elongation in the Baok-variety proceeds much more slowly than in the Untung-variety. In the latter the rate of elongation depends on daylength.

Though the lengths of the blades vary, the sheaths are of increasing lengths, that of the highest or flag-leaf is longest of all. It is leathery, with a very smooth interior surface, which facilitates the emergence of the mature panicle. Transplanting and other disturbances influence the lengths of blades and sheaths of plants grown in the field.

5.4. *Bud development and tillering in the first vegetative period.* The future axillary bud originates as a group of meristematic cells between

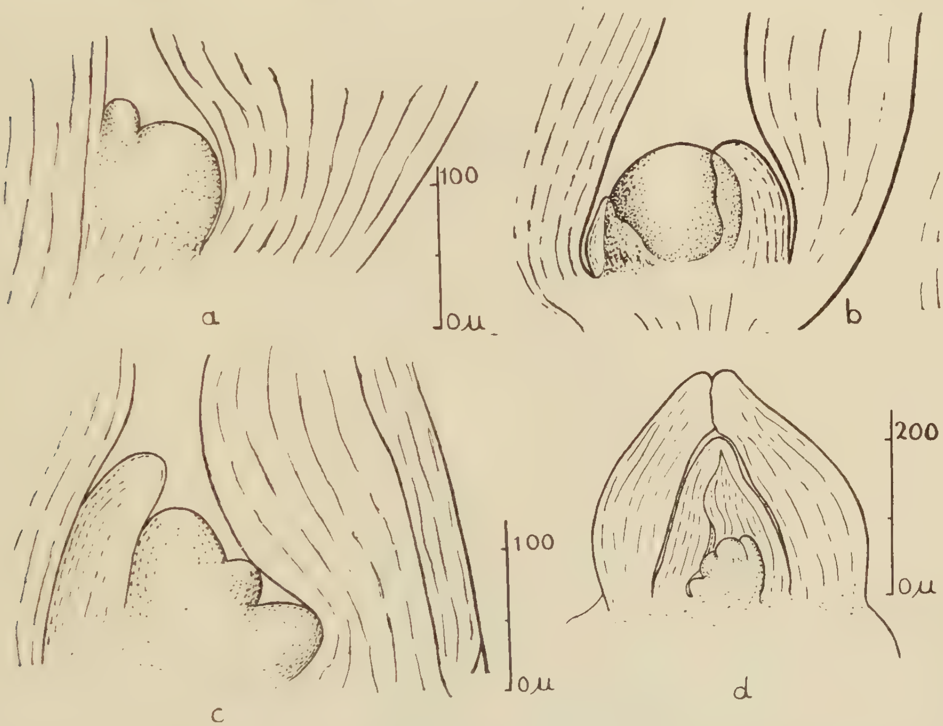

Fig. 7. Axillary buds. *a:* with primordium of the prophyll; *b:* with the prophyll developed; *c:* with prophyll and primordium of the 1st leaf; *d:* with prophyll and 2 leaf primordia.

the growing point and a newly formed leaf primordium; a seedling 6 days old showed 4 axillary buds (fig. 5). Each growing point is soon surrounded by the closed prophyll and underlying leaf initials (fig. 7 and 8). KUILMAN [21] as well as RAMIAH and NARISIMHAM [36] consider the moment an elongating bud appears above the ligule of the subtending leaf as the beginning of the process of tillering. In fact the moment at which bud elongation starts cannot be detected without dissecting the plant and even then it is difficult to say if a bud has entered the grand period of growth or not. Here it must be remarked that during the vegetative period the number of buds smaller than 4,5 mm increases rapidly. In all plants, however, only a few buds between 4,5 mm and about 60 mm could be detected and also the number of shoots developed out of the elongated

buds is relatively small. Probably the buds, initiated at an increasing rate, grow slowly until they reach a length of about 4,5 mm, when they enter

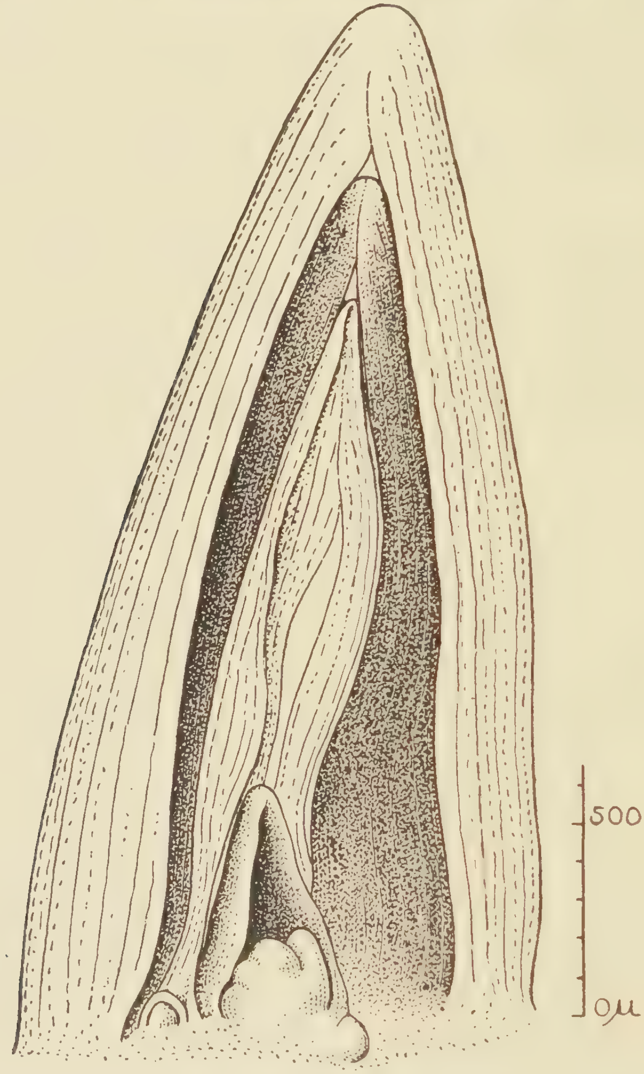

Fig. 8. Axillary bud 1/6 of a 27 days old plant of the Untung-variety, with 4 leaf primordia under the prophyll and the 5th leaf just visible.

the grand period of growth. They quickly pass the stages between 4,5 mm and 60 mm, which explains the small number of these elongating buds in each plant. After elongation the prophyll is split open by the first leaf and the bud can be considered as a shoot. The number of shoots increases only slowly. Fig. 9 shows the relation between the number of buds before the grand period of growth, the number of buds in this period and the number of shoots found in plants of the Baok-series from the 3rd till the 46th day.

In buds with a length of 3,5 mm to 4 mm 3 leaf primordia are found

under the prophyll (table 1, 44 days). At the time a 4th primordium is split off, the prophyll starts elongating, immediately followed by the 1st

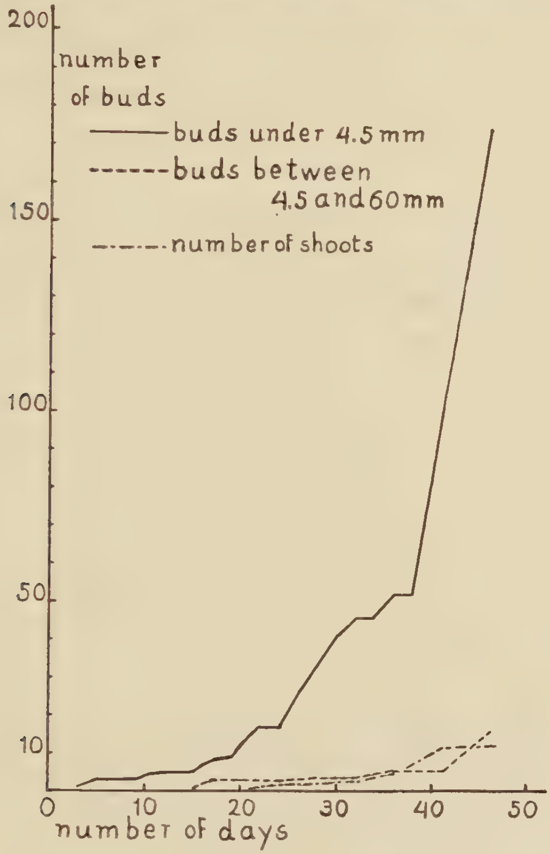

Fig. 9. Number of buds under 4,5 mm, number of elongating buds and number of shoots found during the first part of the vegetative period in the Baok-series.

leaf. The younger primordia are left behind, surrounding the growing point (table 1, 49 days). The development of the primordia of the buds is similar to those of the main stem.

TABLE I

Lengths in mm of the primordia of buds in the vegetative and in the generative period

primordia present	length in mm of primordia of buds from plants of:			
	44 days	49 days	76 days	84 days
prophyll	3.600	4.000	3.600	4.000
1st leaf	3.270	2.700	3.200	2.000
2nd leaf	0.550	1.275	1.000	1.450
3rd leaf.	0.160	0.350	0.450	0.900
4th leaf.		split off	0.180	0.540
5th leaf.			split off	0.270
6th leaf.				split off

In the Untung-variety as a rule the axillary bud of the first leaf does not develop or it elongates slowly to an extent of only 1 or 2 mm. Its position just underneath the pointed stem base seems to be unfavourable. The bud 1/2 only develops occasionally (KUILMAN [21]), but by dissecting the plants it became evident that the rate of growth is low. If it becomes visible outwardly, it follows the younger bud 1/3, which has a somewhat higher rate of growth and appears about the 18th day. At approximately the same time the bud 1/4 emerges from the sheath of the fourth leaf, having a still higher rate of growth. Within a few days 1/2, 1/3 and 1/4 may all emerge. All other buds expand at the same rate as 1/4 and appear from now on at regular intervals of 3 or 5 days, the same interval as that between the maturity of two successive leaves. Actually there is no pause

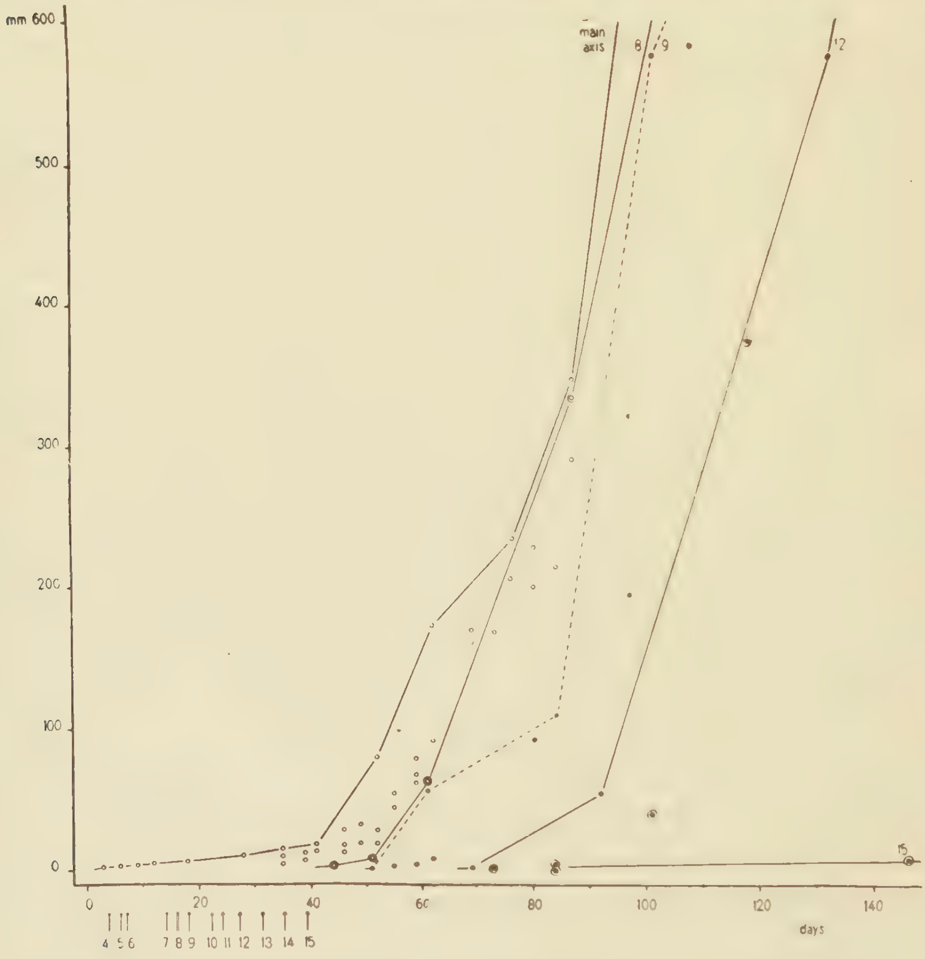

Fig. 10. Elongation of the main stem and its laterals of the July-series of the Untung-variety. Abcissa: number of days after germination. Ordinate: height in mm. ↑ time of initiation of the laterals with their number. -o- main stem; laterals: ◯ 1/4–1/7; circle with circle = 1/8; --●-- 1/9; ● 1/10 and 1/11; –●– 1/12; circle with point = 1/13 and 1/14; line, circle with point, line = 1/15.

in the process of tillering at this early stage as has been asserted by KUILMAN [21], but only an increased rate of elongation (fig. 10). In the Baok-variety the buds 1/2, 1/3, 1/4 and the higher ones develop at equal rates (fig. 11), though on the whole more slowly than in the Untung-variety (KUILMAN [22]).

Fig. 11. Elongation of the main stem and its laterals of the Baok-series. Abcissa: number of days after germination. Ordinate: height in mm. ↑ time of initiation of the laterals with their number. –o– main stem; laterals: ○ 1/4—1/8; circle with circle = 1/9; --●-- 1/10; ● 1/11 and 1/12; –●– 1/13; circle with point = 1/14 and 1/15; line, circle with point, line = 1/16; ♂ laterals younger than 1 16.

Which of the many buds initiated will reach the 4-leaved stage necessary for elongation cannot be predicted. Generally those of a lower order are earlier in shooting than those of a higher order; those in the middle part of the stem have a better chance than those at the base or at the top. Environmental conditions are of great influence: the amount of nutrients, lodging, damage by insects may define the rate and the manner of tillering, unlike the rate of leaf initiation, which bears a constant character. In the vegetative period the ratio of the number of leaves and leaf initials to the number of shoots and axillary buds present at a particular moment may be an index of the degree of tillering. Plants of the same age grown in different soils showed a different index.

(To be continued in next number of "Proceedings")

BOTANY

TRANSGRESSIEVE VARIABILITEIT EN TRANSGRESSIE-SPLITSING. III

DOOR

G. P. FRETS

(Communicated by Prof. J. BOEKE at the meeting of June 24, 1950)

RESULTATEN.

We vinden transgressieve variabiliteit bij de splitsingsgeneraties na kruising van 2 zuivere lijnen, de I-lijn en de II-lijn, van Phaseolus vulgaris. Bij de lengte en de dikte, die voor de bonen van de I- en de II-lijn zeer verschillend groot zijn, — dit geldt vooral voor de lengte —, komt de transgressieve variabiliteit minder vaak voor dan bij de breedte. De transgressieve variabiliteit van het gewicht is zeer duidelijk. Ze hangt er mee samen, dat er nieuwe variëteiten gevormd worden met de grote lengte en de grote breedte van de bonen van de I-lijn en de grote dikte van de bonen van de II-lijn. Van deze bonen zijn alle 3 afmetingen groot: de formule is L B Th.

Om de transgressieve variabiliteit, die we in ons materiaal gevonden hebben, te kunnen verklaren, is een kleine, voor de hand liggende, verandering nodig in onze formules. De formule voor de bonen van de I-lijn wordt:

$$L_1L_2l_3L_4L_5L_6L_7\ B_1B_2b_3B_4b_5b_6b_7\ th_1th_2Th_3th_4th_5th_6th_7$$

en die van de bonen van de II-lijn:

$$l_1l_2L_3l_4l_5l_6l_7\ b_1b_2B_3b_4b_5b_6b_7\ Th_1Th_2th_3Th_4th_5th_6th_7$$

Als formule voor de bonen van de F_1-zaadgeneratie volgt daaruit:

$L_1l_1L_2l_2L_3l_3L_4l_4L_5l_5L_6l_6L_7l_7B_1b_1B_2b_2B_3b_3B_4b_4b_5b_5b_6b_6b_7b_7Th_1th_1Th_2th_2Th_3$-$th_3Th_4th_4th_5th_5th_6th_6th_7th_7$. En in de F_2 — of volgende splitsingsgeneraties — verschijnen bonen met de formules:

$$L_1L_2L_3L_4L_5L_6L_7B_1B_2B_3B_4b_5b_6b_7Th_1Th_2Th_3Th_4th_5th_6th_7$$

en

$$l_1l_2l_3l_4l_5l_6l_7b_1b_2b_3b_4b_5b_6b_7th_1th_2th_3th_4th_5th_6th_7.$$

d.z. bonen met transgressieve variabiliteit, transgressieve uitsplitsingen.

Bij de hypothese in haar vroegere vorm berust het verschil in de lengte-afmetingen op de werking van 6 lengtefactoren, dat voor de breedte en de dikte op resp. 3 breedte- en 3 diktefactoren. Bij de hypothese, zoals we ze thans veranderen, laten we het aantal factoren, dat aan de verschillen

ten grondslag ligt, onveranderd, doch het is voor de lengte over 7 en voor de breedte en de dikte over 4 factoren verspreid. Deze verandering brengt geen belangrijke consequenties mee voor het theoretische verloop van de kruisingen. Bij de vroegere hypothese onderscheiden we een eerste groep factoren L_1—L_3, B_1—B_3 voor de bonen van de I-lijn en een eerste groep factoren Th_1—Th_3 voor de bonen van de II-lijn en een eerste groep factoren l_1—l_3 en b_1—b_3 voor de bonen van de II-lijn en een eerste groep th_1—th_3 voor de bonen van de I-lijn. Bovendien onderscheidden we een 2de groep factoren L_4—L_6, b_4—b_6 en th_4—th_6 voor de bonen van de I-lijn en een 2de groep l_4—l_6, b_4—b_6 en th_4—th_6 voor de bonen van de II-lijn. Er waren dus 2 groepen van factoren voor ieder van de 3 afmetingen. Deze 2 groepen zijn er ook thans. De eerste groep bestaat thans uit 4 factoren, ze verschilt in 3 factoren. De F_1-kruising is voor de breedte en de dikte voor 4 factoren heterozygoot, voor de lengte voor 7 factoren. We kunnen ook thans met het oog op de 2 groepen van factoren voor ieder der afmetingen, de kruising als een tetrahybride onderscheiden. De vereenvoudigde schrijfwijze voor de formules blijft, die voor de bonen van de I-lijn L_1 L_2 B th, die voor de bonen van de II-lijn l_1l_2 b Th.

We nemen voor onze hypothese 7 paar genen aan in 2 groepen, ieder van 3 factoren, om de grootte-verschillen der afmetingen te verklaren. In de eerste groep zijn de te duiden verschillen over 4 factoren verspreid. Met het oog op de feitelijke gegevens over de afmetingen hadden we kunnen trachten, ons schema iets meer te differentiëren. We deden het niet, omdat daarvoor de gegevens niet voldoende zijn. Het principe wordt door onze formule, menen we, goed uitgedrukt.

Volgens de hypothese in haar vroegere vorm is de formule van de F_1-bonen L_1l_1—$L_6l_6B_1b_1$—$B_3b_3b_4b_4$—$b_6b_6Th_1th_1$—$Th_3th_3th_4th_4$—th_6th_6. Bij volkomen dominantie der grote afmetingen over de kleine zouden dus de F_1-bonen voor de lengte en de breedte de grootte hebben van bonen van de I-lijn, voor de dikte die van de II-lijn. Volgens de hypothese in de vorm van thans, is de formule van de F_1-bonen: L_1l_1—$L_7l_7B_1b_1$—$B_4b_4b_5b_5$—$b_7b_7Th_1th_1$—$Th_4th_4th_5th_5$—th_7th_7. Bij volkomen dominantie van de grote afmetingen over de kleine, zouden hier van de F_1-bonen de lengte en de breedte groter zijn dan de lengte en de breedte van de bonen van de I-lijn, de dikte zou groter zijn dan de dikte van de bonen van de II-lijn. Er zou transgressieve variabiliteit zijn van alle 3 afmetingen. De afmetingen vertonen slechts een zeer onvolledige dominatie van de grote over de kleine. Bovendien is er de invloed van de moederlijke zaadhuid.

Dat we ter verklaring van onze resultaten over transgressieve variabiliteit onze hypothese veranderen en in onze formules een enkele factor meer opnemen, biedt theoretisch geen moeilijkheid. We stellen erfelijkheidsformules op, aan de hand van bij kruising blijkende verschillen. Volgens het principe van allelomorphe paren erffactoren (BATESON) en de chromosomen-theorie van MORGAN, hebben bij vormen, die zich kunnen kruisen, de 2de oudervorm plaatsen beschikbaar voor genen (A) van eigenschappen

van de eerste oudervorm, die de 2de niet bezit (a) en omgekeerd. Waar we, om de transgressieve variabiliteit te duiden voor de bonen van de I-lijn een 7de L-factor moeten aannemen, is daarbij mede gesteld, dat de bonen van de II-lijn een plaats (l_7) voor de 7de lengtefactor beschikbaar hebben.

Er is ook geen bezwaar tegen, om een lengte-factor meer aan te nemen met het oog op het mogelijke aantal genen. JOHANNSEN (1926, p. 435 en 1913, p. 387) drukt de "Gesamt-Genotypen" uit door de formule (A + B + C + D + E) + X. NILSSON–EHLE neemt deze voorstelling van JOHANNSEN over. Hij zegt: (1909, p. 12): "Dass die Organismen Aggregate von selbständigen Einheiten sind, ist eine Auffassung, die sich durch die Forschungen MENDEL's und seiner Nachfolger immer mehr befestigt hat". MORGAN (1916, p. 143) berekent, dat omstreeks 7500 erffactoren het kiemplasma van Drosophila samenstellen. Van de het verst geanalyseerde plant, Antirrhinum majus, zegt BAUR (1930, p. 105), dat de vele duizenden duidelijk verschillende rassen steeds weer andere combinaties van een betrekkelijk klein aantal erffactoren voorstellen. HEYMANS (Spec. psychol., hfdst. erfelijkh.) vindt in zijn uitkomsten een bevestiging van de "additieve" opvatting ook der psychische werkingen (I 233; ook II 241, noot). SHERRINGTON zegt (1908, p. 2) ... "it is nervous reactions which par excellence integrates it (d.w.z. the multicellular animal) and constitutes it from a mere collection of organs, an animal individual" (ook p. 7). Aan deze zienswijze sluit wel die van JOHANNSEN aan, uitgedrukt in (1913, p. 382): "Morphologisch-deskriptiv gesehen, zeigt sich also der realisierte Organismus als ein Aggregat von Organen und Geweben; physiologisch gesehen tritt die Koordination dieser Teile schärfer hervor, jedoch mit Beibehaltung der Gliederung des Ganzen in Organe u.a. mehr weniger selbständige Teile."

Wat is die eenheid van het organisme, dat ene in het organisme, waaraan NILSSON–EHLE en HEYMANS voorbijgaan? JOHANNSEN (1926, p. 643) schrijft: "Denken wir uns die meisten der etwa vier hundert verschiedenen Gene, die erbliche "Abnormitäten" bedingen, im Genotypus einer gegebenen Fliege vereint, würde das betreffende Tier, als überhaupt lebensfähig erdacht, doch stets Drosophila megalonaster sein. Eine solche Fliege hat eben das "Zentrale" im Bananenfliegengenotypus" "... Demnach sollte durch die zahlreichen Kreuzungen eine teilweise Analyse des normalen Genotypus erreicht sein, indem also doch das zentrale Wesen des betreffenden Fliegengenotypus keineswegs analystisch aufgelöst wäre." Het "zentrale Wesen"! Met het centrale wezen kan JOHANNSEN de wezenskenmerken bedoeld hebben. Dan wordt soort = wezenskenmerken + de vele andere eigenschappen. SPINOZA defineert (Eth. 2de dl, def. 2, vert. W. MEYER): "Tot het *wezen* van een zaak behoort naar mijne meening datgene, waarmee de zaak staat of valt of wel datgeen, zonder hetwelk de zaak en omgekeerd, datgene wat zonder de zaak zelf niet kan bestaan of begrepen worden". Evenzo (vert. H. GORTER): "Tot het wezen van enig ding zeg ik dat dat behoort door hetwelk als het gegeven is, het ding noodzakelijk gesteld wordt en door hetwelk als het opgeheven is, het ding noodzakelijk wordt opgeheven; of dat, zonder hetwelk het ding, en omgekeerd wat zonder het ding, noch zijn, noch begrepen kan worden".

De vraag naar wat van de soort overblijft als alle eigenschappen worden weggedacht, als de analyse volledig is, als, zoals MORGAN voor Drosophila berekent, het gehele aantal genen van het kiemplasma bekend zal zijn, is een abstracte vraag. Hier heeft het wijsgerig onderzoek van het bewustzijn door SPINOZA, KANT, HEGEL en BOLLAND verricht, verheldering gebracht. Het is het gebied ook van de phaenomenologie van de geest van HUSSERL en HEIDEGGER.

Als we ons denken richten op ons bewustzijn, vinden we ons denken, maar steeds als het gedachte en in het gedachte ben ik me bewust: het ik gaat met het gedachte

mee. Het gedachte is steeds doordrongen van het denken. Het bewustzijn is één en onderscheidt het continue denken en de verscheidenheid van het gedachte. Dit principe van ons bewustzijn, het ongescheiden onderscheiden zijn van denken en het gedachte, van subject en object, passen we op alle ervaring toe. Alle waarneembaarheid is mede bepaald door denkbaarheid. Er is waarneembaarheid, omdat ze denkbaarheid is. De natuur is ook denkbaarheid. Er is aangepast zijn van denkbaarheid en waarneembaarheid. Zo ook doet in onze voorstelling van een soort en haar eigenschappen het principe van ons bewustzijn, d.i. het één zijn van het denken en het gedachte mee. De vraag wat er overblijft, als we van een soort alle eigenschappen wegdenken, is een vraag, die staat buiten het besef van de aard van ons bewustzijn. We kunnen de soort niet van zijn eigenschappen scheiden. We behouden dan een lege abstractie. Zo leert BOLLAND, meen ik, in zijn Collegium logicum (1904). In het licht van deze beschouwing, moeten we ook "the integrative Action of the Nervous System" van SHERRINGTON plaatsen.

Het aantal polymere factoren, dat wordt aangenomen, is verschillend groot. Reeds MENDEL wees er tentatief op, dat de graadverschillen van lila in zijn proeven met Phaseolus door de werking van enige eenheden verklaard konden worden. NILSSON–EHLE (1908) wil, theoretisch, 4 eenheden aanemen, om de scala van kleurverschillen bij zijn proeven over de rode en witte kleur van de tarwekorrel te verklaren. LANG (1911a en b) zet ten opzichte van de proeven over de erfelijkheid van de oorlengte door CASTLE uiteen, dat, als hier de werking van 3, 6 of 12 genen aangenomen wordt, —12 genen voor de huidkleur van negers (1911b) —, het resultaat de gevonden intermediaire erfelijkheid is. Hij geeft er de naam "polymerie" aan (1911b). TINE TAMMES (1911, p. 229 e.v.) meent n.a.v. haar kruisingsonderzoekingen over de lengte van het zaad van vlassoorten (p. 203), dat 4 of 5 polymere factorenparen haar resultaten verklaren kunnen. Zij vond, naast de ontwikkeling van een eigenschap bevorderende, ook de ontwikkeling remmende factoren.

Belangrijk zijn ook de resultaten van Amerikaanse onderzoekers en hun opvattingen. EAST (EMERSON en EAST, 1912, p. 11) heeft in 1910 een overeenkomstige theorie geformuleerd als NILSSON–EHLE en onafhankelijk van hem als resultaat van zijn maïsonderzoekingen over de gele kleur in het endosperm.

EMERSON en EAST menen, dat het verschil van de oudervormen van hun maïsrassen, in verband met de resultaten van hun kruisingsproeven, door de werking van ongeveer 15 erffactoren verklaard zou kunnen worden (1912, p. 95). Bij maïs (EAST en HAYES, 1911) zijn in het geheel meer dan 300 genen vastgesteld. PUNNETT (gecit. EMERSON en EAST) spreekt voor de verklaring van zijn resultaten over de studie van de erfelijkheid van het gewicht van kippen van 50 of meer factoren en voor andere kruisingen van 2 of 3 maal zoveel factoriële verschillen.

SHULL (1914) vindt enkele gevallen van duplicate genen, dus van verdubbeling van het gen voor eenzelfde eigenschap (duplicate genes and plurality of genes). Hij bespreekt uitvoerig de polymerietheorie en aanvaardt ze. Naar zijn mening geeft de samenwerking van de groei bevorderende en de groei remmende factoren ons een goede voorstelling van de verhoudingen in alle organismen.

Het experimentele bewijs van de polymerietheorie is zwak. Dit is ook de mening van EMERSON en EAST (1912). Er zijn maar weinig experimentele feiten. Experimenten, waar de verhoudingsgetallen van F_2-uitsplitsingen in F_3, 1 : 3, 1 : 15, 1 : 63, op de werking in deze uitsplitsingen van resp. één, 2 en 3 paar erffactoren berusten en waar uit het voorkomen van de verschillende F_3's te besluiten is tot een elkaars werking cumulerend resultaat, zijn er weinig onder de experimenten van NILSSON–EHLE. En zijn uitspraken erover zijn meestal onzeker. Het zekerst laat hij zich uit over de kleur van de tarwekorrel met de getalverhouding in F_3 van 1 : 63. Ook over de kaflengte van haver en over de internodiën-lengte van tarwe, waar een remmende factor en 2 verlengende factoren konden worden vastgesteld.

Alle andere experimenten van NILSSON–EHLE over erfelijkheid van eigenschappen met kwantitatieve verschillen, die alle belangrijk zijn, dragen bij, om grondslag te vormen voor de polymerietheorie, in zoverre hier gevonden wordt, dat in een rij continue variaties, onderling verschillende variaties aanwezig zijn, die erfelijk zijn. Zo vindt hij erfelijkheid van kwantitatieve verschillen (afmetingen), van verschillen van graad (kleur) en vooral ook bij physiologische en pathologische eigenschappen (winterhardheid, roest, rijpheid). De erfelijkheid stelt N.E. vast door de F_3's te kweken en de gemiddelden te berekenen.

Op grond van zijn kruisingsonderzoekingen bij maïs en tabak, waar hij intermediaire erfelijkheid van F_1 vindt, noemt EAST (1916) de volgende kenmerken van de erfelijkheid ven eigenschappen met kwantitatieve verschillen: 1. F_1 is uniform, 2. F_2's van verschillende F_1-zaden zijn dezelfde, 3. de variabiliteit van F_2 is veel groter dan van F_1, 4. als beschikt wordt over een grote F_2-opbrengst, dan moeten de grootouderlijke uitgangsvormen erin voorkomen, 5. overschrijding zal daarbij ook voorkomen, 6. de F_3's gekweekt uit F_2 zijn zeer verschillend wat betreft gemiddelden en krommen, 7. de F_3's hebben een onderling verschillende variabiliteit, die ligt tussen die van F_2 en van de oudervormen en 8. Generaties na F_2 hebben nooit een grotere variabiliteit dan die van de populatie, waaruit zij voortkwamen.

Het lijkt mogelijk, dat onder erfelijkheid door polymere factoren, verschillende ervaringen worden samengevat, dat nl. de vele ervaringen over intermediaire erfelijkheid, waarbij F_2 en volgende generaties zich voordien als een continue rij van variaties, waarin verschillende erfelijke variaties kunnen worden aangetoond (ook bij de mens) te scheiden zijn van de ervaringen over mono-, di- en trihybride erfelijkheid door zelfstandige, identieke of niet-indentieke genen voor eenzelfde eigenschap.

Van belang voor deze overweging is misschien ook de vraag, in hoeverre polymere factoren verband houden met polyploidie (DE HAAN, 1947). Van onderzoekingen over polyploidie noem ik die van BLAKESLEE, BELLING en FERNHAM (1923). Zij definiëren tetraploidie als verdubbeling van het diploide aantal chromosomen in de cellen, die plant of dier samen-

stellen en menen, dat verdubbeling van chromosomen van invloed kan zijn geweest voor de oorsprong van de verschillende vormen in de natuur De auteurs schrijven over hun resultaten bij tetraploide Datura's (p. 368): "Hier kan splitsing, de getalverhouding 15 : 1 geven. We zouden hier dan waarschijnlijk spreken van "duplicate genes" (blz. 20).

In zijn verschillende publicaties (1915—1932) spreekt MORGAN zich uit voor de polymeriehypothese en haar grote betekenis als de mendelistische verklaring van een grote groep erfelijkheidsverschijnselen, bespreekt ook zijn eigen experimenten met de notch-eigenschap van de vleugelrand van Drosophila. In zijn publicatie van 1926 behandelt hij de tetraploidie e.a. en polyploidie. Bij dieren zijn slechts 3 gevallen van tetraploidie bekend. MORGAN spreekt niet over polyploidie in verband met polymere factoren. Ook bij HONING en bij PRAKKEN (1950) vind ik het niet behandeld.

H. DE HAAN (1947) acht polyploidie een zeer veelvuldig optredende oorzaak, die de grondslag is voor het ontstaan van polymerie. Hij noemt de onderzoekingen van MÜNTZING (1937) en van LAMPRECHT (1947). MÜNTZING wijst er op, dat de meest typische gevallen van polymere factoren gevonden zijn in polyploide soorten (tarwe en haver, NILSSON–EHLE). LAMPRECHT (1947), die bij Pisum erfelijkheid door polymere factoren vindt, acht het mogelijk, dat P. sativum ontstaan is uit een autotetraploide vorm na recombinatie en verlies van chromosomen.

Misschien is er verband van polyploidie en de gevallen van "duplicate" en "triplicate genes" (SHULL, EAST, NILSSON–EHLE), doch behoort de grote groep van gevallen van intermediaire erfelijkheid door polymere factoren, waartoe we besluiten op grond van het voorkomen van erfelijke variaties in een materiaal met continue variabiliteit (zoals we voor de erfelijkheid van vele eigenschappen met kwantitatieve verschillen bij de mens aannemen) niet hiertoe.

Transgressieve variabiliteit laat zich goed als resultaat van de werking van polymere factoren opvatten. NILSSON–EHLE wijst op grond van de ervaringen bij zijn vele experimenten er op, dat naast polymere factoren, ook modificatiefactoren en nevenwerkingen van andere factoren kleine erfelijke variaties bij kruisingen geven, de laatste zelfs vaker. Voor het verklaren van de transgressieve variabiliteit is de werking van polymere factoren wel vooral aangewezen; ze biedt zich er als het ware voor aan. Zo is het ook in ons materiaal. We vinden transgressieve variabiliteit van plus- en van minusvariaties, — en dit is van ons materiaal ook belangrijk, — we vinden ze veelvuldiger voor de breedte, die slechts weinig verschilt bij de beide uitgangsvormen, dan voor de lengte, met een groot verschil.

JOHANNSEN (1926, p. 486) deelt het resultaat mee van een kruising van bonen, waarbij hij de erfelijkheid van de lengte en de breedte onderzocht. De lengten van de oudervormen hebben een groot, die van de breedten een klein verschil. Hij vindt transgressieve variabiliteit vooral van de breedte.

EAST, die over erfelijkheid van eigenschappen met kwantitatieve ver-

schillen door zijn onderzoekingen bij maïs en tabak een grote ervaring heeft, vermeldt ook van een kruising, waar de oudervormen weinig verschillen, een grote transgressieve variabiliteit van F_2 ten opzichte van de oudervormen (EAST en EMERSON, 1912, tab. 9, 10 en 12).

De transgressies zijn in het algemeen niet groot. Met het oog op de aanname van erfelijkheid door polymere factoren, zou het van belang zijn, als er in de grootte der transgressies ook nog enige regelmaat zou kunnen vastgesteld worden, m.a.w. de vraag, of er ook nog weer erfelijke transgressies van verschillende grootte te onderscheiden zijn. Zeker behoudt de opmerking van NILSSON–EHLE, dat hij bij zijn kruisingen bijna steeds kleine erfelijke variaties vindt en dat hij niet de uitgangsvormen zuiver terug vindt, haar waarde en moet in onze aandacht blijven.

Onze resultaten geven aanleiding, om nog weer te wijzen op de betekenis van de keuze der erffactoren. Wij hebben genen aangenomen voor de afmetingen en niet voor het gewicht. Het gewicht is een samengestelde eigenschap, terug te voeren tot de afmetingen en het soortelijk gewicht. SAX (1923) bestudeert de erfelijkheid van het gewicht van de zaden van Phaseolus vulgaris en vermeldt ook TSCHERMAK (1922). SAX neemt aan, dat het aantal factoren voor het gewicht, dat bij zijn kruisingen is betrokken, omstreeks 5 of 6 is. Grootte-factoren (size factors) zijn aanwezig in de meeste of in alle chromosomen, schrijft hij. "The size factors in beans are independent in causing increased seed weight and several factors have a cumulative effect". En "the allelomorphic factors in the homozygous conditions have double the effect of a single factor in the heterozygous condition". Er is... "no dominance of size factors". Aldus SAX. Wij vinden enige dominantie. Uit de resultaten van het correlatieonderzoek blijkt, dat de afmetingen in het algemeen tegelijk veranderen, lange individuele bonen van de een of andere bonenvariëteit zijn ook breed en dik en korte individuele bonen zijn ook smal en dun. Ook het voorkomen in ons materiaal van transgressieve variabiliteit in dezelfde richting van meer dan één afmeting bij de bonenopbrengst van een zelfde plant, wijst in deze richting. Uit deze feiten zouden we kunnen willen afleiden, dat het gewicht de primaire eigenschap van de grootte is en dat de afmetingen zich daar naar richten. Zo is het echter niet, zoals blijkt uit de splitsingen na kruising. In de splitsing tonen de afmetingen haar zelfstandigheid. Er is zelfstandigheid en gebondenheid. We moeten daarom de afmetingen als de elementaire eigenschappen kiezen en aan haar uitingen na kruising de genen ten grondslag leggen. De resultaten over transgressieve variabiliteit van het gewicht doen ook goed zien, dat het gewicht een samengestelde eigenschap is; de afmetingen zijn de elementaire eigenschappen.

Transgressie-splitsingen zijn erfelijke transgressieve variaties. Met het oog op de grote niet-erfelijke variabiliteit is het moeilijk, om de erfelijkheid van de transgressie-splitsingen vast te stellen. Daarop wijst ook NILSSON–EHLE. Hij kweekt op elkaar volgende generaties en bepaalt de gemiddelden. De gemiddelden en de distributiekrommen wijzen de erfelijk-

heid goed aan. Zo vinden we het ook in ons materiaal. We bepalen ons hier tot de reproductie van een enkele afbeelding (fig. 17—19). Uit deze

Fig. 17

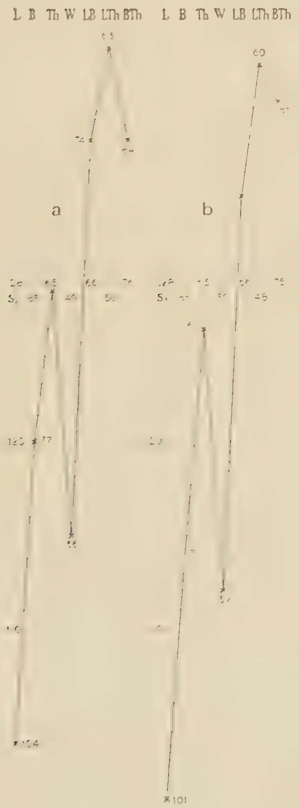

Fig. 19

Fig. 17. Characterogram (b) of the beanyield of pl. 783, F_5-1936 with transgressive variability of plus variations of the mean dimensions and the mean weight, and the characterogram (a) of its initial bean, 7p 2b of pl. 297, F_4-1935. S_1 = standard-characterogram of 1935, S_2 idem of 1936.

Fig. 19a. Characterogram of the initial bean 3 p 5 b of pl. 492, F_5-1936 for pl. 344. b. Idem of the averages of the beanyield of pl. 344, F_6-1937. The mean length and the mean breadth show transgressive variability of minus-variations. Pl. 344, F_6-1937 descends of pl. 492, F_5-1936 (fig. 18b), that also shows this transgressive variability. S_2 standard characterogram of 1936, S_3 id. of 1937.

afbeeldingen, uit tab. 5a—e (1950) en uit andere tabellen blijkt, dat de transgressie-splitsingen tot het genotypisch milieu behoren, waaruit ze voortkomen.

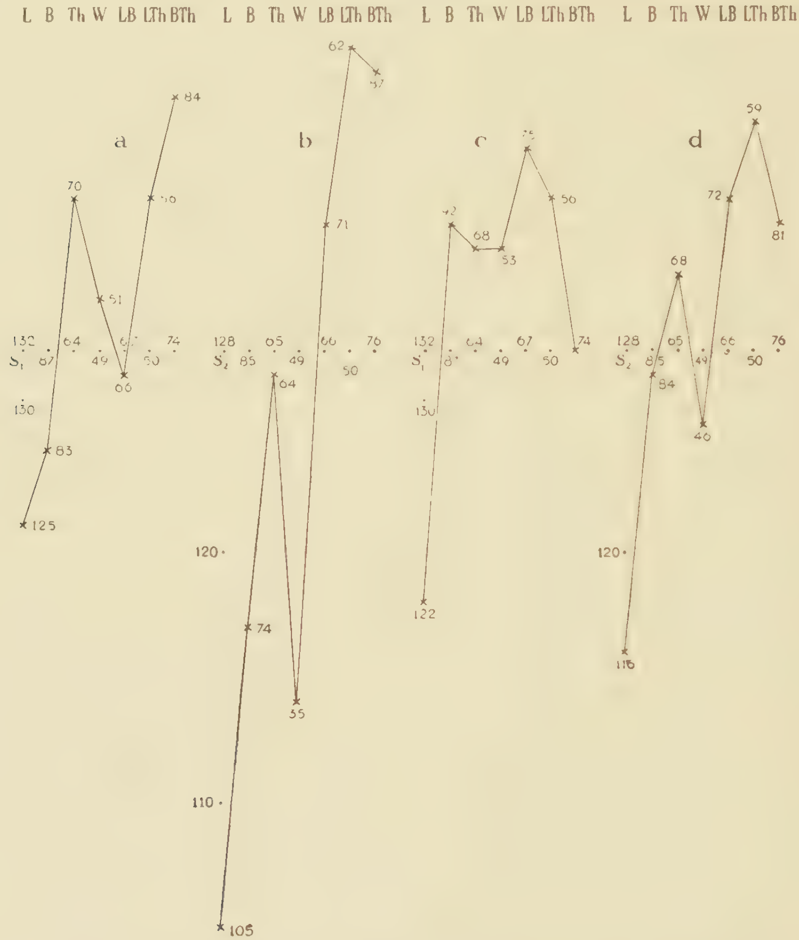

Fig. 18a. Characterogram of the initial bean 5p 2b of pl. 127, F_4-1935 for pl. 492. b. Idem of the mean dimensions and weight of the beanyield of pl. 492, F_5-1936. The mean length and the mean breadth show transgressive variability of minus-variations. c. Characterogram of the initial bean 10 p 1 b of pl. 128 F_4-1935 for pl. 497. d. Idem of the averages of the bean yield of pl. 497, F_5-1936. No transgressive variability. The characterograms of fig. 18a—d resemble each other very much.

EAST (1916, p. 324) schrijft: "... extremes like each parent were not produced ... but the greatest extremes gave the least regression". NILSSON-EHLE (1911, II, S. 8) schrijft: "Es ist aber die Frage, ob die beiden homozygoten Gruppen, die homozygot hohen und die homozygot niedrigen an sich wirklich homogen sind und ganz dieselben Abstufungen wie die konstanten Elternlinien darstellen, oder ob nicht in jeder Gruppe kleinere, erbliche Spaltungsdifferenzen vorhanden sind. De oudervormen van mijn materiaal, de bonen van de I- en van de II-lijn zijn homozygoot voor de onderzochte eigenschappen, hebben een grote niet-erfelijke variabiliteit en tonen een totale regressie. Het maakt de indruk, dat de uiterste variaties van mijn kruisings-

materiaal, die uiterlijk b.v. met de bonen van de I-lijn en de II-lijn overeenkomen, niet totale regressie tonen, hetgeen kan berusten op niet-gehele raszuiverheid van de onderzochte eigenschappen. Zo is verder onderzoek van de erfelijkheid van transgressiesplitsingen nodig.

De mendelistische theorie van erfelijkheid van kwantitatieve eigenschappen door polymere factoren geeft een goede verklaring van de in ons materiaal aanwezige voorbeelden van transgressieve variabiliteit.

Summary.

Transgression segregates are hereditary transgressive variations. A survey is given of transgressive variability of plus- and of minusvariations occurring in the seedgenerations F_5-1936, F_4-1935, F_3-1934, F_2-1933 and F_6-1937 i.e. the offspring of crosses of 1932 and in the seedgenerations F_5-1937, F_4-1936, F_3-1935, F_2-1934 i.e. the offspring of crosses of 1933. The crosses are of beans of the I- and the II-line, which are pure lines of Phaseolus vulgaris.

Transgression segregates occur especially in crosses where the characteristics under examination do not differ much in the parentforms. In the two pure lines of our material this is not the case for the length and in a lesser measure for the thickness. We may therefore expect transgressive variability especially for the breadth. This follows from the polymery theory.

We distinguish transgressive plus- and minus variations and discuss them separately for the 3 dimensions, the weight and the indices.

The average and the distribution curve of a character of the beans of a beanyield give a good expression of the phaenotype of it. We shall make use of them in our survey.

Tabs. 1 and 2 and figs. 1—5 refer to transgressive plusvariations of F_5-1936, tabs. 3 and 4 and figs. 6—8 refer to transgressive minusvariations of F_5-1936. Tab. 5 and fig. 9 refer to transgressive plusvariations of F_4-1935, tab. 6 to transgressive minusvariations of F_4-1935. Tab. 7 and figs. 10 and 11 refer to transgressive plusvariations of F_3-1934, tab. 8 and fig. 12 to transgressive minusvariations of F_3-1934. Tab. 9 and tab. 10 refer to resp. transgressive plus- and ditto minusvariations of F_2-1933.

Tab. 11 and figs. 13—14 and tab. 12 and fig. 15 refer to resp. transgressive plus- and ditto minusvariations of F_6-1937.

Tab. 13 and tab. 14 give summarizing surveys of resp. transgressive plus- and minusvariations of F_5-1936 up to and including F_2-1933 and F_6-1937.

Tab. 15 and fig. 16 and tab. 16 refer to resp. transgressive plus- and minusvariations of F_5-1937.

Tab. 17, 19 and 21 refer to transgressive plusvariations of resp. F_4-1936, F_3-1935 and F_2-1934; tabs. 18, 20 and 22 to transgressive minusvariations of these generations. Tabs. 23 and 24 give a survey of resp.

transgressive plus- and minusvariations of F_5-1937 to and including F_2-1934.

To be able to explain the transgressive variability we have found in our material, we have made a slight and obvious alteration in our formulas. The formula for the I-line becomes $L_1L_2l_3L_4L_5L_6L_7B_1B_2b_3B_4b_5b_6b_7$ $th_1th_2Th_3th_4th_5th_6th_7$ and for the beans of the II-line: $l_1l_2L_3l_4l_5l_6l_7$ $b_1b_2B_3b_4b_5b_6b_7Th_1Th_2th_3Th_4th_5th_6th_7$. Thus transgressive segregates appear in F_2- or succeeding generations.

To the theory of the polymere factors the authors joins a few remarks on the number of factors we can assume here, on the number of mendelizing genes and the fundamental characters of the species and on the total number of genes of the germplasm and the species-notion. Also on polyploidy and polymery. It seems possible to the author, however, that for the heredity of quantitative characters which are explained by the poly. mery theory a distinction must be made between the cases with mono-- di-, trimery where experimentally the numerical ratios 1 : 3, 1 : 15, 1 : 63 are found and the great group of cases where hereditary variations can be cultivated from material which forms a series of continual variations and whereby a more or less arbitrary number of factors 2, 5, 8, 15 and more are assumed.

We mention the following results.

We find the transgressive variability oftenest for the breadth (p. 10).

The transgressive variability of the weight is very clear. It is connected with the fact that new variations are formed with the great length and the great breadth of the beans of the I-line and the great thickness of the beans of the II-line. Of these beans all 3 dimensions are great; the formula is L B Th.

We often find that of the same beanyield not only the length but also the breadth and occasionally all 3 dimensions (and the weight) show transgressive variability. It appears from this that the dimensions vary together and in the same direction. There are, however, also variations, segregates, where a great length occurs together with f.i. a small breadth. In these cases the independence of dimensions is shown.

The weight is a composite character. Others, f.i. SAX, 1923, examined the heredity of the weight of the seeds of Phaseolus. The examination of the dimensions *and* the weight gives more.

We occasionally also find transgressive variability of the indices. The index is a secondary character. The dimensions, though not entirely independent of each other, are the primary characters.

The transgressions we find are in general not great. In view of the assumption of heredity through polymere factors it would be of importance if some regularity could also be established in the extent of the transgressions i.o.w. if again there are different hereditary transgressive variations to be distinguished.

Transgression segregates are hereditary transgressive variations. In

view of the great non-hereditary variability it is difficult to establish the heredity of the transgressive segregates. This is also pointed out by NILSSON-EHLE. He cultivates successive generations and determines the averages. The averages and the distribution-curve show the heredity clearly. We find this too in our material. We limit ourselves here to the reproduction of a single figure (fig. 17—19).

The mendelian point of view as the basis of the polymery theory of the heredity of quantitative characters gives a good explanation of the occurrence of transgressive variability.

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PETROLOGY

PRELIMINARY NOTES ON GLAUCOPHANE-BEARING AND OTHER CRYSTALLINE SCHISTS FROM SOUTH EAST CELEBES, AND ON THE ORIGIN OF GLAUCOPHANE-BEARING ROCKS *)

BY

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(Communicated by Prof. H. A. BROUWER at the meeting of Sept. 30, 1950)

During the year 1949 the author had the opportunity to make a brief study of a collection of crystalline schists from South East Celebes; the investigated samples were collected some tens of years ago by BOTHÉ, HETZEL, and STRAETER of the "Dienst van de Mijnbouw". Since the general results of this study may be valuable to other investigators, they are published in advance in this preliminary note.

The investigated collection comprises more than 170 samples of crystalline schists, most of which have been collected in the Rumbia Mts, in and N. of the Mendoke Mts, and near the lower course of the river La Solo (see map). Some data on the petrology of these regions have already been published by WUNDERLIN (1913).

The investigated crystalline schists belong essentially to three groups,

corresponding to those recognized in a paper on the geology and petrology of the neighbouring island of Kabaena (DE ROEVER 1950):

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1°. rocks that have mainly been metamorphosed in the amphibolite and epidote-amphibolite facies (La Solo region),

2°. low grade dynamometamorphic phyllites and blastopsammitic quartzites (La Solo region), and

3°. strongly recrystallized rocks that have mainly been metamorphosed in the glaucophane-schist facies and that are devoid of relics of older metamorphic facies (Rumbia and Mendoke Mts).

The rocks of the first group, which have mainly been metamorphosed in the amphibolite and the epidote-amphibolite facies, have only been recognized as occurring in a narrow zone along the SW border of a large mass of peridotites and serpentines near the lower course of the river La Solo. They are characterized by the presence of large crystals of amphibole of a blue-green or green colour, and by the occurrence of minerals of the epidote-group (a.o. piedmontite), garnet, biotite, and non-alkaline metamorphic clinopyroxene. The garnet shows alteration into chlorite. Hysterogene pumpellyite was found in two samples. Hysterogene glaucophane and lawsonite, which have so frequently been found in the corresponding rocks of part of eastern Central Celebes (DE ROEVER 1947), have not been observed here. The rocks of this group do not show any features not yet known from eastern Central Celebes or Kabaena.

The low grade dynamometamorphic phyllites and blastopsammitic quartzites have been found in a zone adjoining that of the rocks of the first group on the SW side, in which direction the glaucophane-bearing rocks have been found at a larger distance. The rocks of the second group have apparently been metamorphosed in the greenschist facies.

The investigation of the rocks that have mainly been metamorphosed in the glaucophane-schist facies and that are devoid of relics of older metamorphic facies, from the Rumbia and Mendoke Mts, has yielded interesting results, and indicated a closer analogy between the geology and petrology of Celebes and that of the Alps than was hitherto supposed to exist. Almost sixty of the samples studied were found to contain lawsonite or glaucophane; crossite was recognized in only one of the samples. Though no relics of older metamorphic facies have been observed, almost all samples were recognized to be polymetamorphic, the polymetamorphism being illustrated by the alteration and deformation of the lawsonite and glaucophane. Minerals of the epidote-group and fibrous colourless to light green amphibole were found in many of these rocks. By far the majority of the rocks containing the latter minerals in addition to lawsonite and glaucophane were found to show features indicating a younger age for the minerals of the epidote-group and the fibrous colourless to light green amphibole, and an older age for the glaucophane and lawsonite, viz. alteration of the lawsonite into chlorite, minerals of the epidote-group, sericite, albite, and carbonate, alteration of the glaucophane into chlorite and albite, and the occurrence of rims of colourless to light green amphibole around the glaucophane. Complete pseudomorphs after

glaucophane or lawsonite were found in many of the rocks studied, also in a number of rocks in which no original lawsonite and glaucophane were left. Only in two of the rocks studied the minerals of the epidote-group and the fibrous colourless to light green amphibole were not accompanied by relics of lawsonite or glaucophane, or by pseudomorphs after these minerals. Furthermore the crystals of lawsonite and glaucophane have often been bent or broken. The minerals undoubtedly belonging to the younger phase of metamorphism, viz. the minerals of the epidote-group, the fibrous colourless to light green amphibole, and the chlorite, sericite, albite and carbonate resulting from the alteration of the lawsonite and glaucophane are indicative of the greenschist facies (TURNER 1948).

Almost half of the lawsonite- or glaucophane-bearing rocks studied was furthermore found to contain garnet, while chloritoid was observed in several of these rocks, sometimes together with garnet, lawsonite, and glaucophane. The occurrence of the garnet and chloritoid was found to be independent of the intensity of the younger metamorphism in the greenschist facies, and the mode of occurrence of the minerals mentioned indicated the contemporaneous development of garnet, chloritoid, lawsonite, and glaucophane. The garnet may also have been altered, into chlorite, while the crystals of chloritoid have sometimes been broken. Thus the rocks of the Rumbia and Mendoke Mts apparently belong to a hitherto unknown subfacies of the glaucophane-schist facies, characterized by the occurrence of garnet, chloritoid, lawsonite, and glaucophane, the *garnet-lawsonite-glaucophane-schist subfacies.*

Though most lawsonite- and glaucophane-bearing rocks of the collection were determined as paraschists, a number of orthoschists containing these minerals has also been observed. These orthoschists generally contain relics of igneous titaniferous augite with its characteristic colour. This titaniferous augite was found to have been altered along the rims and in veins into a green, apparently jadeite- and aegirine-bearing clinopyroxene. In the rocks of the Rumbia and Mendoke Mts the alkaline pyroxenes seem to be confined to these orthoschists. Perfect blastophitic structures were observed in several of the clinopyroxene-bearing rocks. In one instance, where the identity of the original felspar was established beyond doubt by the well-developed blastophitic structure, the original felspar was found to have been mainly altered into glaucophane. Other alteration-products of the original ophitic felspar are mainly albite, lawsonite, and chlorite.

The only structural relics of the lawsonite- and glaucophane-bearing paraschists are represented by a fine graphitic banding. In some crystals of lawsonite several folds of these bands have been found enclosed (definitely not S-shaped structures produced by rotation during the growth of the crystals), indicating that a phase of folding occurred before the formation of the lawsonite.

Some of the glaucophane- and lawsonite-bearing rocks were further

found to contain pumpellyite; in one of these the pumpellyite is clearly of a younger age than the glaucophane, as indicated by the zonary structure of crystals of amphibole that protrude from the main body of the rock into pumpellyitic lenses; these crystals show cores of glaucophane and rims and tips of very pale green amphibole.

Before discussing the results of the investigations described here we may first summarize our knowledge of the crystalline schists of eastern Central Celebes and Kabaena (DE ROEVER 1947 and 1950). Arguments were given for the existence of large scale overthrusts in these regions. An older metamorphism in the amphibolite and epidote-amphibolite facies, which is older than the radiolarites and the spilitic igneous rocks of the alpine geosyncline, could be distinguished from a younger metamorphism in the lawsonite-glaucophanite subfacies, which is younger than the radiolarites and the spilitic igneous rocks. The metamorphism in the amphibolite and the epidote-amphibolite facies was considered to belong to a pre-alpine orogenetic cycle, the original rocks consequently being of paleozoic or greater age. The mutual age-relations of the metamorphism in the amphibolite facies and the metamorphism in the epidote-amphibolite facies have not yet been adequately elucidated. The mineral pumpellyite was assumed to belong to a separate metamorphic facies, containing among others colourless amphibole, albite, quartz, and carbonate as other typical minerals. The pumpellyite is younger than the radiolarites and igneous rocks, but was found to be older than the glaucophanitic metamorphism in a number of rocks. Therefore the metamorphism in the pumpellyitic facies was considered as a kind of precursor of the metamorphism in the lawsonite-glaucophanite subfacies. This mineral, however, was also found in a number of rocks without critical minerals of the lawsonite-glaucophanite subfacies, occurring in a zone along the boundary of the region occupied by the rocks affected by the glaucophanitic metamorphism. The metamorphism in the lawsonite-glaucophanite subfacies could further be separated from a younger phase of dynamic metamorphism, which apparently accompanied the paroxysm of the overthrust movements. In Kabaena this younger phase of dynamic metamorphism expresses itself mainly in the deformation of the crystals of the characteristic minerals of the lawsonite-glaucophanite subfacies, the alteration of which is only slight. In Central Celebes alteration of these minerals was only locally observed (DE ROEVER 1947, page 98 and 114), while WILLEMS described the occurrence of deformed crystals of glaucophane in a number of rocks from western Central Celebes (1937). The dynamic metamorphism accompanying the great overthrust movements is possibly of post-Eocene age, since phyllitic rocks with Eocene fossils have been found in northern Celebes (BROUWER 1947, page 15—16). The original material of strongly recrystallized paraschists of the lawsonite-glaucophanite subfacies without relics of older metamorphic facies was furthermore argued to be of mesozoic age. These schists, which are widely

distributed in Central Celebes and Kabaena, are considered as metamorphic geosynclinal deposits of the alpine orogene, and have been compared with the "Bündnerschiefer" or "schistes lustrés" of the penninic overthrust sheets of the Alps. In both regions these schists apparently represent the lowermost tectonic unit(s) exposed. In Kabaena, and probably also in eastern Central Celebes they are covered by a tectonic unit that has only been affected by the youngest phase of dynamic metamorphism. In Kabaena this dynamometamorphic unit in its turn is covered by the rocks of the epidote-amphibolite and the amphibolite facies, which occur at the base of a large mass of peridotites and serpentines. A zonal distribution corresponding to this superposition was found in South East Celebes.

The lawsonite-glaucophanite subfacies, the existence of which was proved by the author, is a subfacies of low grade metamorphism, since the rocks of this subfacies grade into non-metamorphic rocks. Type-minerals of this subfacies in eastern Central Celebes are mainly glaucophane and crossite, jadeite and jadeite-aegirine, lawsonite, quartz, albite, muscovite, chlorite, carbonate, titanite, and hematite. In the slightly stronger metamorphic rocks of Kabaena chloritoid was also found. The lawsonite-glaucophanite subfacies is a.o. characterized by the instability of garnet and minerals of the epidote-group.

As to the origin of the glaucophanitic metamorphism the following may be quoted from the paper on Kabaena (DE ROEVER 1950): "Since the glaucophanitic metamorphism is definitely older than the dynamometamorphism accompanying the paroxysm of the overthrust movements, the glaucophanitic rocks have apparently not been originated under the influence of stresses producing large scale overthrusts, but under conditions governing after the sedimentation of part of the geosynclinal deposits, and before the paroxysm of the overthrust movements. For a metamorphism like that of the lawsonite-glaucophanite subfacies, apparently having its centre in the central part of the geosyncline, we then may conclude to conditions mainly determined by the geosynclinal situation of the metamorphism, with a relatively high hydrostatic pressure and more subordinate thermal influence and shearing stress, i.e. the metamorphism in the lawsonite-glaucophanite subfacies is a characteristic geosynclinal metamorphism, predominantly governed by the hydrostatic pressure inherent to its geosynclinal environment, with more subordinate thermal influence and shearing stress..... The above conclusion for the origin of the rocks of the glaucophane-schist facies is corroborated by two entirely independent facts. Firstly, in the Alps the essential distribution of the glaucophane-bearing rocks is in the penninic overthrust sheets; here the glaucophanitic metamorphism is older, too, than the paroxysm of the orogenetic movements, as indicated by the alteration of the glaucophane into chlorite, albite, and calcite....., and by the occurrence of zones of green amphibole around the glaucophane..... Secondly, many of the minerals of the glaucophane-schist facies are of a relatively high

specific gravity, indicating that hydrostatic pressure was an important factor governing their production."

According to their petrological characteristics and their tectonic position the rocks from South East Celebes that have mainly been metamorphosed in the amphibolite and epidote-amphibolite facies are to be compared with the corresponding rocks from Kabaena and eastern Central Celebes; their original material is considered to be of paleozoic or older age, while the metamorphism in the amphibolite facies and the epidote-amphibolite facies apparently took place during an orogenetic cycle of pre-alpine age.

The phyllites and blastopsammitic rocks of the second group have only been affected by the dynamic metamorphism accompanying the paroxysm of the overthrust movements; their original material is considered to be of mesozoic and possibly also eocene age.

The strongly recrystallized rocks that have mainly been metamorphosed in the glaucophane-schist facies and that are devoid of relics of older metamorphic facies correspond to the paraschists of comparable metamorphism in Kabaena and Central Celebes. So they are considered as metamorphic geosynclinal deposits of the alpine orogene, the original material being of mesozoic age. Main argument for this assumption is the absence of relics of older metamorphic facies, especially of epidote, in the corresponding rocks from Kabaena and Central Celebes. The resemblance to the "Bündnerschiefer" or "schistes lustrés" of the Alps is even more complete in South East Celebes, owing to the presence of orthoschists and owing to the greater intensity of the younger phase of dynamic metamorphism, which intensity may be compared to that of the corresponding metamorphism in the Alps.

The mode of occurrence of the mineral pumpellyite in the glaucophane-schists of South East Celebes leads to the supposition that the metamorphism in the glaucophane-schist facies has been adjoined in space and time by the metamorphism in the pumpellyitic facies: in a number of rocks from eastern Central Celebes the pumpellyite was found to be older than the glaucophanitic metamorphism, further it was also found there in rocks that have not been affected by the glaucophanitic metamorphism and that have been collected in a zone along the boundary of the region occupied by the glaucophanitic rocks, and lastly it was found as a younger mineral in the glaucophane-schists from South East Celebes. During the production of the mineral pumpellyite the confining pressure was probably lower than during the metamorphism in the glaucophane-schist facies; the temperature was probably also lower, while the relative importance of stress is insufficiently known.

The garnet-lawsonite-glaucophane-schist subfacies is apparently the adjoining higher metamorphic equivalent of the lawsonite-glaucophanite subfacies, which in Celebes represents the lowest grade of glaucophanitic metamorphism. Both subfacies are characterized by the instability of epidote, the latter also by the instability of garnet. The garnet-lawsonite-

glaucophane-schist subfacies may also be termed garnet-lawsonite subfacies of the glaucophane-schist facies; for the lawsonite-glaucophanite subfacies the name of lawsonite-chlorite subfacies may also be used, which, however, need not imply that chlorite is unstable in all types of rocks of the garnet-lawsonite subfacies. With rising intensity of the glaucophanitic metamorphism chloritoid apparently appears before garnet: chloritoid has not been found in the rocks with a very low grade of glaucophanitic metamorphism from eastern Central Celebes, whereas in the stronger metamorphic rocks of Central Kabaena chloritoid, but no garnet has been found, and whereas chloritoid and garnet appear together in the strongest metamorphic glaucophane-schists yet described by the author. The jadeite-rich alkalipyroxenes and crossite seem to be confined to the rocks of the subfacies with a lower grade of metamorphism. This is indicated by the important role of these minerals in eastern Central Celebes, and their comparative rareness in Kabaena and South East Celebes; the association of older crossite with younger glaucophane in a number of rocks from eastern Central Celebes points in the same direction. Jadeite and other jadeite-rich alkalipyroxenes are apparently formed during low grade glaucophanitic metamorphism, a statement that may also hold true for the occurrences of jadeite in other parts of the world. The low grade metamorphic origin of jadeite was already mentioned in a former paper (DE ROEVER 1947, page 164); the low temperature of origin has been confirmed by YODER (1950).

Glaucophane-schists with garnet and chloritoid, apparently belonging to the garnet-lawsonite-glaucophane-schist subfacies, have already been described from other localities in Celebes, too (a.o. DE ROEVER 1947, page 114).

As to the origin of the glaucophane-bearing rocks some additional remarks may be made, especially since TURNER (1948) doubts the existence of a separate glaucophane-schist facies.

The metamorphism in the glaucophane-schist facies has its main development in the metamorphic geosynclinal deposits of the deeper central parts of younger orogenes, like that of Celebes. There it is a genuine regional metamorphism. On the other hand glaucophane, etc. may be of irregular distribution in the peripheral or upper parts of these orogenes, as e.g. in part of eastern Central Celebes and in California (TALIAFERRO 1943). Though the rocks of California have not been studied from a facial point of view (occurrence of zonary amphiboles!) TALIAFERRO has sufficiently proved the importance of metasomatic processes for their formation; in California the development of glaucophane, etc. instead of the normal products of such metasomatic processes, like albite, etc. during or after this additive metamorphism, in the chemically adapted rocks, however must also have been controlled by physical factors, as indicated by the results of the investigations of the present author.

Though arguments for the existence of a separate glaucophane-schist

facies have already been given in a former paper (DE ROEVER 1947), this question here may be dealt with in a more elaborate way. The occurrence of lawsonite is confined to regions with glaucophane-bearing rocks, while lawsonite and glaucophane have been found in intimate association in numerous metamorphic rocks of very different character and varying chemical composition (original igneous and sedimentary rocks as well as previously existing metamorphic rocks). In the rocks from Celebes and Kabaena the lawsonite is not accompanied by contemporaneous anorthite, zoisite or clinozoisite, but it appears instead of these minerals. Further the absence of glaucophane or allied minerals in lawsonite-bearing rocks is apparently due either to the chemical composition of the rocks in question, or to alteration of the glaucophane, etc. So it is not only the occurrence of glaucophane alone that we have to explain, but also that of the mineral association lawsonite-glaucophane. This gives an entirely different aspect to the problem, since lawsonite, unlike glaucophane, is not of a peculiar chemical composition. The occurrence of such a special lime-mineral of normal composition in intimate association with glaucophane is not explained by the assumption of an essentially metasomatic or allied origin of the glaucophane, which therefore has to be rejected. On the contrary, the development of lawsonite instead of the chemically almost similar or resembling minerals anorthite, zoisite or clinozoisite must be due to special physical conditions, i.e. lawsonite must belong to one or more separate metamorphic subfacies, in which neither anorthite nor zoisite and clinozoisite appear as stable minerals. The intimate association of lawsonite with glaucophane described above indicates definitely that glaucophane is a stable mineral of these separate subfacies. In these subfacies glaucophane and allied alkalipyroboles appear as wide-spread minerals. Should the development of glaucophane be essentially controlled by metasomatism or allied causes, then metasomatism would be abnormally important in the rocks of these special subfacies as compared to those of other subfacies, which is very improbable. So the development of glaucophane in lawsonite-bearing rocks of suitable chemical composition may be assumed to be controlled by special physical conditions. So far we have only been dealing with glaucophane that occurs in intimate association with lawsonite. It would be the merest chance when the stability conditions of lawsonite and glaucophane were exactly similar to each other; therefore we may expect these minerals to have different areas of stability. So the glaucophane of regions without lawsonite may safely be assumed to have a mode of origin comparable to that of the glaucophane of lawsonite-bearing regions. Another strong argument for the essential unimportance of metasomatism for the development of glaucophane is furnished by the fact that most glaucophane-schists are not specially sodic rocks. The arguments mentioned imply the conclusion that the development of glaucophane in rocks of suitable chemical composition is controlled by special physical conditions, i.e. that there exists a separate

glaucophane-schist facies. Similarly as in other metamorphic facies metasomatism according to the principle of enrichment in the most stable constituents has occurred in the rocks of the glaucophane-schist facies. In the deeper central parts of younger orogenes like that of Celebes the glaucophane-schist facies is of regional distribution; near the boundaries of its stability-area, however, its distribution is irregular owing to the unequal distribution of the controlling physical factors.

Since the occurrence of the critical minerals of the glaucophane-schist facies, like lawsonite and glaucophane, is confined to chemically suitable rocks, these minerals may be rather scarce, like in the western part of eastern Central Celebes (excepting the Poso fault trough); many other type-minerals correspond with those of other facies, like quartz, albite, muscovite, chlorite, and calcite, so that the distribution of the glaucophane-schist facies is often obscured. The distribution of the glaucophane-schist facies furthermore may be difficult to recognize on account of the occurrence of minerals originated during a younger phase of dynamic metamorphism, like in the Alps and South East Celebes. In fact, the mode of origin of the glaucophane-schists contended by the author implies that the rocks of the glaucophane-schist facies are even generally influenced by a younger dynamic metamorphism and therefore are always more or less polymetamorphic. In many cases, however, this younger phase of metamorphism has mainly given rise to a deformation of the glaucophanitic minerals, whereas newly developed minerals are of limited occurrence.

For the reasons already given in a former paper (DE ROEVER 1950), which have been quoted above, the metamorphism in the glaucophane-schist facies is considered to be a characteristic metamorphism of geosynclines of the alpine type, the development of lawsonite and glaucophane being mainly governed by the relatively high confining pressure inherent to the geosynclinal environment, with more subordinate thermal influence and shearing stress. The age of the metamorphism in the glaucophane-schist facies is confined as yet to the period between the oldest phase of folding of the geosynclinal deposits (folded inclusions in lawsonite from South East Celebes) and the paroxysm of the overthrust movements. Further data on the age of this metamorphism may be provided by the occurrence of metamorphic and detrital glaucophane in fossiliferous rocks and formations. The author has the intention to make a survey of all literature on lawsonite- and glaucophane-bearing rocks in order to test and elaborate his conclusions.

Though relatively high confining pressure is considered to be an essential factor in the production of lawsonite and glaucophane, the influence of elevated temperature and shearing stress cannot be denied, the former being implied by the depth of formation, the latter by the schistosity of many glaucophane-bearing rocks. The development of critical minerals of the glaucophane-schist facies in very low grade metamorphic rocks, which grade into non-metamorphic rocks, indicates that only the relative

importance of the confining pressure has been the controlling factor in the production of these minerals, and not its absolute magnitude. Therefore in a diagram like that of ESKOLA (1939, page 345) the glaucophane-schist facies may be placed obliquely below the appropriate other facies, the confining pressure being lower for low grade glaucophanitic metamorphism, and higher for the corresponding high grade metamorphism. The metamorphism that produces blue-green amphiboles may be transitional between the glaucophanitic metamorphism and the "normal" regional metamorphism; here the role of stress may be larger than in the production of glaucophane-bearing rocks, while the confining pressure may have been of less importance. The apparently rare occurrence of glaucophane-bearing rocks in paleozoic orogenes and still older mountain systems may be due to the fact that the folding of their geosynclinal deposits has attained a smaller depth.

Concluding these general remarks the author wishes to state that his conclusions have been given in a rather too precise way for the sake of clearness; it cannot be doubted that complications may exist, and that e.g. the formation of glaucophane must have been at least of slightly different age in different parts of the same orogene.

The metamorphic history of the eastern part of Celebes and of Kabaena may now be briefly summarized. During an older orogenetic cycle paleozoic or older rocks were metamorphosed in the amphibolite and epidote-amphibolite facies. After the origination of the alpine geosyncline the geosynclinal deposits were affected by at least one phase of folding before the metamorphism in the glaucophane-schist facies. This metamorphism is thought to have been preceded, laterally accompanied, and followed by metamorphism in a facies characterized by the occurrence of pumpellyite. The pumpellyitic and the glaucophanitic metamorphism may be considered to represent one prolonged phase of metamorphism, which occurred when the geosynclinal deposits attained their greatest depths. The glaucophanitic metamorphism seems to be distinctly separate from the youngest dynamometamorphism in the greenschist facies that accompanied the paroxysm of the large scale overthrust movements, since blue-green amphiboles of intermediate age have not been found.

At the end of this paper some brief remarks may be made on the comparison of the metamorphism of Celebes with that of other islands of Indonesia. The occurrence of glaucophane-bearing rocks in a small region in Central Java was already described long ago (NIETHAMMER 1909). The regional importance of glaucophane in the underground of this island is illustrated by its detrital occurrence at several other localities (LOOS 1924, VAN BAREN 1928, DRUIF 1930). Furthermore several rocks of Timor have been found to contain crossite, etc., part of which was formerly considered to be of magmatic origin, e.g. the only igneous rocks known from the lowermost tectonic unit of Timor, the Kekneno-series (DE ROEVER 1940), and the alkalitrachytes of the feebly atlantic Permian differentia-

tion-series of the next higher Sonnebait overthrust sheet, which were formed during the embryonal stages in the evolution of the geosyncline (DE ROEVER 1942). These examples, which are far from complete, may suffice to indicate that a metamorphism in the glaucophane-schist facies has also occurred in other Indonesian islands; metamorphic geosynclinal deposits like those of Celebes and comparable to those of the penninic overthrust sheets of the Alps may therefore be hidden below or near other parts of the island arcs of Indonesia.

Amsterdam, Geological Institute, June 1950.

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ON SOME PHOTOPERIODIC AND FORMATIVE EFFECTS OF COLOURED LIGHT IN BRASSICA RAPA, F. OLEIFERA, SUBF. ANNUA

BY

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Introduction.

Previous investigations in this field show a wide variety of methods and quality of light used. To a great extent the lack of agreement between the results obtained may be attributed to differences in experimental methods, intensities and purity of the light used.

The greater part of the investigators found a maximum for the photoperiodic response in the red part of the spectrum. RASUMOV (1), working with radiation filtered through coloured solutions, found a maximum of photoperiodic effectiveness in the red region in the inhibiting of flowering in short-day plants and in the promoting of flowering in long-day plants. WITHROW and co-workers obtained similar results with both types of plants (2, 3); WITHROW and BENEDICT report one case in which blue radiation was effective (2) but this plant, *Callistephus chinensis*, var. Heart of France, responded equally to all wavelength regions employed. KATUNSKY (4) and KLESHNIN (5) reported that all wavelength regions were effective, provided the intensities were sufficiently high and they only found differences in the degree of effectiveness. These results are in general agreement with those of BORTHWICK and PARKER and co-workers (6, 7) on the determination of the action spectrum for floral initiation in *Xanthium saccharatum*, *Soja max* var. Biloxi, and a variety of barley. They found a minimum of effectiveness in the blue and violet regions and a broad maximum in the red region of the spectrum. The action spectrum curves obtained in this work are very similar, and on the whole confirm the statements made by the authors mentioned above.

FUNKE (8, 9) investigated a large number of plants as to their photoperiodic reaction to different wavelength regions. Using supplementary filtered radiation from incandescent lamps or from daylight, in addition to a short day in normal daylight, he obtained results which led him to divide his material into four groups. In his first group he placed the plants which responded to additional red light in the same way as they did to

additional white light, and which did not respond to additional blue light. The second group contains the plants which reacted similarly to red, white or blue additional radiation. In the third group the plants only reacted to white radiation, whereas blue and red radiation were ineffective. The representatives of the fourth group did not react to red radiation, whereas additional blue radiation had the same effect as white. FUNKE further remarks, that all species belonging to group 4 are *Cruciferae*, whereas none of the *Cruciferae* he investigated belongs to the contrasting group 1. Especially as far as group 4 is concerned, these results are isolated and have never been confirmed. Our investigation was undertaken in the first place to investigate whether FUNKE's results could be reproduced under more strictly controlled conditions.

Material and methods.

The plant used throughout this experiment is a member of the *Cruciferae*. Its Dutch name, "Boterzaad" (butterseed), is given by STARING (10) as representing *Brassica campestris trimestris*, in German called "Sommerrübsen". OUDEMANS (11) calls it *Brassica Rapa campestris* (= *B. campestris* L), whereas WETTSTEIN and ENGLER classify it as *Brassica Rapa oleifera* subf. *annua* and *Brassica campestris annua* respectively (12, 13).

To the photoperiod it reacts essentially as a long-day plant.

The seed obtained was from a land race; the seed for each experiment was taken from one plant so as to obtain material with the smallest possible variation. The seeds were made germinating in sand, after which the plants were transplanted into earthenware pots, and in the last series into flat square earthenware dishes at the rate of two per pot or 16 per dish. The sand into which the seedlings were transplanted received a quantity of artificial fertilizers equivalent to a normal field-manuring. In the first three series the young plants were exposed to 18 hours irradiation daily with white light of about 22×10^3 ergs/cm² sec., until the beginning of the treatment with supplementary light. In the fourth series, the plants received a 10 hour day with known energy during this period. Ten days after transplanting, when the plants had expanded about 4 leaves, the treatment with supplementary light in the different wavelength regions commenced. From this moment, and in the last series also before this moment the plants received a 10 hour day at an intensity of about 22×10^3 ergs/cm²/sec. This light was obtained from a set of twelve 40 W fluorescent tubes, placed closely together. Between 400 and 700 $m\mu$ the spectral energy distribution of the light obtained from these sources is very near that of daylight. It is not necessary to cool the compartment otherwise than with a fan. The plants never received natural day light.

After a 10 hour period in this white light of relatively high intensity, the plants were transferred into the equipment for coloured illumination, described elsewhere (14), to be exposed to additional irradiation with various wave length regions. This irradiation was given 8 hours per day immediately after the 10 hour period in white light. After this 8 hour period, the plants were transferred to a dark case for 6 hours. In the first three series the intensities during the additional irradiation were 3000 ergs/cm²/sec., whereas in the last series the intensities in all compartments were kept constant at 1000 ergs/cm²/sec. (total radiation, *cf.* (14)), by varying the distance between object and lightsource.

In the first three series the white control received 18 hours per day at an intensity

of 22×10^3 ergs/cm²/sec., which the others received only during 10 hours. In the fourth series, the white control plants received an additional radiation with white light at an intensity equal to that of the light the other plants received.

The available equipment did not permit to keep the temperature constant throughout the various series, but within the duration of an experiment the temperature in all the compartments could be kept constant within 1°—2° C. The average temperature in the various series was never lower than 18° C nor higher than 22° C. The humidity was kept as near to 70 % as possible; it did not fall below 60 %.

Photoperiodic effect of the supplementary irradiation.

The data, compiled in Table I and II indicate that blue supplementary light is most effective in prolonging the short day. In all cases plants irradiated with blue supplementary light produce flowers considerably earlier than those exposed to radiation of other wavelength regions and earlier than the white controls, even in the first three series, in which the white controls received a much higher light intensity than the other plants.

TABLE I

Photoperiodic response of *Brassica Rapa* var. to light of various wavelength regions, at an intensity of 3000 ergs/cm²/sec. (total radiation, *cf* (14)), supplementing a short day. Numbers of days from beginning of treatment until appearance of first flower bud and first flower.

	First series	Second series	Third series	
	flower bud	flower	flower bud	flower
White [1])	14	23	11	21
Violet	16	30	13	27
Blue	12	20	9	18
Green	24	32	—	—
Yellow	27	31	22	46
Red	21	41	26	43
Infrared	21	36	24	38
Dark	27	34	—	68

[1]) White supplementary light at an intensity of 22×10^3 ergs/cm²/sec.

TABLE II

Photoperiodic response of *Brassica Rapa* var. to light of various wavelength regions, at an intensity of 1000 ergs/cm²/sec. (total radiation, *cf* (14)) supplementing a short day. Numbers of days after beginning of treatment when flower buds, resp. flowers were observed in 50 % of the plants.

	Flower buds	Flowers
White	15	28
Violet	13	25
Blue	12	22
Green	> 37	—
Yellow	> 37	—
Red	> 37	—
Infrared	23	36
Dark	> 37	—

It was already remarked that, in the initial stage of development the plants of the first three series received a number of long-day cycles, *viz.* 15 in the first series, 12 in the second, 2 in the third, after being transplanted into their final places. Therefore, in a subsequent series we examined the growing points microscopically in order to estimate whether the initial period with long days might already induce the initiation of inflorescence primordia. Notwithstanding the fact that in this series the supplementary illumination in the white compartment was only about 10^3 ergs/cm² instead of 22.10^3 ergs/cm² sec. in the preceding series, after 8 cycles, 5 plants out of 5 had inflorescence primordia (Table III, Plate 1*B*). This indicates that in the first three series the pretreatment may have lead to a considerable induction.

TABLE III

Effect of supplementing a short day with light of various spectral regions on the initiation of inflorescence primordia in *Brassica Rapa* var. Intensity of supplementary light: 1000 ergs/cm²/sec. (total radiation, *cf.* (14)). Number of plants with inflorescence primordia per 5 plants.

White	2	3	5
Violet	1	3	4
Blue	1	2	4
Green	—	—	1
Yellow	—	1	1
Red	1	2	4
Infrared	—	1	3
Dark	—	—	—
Number of days after beginning of treatment	2	5	8

Notwithstanding this possible leveling influence, also in the first series the subsequent treatment with colours yielded very distinct differences. Table IV, *e.g.*, shows the stem length reached 27 days after the beginning of the treatment, in the 1st, 3rd and 4th series. It is seen that the group dark, green, yellow, red remains relatively short, whereas infrared, violet, white, and especially blue, become much longer. It is striking that in the first series the plants of the short group are much longer than in the third series, which, according to our further experience is to be ascribed to the inducing effect of the long days in white light at the beginning of the experiment which amounted to 15 cycles in the 1st series and only 2 in the 3rd series. The plants of the 4th series are again shorter, owing to the lack of long white days throughout. Plate 1A gives a general view of the 3rd series, 30 days after the beginning of the treatment with coloured light. In this stage flowering had proceeded a good deal already in blue and white additional illuminations, and just started in violet and infrared. The plants which had received red, yellow, or green were not much different still from those receiving darkness in addition to the 10-hour day.

TABLE IV

Effect of supplementing a short day with light of various wavelength regions on stem elongation in *Brassica Rapa* var. Length of stem in cm, 26 – 27 days after beginning of treatment.

	First series 27 d. [1]	Third series 27 d. [1]	Fourth series 26 d. [2]
White	47.0	32.5	6.8
Violet	40.0	35.0	11.7
Blue	61.0	51.0	26.1
Green	21.0	4.0	2.3
Yellow	20.5	4.5	2.3
Red	21.5	2.5	2.0
Infrared	34.0	19.5	14.1
Dark	20.0	2.0	2.2

[1]) At an intensity of 3000 $ergs/cm^2/sec.$ (white 22000 $ergs/cm^2$ sec.).
[2]) At an intensity of 1000 $ergs/cm^2/sec.$

The fourth series which had not received any "long" day in white light beforehand, showed a similar behaviour (*cf.* Plate II, photographed after 38 days). The plants, supplemented with blue light are in full flowering, violet, white and infrared have just started. Red, yellow, green and dark are completely vegetative, and evidently less stretched than those pictured on Plate IA, which received a few long days beforehand (*cf.* also above). Very remarkable is the tall stature of the infrared plants which, before flowering, are already much longer than those in blue, violet and white (one specimen of the infrared group just starts flowering).

TABLE V

Effect of supplementary light of various wavelength regions on stem elongation in *Brassica Rapa* var. Length of stem in cm after various numbers of days, fourth series.

	Length of stem in cm			
White	1.6	—	6.7	17.3
Violet	2.5	4.2	11.7	27.4
Blue	5.0	13.7	26.1	35.6
Green	1.5	2.0	2.3	3.5
Yellow	2.0	2.1	2.3	4.6
Red	1.6	1.9	2.0	6.1
Infrared	4.0	6.4	14.1	43.1
Dark	1.4	2.0	2.0	3.0
Number of days	8	16	26	38

Table V and Fig. 1 give the length of the stem for this series, averaged over 10 plants, on various days. It is seen that the average length in infrared, *e.g.* after 40 days, has surpassed even that in blue, nothwith-

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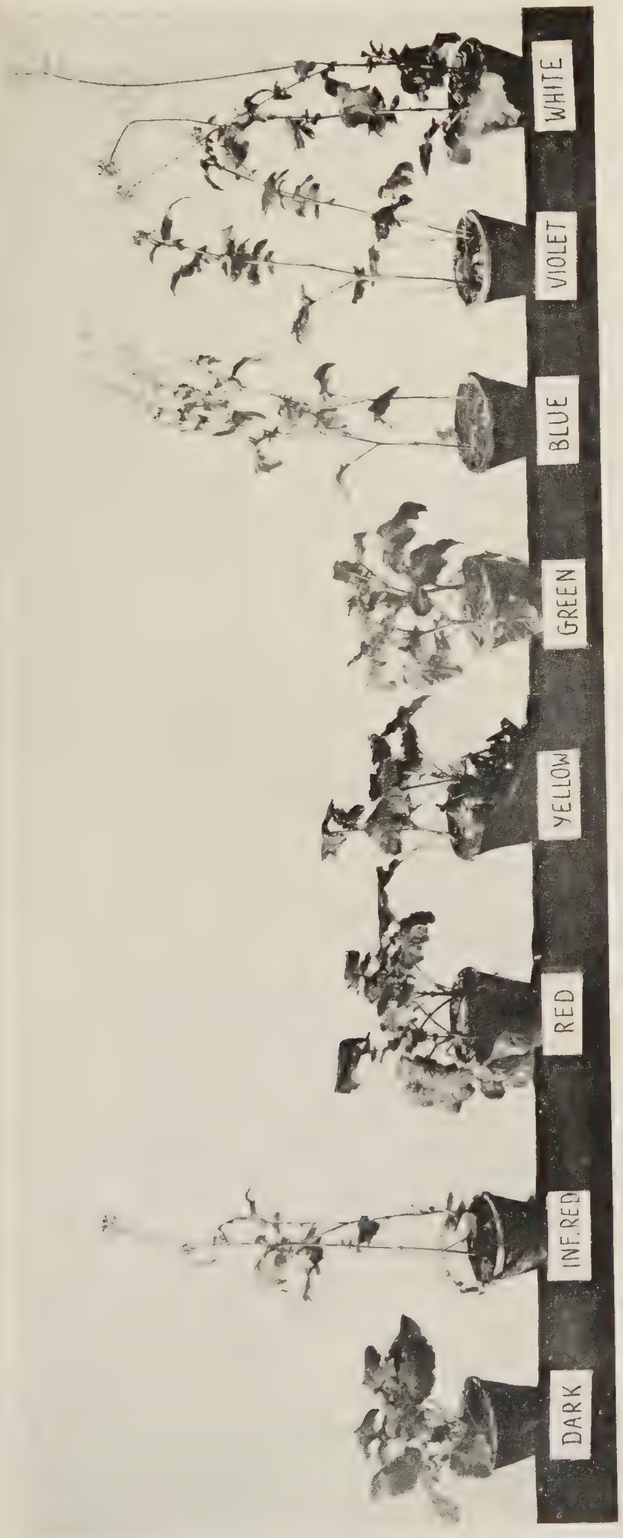

Plate 1. *A.* *Brassica Rapa* f. *oleifera* subf. *annua*, 31 days after beginning of treatment with coloured light (3rd series).

B. Microphotographs (× 200) of growing points of *Brassica Rapa* f. *oleifera* subf. *annua*. Left: vegetative, right: inflorescence primordium.

Plate 2. *Brassica Rapa* f. *oleifera* subf. *annua*, after 38 days of treatment with supplementary light at an intensity of 1000 ergs/cm^2/sec. total radiation.

standing the % flowering was still much less. Another feature, which was also evident in the first three series, is that the infrared region definitely promotes flowering as compared, *e.g.*, with the visible red, yellow and green. Whereas in the third series the aspect of the white-supplemented plants resembles the blue ones, in the fourth series it is more

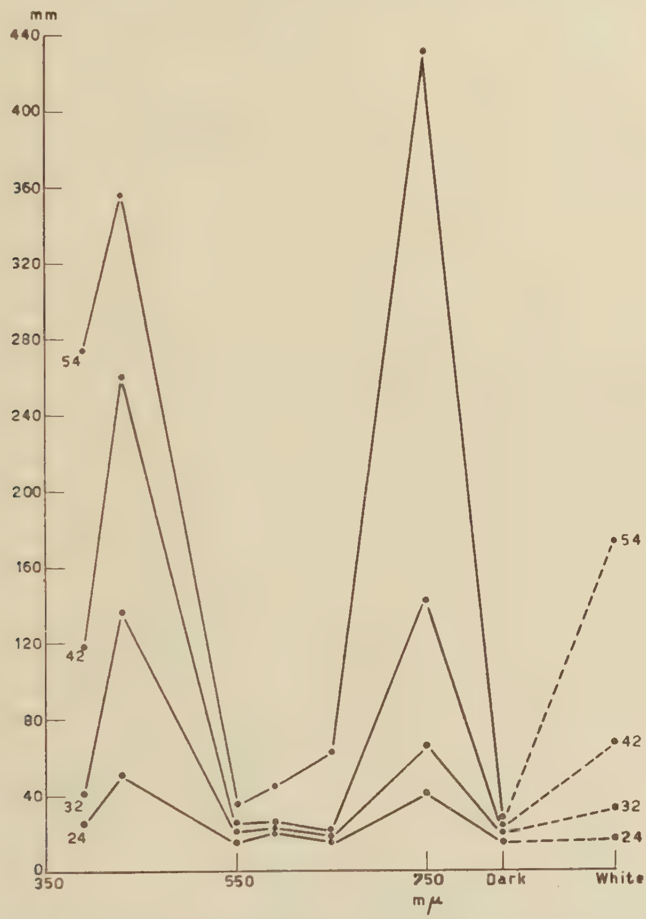

Fig. 1. Effect of supplementary light of various spectral regions (*cf* (14)) on stem elongation in *Brassica Rapa* var. Intensity of supplementary light: 1000 ergs/cm²/sec. total radiation. Numbers give plant age in days, treatment with supplementary light started when plants were 16 days old.

like the violet ones, which, probably, is due to the fact that the amount of white light supplied additionally was much higher in the third series.

It should be emphasized especially that the infrared compartment did not have a higher temperature than the other ones. Furthermore, owing to the relatively low intensities applied, in all compartments the leaf temperature, measured by a thermocouple applied between a double-folded leaf, indicated a leaf temperature of about 0.2 — 0.3° C below that of the air in the compartment.

The marked difference in aspect of the plants in the red and infrared compartments is the more striking because in the red compartment there is a rather considerable amount of near infrared radiation. One would presume that either the action of the near infrared is suppressed by red, or that the active infrared region is situated at longer wavelengths.

The data of Tables II and III indicate that there is some difference in response whether the formation of inflorescence primordia or the actual flowering is observed.

The red radiation seems to propagate the initiation of the inflorescence primordium, but it seems to prevent the subsequent development into flower primordia and flowers, whereas the further development of the inflorescence primordia is very rapid in blue, violet, white and infrared radiation.

Some observations on formative influences of the various wavelength regions.

A formative influence was manifest in two ways: the wavelength regions affected the stem elongation and the form of the leaves. The plant species used in our experiments produces a rosette of leaves when in vegetative condition, the leaves are approximately egg-shaped with long petioles. After a photoperiodic induction the stem begins to elongate, the later leaves are much smaller, sessile, and in the form of a narrow triangle. Hereafter these leaves will be referred to as vegetative and generative leaves respectively.

It appears from our data that in flowering plants the total growth of petioles is less than in non-flowering plants, which is due to the fact that the generative leaves before and after development of the seedstalk have no petioles. The leaf length continues to grow substantially in all cases, but owing to the fact that the last generative leaves are small, and because the vegetative plants continue to grow longer, they have more total leaf length growth, and also larger leaf area (*cf.* Table VI). The total growth

TABLE VI

Effect of supplementary light of various spectral regions on number and dimensions of leaves, and total leaf area of 10 plants in *Brassica Rapa* var. Duration of treatment 40 days. Intensity of supplementary light: 1000 ergs/cm²/sec.

	Total number of leaves	Average dimensions per leaf in mm	Average leaf area in mm²	Total leaf area in cm²	Ratio $\frac{\text{length}}{\text{breadth}}$
White . . .	67	57.2 × 38.0	2180	1400	1.51
Violet . . .	43	50.8 × 32.0	1680	722	1.55
Blue. . . .	61	41.7 × 24.1	960	582	1.72
Green . . .	66	54.5 × 41.0	2240	1480	1.33
Yellow. . .	53	62.4 × 49.7	3100	1640	1.25
Red	47	58.0 × 43.2	2500	1170	1.34
Infrared . .	54	52.7 × 38.0	2040	1100	1.36
Dark . . .	57	62.2 × 48.4	3000	1710	1.30

in leaf length is affected in the first place by the date of flowering. The earliest flowering plants have the narrowest average leaves — owing to the shape of the generative leaves, consequently the ratio total leaf length / total leaf breadth is highest in these cases (Table VI). This also appears from fig. 2, in which it is clearly shown that for the 5th — 7th leaf the ratio length / breadth *e.g.* in the blue plants is much higher than in the yellow ones, whereas for the 1st — 3rd leaf this ratio is much the

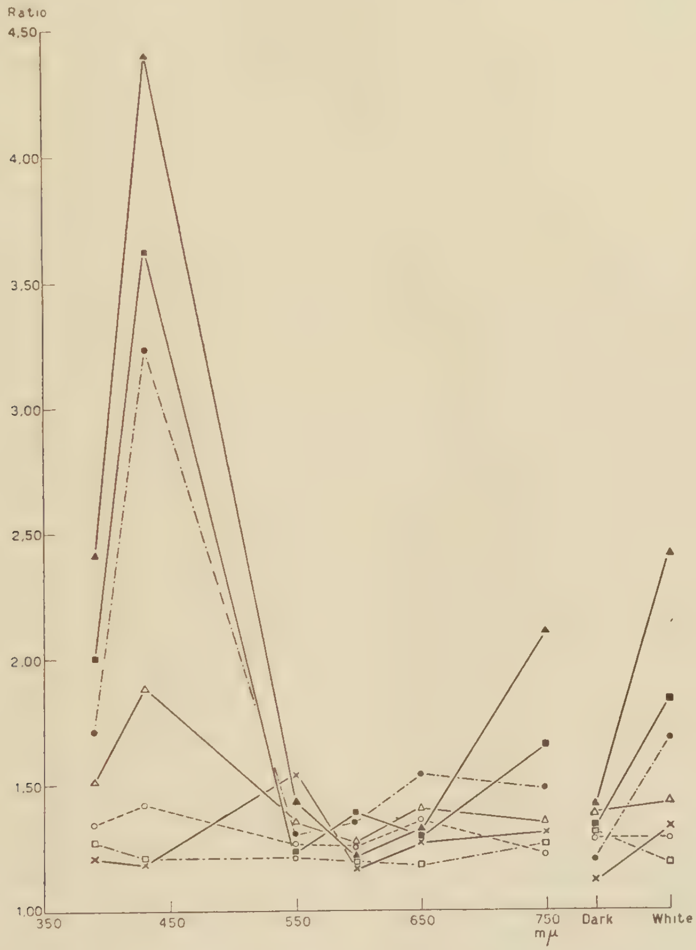

Fig. 2. Effect of supplementary light of various spectral regions (*cf* (14)), at an intensity of 1000 ergs/cm²/sec. total radiation, on the ratio leaf length/leaf breadth in *Brassica Rapa* var. Average of 16 plants. Experiment started Feb. 25, 1950; measured April 4, 1950 (38 days after beginning of treatment with coloured light).
× = 1st, □ = 2nd, ○ = 3rd, △ = 4th, • = 5th, ■ = 6th, ▲ = 7th leaf.

same in both cases. There is some formative effect of the various treatments on the shape of the very first leaf of the plants, before photoperiodic induction could have any effect. The plants in blue and violet irradiation have very nearly round first leaves, whereas especially in green

light the first leaf is more oblong. There is reason to consider the influence of the infrared radiation as being of a nature somewhat different from that of the influences of blue, violet and white light. The plants in infrared initiate their inflorescence primordia fairly late, but begin to elongate their internodes immediately. This elongation does not coincide with a formation of generative leaves as is the case in blue, violet and white light, so that the plants have a different aspect as is shown especially on Plate 2. ROODENBURG, on various occasions (*cf.*, *e.g.* (15)) has expressed the opinion — founded upon more or less indirect evidence — that the near infrared has a strongly elongating effect upon plants. This is thus confirmed directly in the present experiments.

Discussion.

Though FUNKE does not seem to have investigated *Brassica Rapa* f. *oleifera* subf. *annua*, there is no doubt that it belongs to his fourth group of plants, which do not react to red supplementary light, but have a strong response to the blue and violet wavelength regions, as far as their photoperiodic reaction is concerned. The present data do not permit to explain this behaviour which is different from that of many other plants studied (*cf. e.g.* 6, 7). The marked effect of the infrared irradiation found in these experiments seems to be due to photoperiodic induction only in part since the behaviour of these plants is somewhat different. At present this subject is being investigated further, together with a further investigation into the other groups distinguished by FUNKE.

The microscopical examination shows that inflorescence primordia are initiated fairly early in all cases, although there is a difference in the time of initiation between the plants in the various coloured supplementary illuminations. Greater differences occur in the development of the initiated primordia. Neither this development, nor the elongation of the stem and the development of a seedstalk seem to be promoted by red, green or yellow irradiation. The effectiveness of the blue and violet regions of the spectrum, combined with the inactivity of the rest of the visible spectrum might suggest carotenes or similar pigments to be the initial receptors of the photoperiodically active radiation. However, the photoperiodic effect of the infrared radiation then still remains obscure.

Summary.

Brassica Rapa oleifera subf. *annua* has been exposed to irradiation in different wavelength regions supplementary to a short-day illumination with day-light fluorescent tubes under constant conditions.

There is a strong photoperiodic response to blue, violet and infrared irradiations, whereas green, yellow and red irradiations have no effect on the photoperiodical behaviour of this plant. Additionally, the irradiations had formative influences on the leaves and the elongation

of the stem. A considerable elongation of the stem, much before flower formation, was observed only in the infrared irradiation. The effect can not be due to increase of temperature, neither in the compartment, nor in the leaves.

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ELEKTRONENMIKROSKOPISCHE UNTERSUCHUNG DER PFLANZENZELLEN

(EINFLUSS VERSCHIEDENER FIXATIONSFLÜSSIGKEITEN AUF DIE ZELLE)

VON

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I. *Einleitung*

Nachdem VON ARDENNE (1939) mit seinen ersten elektronenmikroskopischen Aufnahmen 0′ 5 μ dicker Gewebeschnitte gezeigt hat, dass Schnitte unterhalb 1 μ technisch herzustellen sind, haben sich mehrere Forscher mit dieser Frage weiter beschäftigt und das Problem auf verschiedene Weise gelöst: CLAUDE & FULLAM (1946), PEASE & BAKER (1948), BRETSCHNEIDER (1949), NEWMAN, BORYSKO & SWERDLOW (1949), DANON & KELLENBERGER (1950). Unserer Erfahrung gemäss ist heute bereits die einwandfreie Herstellung bis 0′ 2 μ dicker Schnitte und deren erfolgreiche elektronenmikroskopische Untersuchung als gesichert zu betrachten. Nach diesen rein technischen Fortschritten ergeben sich nun in erster Linie prinzipielle Fragen die das Objekt selbst und die Interpretation der Befunde betreffen. Einer der ersten Fragen ist dann in Anbetracht der hohen Vergrösserungen die Güte des Fixationszustandes unserer Objekte. Von ihr hängt nl. zum Grossteil der Wert der Interpretation ab. In der vorliegenden Veröffentlichung dokumentieren wir zunächst die vergleichende elektronen-optische Untersuchung des Einflusses verschiedener Fixationsflüssigkeiten auf ein und dasselbe Gewebe, nl. dem Meristem der Wurzelspitze von *Allium cepa*. Hinsichtlich der Technik verweisen wir auf unsere früheren Mitteilungen (BRETSCHNEIDER, 1949—1950).

II. *Technische Bemerkungen*

Die jungen Wurzelspitzen von ca 2 Millimeter Länge wurden mit verschiedenen Fixierungsflüssigkeiten nach den herkömmlichen histologischen Angaben fixiert, in dem von uns beschriebenen (L. H. BRETSCHNEIDER,

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1950, *a*) Wachsparaffin eingeschlossen und mit dem "Cambridge Rocking Microtome" bei 0° C bis 4° C längsgeschnitten. Die 0′ 2 bis 0′ 4 μ dicken Schnitte wurden nach Entparaffinieren mit Xylen mit den Delfter Elektronenmikroskopen bei einer Spannung von 110 kV untersucht. Für weitere technische Einzelheiten verweisen wir auf unsere früheren Veröffentlichungen (BRETSCHNEIDER, 1949—1950, *a*, *b*).

III. *Einfluss verschiedener Fixationsmittel auf die Feinstruktur des Zellplasmas*

Mehr noch als die Lichtmikroskopie stellt die Elektronenmikroskopie hohe Anforderungen an die Fixation biologischer Objekte. Eine Fülle von Fixationsmitteln aus der histologischen Technik stehen der Elektronenmikroskopie zur Verfügung und es ist zu prüfen welche davon den hohen Anforderungen entsprechen. Wir haben, um zu einem richtigen Urteil zu kommen, verschiedene Fixationsmittel an ein und demselben Objekt und bei gleicher Vergrösserung von 12000 × und bei gleicher Spannung von 100 kV untersucht. Die betreffenden Fixationsmittel sind mit ihren Bestandteilen in der Tabelle I vereinigt, während Bildausschnitte aus den elektronen-optischen Aufnahmen in der Tafel 1 zusammengefasst sind.

Wir haben eine Pflanzenzelle aus dem Meristem gewählt, weil sie 1) viel Hyaloplasma und wenige paraplasmatische Einschlüsse besitzt, 2) durch den Besitz von Zellwänden die Diffusionsschwierigkeiten der Fixationsflüssigkeiten erhöht und dadurch den Fixationserfolg oder Misserfolg noch deutlicher wiedergibt, 3) durch die rege Teilungsfähigkeit des Gewebes die Untersuchung von Mitosestadien fördert. Wir haben nun zur Beurteilung der Güte eines Fixationsmittels die Teilchengrösse des denaturierten Plasmas genommen, weil anzunehmen ist, dass die Eiweissmoleküle ihre grösste Dispersität — also die kleinste Teilchengrösse — im lebenden Plasma besitzen, während jede Vergröberung durch die Fixation als Teilchenvergrösserung zum Ausdruck kommen muss.

In der Tafel 1 sind fortlaufend von links oben nach rechts unten die Aufnahmen nach ihrer Teilchengrösse angeordnet, während in der Tabelle I die mittlere Dicke und Länge der Teilchen eingetragen wurde. Die geringe aber sehr gleichmässige Teilchengrösse der ersten 6 Aufnahmen musste bei dieser Vergrösserung mit der Luppe bestimmt werden. Bei einer genauen Untersuchung ergibt sich, dass das Grundplasma aus einem regelmässigen Geflecht feiner Proteinfilamente besteht, deren Dicke ca 160 Å beträgt. Sie bilden durch ihre geometrische Anordnung ein Netzmuster welches schematisch in der Figur *A* wiedergegeben wurde. In der einen Richtung betragen die Abstände ca 620 Å in der Senkrecht daraufstehenden ca 420 Å. Das Netzmuster lässt sich zurückführen auf ein annäherend hexagonales System von Haftpunkten mit welchen die Proteinfilamente zusammenhängen. Diese Filamente besitzen demnach eine bestimmte Periodizität in ihrem Verlauf, auf welche wir in unserer nächsten Mitteilung noch eingehender zurückkommen werden.

TABELLE I

Teilchengrösse in Ångström	Osmiumtetroxyd	Sublimat	Phosphor Wolframsäure	Natrium sulfat	Trichloressigsäure	Sulfosalicylsäure	Pikrinsäure	Kaliumbichromat	Chromsäure	Essigsäure	Chloroform	Formol	Alkohol	
160 × 600	×							×	×					Champy
160 × 600								×				×		Regaud-Kopsch
160 × 600							×			×		×		Bouin
200 × 800												×		Formol
200 × 800		×		×				×				×		Helly
200 × 800	×									×				Flemming
200 × 800		×			×							×		Romeis
400 × 1600													×	Alkohol
600 × 1600										×	×		×	Carnoy
400 × 2500										×				Essigsäure
600 × 3000		×								×			×	Schaudinn
800 × 3000		×		×				×		×				Zenker
800 × 3500						×								Sulfosalicylsäure
1600 × 4500			×										×	Rawitz
nur an	×													Osmiumtetroxyd
tierischen	×	×												Apathy
Zellen erprobt		×					×			×			×	Lenhossek

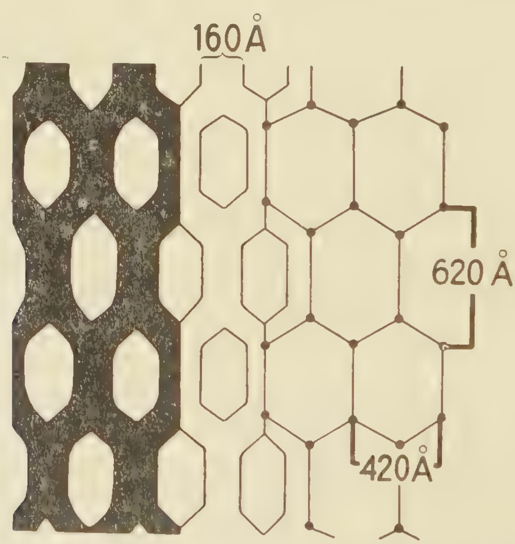


Fig. A. Schema der Grundnetzstruktur des Plasmas nach Fixation mit der Champy'schen oder Regaud'schen Fixationsflüssigkeit. Scheme of the ground-pattern of the plasm after fixation with Champy's or Regaud's fluid.

Es ist ziemlich wahrscheinlich dass dieses zarte und regelmässige Strukturmuster das nichtvergröberte Niederschlagsbild der ursprünglichen Ultrastruktur des Cytoplasmas darstellt. Es tritt deutlich auf nach den beiden besten Fixationsgemischen die wir bisher kennen, dem Gemisch von CHAMPY (Tafel I/1) und dem von KOPSCH–REGAUD (Tafel I/2). Beide sind hinsichtlich ihres Resultates und dem Fixationsprinzip ihrer Komponenten gleichwertig, unterscheiden sich aber in ihrer Zusammensetzung. Sowohl das Osmiumtetroxyd als auch das Formol besitzen die folgenden gemeinsamen Eigenschaften: 1) sie sind wenig fällungsfähig gegenüber Eiweissen, 2) führen aber die Plasmastrukturen in einen nachher unlöslichen Zustand über, 3) wirken homogenisierend d.h. erzeugen die kleinste Teilchengrösse, 4) führen nebst Eiweissen auch Lipoide und Fette in weniger löslichen Zustand über, 5) sie fixieren sowohl den Kern als das Plasma in gleich günstigem Sinne. Unterstützt werden diese Eigenschaften noch überdies durch die Kombination mit Kaliumbichromat oder Chromsäure. Auch diese beiden kennzeichnen sich durch ihre Eigenschaft Proteine und Lipoide in schwer lösliche Verbindungen über zu führen und ergänzen dadurch die gleiche Wirkung des Osmiumtetroxydes oder Formols, wobei sie dafür Sorge tragen, dass noch nicht gefällte Anteile ebenfalls unlöslich gemacht werden. Gegenüber Kern und Cytoplasma verhalten sich die beiden Chromverbindungen in ihrer Fixationswirkung aber antagonistisch. Die Chromsäure fixiert den Kern gut aber das Plasma schlecht, während das Kaliumbichromat umgekehrt den Kern schlecht und das Plasma gut fixiert. Unterdrückt werden die schlechten Wirkungen durch die Kombination mit dem Formol (KOPSCH–REGAUD) oder OsO4 (CHAMPY) während sich beim Gemisch von CHAMPY überdies noch durch Vereinigung von Chromsäure mit dem Kaliumbichromat die antagonistischen Wirkungen dieser beiden aufheben. Hinsichtlich der "färbenden" Wirkung von Metallen mit einem höheren Atomgewicht in der Elektronenmikroskopie verdient die KOPSCH–REGAUD'sche Fixationsflüssigkeit mit seinem Formol als rein organischer Stoff und dem Kaliumbichromat (K_{39} Cr_{52}) den Vorzug gegenüber der osmiumhältigen (Os_{190}) von CHAMPY. Die nichterwünschte Anfärbung durch Osmiumtetroxyd kann in vielen Schnitten durch die hohe Elektronenstreuung störend wirken und eine nachträgliche Färbung z.B. mit Phosphor-Wolframsäure beeinträchtigen. Die erzielte günstige Durchstrahlung sehr dünner Schnitte wird dann durch die stärkere Elektronenstreuung des OsO_4 wieder eliminiert.

Für die weitere Beurteilung der folgenden Fixationsmittel legten wir die durch die ersten beiden erzielte Teilchengrösse als Maaszstab an. Das Strukturbild des Plasmas nach der Fixation mit dem BOUIN'schem Gemisch zeigt noch die gleiche Teilchengrösse, kennzeichnet sich aber durch eine gewisse schrumpfende Wirkung, wobei die Grundnetzstruktur zusammengeschoben wird. Dadurch kann man sie nur an den besonders dünnen Teilen sehen. Dieses Formol- und Pikrinsäure-hältige Gemisch hat sich

auch für tierische Gewebe elektronen-optisch als noch ziemlich zureichend erwiesen (BRETSCHNEIDER, 1949 -1950). Es stellt selbst noch den periodischen Aufbau (Querstreifung) gewisser Eiweissketten dar (Cnidocystenmembrane) obwohl es kein Schwermetallsalz besitzt. Die groben Artefakte welche die mit BOUIN'schem Gemisch fixierten Gewebe in der Veröffentlichung von NEWMAN c.s (1949) aufweisen, sind unserer Erfahrung gemäss nicht durch die Fixation, sondern durch eine zu rasche und nicht schonende Nachbehandlung in der Alkoholreihe entstanden, vielleicht auch während der Einbettung im Metacrylat.

Die fixierende Wirkung des Formols ergab bereits eine gewisse Vergröberung der Grundnetzstruktur, obwohl an vielen Stellen noch eine deutliche Netzstruktur im Plasma zu sehen ist. Die 10 % Lösung fixierte besser als die 4 %. Im Vergleich mit den vorhergehenden 3 Fixationsmitteln bei welchen die Netzstruktur eine kontinuierliche Anordnung zeigt, entstehen beim Formol grössere Lücken indem die Basisstruktur wahrscheinlich örtlich bricht oder zum Teil aufgelöst wird. Bekanntlich beruht die Formolfixation auf der Methylierung der Eiweisse wobei u.a. Säuren auftreten wie Methyleneiweisskörper mit ihren wirksamen Karboxylgruppen (SPATZ, 1923) und Phosphorsäure (WEIL, 1930) welche durch Hydrolyse von Phosphatiden entsteht. Durch eine teilweise Lösung von Eiweissen nimmt das Formol während der Fixationsdauer den Charakter einer Pufferlösung an und steigt die pH von 3′3 auf 4′7. Durch die sich anschliessende Alkoholbehandlung werden aber etwa gelöste oder verlagerte Substanzen wieder niedergeschlagen. Auch könnte eine Strukturvergröberung durch die leichte quellende Wirkung des Formols zustande kommen und nachher im Alkohol fixiert werden. Jedenfalls kommt es, wie die Aufnahmen zeigen, zu einer gelinden Strukturvergröberung wobei aber der Diameter der an der Basisstruktur beteiligten Proteinbündel noch nicht so stark vergrössert ist als bei den nachfolgenden Fixationsmitteln. Recht sinnfällig wird bei einem Vergleich der Abbildung 2 mit 4 in der Tafel 1 die eingreifende Verbesserung der Strukturfixation bei der Kombination des Formols mit dem Kaliumbichromat im REGAUD'schen Gemisch. Abgesehen von der leichten Vergröberung der Feinstruktur taugt aber das Formalin noch ziemlich gut für die elektronenmikroskopische Untersuchung. Es hat als rein organisches Fixationsmittel den grossen Vorteil dass keinerlei Imprägnation oder Anfärbung durch Metalle stattfindet und dass sich viele komplexere Zellbestandteile, wie die Kernstrukturen, Mitochondrien, Fibrillen u.a. gut erhalten.

Zu dem letzten von uns untersuchten Fixationsmittel das noch einigermassen die Grundnetzstruktur zur Darstellung bringt, gehört das von HELLY eingeführte *Zenker-Formol*. Es besitzt gleich dem REGAUD'schen Gemisch Formol und Kaliumbichromat und verdankt diesen beiden seine bessere Wirkung als das nahverwandte Gemisch von ZENKER (Essigsäure) (Tafel 1 no 13) denn das Sublimat als weitere Komponente ergab sich als ein schlechtes Fixationsmittel. Das HELLY'sche Gemisch steht dem Formol

insoferne nach, als die Grundnetzstruktur stärker zersprengt ist. Ausserdem machen sich Quecksilberniederschläge störend bemerkbar. Das HELLY'sche *Zenker-Formol* kann aber für schwächere Vergrösserungsbereiche noch gute Dienste leisten und wurde mit Erfolg von NEWMAN c.s. (1949) für die Niere gebraucht.

Alle nun folgenden Fixationsmittel (7 bis 15) sind ihrer hohen Teilchengrösse wegen für die Elektronenmikroskopie zu vermeiden. Ihr Einfluss auf das elektronen-optische Bild zeichnet sich aus: 1) durch den völligen Mangel einer Grundnetzstruktur, 2) durch eine in der angegebenen Richtung 7—15 zunehmenden Vergröberung der dargestellten Teilchen von 400 Å bis 8000 Å, 3) durch eine oft weitgehende Verlagerung selbst grösserer Strukturen, 4) durch Lösung bestimmter Anteile. Da das Plasma aus hydrophilen Mischkolloiden besteht, kommt es bei der schlechten Fixierung zu einer ausgedehnten Koazervation wobei durch die Scheidung in kolloidreiche und kolloidarme Anteile eine mehr oder weniger grobe Ausflockung stattfindet und schliesslich durch Zusammensintern der viskösen Koazervattropfen eine eingreifende Aggregation eintritt. Wenn man untersucht welche Fixationsmittel in dieser Reihe liegen, dann kann man feststellen, dass merkwürdigerweise stark eiweissfällende Stoffe gerade am ungeeignetsten sind für die Erhaltung der aus Eiweissen bestehenden Grundstruktur. Hierzu gehören Trichloressigsäure des ROMEIS'schen Gemisches (8), Sulfosalicylsäure (14), Phosphor-Wolframsäure des RAWITS'schen Gemisches (15), Sublimat im Gemisch von ZENKER (13) und SCHAUDINN (12). Nicht die stärksten Fällungsmittel für Eiweiss, sondern die sehr schwach fällenden wie Osmiumtetroxyd und Formol erhalten die Grundstruktur durch ihre schonende Wirkung am besten. Auch nicht die rasch diffundierenden Stoffe wie Essigsäure (11), Trichloressigsäure (8), Chloroform im Gemisch von CARNOY (10) oder Alkohol (9) ergeben gute Resultate; weder allein noch als Zusatz zu weit besseren Fixationsmitteln taugen sie in der Elektronenmikroskopie. Schon weil diese schnell diffundierenden Stoffe im Laufe der Fixation den weniger diffusiblen Komponenten vorauseilen, wird die günstige Wirkung aufgehoben. Ein treffendes Beispiel liefert das Flemming'sche Gemisch (7) das trotz der OsO4 und Chromsäure ein weit gröberes Fixationsbild ergibt als das essigsäurelose Gemisch von CHAMPY (1). Für die Erhaltung feinerer Strukturen untauglich ist die von NEWMAN c.s. (1949) ebenfalls am Meristem der Zwiebelwurzel angewandte Chromessigsäure. Als völlig untauglich erwies sich das Gemisch von RAWITZ (15) obwohl es die in der Elektronenmikroskopie als Färbemittel so geschätzte Phosphor-Wolframsäure enthält. Zu der sehr groben Ausflockung dieser Lösung — diese Säure ist eins der heftigsten Fällungsmittel für Eiweisse — gesellt sich noch die anfärbende Wirkung des schweren Wolframiums (W_{184}), wodurch eine ausgesprochene Silhouettewirkung im elektronenmikroskopischen Bild zustande kommt. Ebenso wie das Formalin allein angewandt nicht die günstigste Fixation entfaltet, sondern erst im Verein mit Chromverbindungen, erwies sich — jedenfalls

für botanische Objekte — das Osmiumtetroxyd allein wegen seiner geringen Diffusion als untauglich, für dünne zoologische Objekte hat es sich als sehr wertvoll erwiesen.

IV. *Das elektronenoptische Bild der Meristemzelle nach der Formolfixation*

In einer unserer früheren Publikationen (Bretschneider, 1950 *a*) haben wir eine Aufnahme der Meristemzelle der Zwiebel nach der Kopsch-Regaud'schen Fixation veröffentlicht. Die folgenden Abbildungen zeigen das selbe Gewebe nach der Formolfixation wobei wir vorallem die sich teilende Zelle berücksichtigen. Diese Aufnahmen stellen mit allen ihren differenzierten Grautönen das "unbeeinflusste" Streuungsbild des Gewebes dar, wobei lediglich die von Natur aus vorhandenen Dichteunterschiede seiner Strukturen und Substanzen bestimmend wirkten bei dem Zustandekommen des elektronenoptischen Bildes. So sehen wir in der Übersichtsaufnahme Abbildung 2 wie sich deutlich die dichteren Mitochondrien vom weniger dichten Plasma abheben, wie sich in den weitaus dichteren Interkinesekernen die feinverteilte Chromatinstruktur von der weniger dichten Karyolymphe abhebt, während die Nukleolen die grösste Dichte besitzen. Das Cytoplasma der sich teilenden Zelle in der Mitte des Bildes zeichnet sich aus durch eine grössere Dichte als dasjenige der anderen Zellen. es ist frei von Vakuolen und lässt die Anordnung der Chromosomen und Mitochondrien an der Peripherie der Teilungsspindel sehen. In der gleichenelektronenoptisch aber stärker vergrösserten Zelle der Abbildung 3 erkennt man eine lineare Anordnung des Grundnetzes in der Längsachse der Teilungsspindel. Im Grundplasma eingebettet finden wir eine geringe Zahl dünner Fibrillen deren Durchmesser 40 bis 60 mμ misst. Sie scheinen sich unmittelbar aus den Filamenten des Grundnetzwerkes durch Verschmelzung zu bilden und zeigen einen perlschnurartigen Aufbau indem globulare und filare Teile miteinander abwechseln. Zahlreiche Granula welche 40—100 mμ messen erfüllen die Spindel und bilden im Spindeläquator dickere Tropfen. Aus der zentralen Anhäufung dieser Tropfen entsteht die zukünftige Zellplatte als Scheidewand der beiden Tochterzellen. Die noch kompakten zentripetalen Chromosomenenden lassen eine glatte Begrenzung erkennen. An ihnen heften sich — wie man sieht — die in der Längsrichtung orientierten Plasmafilamente fest. Die unmittelbare Umgebung der Chromosomenenden ist strukturärmer d.h. wasserreicher, während die Spindelmitte sehr dicht ist. Die Mitochondrien sind stäbchenförmig und ca 300 mμ breit und bis 2 μ lang. Sie sind nicht homogen sondern zeigen dichtere und weniger dichte Teile. Die nach der Zellmitte orientierten Enden der Anaphasechromosomen waren nicht durchstrahlbar, dahingegen bemerkt man in den peripheren Teilen bereits eine gewisse Auflockerung. Besonders im zweiten Chromosom linksoben sehen wir die elektronenoptische Projektion der Chromonemaspirale als dunkleres Zickackband während sich die Chromosomenmatrix als eine weniger dichte Hülle abbildet.

In einem folgenden Schnitt durch eine Meristemzelle in der Prometaphase, Abbildung 4, erkennt man aus der Form die gepaarten Chromonemata. Trotz der grossen Dichte kann man doch eine weniger kompakte Grundmasse — die Matrix — unterscheiden in welcher die stark gestaute Chromonemaspirale als weit dichtere Struktur eingebettet liegt. Um manche Chromosomen ist das umgebende Cytoplasma strukturärmer (Mitte rechts und links) und daher lichter auf dem Bilde. Die Grundnetzstruktur zeigt in dieser Zone eine radiäre Anordnung. Vielleicht vollzieht sich im lebenden Zustande in dieser wasserreicheren Hülle die Verschiebung der Chromosomen zur Äquatorialplatte. Das Spindelplasma lässt einige 60 mμ dicke Fibrillen erkennen und zeigt im Übrigen eine ziemlich gleichmässige und dichte Netzstruktur. An verschiedenen Orten (siehe Pfeile) erkennt man fibrilläre Strukturen in welchen in ziemlich regelmässigen Abständen reihenweise Granula angeordnet sind. Wir finden diese 60 bis 80 mμ grossen Granula auch in den Stosspunkten des Grundnetzes. Als weitere Elemente finden wir einzeln oder gruppenweise vereinigt ca 100 mμ grosse dichtere Granula (g) und an der Peripherie des Phragmoplasten die Mitochondrien und die Benzley-Bowen'schen osmiophielen Körperchen (p). Man erkennt diese dem Golgiapparat homologen Gebilde an ihrer dichteren (osmiophielen) Aussenzone. An verschiedenen mit (n) bezeichneten Orten sieht man in dieser Aufnahme schön die regelmässige Anordnung der Grundnetzstruktur.

Nach erfolgter Zellteilung werden zwischen den dünnen primären Zelloberflächen Pektinstoffe als Zellwandmaterial abgeschieden. Man sieht diesen Vorgang in der Abbildung 5, in welcher das Pektin als eine homogene wenig dichte Masse in den Interzellularen abgesetzt wird.

V. *Diskussion zu den bisherigen Ergebnissen der elektronenoptischen Schnittuntersuchung*

Obwohl sich die Elektronenmikroskopie von Gewebeschnitten noch in ihrem Beginnstadium befindet und ihr manche Mängel anhaften, bestrebt sie sich doch gegenüber der Lichtmikroskopie einige wesentliche Fortschritte zu zeitigen. Es ist darum schon am Beginn erwünscht Unzulänglichkeiten und Fortschritte im elektronenoptischen Verfahren kritisch zu beurteilen.

Ausser den hier besprochenen Fixierungsflüssigkeiten liessen sich natürlich noch viele andere in der Lichtmikroskopie gebräuchliche Fixationsmittel elektronenoptisch testen. Wir glauben aber dass sich der alte schon so oft geäusserte Wunsch der Histologen die vielen verschiedenen Fixationsmittel vielleicht einst durch ein oder einige wenige "Standardmittel" ersetzen zu können, in der Elektronenmikroskopie nun verwirklichen lässt. Zwei wichtige Momente tragen nämlich in der Elektronenmikroskopie zu einer tiefgreifenden Einschränkung der grossen Zahl bei: erstens die Vorbedingung einer kleinst möglichen Teilchengrösse als Ausdruck einer annäherend naturtreuen Fixation und zweitens das Wegfallen der

in der Histologie berechtigten Anforderung an ein Fixationsmittel das fixierte Objekt mit verschiedenen Farbstoffen darstellen zu können. Nur die erste Bedingung muss in der Elektronenmikroskopie verwirklicht sein.

Aus verschiedenen Gründen ist schon heute am Beginn unserer elektronenoptischen Schnittuntersuchung eine Klärung der Fixationsfrage erwünscht. Die Elektronenmikroskopie ermittelt nämlich einen Grossteil des geförderten Tatsachenmaterials durch "Messung". Diese hat dann erst Sinn wenn man sie vergleichend auswertet indem man sie mit andere gemessene Dinge in Beziehung bringt. Es wäre nun verfehlt Gemessenes miteinander zu vergleichen das durch irgend eine technische Ursache z.B. Fixation nicht vergleichbar ist. Je mehr Fixationsmittel wahllos durch die verschiedenen Untersucher verwendet werden, desto unvergleichbarer werden die Resultate ausfallen, weshalb eine gewisse Normalisierung auf diesem Gebiete sehr erwünscht wäre.

Wie aus dieser vergleichenden Untersuchung von 15 gangbaren Fixationsmittel hervorgeht, kommen nur das Osmiumtetroxyd, Formol, Kaliumbichromat und die Chromsäure aus dieser Reihe in Frage. Möglicherweise kommen zukünftig noch andere neue oder erprobte hinzu welche den gestellten Anforderungen der kleinsten Teilchengrösse und der Erhaltung der Grundstruktur des Plasmas entsprechen. Nur insoferne die besten Fixationsmittel aber übereinstimmen in diesen Eigenschaften sind die betreffenden Befunde untereinander vergleichbar.

Warum die elektronenoptische Prüfung der Fixationsmittel wie wir sahen so kritisch ausfällt und viele in der Histologie geschätzte Fixierungsflüssigkeiten ablehnt, findet seinen Grund in der geringen Dicke der elektronenmikroskopischen Schnitte. Sie sind nämlich rund 25 bis 50 mal dünner als jene der Lichtmikroskopie und dies hat zur Folge dass sich die unvermeidlichen Artefakte bereits viel früher geltend machen als beim viel dickeren histologischen Schnitt. Denkt man sich 25 bis 50 elektronenmikroskopische Schnitte übereinander projektiert, dann werden sich in diesen Schichten Lücken und Materie derart oft decken, dass auch dann noch ein brauchbares lichtmikroskopisches Bild zustande käme, wenn jeder elektronenmikroskopische Einzelschnitt an sich schlecht ist. Es ist darum jetzt erwünscht die Güte eines Fixationsmittels elektronenoptisch zu prüfen weil nur an solch dünnen Schnitten der Erfolge der Fixation befriedigend beurteilt werden kann.

Zur Frage der Schnittdicke möchten wir noch erwähnen, dass wir auf verschiedenem Wege optisch die Dicke mancher unserer Schnitte bestimmt haben, weil diese nicht immer übereinstimmt mit der beim Schneiden eingestellten Mikrotomskala. Die Schnittdicke ist nämlich das Resultat sowohl der mechanischen Fortbewegung des Objektes durch das Mikrotom als auch der beim Schneiden gerade herrschenden Temperaturkonstellation. Geringe Temperaturschwankungen führen schon zum Zusammenziehen oder Ausdehnen des Objektes und seines Einschlussmittels, sodass nur bei einem Temperaturgleichgewicht zwischen Messer, Block und

Umgebung die Schnittdicke garantiert ist. Es gibt neuerdings aber auch ein Schneideverfahren nämlich das von NEWMAN c.s. (1949) bei welchem die Schnittdicke unterhalb 1 μ erzielt wird durch abwechselndes Erwärmen und Abkühlen des in Metacrylat eingeschlossenen Objektes. Wir geben aber der weit bequemeren und zuverlässigeren Schnittführung auf mechanischem Wege den Vorzug.

Die Erfahrung lehrt, dass man durch Vergleich der photographischen Schwärzung in Aufnahmen mit getesteter Schnittdicke auf diejenige anderer Aufnahmen schliessen kann. Einige Angaben in der Literatur über erreichte Schnittdicken von 0′ 1 μ sind unserer Erfahrung nach nur mit Vorbehalt zu betrachten. Diese Dicke ist sicherlich ein erstrebenswertes Ziel in der Elektronenmikroskopie, doch muss mit ihr überdies die Erhaltung der topographischen Verhältnisse im Schnitt einhergehen. Die Schwierigkeiten in dieser Hinsicht nehmen aber mit abnehmender Schnittdicke rasch zu. Schon bei unseren 0′ 3 und 0′ 2 μ dicken Schnitten erfahren wir öfters, dass der Widerstand zwischen Messer und Objekt nanche Teile des Schnittes verlagert und dadurch Lücken entstehen, die bei der weiteren Verarbeitung beim Strecken und Aufziehen auf den Trägerfilm noch verstärkt werden.

Vergleicht man das elektronenoptische Bild mit dem lichtoptischen, dann fällt vorallem auf, dass nach der reinen Formolfixation im Grossen und Ganzen das elektronenmikroskopische Bild in seinen Nuancierungen mit dem herkömmlichen gefärbten Bild in der Histologie übereinstimmt, obwohl es doch weder imprägniert noch tingiert ist. Im elektronenoptischen Bild bestimmen aber lediglich die Unterschiede in der Dichte d.h. dem Grad der molekularen Verteilung der verschiedenen organischen Substanzen in den Strukturen den photographischen Kontrast. Berechnungen ergaben, dass der höhere Phosphorgehalt in den Kernstrukturen prozentuell zu gering ist um als Massenunterschied eine sichtbare Rolle beim Entstehen der photographischen Schwärzung zu spielen. Wäre der Schnitt dicker oder das Objekt mit Schwermetallen behandelt (Tafel 1 Figur 15) oder die Spannung während der Aufnahme geringer, dann entstünde von dem selben Objekt ein Schattenbild. Will man also ein befriedigendes differenziertes Bild erhalten, dann müssen sowohl Fixation, Schnittdicke und Spannung in einer gewissen Korrelation zu einander stehen (siehe auch BRETSCHNEIDER 1949—1950 *a*, *b*). Der ungefärbte mit dem Lichtmikroskop betrachtete Schnitt zeigt gegenüber dem elektronenoptischen bekanntlich weit geringere Kontraste und muss daher immer gefärbt werden. Aus der ähnlichen Kontrastverteilung beider Bilder kann man den Schluss ziehen, dass die an sich schon dichteren Strukturen dabei mehr Farbstoff aufnehmen und dann satter gefärbt erscheinen. Durch Salzbildung zwischen Farbstoff und Struktur und durch das Differenzieren bei der Weiterbehandlung werden Farbstoffe aber fester gebunden respektive wieder abgegeben, sodass das gefärbte Präparat nie das reine Bild der Strukturdichte mehr darstellt, wie dies beim elektronenoptischen

der Fall ist. Man kann das elektronenoptische Bild noch besser vergleichen mit dem Phasenkontrastbild in welchem ebenfalls ohne Färbung Kontraste auftreten und zwar durch Unterschiede in der Dicke und der Brechzahl der Strukturen. Aus den hier wiedergegebenen Bildern können wir also den Schluss ziehen, dass Mitochondrien, Chromatin und Nukleolen ein sehr dichtes molekuläres Gefüge mit kleinen intermicellaren Lücken besitzen müssen, die intercellulare Cellulose dahingegen ein sehr lockeres Gefüge. Ein solcher Schluss liesse sich aus der lichtmikroskopischen Betrachtung eines ungefärbten Schnittes nicht ziehen.

Die Schwierigkeiten der elektronenmikroskopischen Schnittuntersuchung liegen heute nicht so sehr mehr im Anfertigen der Schnitte, sondern in der Interpretation der Ergebnisse. Durch die sprunghafte Verlegung des optischen Auflösungsvermögens von 200 mμ des Lichtmikroskopes auf 2 mμ des Elektronenmikroskopes wird uns nun eine Strukturtreppe zugänglich gemacht über welche man bisher nur theoretische Überlegungen anstellen konnte. Schon bei der 12000 fachen Vergrösserung der hier abgebildeten Aufnahmen kommen die Plasmafilamente des Grundnetzes mit einem Durchmesser von bloss 16 mμ zur Darstellung. Das im Lichtmikroskop nach Formol oder Osmiumgemischen völlig homogen erscheinende Hyaloplasma löst das Elektronenmikroskop in seine Ultrastruktur auf, die aus einem Netzwerk mit einer Maschenlänge von ca 60 mμ besteht. Es lassen sich in ihr deutlich globuläre und fibrilläre Gebilde als einfachste Bauelemente unterscheiden, die also wesentlich kleiner sind als die 200 mμ der lichtmikroskopischen Auflösungsgrenze. Die Frey–Wyssling'sche Voraussage (Frey–Wyssling, 1938) dass das Cytoplasma aus Polypeptidketten besteht die miteinander zu einem Molekulargerüst verwachsen, während die Seitenketten durch "Haftpunkte" miteinander in Beziehung treten, wird elektronenmikroskopisch bestätigt. Aus der gefundenen Dicke von 16 mμ für die Netzfilamente ist zu schliessen, dass es sich um mehrere parallel gestellte Eiweissketten handelt und nicht um Einzelketten. Aus der sehr regelmässigen Konfiguration der Netzstrukturen (Fig. *A*) ist zu schliessen, dass die von Frey–Wyssling vorausgesagten Haftpunkte selbst eine strengere Anordnung besitzen als man ursprünglich dachte und vielleicht ihre Ursache finden in dem periodischen Aufbau des Eiweissmoleküls. Die regelmässige Gitterstruktur erinnert schon stark an einen pseudokristallinen Zustand des Grundplasmas wobei die Gitterelemente und die Kohäsionskräfte aber von einer höheren Ordnung sein müssen. Wir kommen in einer nächsten Publikation über den Feinbau des Neurons hierauf noch näher zurück, weil dort die Ausbildung eines periodischen Musters des Grundcytoplasmas noch eindringlicher ist.

Eine besondere Form von parallelisierten Molekülketten finden wir in der Feinstruktur der Spindel. Die meist so ausgesprochen grobfaserige Struktur der Teilungsspindel im gefärbten histologischen Präparat fehlt im elektronenoptischen Bild. An seiner Stelle sehen wir das feine parallel

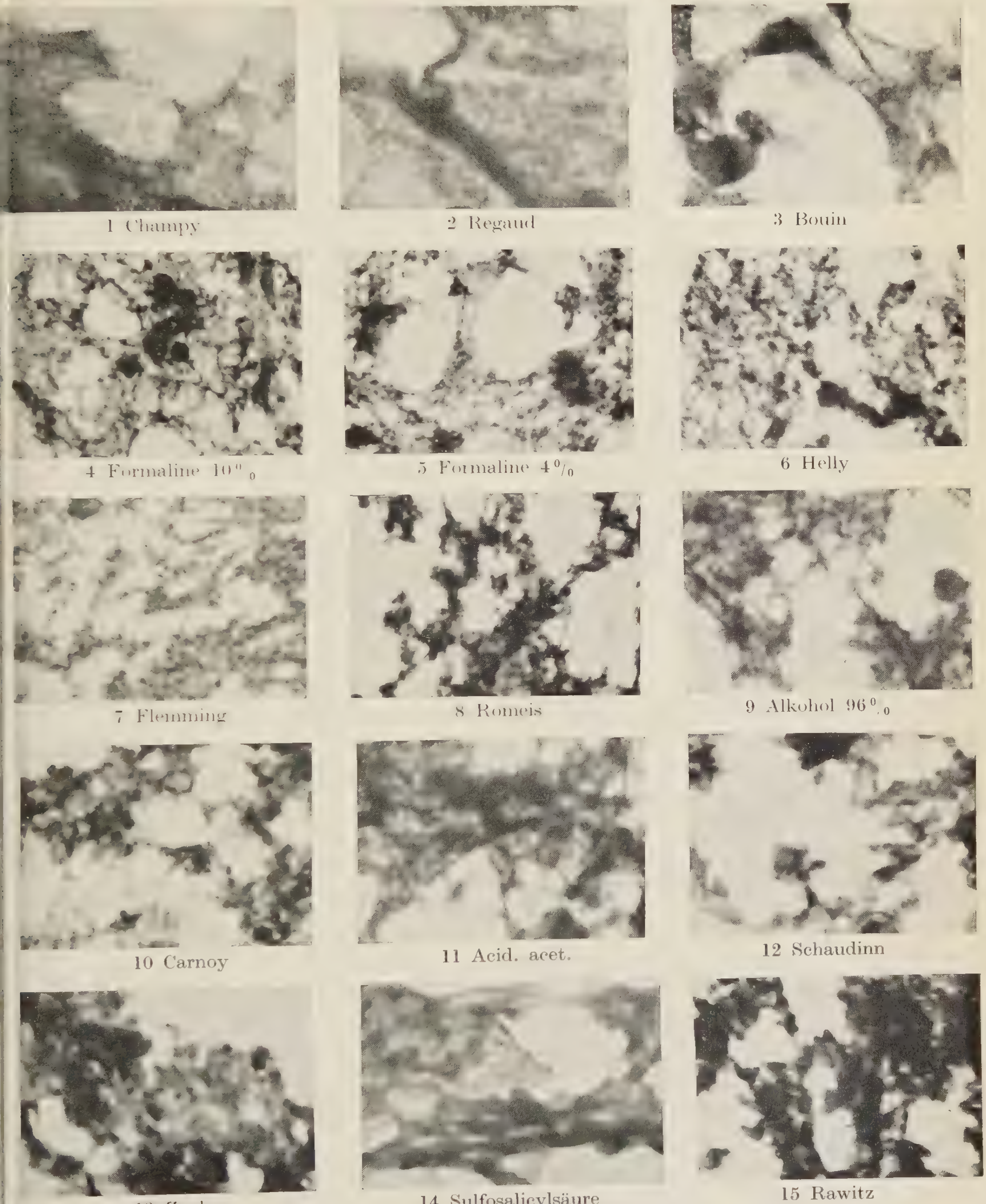


Fig. 1. Cytoplasma der Meristemzellen in der Wurzelspitze der Zwiebel. Bildausschnitte nach verschiedener Fixation aber gleicher elektronen optischer Vergrösserung von 12.000 ×. Spannung 100 kV. Cytoplasm of the meristemcell of the root-top cells in onion. Pictures after different fixations fluids but at the same magnification of 12.000 × and 100 kV.

gerichtete Grundnetzwerk und in ihm nur einige 60—80 mμ dicke Fibrillen. Dieser Parallelordnung verdankt die Spindel ihre optische Anisotropie, welche W. SCHMIDT (1937) als positive Doppelbrechung beim lebenden Seeigelei beobachtete. Wahrscheinlich kommt die grobfaserige Struktur im gefärbten Präparat teils durch eine schlechte Fixierung, teils durch optische Täuschung zustande, wobei die an sich lichtoptisch nicht sichtbaren parallel gerichteten Plasmafilamente einen faserigen Aufbau "vortäuschen".

Wir können also feststellen, dass uns vorläufig die Elektronenmikroskopie von Schnitten die gestalltlichen und topographischen Verhälltnisse der Feinstruktur der Zelle auflöst, dass wir aber zunächst über den Chemismus und die Bedeutung dieser Strukturen weiter noch nichts aussagen können.

Gegenüber den Fortschritten der elektronenmikroskopischen Schnittuntersuchung stehen aber auch gewisse Unzulänglichkeiten, welche die Lichtmikroskopie bisher besser zu lösen vermochte. Wir meinen hier die umfangreiche cytochemische Seite der Histologie, die durch spezifische Farbreaktionen den Chemismus der Zelle abtastet und zu weitgehenden Schlüssen führt. Eine der Unzulänglichkeiten ist b.w. das Misslingen einer klareren Darstellung des Chromosomenfeinbaues auf Schnitten. Wie aus den hier abgebildeten Aufnahmen hervorgeht, besitzen die Chromosomen eine sehr hohe Dichte. Das Metaphasechromosom zeichnet sich durch seine starke Spiralisation aus, wodurch die gestauten Windungen eng aneinander liegen. Die Projektion dieser übereinanderliegenden Strukturen infolge des grossen Fokusraumes des Elektronenmikroskopes verstärkt die Elektronenstreuung, während überdies die umhüllende Matrixsubstanz des Chromosomes sprunghafte Unterschiede in der Strukturdichte aufhebt. Da der Anteil des an sich nicht schweren Phosphormoleküls (P_{31}) in der Nukleinsäure nur 11 % beträgt und diese nur einen Bruchteil der Chromosomenmaterie ausmacht, kann der Phosphorgehalt als Massenunterschied nicht Schuld tragen an der starken Streuung. Auch bei einer Schnittdicke von bloss 0′ 2 μ können wir nur einen ca 160 mμ dicken dunkleren zentralen Teil — wahrscheinlich das Chromonema — unterscheiden. Die Matrix in welcher das Chromonema eingebettet ist, scheint weniger dicht und homogen zu sein. Bei solchen dichten Strukturen ist in dieser Hinsicht die Lichtmikroskopie mit ihren spezifischen färberischen Reaktionen der Elektronenmikroskopie noch immer überlegen. Weder eine Anfärbung mit OsO_4 noch mit Phosphorwolframsäure verbesserte das Bild, denn beide erhöhen ihrerseits wieder die Elektronenstreuung. Unseren elektronenoptischen Untersuchungen am Spermium gemäss (BRETSCHNEIDER, 1949 *c*; 1950 *c*) ist zu erhoffen dass auch in der Schnittuntersuchung bei solch dichten Strukturen, wie es das Chromosom besizt, ein teilweiser spezifischer Abbau der Struktur zu einem Erfolg führen könnte. Unsere bisherigen Bestrebungen in dieser Hinsicht führten noch nicht zu dem gewünschten Erfolg, da nach der Fixation mit Formol oder

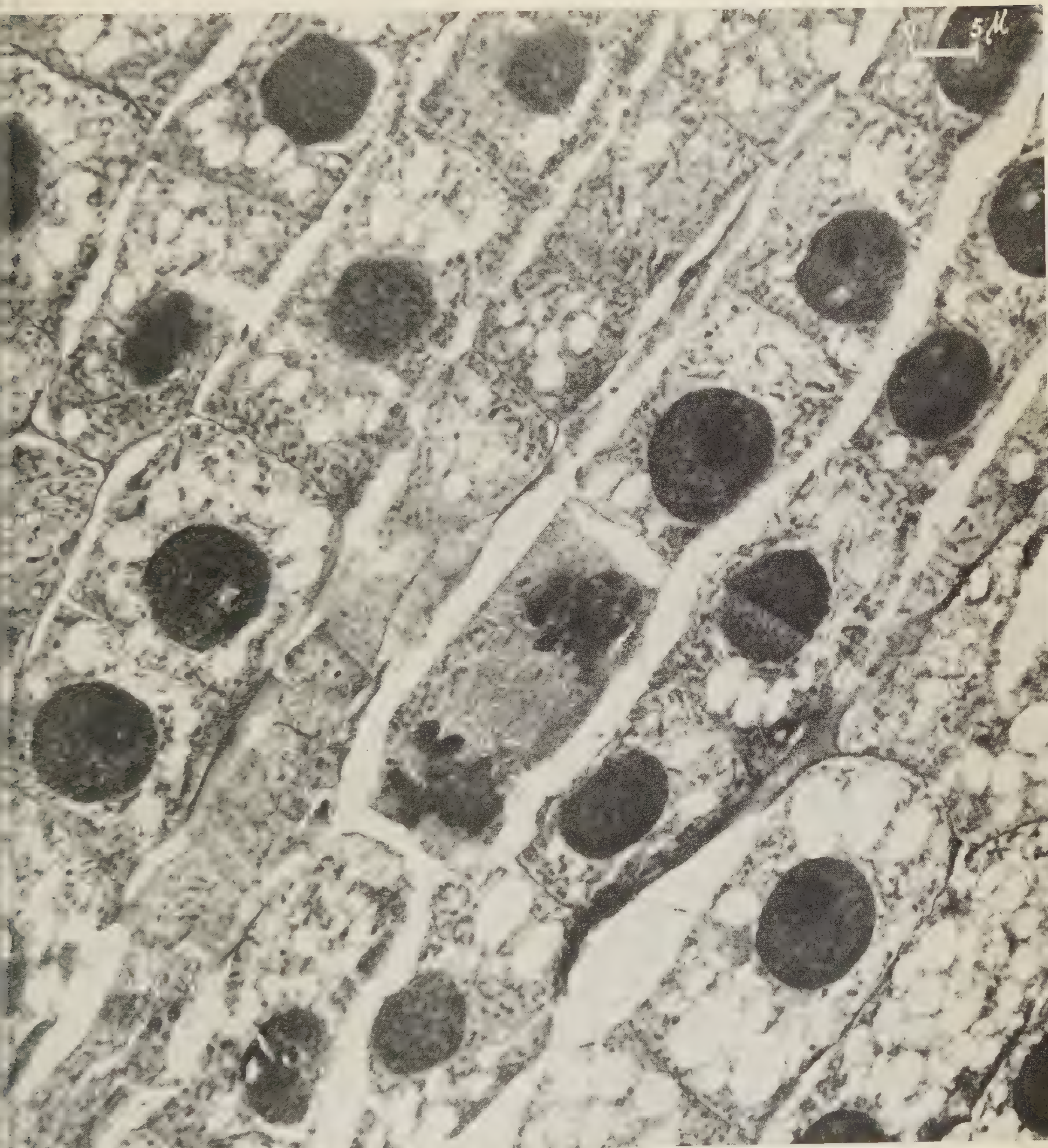
Osmium die organischen Substanzen zum Grossteil unlöslich werden. Am Spermium konnten wir seinerzeit diese Reaktionen ohne vorhergehende Fixation ausführen, während in der Lichtmikroskopie in solchen Fällen neutralere Fixationsmittel wie Alkohol u.a. angewandt werden, die für die Elektronenmikroskopie aber untauglich sind. Wir erhoffen uns daher mehr Erfolg von dem Ersetzen einer Fixation durch das "freezing-drying" Verfahren, bei welchem bekanntlich keine Denaturation der Baustoffe eintritt. Unsere Untersucuhungen werden nun in dieser Richtung fortgesetzt, da die chemische Erfassung vieler Strukturen nur durch Ausschaltung einer Fixation elektronenoptisch durchzuführen ist.

VI. *Zusammenfassung*

Diese Basisuntersuchung bezweckt aus 15 gebräuchlichen Fixationsmitteln die für die Elektronen-mikroskopie wertvollsten heraus zu suchen. Als Indikator für die Güte eines Fixationsmittels wurde die Teilchongrösse des fixierten Plasmas und seine Grundnetzstruktur gewählt. Die kleinste erzielte Teilchengrösse und die beste Erhaltung der Grundnetzstruktur wurden als die dem Leben am nächsten stehenden aufgefasst. Es stellte sich heraus, dass nicht etwa die starken eiweissfällenden Stoffe eine gute Fixation ergeben wie Essigsäure, Trichloressigsäure, Sublimat, Phosphor-Wolframsäure, Sulfosalicylsäure, sondern gerade die schwachfällenden wie das Osmiumtetroxyd und das Formol. In Kombination mit der Chromsäure und dem Kaliumbichromat in den Gemischen von Champy oder Kopsch–Regaud stehen sie an erster Stelle. Sie lassen im Hyaloplasma eine Grundnetzstruktur unterscheiden aus 160 Å dicken Proteinfilamenten welche ein regelmässiges annährend hexagonales Muster mit Achsenlängen von 420 Å bei 620 Å bilden. An zweiter Stelle stehen dann reine Lösungen des Formols (für tierische Zellen auch die des Osmiumtetroxydes). Hernach folgen noch das Gemisch von Bouin und das Zenker-Formol nach Helly. Alle übrigen Fixationsmittel (7–15) erwiesen sich als untauglich für die Elektronenmikroskopie. Die Untersuchung wurde an ein und demselben Gewebe und zwar am Meristem der Zwiebelwurzel ausgeführt. Es wird auf den Wert des reinen Streuungsbildes nach der Formolfixation an der Hand einiger Aufnahmen sich teilender Zellen gewiesen. Da die Elektronen-mikroskopie viele ihrer Befunde durch Messung ermittelt, aber die Teilchengrösse der Struktur durch die Fixation bestimmt wird, wäre zweeks Vergleichung verschiedener Befunde eine Beschränkung auf einige wenige der allerbesten Fixationsmittel sehr erwünscht.

Summary

This basic investigation on the influence of 15 usual fixation fluids was made to find the most useful fixation fluid for electron-microscopy. For the critical examination we took as an indicator the size of the particles and the ground-pattern of the cytoplasm. The smallest size of particles



2. Schnitt durch das Meristem der Wurzelspitze von der Zwiebel. Fixation mit 10 % Formol, Schnittdicke '3 μ, Vergrösserung 2.000 ×, Spannung 100 kV. Section through the meristem of the root-top in onion. Fixation 10 % formalin, thickness about 0'3 μ, magnification 2.000 ×, emission voltage 100 kV.

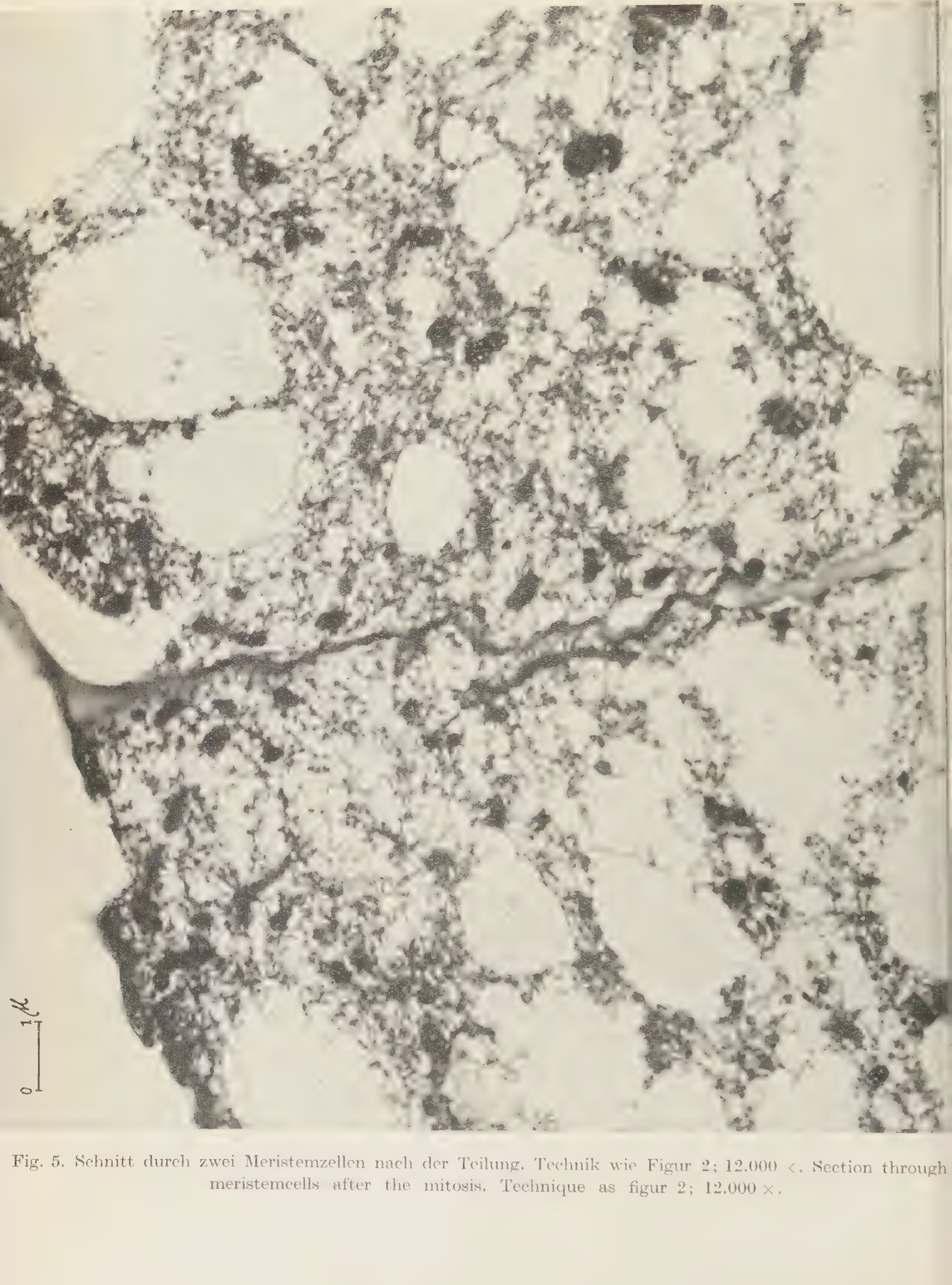


Fig. 5. Schnitt durch zwei Meristemzellen nach der Teilung. Technik wie Figur 2; 12.000 ×. Section through meristemcells after the mitosis. Technique as figur 2; 12.000 ×.

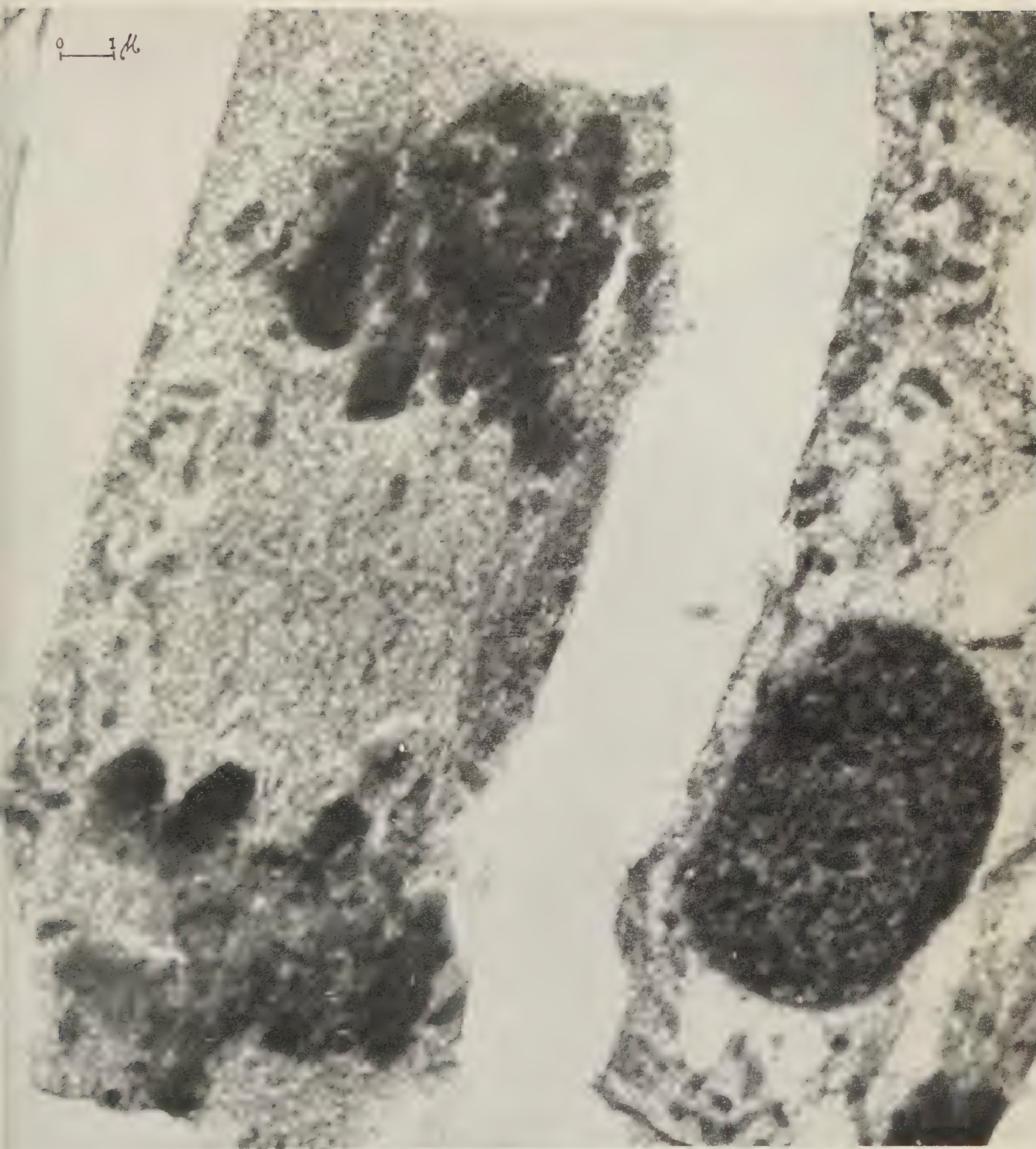
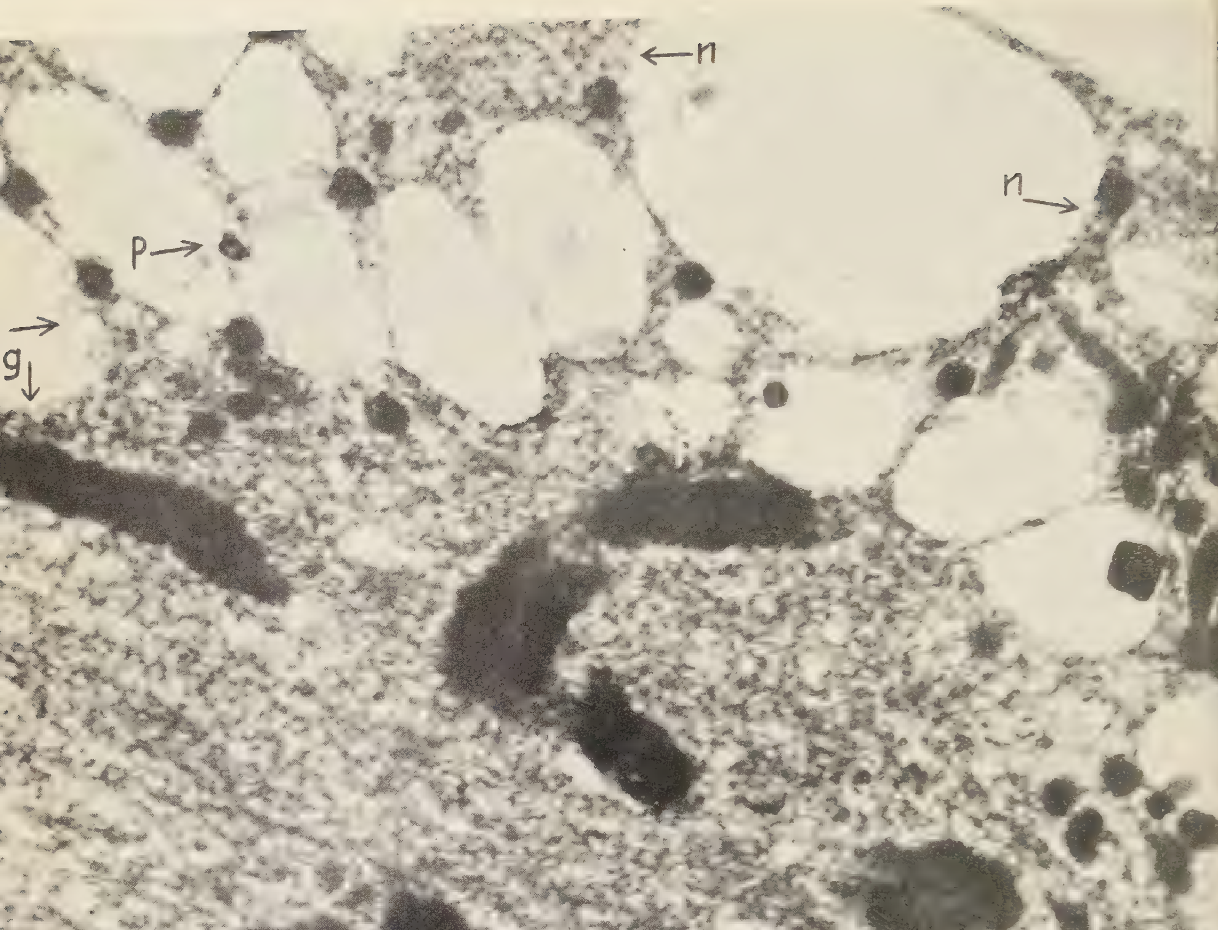


Fig. 3. Schnitt durch eine Meristemzelle im Anaphasestadium der Teilung. Technik wie in Figur 2; 8.000
Section through a meristemcell. Anaphas of the mitosis. Technique as in figur 2; 8.000 ×.



. 4. Schnitt durch eine Meristemzelle im Prometaphasestadium der Teilung. Technik wie Figur 2; 12.000 ×.
Section through a meristemcell. Prometaphas of the mitosis. Technique as figur 2; 12.000 ×.

and the best conservation of the ground-structure was considered as being most similar to the living state. The substances which coagulate the proteins very strongly, for example acetic acid, trichloracetic acid, phosphotungstic acid, sublimate, sulfosalicylic acid, are not the most favourable fixation fluids but on the contrary, the less coagulating substances as OsO_4 and formalin. These two fixation fluids give, in combination with chromic acid and potassium bichromate, as in CHAMPY's fluid (plate I/1) and KOPSCH-REGAUD's fluid (plate I/2) the most excellent fixation for the electronmicroscopic investigation of cells. After fixation with them we found in the hyaloplasm a ground-network of thin protein-filaments with a diameter of about 160 Å forming an almost regular pattern, fig. *A*. In second place follow pure solutions of formol (plate I/4 and 5) and for animal cells also OsO_4; in the third place the fixation fluids of BOUIN (plate I/3) and HELLY (plate I/6). All the following fixation fluids are useless for the electron-microscopy of cells. The investigations were made on the meristem of the root top of the onion. The figures plate 2, 3 and 4, of cells in mitosis after fixation with formol give an impression of the electron-optic picture produced only by means of electron scattering and without electron staining. The electron-microscopy measures various structures of its objects. Therefore, it is desirable for comparable results to use only some of the best fixation fluids, because the size of the particles depends on the fixation fluids.

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INDUCED ROTATION OF CLEAVAGE SPINDLES IN LIMNAEA STAGNALIS L.

BY

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Introduction:

The present investigation was induced by the work of F. H. SOBELS (1948). During his investigations on the influence of thiourea on the development of *Limnaea stagnalis* he observed occasional rotations of cleavage spindles, especially when decapsulated eggs were treated with a 0.75 % thiourea solution.

Similar phenomena have been mentioned by HÖRSTADIUS (1928) in eggs of *Paracentrotus lividus*, after shaking or meridionally cutting the eggs and even after exposure to diluted seawater, and by CONKLIN (1938) in eggs of *Credidula plana* after treatment with low temperatures. Both authors suggest the possibility that the factors governing the direction of cleavage spindles are relatively independent from the other developmental processes, and may cause the cleavage spindle to assume the correct position according to the age of the egg, even if one or more cell divisions or even nuclear divisions did not occur due to any form of retardation of development. This hypothesis explains e.g. the often observed so-called "premature micromere-formation", in which, in spiral cleavage, micromeres are formed, not from a 4-blastomere stage, but from a 2-blastomere or even an uncleaved stage. HÖRSTADIUS has been able to explain nearly all the abnormal cleavages experimentally obtained by him with this hypothesis, combined with the known facts about the micromere-forming potencies of the vegetative plasm of the Echinoderm egg. The hypothesis can also be applied to the greater part of CONKLIN's abnormal cleavages, but in a number of cases the abnormalities are so irregular that he is forced to assume a direct disorienting effect of the cold treatment on the cleavage pattern.

The same hypothesis has been applied by SOBELS (1948) to the spindle rotations observed in *Limnaea*. The purpose of the present investigation has been, therefore, to give an answer to the following questions:

1. Is spindle rotation always coupled with retardation of development?
2. Does spindle rotation occur more or less simultaneously with the normal changes in the direction of cleavage spindles in the control eggs?

Methods:

Egg-masses were obtained by stimulating the snails with *Hydrocharis*, as indicated by RAVEN and BRETSCHNEIDER (1942).

Only decapsulated eggs were used and thiourea was applied in a concentration of 0.75 %. As a solvent a solution of 0.04 % $CaCl_2$ was used, in order to provide the necessary $Ca^{\cdot\cdot}$-ion concentration (HUDIG 1946).

The eggs were observed in the experimental solution till after the 3rd cleavage; the only variable factor has been the moment at which they were put into the solution. The control eggs were kept in a 0.04 % $CaCl_2$ solution. All eggs were kept at a constant temperature of 23° C.

In addition to the experiments with thiourea a number of eggs were treated with different concentrations of sucrose, dissolved in 0.04 % $CaCl_2$, and some experiments concerning the influence of low temperatures were carried out.

Experimental results:

A. Thiourea

1. *Abnormalities and types of rotation*

a. In general we observe that the blastomeres are always more rounded than in normal development, which makes it impossible to establish exactly the stage of development according to RAVEN's normal table (1946).

b. First cleavage. An equatorial cleavage, often observed by CONKLIN in *Crepidula*, never occurred in these experiments. Rotation of the first cleavage spindle apparently occurs only in cases in which inhibition is so strong that the egg remains uncleaved [1]) (cf. SOBELS, p. 905).

c. Second cleavage. In most of the rotation-types described, rotation occurs in a plane parallel to the original plane of the first cleavage. It is impossible to tell the direction of the rotation, as in most cases it is unknown which pairs of blastomeres originate from the same $\frac{1}{2}$-blastomere.

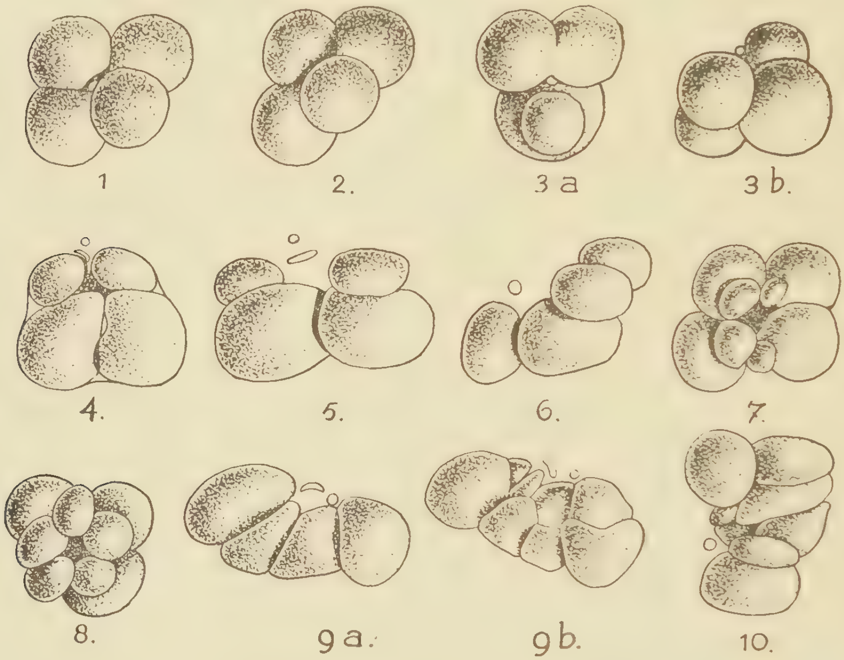
A series of separate types, established according to the rotation-angle, is illustrated by figs. 1—4. The type of fig. 1 is reversible; it consists, at most, of a slight exaggeration of the normal obliquity of the second cleavage spindles, and has, therefore, not been considered a rotation in the analysis of the experimental figures. The types of figure 2, 3 and 4 will henceforth be indicated as I, II and III. Type I (fig. 2) is formed by eggs, in which one of the blastomeres is distinctly lying above the plane of the 3 other ones. Type II is the rotation described by SOBELS, in which one of the blastomeres divides nearly meridionally and the other one in an equatorial plane [2]). In type III both blastomeres apparently divided

[1]) In *Crepidula* the case is similar, as the equatorial two-cell stages seldom develop any further (cf. CONKLIN, p. 188—189).

[2]) As to the origin of type II it is remarkable that I never, in about 20 observations of this type, observed it to originate from a 3-cell stage, in the way SOBELS described.

equatorially, as indicated by the position of the polar bodies. It is observed often, though not in all cases, that the animal blastomere of a rotated pair is smaller, the vegetative one bigger than the blastomeres of the non-rotated pair.

Type III has some variants, which prove that the planes of rotation are not always parallel to the first cleavage plane (fig. 5) [3]), whereas the positions of the second cleavage planes may vary (cf. fig. 9*a* in which the



Limnaea stagnalis. Abnormal cleavage stages.

Fig. 1. Exaggeration of obliquity of second cleavage spindles.
Fig. 2. Rotation-type I.
Fig. 3. Rotation-type II.
Fig. 4—6. Rotation-type III.
Fig. 7—8. Abnormal 8-cell stages derived from normal 4-cell stages.
Fig. 9. Third cleavage of type-III egg.
Fig. 10. Eight-cell stage derived from type-III egg.

animal blastomeres are bigger than the vegetative ones). The thoroughly disorienting effect of thiourea on the second cleavage is most evident in a type as shown in fig. 6, and in the T-shaped eggs, observed by myself as well as by SOBELS. All these types are indicated in the experiments as type III.

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[3]) It is remarkable that, in my observations, only the animal blastomeres lose contact.

d. Third cleavage. The third cleavage of normal 4-cell stages can show several abnormalities. The normal dexiotropic cleavage-angle of 45° can be reduced, even to 0°. This is often accompanied with a reduction of the number of micromeres. Some other possibilities are shown by figs. 7 and 8.

e. Third cleavage in rotated 4-cell stages. Type I is capable of an entirely normal dexiotropic third cleavage. There is no evidence of a third cleavage of type II [4]). Type III, however, is capable of giving rise to rather regular, but also to very irregular 8-cell stages, a process illustrated by fig. 9. A more irregular 8-cell type is shown in fig. 10. These eggs often consist of less than 8 cells, never of more. Their most remarkable feature is the fact that nearly always all cells are situated in one plane. Possibly gravity causes a more or less horizontal position of the cleavage spindles by flattening the blastomeres of the 4-cell stage.

The four cell-divisions giving rise to an 8-cell type have been observed in a number of cases to occur perfectly simultaneously.

2. *Experiments*

Each batch (batches numbered 1—16) was divided into four lots of 5—10 eggs each (lots numbered *a—d*), which were transferred to the solution at different moments between laying and first cleavage.

For the lots five subjective degrees of retardation as compared with the control eggs [5]) were established: no, slight, moderate, rather strong and strong retardation. The last degree was left out of account with regard to rotation, as in those lots most of the eggs are arrested in development before reaching the second cleavage.

The rotation is given as the percentage of the total number of eggs of the lot. In this percentage types I, II and III are counted together.

The results do not lend themselves to statistical treatment. They can only point to possibilities, principally because the number of figures is too small.

a. Retardation of development. In general it is observed that the degree of retardation in lots of the same batch decreases as the lots have been placed into the solution at a later moment. Two batches show the opposite behaviour. Different batches show great variations in susceptibility, as far as retardation is concerned.

b. Rotation during second cleavage. With regard to rotation we also find great variations in susceptibility. Seven out of sixteen batches used do not show any rotation at all. They will be altogether left out of account.

If rotation occurs, the percentages in the lots of one batch do not differ very much, no matter what the degrees of retardation of the different lots are. The percentages in different batches may, however, show considerable differences. The following figures may serve as an example (Table I).

[4]) Even if such a cleavage occurred it would be very difficult to interpret.

[5]) The exact point of time of a cleavage in a certain lot was taken to be the moment at which 50 % of the eggs had cleaved.

TABLE I

Degree of retardation (* = no, ** = slight, *** = moderate, **** = rather strong) and percentage of eggs showing rotation of second cleavage spindles in different lots (*a*, *b*, *c*, *d*) of 4 batches (3, 9, 10, 15).

Batches	Lots			
	a	*b*	*c*	*d*
3	**** 10 %	—	** 10 %	—
9		**** 10 %	**** 10 %	** 20 %
10	*** 30 %	*** 50 %	*** 40 %	—
15	** 75 %	** 60 %	** 75 %	—

To find an answer to the question, whether spindle rotation is always coupled with retardation of development, we have been looking for cases in which rotation occurred in an individual egg which did not show any delay of cleavage as compared with the controls, and which belonged to a lot which was not delayed as a whole either. Only in that case we can be sure that it does not concern an egg which is one of the first to cleave in its lot, and in control solution would have cleaved some time before the average time of cleavage of the controls. The following cases may be considered (Table II).

TABLE II

Degree of cleavage delay (cf. tab. I) in individual eggs showing rotation of second cleavage spindle

Lot	Degree of retardation of the lot	Rotation type	Indiv. delay of 2d cleavage	
2*a*	**	II	< 10 min.	no. 8 to cleave of lot of 10
9*d*	**	II	0 min.	no. 4 to cleave of lot of 10
15*a*	**	II	0 min.	no. 2 to cleave of lot of 8
15*b*	**	III	< 5 min.	no. 1 to cleave of lot of 5
15*c*	**	III	< 10 min.	no. 4 to cleave of lot of 8
16*d*	*	II	< 10 min.	no. 4 to cleave of lot of 8

Concerning the question whether spindle rotation occurs simultaneously with the changes in the direction of the spindles in the control eggs, it must be remarked that a second cleavage leading to a rotation type was never observed to appear at the same time as the third cleavage of the controls. Rotationtype second cleavages may appear *at any time between* the second and the third cleavages of the controls, but most frequently about 10 to 60 minutes after the second cleavage, i.e. 75–135 minutes before the third cleavage of the controls.

c. Rotation during third cleavage. We only consider the types in which the normal cleavage-angle has been reduced to 0°. These types only occur in lots with no, slight or moderate retardation. Out of about 10 third cleavages observed in lots with rather strong or strong retardation, no one

was of this type, though most of them were either abnormal or showed smaller rotation-angles. Of 4 of the types "third cleavage 0°" it is certain that they were not delayed in comparison with the controls, nor were the lots to which they belonged.

B. Sucrose

In order to investigate whether osmotic influences play a part in the induction of spindle rotation, a series of parallel experiments were carried out, in which, besides a 0.75 % solution of thiourea, a sucrose solution of the same osmotic pressure, i.e. of 3.3 %, was used, together with some other solutions of lower concentrations.

1. *Retardation of development*

In 3.3 % sucrose, retardation nearly always was stronger than in 0.75 % thiourea. The degree of retardation always was "strong". In lower concentrations of sucrose, i.e. from 3 % downwards, retardation nearly always was less than in 0.75 % thiourea. Mostly, and in any case from 2.5 % downwards, there was no retardation at all.

2. *Abnormalities and rotations*

It is remarkable that in 3.3 % sucrose solution the eggs behaved much more like those in hypertonic solutions (cf. Grasveld 1949) than they did in 0.75 % solution of thiourea. The blastomeres were more rounded, and often lost contact especially after the 3rd cleavage. Sometimes a pointed shape of the first polar body was observed.

All three rotation-types occur in sucrose solutions of 2.4 % to 3.3 %, that is at all degrees of retardation. On two occasions the formation was observed of a radially symmetrical 3-cell type, possibly the result of the formation of a triaster during karyokinesis.

All three eggs of type II observed later showed a retrogression to a type in which the pairs of blastomeres formed an angle smaller than 90°. This phenomenon may be connected with the loose mutual attachment of the blastomeres.

C. Cold treatment

The eggs were kept at temperatures of -- 2° to + 6° C, during periods varying from 6 to 36 hours. No spindle rotation could be induced in this way, the only result being a total arrest of development during treatment, and a retardation of development afterwards. Very much inhibited eggs showed a considerable increase in volume. The cleavage cavity always was larger than normally.

Discussion:

The most important fact shown by this investigation is that spindle rotation is not an effect which is specific for thiourea, but that it can also

be produced by sucrose. This leaves open the possibility that spindle rotation is an osmotic effect. In contradiction with CONKLIN's results in *Crepidula*, cold treatment failed to induce spindle rotation in *Limnaea*.

Of the two questions put in the introduction to verify SOBELS' explanation of spindle rotation, the second one can be answered in the negative. In 6 out of 8 batches concerned spindle rotation types appared 75 minutes or more before the third cleavage of the controls.

The answer to the first question also seems to be negative, according to the data given in table II.

Other considerations speaking against SOBELS' explanation are:

Firstly: No rotation-types have been observed by me to originate from intermediate 3-cell stages, as described by SOBELS.

Secondly: The 8-cell types originating from rotation-type III remind us very much of a 3rd cleavage, especially as the four cell-divisions have been observed to occur simultaneously. According to SOBELS' explanation one would expect a transitory 6-cell stage, representing a "half, premature, fourth cleavage", soon followed by further cleavages.

Thirdly: The synchronicity of cleavage divisions can be disturbed without spindle rotation. This has also been observed by SOBELS himself.

Lastly: Disorientations of the cleavage spindles are not restricted to rotations in a plane parallel to the original first cleavage plane. This has been observed by SOBELS as well as by me (cf. fig. 5).

We must take into account the possibility that the more pronounced rounding off and the resulting looser mutual attachment of the blastomeres in thiourea and sucrose plays a part in the formation of rotation-types. In those cases, however, where the animal cell of a rotated pair is smaller than the vegetative one, we can be sure that a spindle rotation has taken place: the difference in cell size can easily be explained by the fact that the nucleus of a $\frac{1}{2}$-blastomere is situated nearer the animal side of the cell. The direction of the spindle is the primary factor, the formation of a "micromere" is the effect [6]).

In the type of fig. 8 we also see that the meridional cleavage divisions result in pairs of blastomeres of equal size, whereas the "equatorial" cleavage divisions result in pairs of blastomeres of unequal size.

Summary:

1. Eggs of *Limnaea stagnalis* were cultured in a solution of thiourea of 0.75 % and in sucrose solutions varying between 2 % and 3.3 %.
2. In thiourea as well as in sucrose solutions retardation of development occurs, the degree of which decreases as the eggs are transferred to the solution at a later moment.
3. Rotation of second cleavage spindles occurs in thiourea as well as

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[6]) This of course applies to all cases of so-called "premature micromere formation".

in sucrose, and involves either one or both blastomeres. The third cleavage in thiourea may show an alteration in the normal cleavage angle of 45°.

4. Treatment with low temperatures failed to induce spindle rotation.

5. Spindle rotation seems not always to be coupled with retardation of development. Moreover in most cases spindle rotation during second cleavage occurs 75 minutes or more before the third cleavage of the controls. It is, therefore, not likely that spindle rotation in *Limnaea* can be compared with the so-called "premature micromere formation", observed in *Paracentrotus* and *Crepidula*.

6. Thiourea has a disorienting effect on the directions of the cleavage divisions and the situations of the cleavage planes in second and third cleavage. Spindle rotations during second cleavage are not restricted to planes parallel to the first cleavage plane.

7. In the cases where rotations occur, their number seems not to depend on the degree of retardation, but only on the nature of the batch to which they belong. Both as regards retardation and spindle rotation the batches show great individual differences in susceptibility.

In am much indebted to Prof. CHR. P. RAVEN for suggesting this investigation, for his continuous interest in the work and his valuable advice in the preparation of this paper.

I also wish to thank Dr P. D. NIEUWKOOP for his valuable and inspiring assistance.

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NOTES ON PSEUDOXENODON INORNATUS (BOIE) AND PSEUDOXENODON JACOBSONII LIDTH

BY

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(Communicated by Prof. H. BOSCHMA at the meeting of Sept. 30, 1950)

Two species of the genus *Pseudoxenodon* Blgr. are known from the Indo-Australian Archipelago, viz., *Pseudoxenodon inornatus* (Boie) from Java and *Pseudoxenodon jacobsonii* Lidth from Sumatra. Both species are rare in collections. Of *P. inornatus* a small number of specimens is known, and of *P. jacobsonii* the holotype seems to be the only specimen that reached a museum. Recently the Amsterdam Zoological Museum received from Mr J. C. BAUWENS four snakes from Java, two of which proved to belong to *Pseudoxenodon inornatus* (Boie). As only very little is known about the variability of this species, some notes on these specimens may be published. For comparison I examined the holotype and a juvenile specimen in the Leiden Museum. Moreover, the holotype of *Pseudoxenodon jacobsonii* Lidth has been re-examined, and some notes on this specimen are appended.

Pseudoxenodon inornatus (Boie)

Elops sp., KUHL & VAN HASSELT, Alg. Konst- en Letter-Bode, no. 7, 100 (1822).

Elaps sp., KUHL & VAN HASSELT, Isis, pt. 4, 473 (1822); KUHL, Bull. Sci. nat. Géol. (2e Sect. Bull. Sci. Industr.), 2, 80 (1824).

Xenodon inornatus BOIE, Bull. Sci. nat. Géol. (2e Sect. Bull. Sci. Industr.), 9, 238 (nom. nud.) (1826); BOIE, in SCHLEGEL, Isis, 19, 293 (nom. nud.) (1826); H. BOIE, in F. BOIE, Isis, 20, 541 (1827); WAGLER, Nat. Syst. Amph., 272 (1830); SCHLEGEL, Essai Physion. Serp., vol. 1, 139 (1837) and vol. 2, 89, pl. III figs. 10—12 (1837); SCHLEGEL, Essay Physiogn. Serp. (transl. by TRAILL), pp. 138, 254, pl. I fig. 4 (1843); FITZINGER, Syst. Rept., 28 (1843); SCHLEGEL, Handl. Dierk., vol. 2, 45 (1858); JAN, Elenco Sist. Ofidi, 56 (1863) (Arch. Zool. Anat. Phys., 2, 319 (1863)); JAN & SORDELLI, Icon. Gén. Oph., vol. 2, pt. 19, pl. V fig. 2 (1866).

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[*Pseudoxenodon*] *inornatus*, POPE, Rept. China, 139 (1935).

Ps[*eudoxenodon*] *inornatus*, BOURRET, Serp. Indochine, 2, 110 (1936).

The species was described by H. BOIE (in F. BOIE, 1827, p. 541) as *Xenodon inornatus*, and it was stated to come from Java; no exact locality in the island was mentioned. Even ninety years later when DE ROOIJ (1917) published a survey of the Indo-Australian snakes, no exact localities in Java were mentioned. DE HAAS (1950, p. 512) mentions that of a number of species, among which *Pseudoxenodon inornatus*, "exact places where they have been found are now known", but no locality is mentioned in his paper. I did not find any locality mentioned in literature, and it may be that DE HAAS based his statement on a specimen in the Buitenzorg Museum. However this may be, it is at present possible to establish the definite type-locality; the specimens collected by Mr J. C. BAUWENS provide us with a second Javan locality.

In BOIE's manuscript notes for his unpublished Erpétologie de Java I found the description of *Xenodon inornatus*, which contains a reference to the manuscript name given to this species by KUHL & VAN HASSELT. A further search in the manuscript notes left by KUHL provided me with a brief description of the holotype, and in this the locality is mentioned as Tjihandjawar (at the foot of Mt. Pangerango, W. Java). Thus the type-locality of *Pseudoxenodon inornatus* has been definitely established. This species is one of the new species of *Elaps* mentioned by KUHL & VAN HASSELT (1822*a*, p. 100: *Elops*; 1822*b*, p. 473) and by KUHL (1824, p. 80).

A second specimen in the Leiden Museum was collected in Java by S. MÜLLER; no reference to it can be found in MÜLLER's manuscript notes, and therefore, the exact locality cannot be ascertained.

The two specimens which Mr J. C. BAUWENS sent to the Amsterdam Museum both came from the Sumadra Estate near Garut, West Java.

The four specimens examined by me may be described as follows.

♂, Holotype, Tjihandjawar, W. Java, leg. KUHL & VAN HASSELT, 1821, Mus. Leiden, reg. no. 233.

Maxillary teeth 16 + 2; the anterior 16 gradually increasing in length from anteriorly to posteriorly, followed by two strongly enlarged teeth. There is no marked interval between the last of the small teeth and the first strongly enlarged tooth: in fact the distance between these teeth is not larger than between two of the small teeth. Scales in 19 rows on the neck and at mid-body, in 15 rows in front of the vent. In the anterior half of the body the scales are placed in distinctly oblique series, on the posterior half of the body the scales are placed in longitudinal series. In the anterior half of the body the three mid-dorsal rows are keeled, while in the posterior part of the body all scales, except those of the outer row, are keeled. Ventrals 120, anal divided, subcaudals 36/36 + 1. Three preoculars. On the right the upper and middle preocular are large and about equal in size; the lower preocular is very small, it is wedged in between the middle preocular and the 4th upper labial. On the left the upper and lower preocular are large and of about equal size; they are separated by the much

smaller middle preocular, which is square. Three postoculars. Upper labials 8, the 4th and 5th border the orbit. Two anterior temporals followed by four scales; three of these latter are in contact with the upper anterior temporal, and only one is in contact with the lower anterior temporal. Lower labials 9, four of which are in contact with the lower anterior chinshields; the lower labials of the first pair form a suture behind the symphysial.

Frontal only slightly longer than broad (not quite 1.2 times), about 0.8 times the length of the parietals, about 1.6 times its distance from the rostral, and 1.1 times its distance from the tip of the snout. The prefrontals are larger than the internasals; the length of the prefrontal suture is about 1.4 times that of the internasal suture. Loreal slightly more than 1.1 times as high as long. Rostral 1.8 times as broad as high. The chinshields are short and broad; the posterior about 1.4 times as long as the anterior.

The total length of the type is given by SCHLEGEL (1837, p. 89) as 470 mm, tail 70 mm. In the preserved specimen I make the total length to be 464 mm, the tail 74 mm.

The coloration has faded, but a $\wedge$-shaped marking on the neck is still visible. There are faint indications of rhomboidal markings on the back. A whitish line is seen along the sides of the posterior part of the body and along the sides of the tail, close to the ventrals. KUHL's notes give the following description of the coloration: "Color omnis lateris superioris rufus ad spinam obscurior, cum figuris rhomboidalibus concotenatis pallide fuscis uniserialibus. Squamarum interstitiis hic inde flavidis. Latere inferiori rufescent: albido, capite colloque infra antem luteis. Ad corporis posterioris latere lineola dilluta albida inter abdomen et dorsum, ad caudam antem circumscripta albescent: flava."

1 juv., Java, leg. S. MÜLLER, Mus. Leiden, reg. no. 234.

Maxillary teeth 17 + 2, a very small interspace. Scales 19, 19, 15; ventrals 119, anal 1/1, subcaudals 37/37 + 1. Right 4 preoculars; the second from above small and square; the lower preocular very small. Left 3 preoculars, the upper two large and equal in size, the lower very small. Three postoculars. Two anterior temporals, followed by four scales; two of these scales in contact with the posterior border of the upper anterior temporal, the two others in contact with the posterior border of the lower anterior temporal. Upper labials 8, the 4th and 5th bordering the orbit. Lower labials 10 on the left side (5 bordering the anterior chinshield), 9 on the right (4 bordering the anterior chinshield).

Frontal slightly broader than long, its length contained 0.97 times in its width, 0.6 times the length of the parietals, 1.2 times its distance from the rostral, and 0.9 times its distance from the tip of the snout. The prefrontals are larger than the internasals; the prefrontal suture is 1.3 times the internasal suture. Loreal not quite 1.1 times as high as long.

Rostral 1.6 times broader than high. Posterior chinshields 1.2 times as long as the anterior.

Traces of the umbilicus at the 12th to 14th ventral in front of the anal.

Total length 210 mm, tail 29 mm.

The coloration has faded to some extent, still the colour pattern is discernable. The specimen has a /\-shaped black marking on the neck with its tip on the occiput. The anterior part of the back shows dark oblique bars which in some places appear to be alternating. The posterior part of the back shows rhomboidal blackish markings with a pale centre. A whitish line along the sides of the posterior part of the body and along the tail, close to the ventrals.

♂, Sumadra Estate near Garut, W. Java, 2700 ft., August 1949, leg. Mr J. C. Bauwens, Zool. Mus. Amsterdam.

Maxillary teeth 19 + 2, no interval. Scales 19, 19, 15; ventrals 123, anal 1/1, subcaudals 38/38 + 1. Right 2 preoculars; left 3 preoculars, the lower very small. Three postoculars. Right 8 upper labials, left 7; on both sides the 4th and 5th bordering the orbit. Temporals 2 + 3. Right 9 lower labials (4 in contact with the anterior chinshield), left 10 lower labials (5 in contact with the anterior chinshield); lower labials of the first pair forming a suture behind the symphysial.

Frontal slightly longer than broad (not quite 1.2 times), about $\frac{3}{4}$ the length of the parietals, 1.4 times its distance from the rostral, about 0.9 times its distance from the tip of the snout. Prefrontals larger than internasals; the prefrontal suture is about 2.7 times the internasal suture. The loreal is nearly 1.1 times as high as long. Rostral 1.6 times as broad as high. Posterior chinshields 1.3—1.4 times as long as the anterior chinshields.

Total length 741 mm, tail 110 mm; the specimen largely exceeds the greatest length (470 mm) mentioned for this species by de Rooij (1917, p. 57).

The colour of the preserved specimen is a uniform dark lead grey, without any distinct markings. It is possible that the very dark colour is partly due to the fluid in which the specimen has been preserved. The sides of the tail show a very narrow undulating white line, which is partly broken up into a series of short white lines. The ventral surface is greyish, except for the throat, which is cream white. The upper part of the upper labials is grey, the lower half is cream coloured, with dark borders to the sutures. Greyish borders are present on the sutures between the lower labials and all around the chinshields.

One hemipenis was evaginated; the other was dissected from the specimen for closer study. The hemipenis is forked, the bifurcation takes place at about 2/3 of its length. The lobes are unequal in length. Length of dissected hemipenis 38 mm. The basal 14 mm are covered by very small spines; towards the bifurcation the spines become gradually larger. The

proximal part of each lobe is also covered with spines. At 7 mm from the top (measured on longest lobe) the hemipenis is covered by oblique folds with spiny scallops; very low longitudinal folds connect the strong oblique folds. The last 3 mm are covered by calyces. The sulcus spermaticus is bifurcate almost from the base.

The anterior border of the heart is situated at the level of the 21st ventral, at 115 mm from the tip of the snout. There is only a single lung, which has an anterior diverticulum which reaches to 22 mm in front of the opening of the trachea into the lung, and to $11\frac{1}{2}$ mm in front of the heart. This anterior diverticulum has a distinct alveolar pattern. The alveolar pattern of the lung extends posteriorly to 65.3 mm from the orifice of the trachea.

♀, Sumadra Estate near Garut, W. Java, 3000 ft., August 1949, leg. Mr J. C. Bauwens, Zool. Mus. Amsterdam.

Maxillary teeth 19 + 2, no interval. Scales 19, 19, 15; ventrals 122, anal 1/1, subcaudals 37/37 + 1. Three preoculars; the upper two have fused for the greater part, and at a first superficial examination it might be said that one very large preocular with below it a very small one are present. Three postoculars. Upper labials 8, the 4th and 5th bordering the orbit. Two anterior temporals followed by four scales. Nine lower labials, of which the anterior four border the anterior chinshields. The lower labials of the first pair form a suture behind the symphysial.

Frontal only very slightly longer than broad (about 1.02 times), about 0.7 times as long as the parietals, 1.3 times as long as its distance from the rostral, very slightly shorter than its distance from the tip of the snout (0.98 times). Prefrontals larger than internasals; the prefrontal suture 1.7 times the internasal suture. Loreal nearly 1.2 times as high as long. Rostral 1.6 times as broad as high. The posterior chinshields nearly 1.3 times as long as the anterior.

Total length 444 mm, tail 69 mm.

General colour pale brownish. Neck with a faint $\wedge$-shaped marking, followed by an area in which the scales have white borders. Vertebral region darker than sides with here and there rhomboidal markings, each of which has a pale centre. Sides with dark oblique lines.

The specimen was captured in the grass of a tea-plantation.

In all four specimens the scales on the anterior part of the body are placed in oblique series, while they form longitudinal series on the posterior half of the body. In the anterior part of the body only the three scale rows on the vertebral region are keeled, while in the posterior half of the body all scales, except those of the outer row, are keeled.

Boulenger (1893, p. 270), de Rooij (1917, p. 56), Bourret (1936, p. 110) and Smith (1943, p. 311) mention that in the genus *Pseudoxenodon* the strongly enlarged maxillary teeth are separated from the smaller

anterior teeth by an interspace. In three of the four specimens of *Pseudoxenodon inornatus* examined by me this is not the case. In these the distance between the last small tooth and the first enlarged tooth is not larger than that between two of the small teeth. POPE (1935, pp. 141, 145, 148, 155, 156) in describing the Chinese species of *Pseudoxenodon* mentioned the interval as varying from small to very small, not appreciable, and very small or absent. Therefore, the presence of a space between the small and enlarged teeth is of little use as a character for this genus.

Pseudoxenodon jacobsonii Lidth

Pseudoxenodon jacobsonii VAN LIDTH DE JEUDE, Zool. Med. Mus. Leiden, **6**, 239, 240 (1922).
Pseudoxenodon jacobsoni, WERNER, Sitz. ber. Ak. Wiss. Wien, mathem. naturw. Kl., Abt. I, **134**, 52 (1925); WERNER, Zool. Jahrb., Syst., **57**, 32, 33 (1929).
[Pseudoxenodon] jacobsoni, POPE, Rept. China, 139 (1935).
Ps[eudoxenodon] jacobsoni, BOURRET, Serp. Indo-chine, **2**, 110 (1936).

The species was described from a single specimen by VAN LIDTH DE JEUDE (1922, p. 240). Re-examination of the type makes it possible to add some information to that supplied by the original description, e.g., VAN LIDTH DE JEUDE did not mention the number of maxillary teeth nor the length of the specimen; the description of the coloration gives the impression that the back is uniformly greyish, but this is not the case. The holotype may be redescribed as follows.

♀, Holotype, Serapai, Korintji, Sumatra, July 1915, leg. E. JACOBSON, Mus. Leiden, reg. no. 4093.

Maxillary teeth 16 + 2, the anterior 16 gradually increasing in size, and separated by an extremely small interspace from the two strongly enlarged posterior teeth. The interspace is only very slightly larger than that between two of the anterior teeth. The number of maxillary teeth in this specimen and that in the holotype of *Ps. inornatus* is the lowest recorded for this genus.

Scales 19, 19, 15; those on the anterior half of the body in strongly oblique series, on the posterior part of the body in longitudinal series. In the anterior part of the body the scales of the vertebral three rows are keeled, while in the posterior part of the body all scales, except those of the outer row are keeled. Ventrals 145, anal 1/1, subcaudals 36/36 + 1. One preocular, three postoculars. Upper labials 7, the 3rd and 4th bordering the orbit. Temporals 2 + 2; the upper anterior is small, the lower large; in the posterior pair the inverse is the case. On the right 10 lower labials of which the anterior five are in contact with the anterior chinshield; on the left 9 lower labials, the anterior four of which are in contact with the anterior chinshield. The lower labials of the first pair are in contact behind the symphysial.

The frontal is broader than long; its length is 0.9 times its width, 0.6

times the length of the parietals, 0.9 times its distance from the rostral, and 0.7 times its distance from the tip of the snout. The prefrontals are larger than the internasals; the prefrontal suture is 1.75 times the internasal suture; the prefrontals are shorter than the frontal.

Left 2 superposed loreals; right a single loreal, which is nearly 1.4 times as high as long. Rostral twice as broad as long. Chinshields short and broad; the posterior 1.3 times as long as the anterior.

Total length 1082 mm, tail 122 mm.

Upper surface greyish, a very distinct /\-shaped black marking on the neck, its apex on the occiput. Back with blackish rhomboidal markings with a paler centre. The anterior sides of the rhombs may be continued on the sides as more or less distinct oblique bands. The sides of the tail with a whitish line on the adjoining borders of the outer row of scales and the ventrals. Throat yellowish; ventral surface of the body anteriorly yellowish with irregular, more or less diffuse dark grey markings; these markings become larger posteriorly, and the posterior part of the ventral surface as well as the lower surface of the tail are uniformly dark grey.

The colour pattern of the back in this fully adult female is essentially the same as that found in the smaller specimens of *Pseudoxenodon inornatus* examined by me. The remark by WERNER (1925, p. 52) that the two species from the Sunda Islands are uniformly coloured on the back in adult specimens, is incorrect.

The stomach of this specimen contained the hind limbs of a frog; the rectum contained remains of insects, inter alia of Coleoptera. I am not sure whether these insect remains imply that this snake feeds on beetles, or rather that these remains represent the stomach contents of the frogs eaten by the snake.

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THE REACTION OF DIPHENYLDIAZOMETHANE WITH SULPHUR DIOXIDE

BY

H. J. BACKER AND H. KLOOSTERZIEL

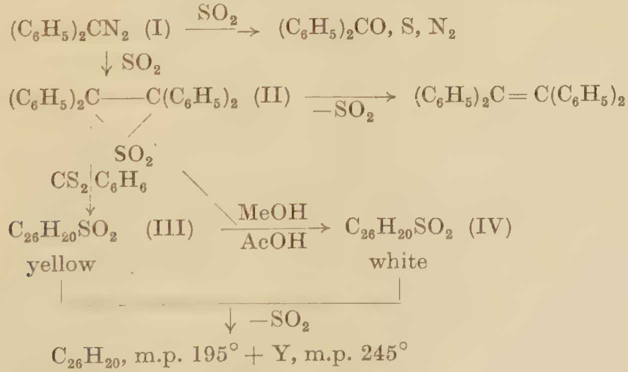
(Communicated at the meeting of October 28, 1950)

STAUDINGER and PFENNIGER [1]) examined in 1916 the reaction of diphenyldiazomethane (I) with sulphur dioxide. In the presence of SO_2 in excess, benzophenone, nitrogen and sulphur are formed. An excess of diphenyldiazomethane, however, leads to a compound $C_{26}H_{20}O_2S$, which they regard as tetraphenylethylenesulphone (II), because it splits, on heating, into SO_2 and tetraphenylethylene.

When heated with indifferent solvents (CS_2, benzene) the sulphone II is transformed into a yellow isomer (III). In the presence of methyl alcohol or acetic acid, however, it gives a white isomer (IV), which may also be obtained from the yellow product by means of the same solvents.

The two isomers (III, IV) are also sulphones. When heated "in vacuo", they lose SO_2 and give rise to the formation of a hydrocarbon $C_{26}H_{20}$ (m.p. 195°), isomeric with tetraphenylethylene; its structure was not established. By heating the yellow sulphone for a long time in boiling acetic acid, S. and P. got a product melting at about 245°, which was not analysed.

In 1930 BERGMANN [2]) synthesized 9,9-diphenyl-9,10-dihydroanthracene (V). In a foot note he suggested, that it might be identical with STAUDINGER's hydrocarbon (m.p. 195°).



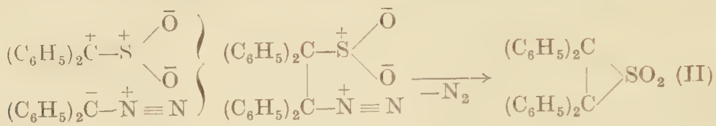
[1]) H. STAUDINGER and F. PFENNIGER, Ber. 49, 1941 (1916).
[2]) E. BERGMANN, Ber. 63, 1628 (1930).

STAUDINGER studied the reaction of diphenyldiazomethane with SO_2 in the hope of obtaining diphenylsulphene:

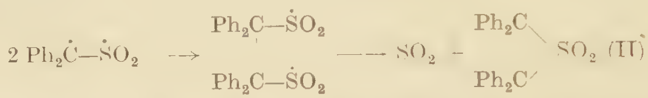
$$(C_6H_5)_2CN_2 + SO_2 \rightarrow (C_6H_5)_2C = SO_2 + N_2$$

This hypothetical sulphene might give rise to two reactions:

A. By virtue of the electron attracting power of the sulphonyl group, the sulphene may show a limiting structure with a positive charge on the central carbon atom. This might combine with the negative carbon atom of a second molecule of diazo compound in its most probable limiting structure; then the product might lose nitrogen and give the sulphone II:



B. Another possibility is a combination of two sulphene molecules in their bi-radical form, followed by the loss of SO_2:



Tetraphenylethylenesulphone (II) has several reasons for being unstable.

1. The ethylenic group carries four phenyl groups.
2. It belongs to an unstable ring of three atoms.
3. The sulphur atom is strongly electro attracting.
4. The molecule has a plane of symmetry vertical to the ethylenic bond.

Thus all conditions are present for breaking this bond and forming a bi-radical: $Ph_2\dot{C}\text{-}SO_2\text{-}\dot{C}Ph_2$ (VI).

Formula VI is only one type of the possible limiting structures. One of the unpaired electrons or both may occupy the *o* or *p* position in the available phenyl groups. Among these there are several equivalent structures of type VII. If the two electrons form together a new bond, a stable five membered ring is formed; the product will be 1,1,3-triphenyl-1,8-dihydro-isothianaphtene-2,2-dioxide (III). The presence of three conjugated double bonds and the conjugated phenyl group explain the yellow colour of this isomer.

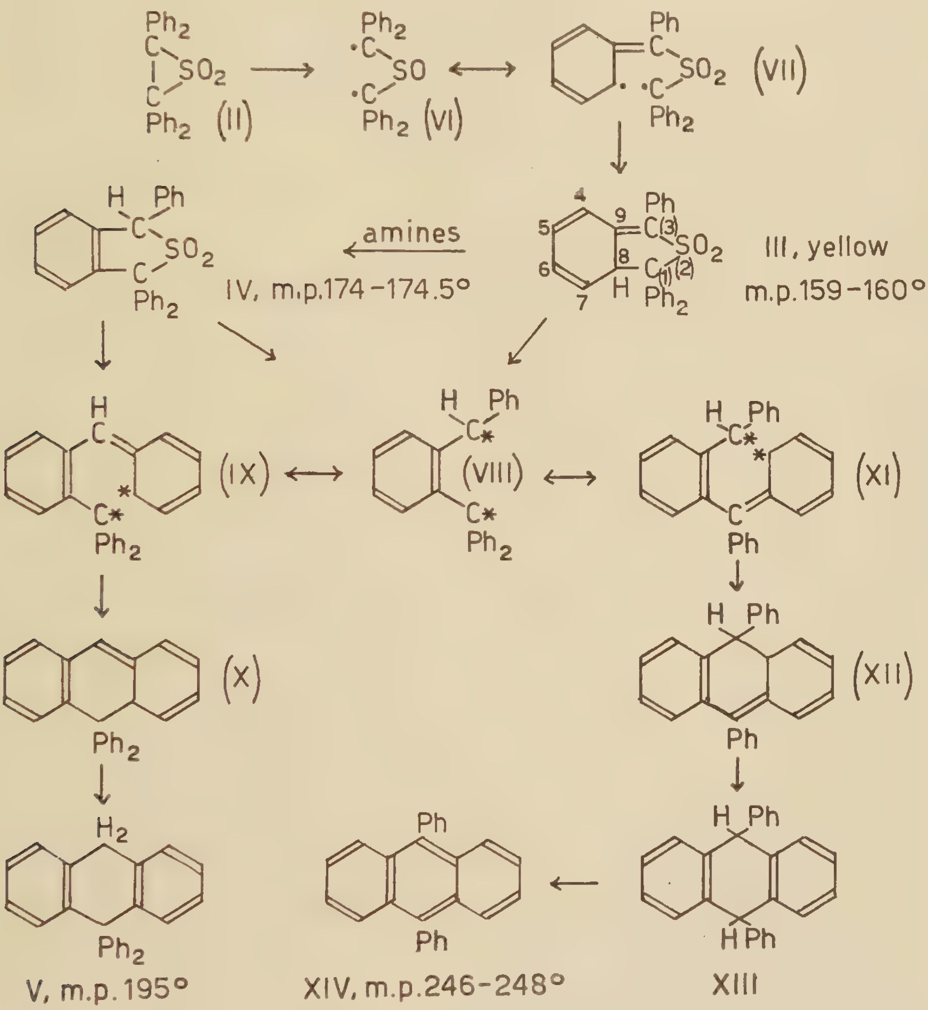
This compound (III) can stabilize itself by aromatisation of the hexadiene ring. This prototropic three carbon tautomerism will occur in polar solvents. STAUDINGER used methyl alcohol or acetic acid. A much better isomerizing agent is an amine. The product has recovered the original benzene ring and the conjugation has disappeared. The new isomer (IV) is colourless.

When heated, this isomer (IV) loses SO_2 and gives the intermediate compound VIII; in the formula the two asterisks represent unpaired electrons (bi-radical) or possibly a positive and a negative charge.

Formula VIII is only one representative of a number of mesomeric forms. The forms IX and XI can close to give six-membered rings. The products X and XII will isomerise to stable anthracene derivatives, respectively:

V, 9,9-diphenyl-9,10-dihydroanthracene,

XIII, 9,10-diphenyl-9,10-dihydroanthracene.



Compound V is the hydrocarbon $C_{26}H_{20}$ (m.p. 195°), obtained by STAUDINGER[1]) and prepared by BERGMANN[2]) by a different method.

Compound XIII wil easily lose a molecule of hydrogen and pass into 9,10-diphenylanthracene, for which substance melting points from 240° to 250° have been recorded. This may be STAUDINGER's compound *Y* (245°).

By decomposing IV "in vacuo" we have got a few crystals of V (m.p. 195°), 6 % of XIII and 65 % of XIV (m.p. 246—248°).

By their melting point and crystalline form the latter compounds (XIII, XIV) were identified with the products which HAACK [3]) obtained by reducing 9,10-diphenylanthracene. The structure of XIV was further proved by chlorination to 9,10-dichloro-9,10-dihydroanthracene [4]) and by oxidation to *o*-dibenzoylbenzene [5]) (mixed melting point).

The experimental part of this research will be published elsewhere.

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ZOOLOGY

THE EFFECT OF INSEMINATION ON THE UTERINE EPITHELIUM OF ELEPHANTULUS

BY

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(Communicated at the meeting of Oct. 28, 1950)

When studying the post-partum involution of the uterus of Elephantulus, during which process the reformation of the uterine epithelium is a salient feature, I remembered having noticed a similar behaviour of the epithelium after insemination. This, of course, motivated me to study again those series of sections of the ovary and uterus in which spermatozoa had been found. It then appeared that, although superficially similar, the reaction of the uterine epithelium after insemination is totally different from that during involution. As Elephantulus ovulates and copulates shortly after parturition, when the involution of the uterus is not yet completed and thus the one phenomenon may obscure the other, for the present study only uteri of animals coming out of anoestrus or of young animals that had just become sexually mature were selected, and of these there were 18 in the collection.

During the post-partum involution, the uterine epithelium, that even in late pregnancy forms a continuous lining of the lumen, is enormously reduced in area and as it had already a certain height during pregnancy, being then cuboidal or even cylindrical, there would be far too much epithelium for lining the small lumen finally reached, if no special measures were taken. A detailed account of this involution will be given elsewhere (to appear in Acta Zoologica). It may be sufficient to say here that the uterine epithelium during the contraction is thrown into numerous high folds. The bases of these folds fuse to form the new lining and all the rest is thrown off as a necrotic mass into the lumen and removed from there to the exterior.

At the approach of oestrus many mitotic divisions in the uterine epithelium can be observed (VAN DER HORST and GILLMAN, 1941). The result of this activity is that at the time of oestrus and before copulation has taken place, the epithelium is very high and crowded with nuclei that are arranged at different levels (fig. 2*a*). Immediately after insemination this epithelium forms numerous outgrowths of an irregular form (fig. 1). They break up into individual cells and these loose cells, now spherical in form,

float into the uterine lumen where they become entangled in a network of fibres. Also a number of spermatozoa can be discerned between these fibres which apparently are the result of a coagulation of the spermatic fluid. Here and there the peripheral part of the epithelium is desquamated

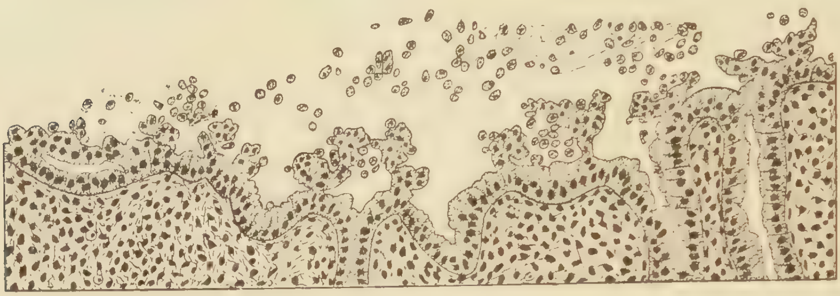
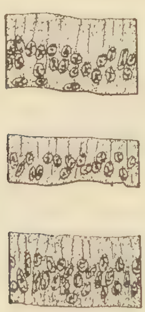


Fig. 1. The uterine epithelium of Elephantulus shortly after insemination. × 150.

as a whole, also this will break up into individual cells. This process is not localised, but occurs over the whole length of the uterine horn and is obviously a result of irritation exerted by the spermatic fluid on the epithelium.

The uterine epithelium in this way loses a considerable number of its constituent cells and the result is striking. During early pregnancy, and before the uterus begins to swell, the epithelium instead of being high columnar as before, has become much thinner, being formed of low columnar cells with the nuclei arranged neatly in a single layer (fig. 2*b*).



a. The uterine epithelium just before copulation,

b. the epithelium after fertilisation,

c. the epithelium after a sterile ovulation. × 250.

Fig. 2.

This result is very obvious when one compares this epithelium with that after a sterile ovulation. In the latter case the epithelium remains very high (fig. 2*c*).

The whole process is of short duration. Already when all the eggs are assembled in the tubal egg chamber, where they are fertilised, the epithelium has nearly resumed its regular appearance, only a few small excrescences are still present, but these will also soon disappear (fig. 3). By this time only a few loose cells have remained in the lumen of the uterine

horns. On the other hand a dense mass of fibres and decaying cells has accumulated in the median uterus, where it forms a real plug. This might be called a vaginal plug, if Elephantulus had a vagina at all which is not the case (VAN DER HORST, 1942). A vaginal plug is known to occur in some

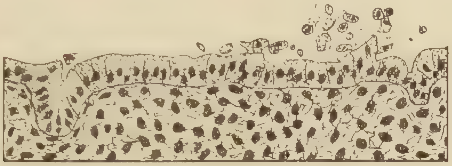


Fig. 3. The uterine epithelium some time after insemination. × 150.

other mammals, particularly in rodents and bats (MEISENHEIMER, 1921). In rodents it is formed by the coagulation of the secretion of the vesicula seminalis that is ejaculated after the seminal fluid proper. This cheese-like substance fills the vagina after copulation, and thus the seminal fluid is retained in the uterus. Whereas in rodents only the male provides the products forming the vaginal plug, in bats the female genital tract is responsible for its occlusion. In Vespertilio and Plecotus this is effected by a secretion to which leucocytes and loose epithelial cells are added, in Vesperugo a proliferation of cells closes the duct. As fertilisation takes place a long time after copulation in bats, this vaginal plug also serves to keep the seminal fluid in the uterus for several months. In Elephantulus both sexes partake in the formation of the "vaginal" plug. Part of the seminal fluid coagulates in the lower uterus as is shown by the presence of sperms in the fibrous mass. To this are added a great many loose cells, desquamated from the uterine epithelium.

The function of the "vaginal" plug of Elephantulus seems to be different from that of rodents or bats. In the former animal the plug is formed only after the spermatozoa have performed their function in fertilising the eggs. Therefore this plug is not formed for the retention of the seminal fluid in the uterus. It is more likely that it inhibits in some way the peristaltic contractions of the uterine wall and thus prevents the early embryos, when they descend, from being ejected from the uterus.

PINCUS and others have demonstrated that the cells of the corona radiata of the rabbit are rapidly dispersed under the influence of the spermatic fluid (PINCUS and ENZMANN, 1936). The active factor was identified as being hyaluronidase, an enzyme that affects the inter-cellular ground substance (see DURAN-REYNALS, 1950). In Elephantulus the egg is liberated not only from the corona radiata but even from the zona pellucida directly at the moment of ovulation. Even so, whatever there is left of the corona radiata breaks up into isolated cells in the Fallopian tube. It is, however, possible or even likely that this same substance, hyaluronidase, which has been found in the spermatic fluid of

several mammals, is responsible for the partial breaking up of the uterine epithelium into loose cells, and thus for the formation of the "vaginal" plug in Elephantulus. May be this can contribute something to the much debated subject of the influence of hyaluronidase on the fertility of mammals.

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MESODERMAL MIXED TUMOR OF THE BODY OF THE UTERUS CONTAINING CARCINOMA

BY

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(Communicated at the meeting of October 28, 1950)

A certain number of the more usual malignant tumors of the uterus, such as the carcinomas, the sarcomas and the malignant leiomyomas, may show a complex structure. This is often caused by variations in the degree of differentiation of the tumor cells or by changes in form or disposition of the cells caused by their environment, e.g. resulting from conditions imposed by the invaded tissues. [1, 2, 3]. In such cases it will usually be possible to recognise these tumors as simple carcinomas, leiomyomas etc., especially when suitable technics are used and different parts of the tumor are examined. There exist, however, other tumors which have multiple neoplastic components; they form a small but interesting group.

Rarest are the trigerminal teratomas, not more than 2 cases having been reported [4]. The existence of another kind of complex tumors, the carcinosarcomas is still being doubted by some authors [5]; according to others 15 acceptable carcinosarcomas of the uterus have been reported [6, 7]. There remain the mesodermal mixed tumors, which are monodermal in origin and are composed of heterotopic cells of various types.

Though more frequent than the carcinosarcomas, the mesodermal mixed tumors of the uterus are still very rare. In the Mayo clinic not more than 2 cases were observed. Likewise FOOT [9] only saw 2 probable cases in 25 years. GLASS and GOLDSMITH [10] collected 94 cases from the world's literature up till 1941. Since then at least 8 more cases have been reported in the American literature [11, 12, 13, 14, 15].

There exists a curious difference in age-incidence between the mesodermal mixed tumors originating in the cervix and those arising in the body of the uterus [16]. Of 38 cervical tumors collected by GLASS and GOLDSMITH [10], 4 were found in the first decade, 6 in the second decade and 88,8 % of the group before the sixth decade. Of 58 tumors of the body of the uterus, 79 % were found after the fifth decade, practically all of them after the menopause.

The tumors are composed of elements foreign to the uterus. In nearly all tumors there occurs myxomatous tissue, which is combined with different tissues, such as cartilage, striated muscle fibers, embryonal

rhabdomyoblasts and sometimes fatty tissue and bone. Histological classification is made depending on whether striated muscle or cartilage predominate. The combination of cartilage and striated muscle fibers is very rare; LIEBOW and TENNANT [11] who restricted their review of the literature to the tumors of the corpus uteri, found only 4 such cases.

There exists no agreement as to which criteria have to be satisfied before an uterine neoplasm can be accepted as a mesodermal mixed tumor. This follows very clearly from the number of cases accepted as such by the different authors. LIEBOW and TENNANT [11] collected 68 tumors originating in the body of the uterus; GLASS and GOLDSMITH [10] found 58 such cases, whereas LEBOWICH and EHRLICH [12] and EHRLICH [13], following the pathological criteria of LÄWEN [17], admitted only 13 cases, to which they added 2 new ones. They rejected such cases as those of GLYNN and BLAIR BELL [18], which contained only striated muscle fibers and which they consider in agreement with the authors as rhabdomyosarcomas. Only such cases which contain striated embryonal muscle tissue in combination with one or more heterologous elements must, in their opinion, be considered mesodermal mixed tumors. LIEBOW and TENNANT [11] found in the literature 19 cases containing striated muscle fibers, to which they added one case. Of these 20 cases only 9 were accepted by LEBOWICH and EHRLICH [12]. Conversely 3 cases, which according to LIEBOW and TENNANT [11] did not contain striated muscle fibers, were accepted by LEBOWICH and EHRLICH [12], who stated that they contained embryonal muscle cells. Probably they were satisfied that the cells described or depicted by the authors were embryonal rhabdomyoblasts, but still without cross striations.

Mesodermal mixed tumors may sometimes be associated with carcinoma or contain epithelial formations which must be considered as malignant. LIEBOW and TENNANT [11] list 8 of such cases; in 3 cases the tumors also contained striated muscle fibers. LEBOWICH and EHRLICH [12] list 2 cases; in the case of EHRLICH [13] a portion of the endometrium adjacent to the tumor was the seat of the usual type of adenocarcinoma of the body of the uterus. In this case areas of carcinoma and sarcoma did not appear to fuse. In the case of REEB and OBERLING [19] a carcinomatous polyp of the fundus uteri coexisted with a mesodermal mixed tumor.

In 8 of the cases collected by GLASS and GOLDSMITH [10] the tumor was associated with uterine myomas. Two of these cases originated in the cervix and six arose in the body of the uterus.

The prognosis of the mesodermal mixed tumors is uniformly bad, regardless of the therapy.

Case Report

A 54 years old virgo was admitted into the hospital because of a tumor which protruded into the vagina. Clinical examination showed that the tumor arose from the corpus uteri. The tumor was removed as far as

possible and sent to the laboratory for microscopic examination. It measured 10 × 5 × 4 cm, its surface was slightly nodular. The consistence was soft; on the cut surface the color was whitish-gray.

A frozen section diagnosis of mesodermal mixed tumor was made and a complete hysterectomy and bilateral salpingo-oöphorectomy were performed (M. J. Hugenholtz). The post-operative course was uneventful.

The second specimen consisted of an uterus and adnexae. The ovaries were small and did not show anything particular. The tubae were thin. The uterus was enlarged, the lumen was occupied by a firm tumor measuring 8 × 7 × 4 cm, which arose from the fundus of the uterus. The most distal part of the tumor was soft and irregular of surface, whereas the more proximal part was smooth and covered by thin endometrium. On the cut surface the firm part of the tumor was finely nodular; the distal part had the same aspect as the first specimen. There was no clear line of demarcation between the two kinds of tussue. The cercical endometrium was smooth and pale.

Microscopic Examination

Thin blocks of tissue were fixed in Bouin's fluid and embedded in tissuemat by Péterfi's methylbenzoate-celloidin method. The sections were stained with Masson's tetrachrome, with the azan-method, with the periodic acid-Schiff's reagent technic, with buffered acid alizarine blue (Thüringer [20]), followed by differentiation in phosphomolybdic acid, with Heidenhain's iron-haematoxylin, with Held's molybdic acid haematoxylin, with Mallory's phosphotungstic acid haematoxylin, with Gomori's method for the impregnation of reticulum, with haematoxylin-picrothiazin, with orcein, with Bodian's stain for nerve fibers and with haematoxylin-azophloxin.

First specimen

The tumor was very polymorphous and there was much intermingling of the different types of tissues. For convenience these will be described separately.

1. *Myxomatous tissue.* This was present in many areas and was characterized by stellate or triangular cells, with long slender processes. In some places the cells showed numerous mitoses, sometimes 3 or 4 in the field of a 4 mm objective. Another curious fact was that a certain number of the cells contained nuclei with abnormally large nucleoli, which were often surrounded by a clear halo.

Gomori's silver impregnation showed that the cells were surrounded by nets of thin fibrillae.

2. *Sarcomatous tissue.* In many areas there were intertwining strands of closely packed plump spindle cells with more or less darkly staining nuclei and numerous, sometimes multipolar, mitoses. In other places the tumor tissue resembled more reticulum cell sarcoma. Again many nuclei contained very conspicuous nucleoli, often surrounded by a clear halo.

The sarcomatous areas contained very numerous argyrophil fibrillae: in the areas resembling reticulum cell sarcoma they apparently were intracellular.

3. *Rhabdomyoblasts.* These cells were found in many places and were quite conspicuous. Very often they were rounded and had a strongly acidophilic protoplasm. Especially the rounded cells contained distinct fibrillae, which were wound around the nucleus. Here and there the cells showed a vacuolization of the peripheral protoplasm, creating the aspect of the "spider cell". The nuclei contained various amounts of chromatin, but almost always a large nucleolus. Other cells were ribbon-like and contained two or more nuclei. Cross-striation was not very frequently observed; it was easiest demonstrated by the staining with acid-alizarine blue, followed by differentiation by phosphomolybdic acid.

The rhabdomyoblasts were sometimes found in large groups: in other instances they were present as isolated elements. Sometimes small rhabdomyoblasts, recognizable by their strongly acidophilic protoplasm and their fibrillae, but much smaller than the cells of the same type which occurred in groups, were found in the sarcomatous areas. In these places, probably due to pressure from other cells, they were almost always elongated. Their nuclei also had the typical aspect.

The rhabdomyoblasts were either as isolated cells or in small groups of 2 or 3 encased by thick nets of argyrophil fibrils.

Cartilage. This was only found in a very few places and was of a more primitive type, as found sometimes in tumors of bone. Its cells often still showed processes.

The cartilage was found in areas, where a small cell with a strongly staining protoplasm dominated. They were grouped in small aggregates, which were surrounded by argyrophil fibers. No such fibers were found inside these cellgroups. Where cartilage developed, a ground substance was seen to separate the originally closely packed cells and the thick reticulum fibers thinned out and partly disappeared.

Giantcells. These were found scattered in the myxomatous tissue; only rarely they showed mitoses. Many were degenerating and contained large hyaline bodies. Their exact nature could not be ascertained.

Epithelial formations. In many blocks there were areas of typical carcinoma. Sometimes it had the aspect of simple adenocarcinoma, in other places the structure of papillary adenocarcinoma. Everywhere the characteristics of malignancy, such as extreme anaplasia of the cells, giant cells, numerous and often abnormal mitoses and large nucleoli were present. It is important to remark that with the Gomori method no argyrophil fibrils could be demonstrated between the epithelial cells and that in most places still a condensation of reticulum fibers surrounded the epithelium as a kind of basement membrane.

The glandular formations of the adenocarcinoma were often surrounded by sarcomatous tissue, containing enormous numbers of reticulum fibers.

thus showing the typical aspect of a carcinosarcoma. Here too the epithelium was separated from the mesenchymatous tissue by a basement membrane, with which the reticulum fibers were connected.

Apart from the adenocarcinoma there were also more solid cellnests, which resembled epithelium. These nests contained sometimes small lumina; there was no special arrangement or orientation of the cells around these lumina. The cells were closely packed, cellborders were not visible. The size of the cells varied little, although occasionally very large elements could be observed. The nuclei contained large nucleoli; the protoplasm was basophilic. Mitotic divisions were numerous; they were sometimes multipolar. In a few cellnests reticulum fibers were found, though in small numbers.

Most of the cellnests appeared to be sharply delimited and in sections impregnated with silver according to GOMORI's method were seen to be surrounded by a basement membrane-like structure as was also found in connection with the glandular formations of the adenocarcinoma. In some places however, the cells of the periphery of the cellnests were more loosely arranged and merged imperceptably with cells of the surrounding mesenchymatous tissue. Between the more loosely arranged cells reticulum fibers had appeared; in such places there was no basement membrane.

No nerves could be found in sections stained with BODIAN's method, nor was bone observed in the many blocks of tissue examined.

Second specimen

The firm part of the intra-uterine tumor was a typical benign leiomyoma, containing much hyaline connective tissue and, as is often the case in myomas of older women, numerous mastcells.

The distal part of the leiomyoma was infiltrated by a very complex tumor tissue in which various structures could be distinguished. It contained adenocarcinoma, sometimes with highly anaplastic cells and numerous abnormal mitoses, solid nests of carcinoma and very cellular sarcomatous tissue. The impregnation of the reticulum showed that the carcinomatous masses were in most instances sharply delimited by a condensation of the reticulum. The sarcomatous cells were surrounded by a very dense network of reticulum fibers, so that the contrast between the carcinomatous and sarcomatous tissue was striking.

Whereas in the first specimen there were areas with myxomatous tissue and areas poor in cells, alternated with very cellular ones, the tumor tissue in the second specimen was everywhere equally cellular and did not contain myxomatous tissue. Only after a prolonged search was an isolated rhabdomyoblast found.

Sections of the cervix showed an atrophic endometrium. In the myometrium in the region of transition between cervix and corpus a few foci of carcinoma, both adenocarcinoma and the more solid form, without differentiation, were found.

Sections of the ovary showed atrophy without peculiar lesions.

Discussion

Since the tumor contained not only myxomatous tissue and rhabdomyoblasts but also striated muscle fibers and cartilage, it even satisfies the more exacting pathological criteria laid down by LÄWEN [17] and adopted by LEBOWICH and EHRLICH [12], so that the diagnosis of mesodermal mixed tumor is not dubious.

The endometrium of the cervix was completely free from tumor tissue. The tumor was found to be attached to the distal part of a submucous leiomyoma, which in turn was attached to the fundus and protruded into the uterine cavity. In the myometrium just above the cervix a few nests of tumor tissue were found. It is therefore quite improbable that the tumor originated in the cervix and it is reasonable to suppose that it arose in the endometrium covering the leiomyoma or at least in the tissue which had been pushed downwards during the development of this benign tumor.

Although GLASS and GOLDSMITH [10] found in the literature only 8 cases in which a mesodermal mixed tumor was complicated by a leiomyoma, we cannot attach any importance to this finding in the present case. Uterine leiomyomas are so frequent in the native population of Curaçao, that we cannot but consider this complication as a simple coincidence.

Of special interest is the occurrence everywhere in the tumor of tissue which according to all standards must be considered as carcinomatous. As has already been stated in the introduction, this has been observed in only 8 cases. Part of this carcinomatous tissue had the typical aspect of adenocarcinoma, although often very anaplastic. It was more difficult to determine the exact nature of the more solid cell nests, which resembled epithelium. That the cells were cancerous cannot be doubted: the numerous and often atypical mitoses, the abnormally large cells and the atypical nucleoli are in my opinion sufficient proof of this. The fact that they were closely packed and that they were usually separated from the surrounding tissue by a kind of basement membrane does not constitute an absolute proof of their epithelial nature. In connection with this it must be pointed out that structures resembling epithelium also occur in purely mesenchymatous tumors such as synoviomas (HAAGENSEN and PURDY STOUT [21], HARTZ [22]). The fact that in some places the cells of these cellnests apparently assumed properties of mesenchymatous elements makes one doubt if the cells must be considered simply as anaplastic cancerous cells such as occur regularly in the common carcinomas of a high grade of malignancy. MASSON [23] has shown convincingly that in the embryonal adenosarcomas of the kidney small undifferentiated cells, which later give rise to mesenchymatous elements, develop from epithelial formations which sometimes resembled the cellnests observed in the present case.

The fact that the carcinomatous formations were present everywhere in the tumor makes it not very probable that an associated endometrial

carcinoma had invaded a mesodermal mixed tumor. More likely both the epithelial and mesenchymatous formations originate from the same source, although it must be conceded that absolute proof for this is lacking.

In the discussion of their case LEBOWICH and EHRLICH [12] state that "carcinomatous degeneration" is extremely rare in mixed mesodermal tumors and that the sarcomatous tissue which has been described in a certain percentage of cases as arising in the tumor growth proper may have been confused with primitive spindle-shaped myoblasts. According to these authors the exact frequency of sarcomatous degeneration cannot be determined with certainty unless the van Gieson stain is used.

It is certainly true that in the present case rhabdomyoblasts were found developing in areas which without the presence of these cells could only be labeled as sarcomatous, since they contained intertwining bundles of plump spindle cells with nuclei of varying size and with large nucleoli, sometimes surrounded by a clear halo, numerous, sometimes atypical mitoses whereas between the cells a very rich network of fibers could be demonstrated with the GOMORI method. When it is correctly assumed that the rhabdomyoblasts develop from the spindle cells, then the sarcomatous tissue should be classified as rhabdomyosarcoma, since it is composed of cells capable of developing into rhabdomyoblasts. It thus remains a sarcoma, although a sarcoma of a special type [24].

It is not quite clear why the VAN GIESON stain should be necessary for the diagnosis of sarcoma. This stain was formerly considered necessary for distinguishing between connective tissue and unstriated muscle fibers and is in this respect much inferior to the trichrome stains; it only stains collagen selectively and it is not to be expected that undifferentiated cells, which may develop into rhabdomyoblasts, will stain with the picric acid as fully normal and differentiated muscle cells may do.

The myxomatous tissue, which in most papers on mesodermal mixed tumors is only very shortly described and discussed, showed in this case several special features such as an increased number of mitoses and nuclear abnormalities.

It should therefore not be considered as a simple "benign stroma"of a malignant neoplastic "parenchym" but as an integral part of a malignant tumor composed of several neoplastic components.

In many places, especially in the tumor tissue invading the leiomyoma, the typical picture of carcinosarcoma was found. Only after a prolonged search could an isolated rhabdomyoblast be demonstrated in the sarcomatous component. This raises the question of the relationship between the carcinosarcomas and the mesodermal mixed tumors of the body of the uterus. The carcinosarcomas also usually arise as polypoid growths from the body of the uterus after the menopause. It is well known that it may be difficult to recognize in sarcomas the few cells which may give a clue as to its nature, such as a few cells with cross striations, especially when no suitable methods are used for the praeparation of the sections. It is

therefore always possible that tumors diagnozed as carcinosarcomas were really mesodermal mixed tumors. Especially carcinosarcomas containing giant cells should be carefully examined, since according to PURDY STOUT [24] the presence of bizarre giant cells in malignant mesenchymatous tumors definetely removes them from the fibrosarcoma class and means that they are derived from some other specialized cell, such as the lipoblast or the rhabdomyoblast. Since there are indications that the frequency of the carcinosarcomas is increasing, probably under the influence of radiation therapy, more material might become available for study [6, 25].

The histogenesis of both the mesodermal mixed tumors and the carcinosarcomas of the uterus is still an unsolved problem. Many hypotheses have been emitted, none of which can be considered to be very satisfactory. WILMS believed that the tumors arise from undifferentiated mesodermal cells which are displaced during the descent of the WOLFFIAN body. LEBOWICH and EHRLICH [12] think it more probable that the cell rests from which the tumors originate, arise from the MÜLLERIAN ducts after their fusion to form the uterus and vagina. MOREHEAD and BOWMAN [26] assume an origin from pluripotential mesodermal cells which have remained dormant and for some reason assume neoplastic properties. According to MCDONALD and all. [8] the mesodermal mixed tumors arise by a process of dedifferentiation of the stromal cells, after which process differentiation takes place into various types of mesodermal structures.

More recent embryological investigations of GRUENWALD [27] have shown that there exist close relations between the limb primordia and the urogenital ridges. According to GRUENWALD this could serve to explain the occasional occurrence of endometrioses in the extremities. Conversely it might be possible that this relationship has something to do with the appearance in the uterus of tissues such as striated muscle fibers and cartilage, which normally occur in the limbs.

MEYER [28] divided mixed tumors into 3 categories: 1. "Kollisionstumoren", which are primarily two unconnected tumors which get intermingled. 2. "Kombinationstumoren", the different components of which have one common stemcell and 3. "Kompositionstumoren", which are produced by the neoplastic proliferation of the "parenchym" and "stroma" of a single tumor. LISA and all. [6] believe that at least the carcinosarcomas belong to the third group, without offering any evidence for this opinion.

Even in well controled experiments it is very difficult or impossible to explain exactly the formation of sarcomatous tissue from primary carcinomas. STEWART, GRADY and ANDERVONT [29] transplanted spontaneous and induced pulmonary tumors of several strains of inbred mice. During repeated transplantations 11 out of 22 tumors developed a change in the characteristics of the growths, which represented a transition from an

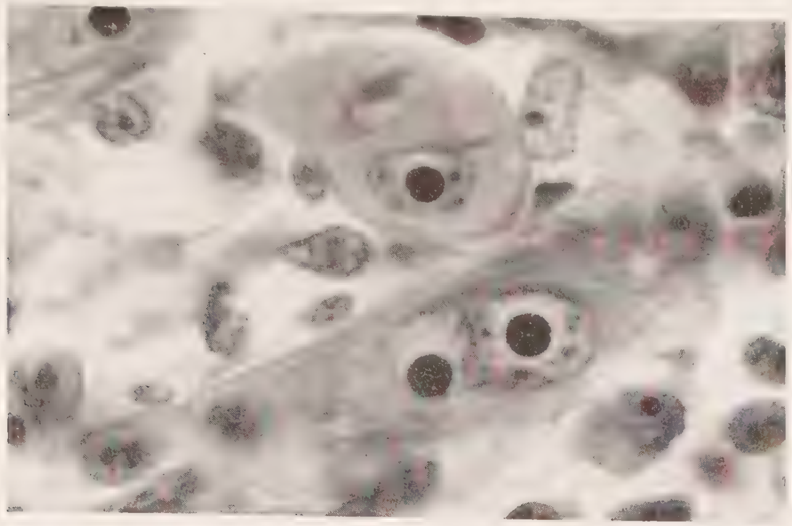

Fig. 1. Rhabdomyoblasts. Note fibrillae and the large nucleoli.

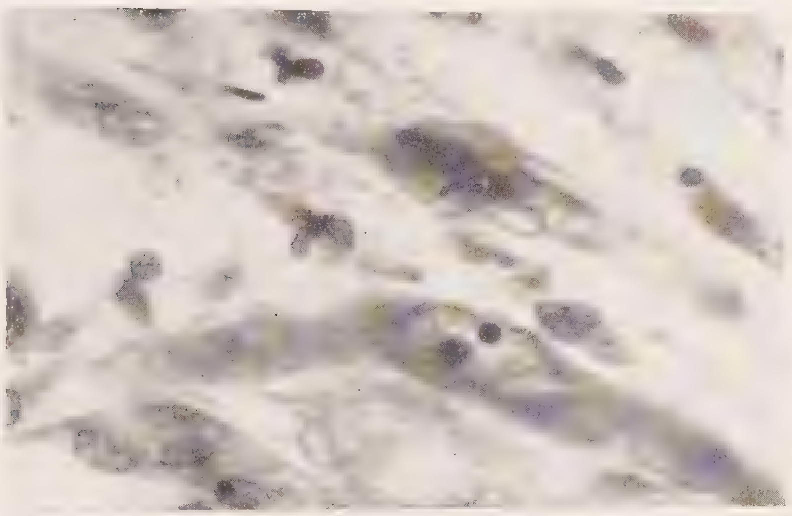

Fig. 3

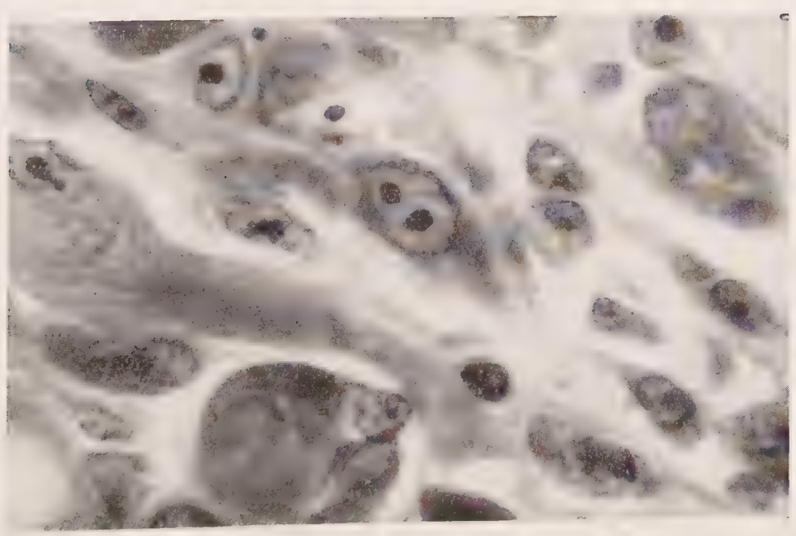

Fig. 4

Fig. 2

Fig. 2–4. Cells with distinct cross striations. Acid alizarine blue-phosphomolybdic acid.

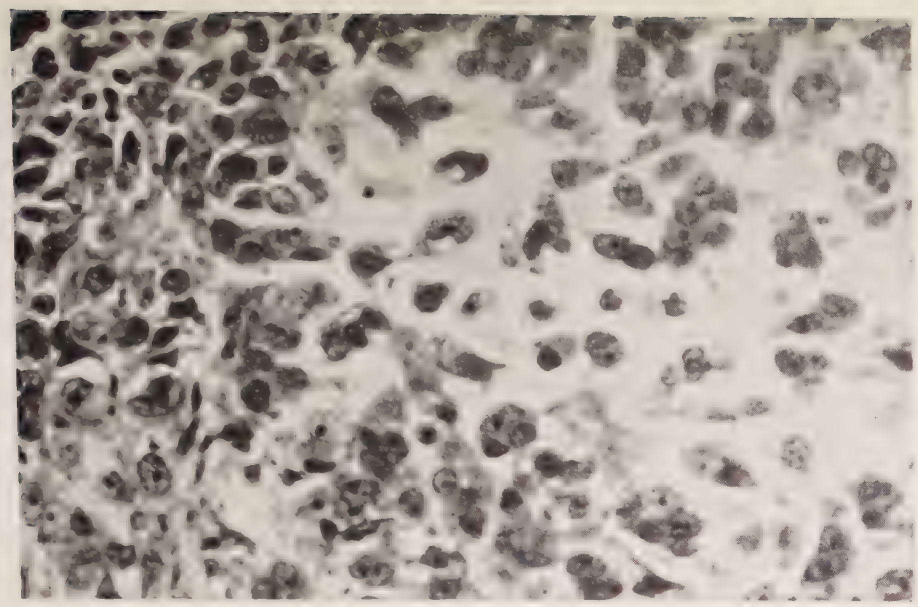

Fig. 5. Primitive cartilage.

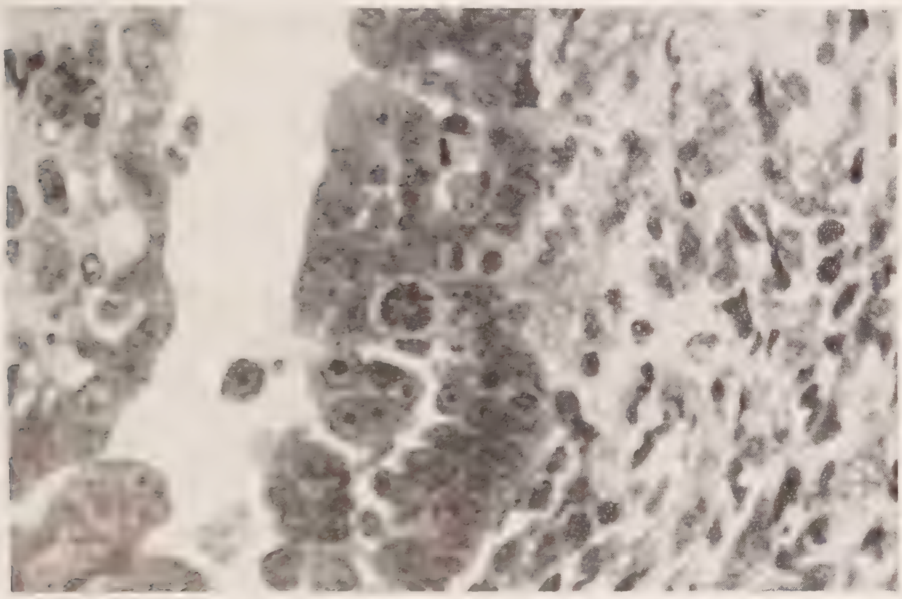

Fig. 6. Carcinoma. Mitoses and large nucleoli.

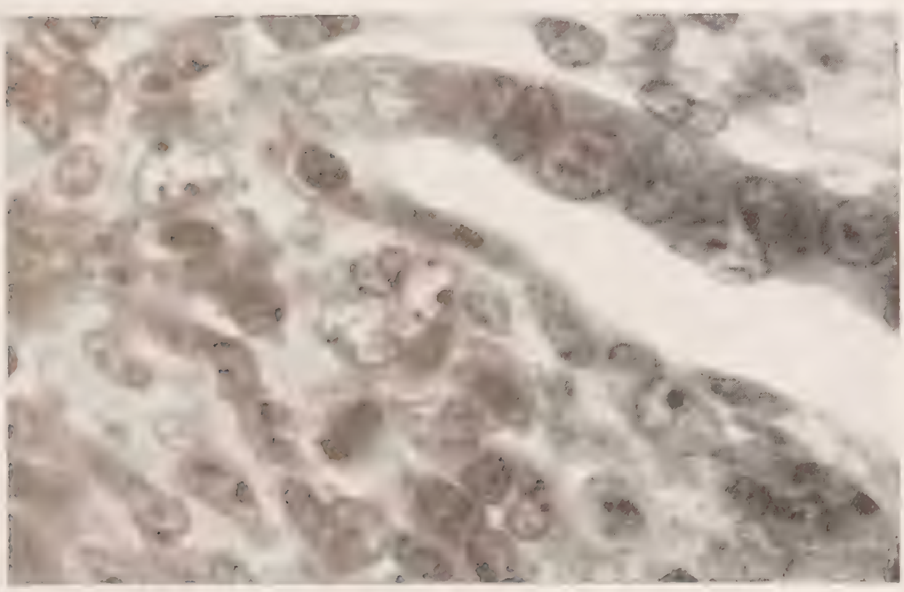

Fig. 7. Carcino-sarcoma. Note mitosis in the sarcomatous tissue.

epithelial pattern to a fibrosarcoma. In part of the lines the tumor transplant had become wholly sarcomatous, while in others the tumor was composed of a combination of sarcomatous and carcinomatous structure. No positive conclusion as to the mechanisms and underlying causes of the sarcomatous transformation was reached, although the various possibilities were carefully considered. According to the authors it might even have been possible that the primary tumors were mixed from the start.

In view of the foregoing it is not surprising that the problem of the histogenesis of the mesodermal mixed tumors has not yet been solved. This solution may eventually be found through the combined efforts of the histopathologists and the experimental embryologists.

Summary

A case of mesodermal mixed tumor of the body of the uterus occurring in a 54 years old virgo, is described. The tumor was attached to a benign leiomyoma of the fundus, which was being infiltrated by the tumor tissue. It protruded into the vagina. In addition to striated muscle fibers, rhabdomyoblasts, cartilage and myxomatous tissue, it contained epithelial structures which satisfied all morphologic criteria of carcinoma. These structures occurred everywhere in the tumor, which in many places presented the typical aspect of carcinosarcoma. In the myxomatous tissue there were nuclear abnormalities and sometimes numerous mitoses. In several places rhabdomyoblasts were seen developing from tissue which had the aspect of a high grade fibrosarcoma. Also there were areas resembling reticulum cell sarcoma. The different theories concerning the histogenesis of the mesodermal mixed tumors and carcinosarcomas and the possible relationship between these tumors are briefly discussed.

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CHEMISTRY

ON THE NATURE OF RECRYSTALLIZATION NUCLEI AND THE ORIGIN OF RECRYSTALLIZATION TEXTURES

BY

W. G. BURGERS AND T. J. TIEDEMA *)

(Communicated by Prof. J. M. BIJVOET at the meeting of Sept. 30, 1950)

On the nature of recrystallization nuclei:

1. Considerable thought has been given to the question of the nature of the nuclei of crystals growing in deformed metals by the process of recrystallization (see for a discussion of the literature up to 1940 BURGERS, 1941). Generally speaking, several authors, though differing perhaps in details of their conception, favour the view that nuclei are small lattice regions already present in the deformed matrix, which, for some reason or other, obtain the faculty to grow during heat treatment. The experimental fact that, in several cases, the lattice orientations of the crystals formed after recrystallization can also be found in the deformed test-piece, has been considered to support this conception (DEHLINGER, 1929; BURGERS, 1942, 1949). Based on the assumption that a lattice region could grow only at the expense of surrounding regions if it is more stable (less strained), it was assumed that those lattice blocks could serve as "potential nuclei" which, being originally in a strained state, on heat treatment suffered some stress-releasing process, which transformed them from "potential" into "actual" growth nuclei (VAN ARKEL, 1930; VAN LIEMPT, 1931). The necessity of such an activation may account for the occurrence of an incubation period before visible growth starts, as observed in recrystallization experiments. Considerations regarding the atomic character of the activation process have been given, as early as 1929 by DEHLINGER (1929, 1933) and, more recently, by BURGERS (1947; 1949). The latter, starting from the assumption that recrystallization was essentially a process of elimination of dislocations, advanced the idea that by a proper elimination of dislocations of "opposite sign" somewhere in the deformed matrix stresses could be reduced locally and so create a "remaining" stress, capable of displacing a dislocation layer between two adjoining domains, initiating crystal growth. It was thought that a lattice block such as *b* in the schematic figure 1, lying in the inflexion point of a S-curved region, was particularly favorably placed for such a local neutralization process, as it is separated from the neighbouring blocks *a* and *c* by dislocations of opposite sign. This conception of the activation process

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and of the most probable nuclear spot is much akin to DEHLINGER's conception of 1929. The "S"-curved regions were considered to be a direct consequence of the occurrence of local disturbances of the glide-process, which produced local rotations (so-called "local curvatures") of the active glide-lamellae about the normal to the glide-direction (see BURGERS, 1934).

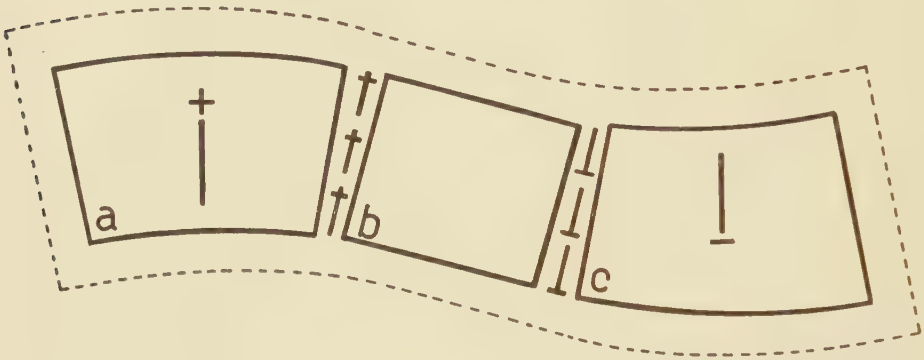

Fig. 1. Three adjoining lattice blocks, which, taken together, can be considered to form a "*S*-curved" lattice region. Block *b* in the "inflexion point" can presumably function as a nucleus for recrystallization.

2. The conception that an elimination of dislocations, involving a release of strain energy, may lie at the root of the process required to "activate" a potential nucleus to growth, was considered to be supported by the phenomenon of "stimulation" of crystal growth. As set forth in various papers (ref. in BURGERS, 1947), in recrystallized aluminium plates crystals may be found of a special "pointed" shape, the occurrence of which can be understood on the assumption that growth of such a "stimulated" crystal starts at the moment that an already growing "stimulating" crystal comes into contact with its nucleus. As it was found that such crystal pairs were mutually oriented as spinel twins (with a precision of less than a minute of arc: GUINIER and TENNEVIN, 1949; May, 1950), a practically perfect fit was possible between the growing crystal and the potential nucleus of the stimulated crystal. This lead to the suggestion that the sudden elimination of dislocations produced when contact was established was the actual cause of the stimulating process and thus constituted a direct example of the growth-activation of a potential nucleus (that of the stimulated crystal), this time not brought about by thermal agitation as such, as in spontaneous nucleation, but in a kind of "artificial" way.

In the light of this phenomenon, a conception of spontaneous nucleation might be conceived, which is somewhat different from that advanced in paragraph 1. From a recent paper by SHOCKLEY and READ (1950) on dislocation models of crystal boundaries (supported by the experimental

work of DUNN and co-workers, 1949; 1950), it follows that the energy content of the boundary layer between two adjoining lattice regions with special mutual orientations is extremely sensitive for slight variations in the orientation of the boundary layer, in this sense that it increases at an infinite rate with deviations from a special position. If then, in a deformed test-piece, adjoining lattice regions happen to be present in such mutual positions, it seems reasonable to assume that on heating a release of strain energy by such slight displacements may occur at their boundary and so transform a potential nucleus into a growth nucleus.

There may be some relation between this conception and that brought forward by KRONBERG and WILSON (1949) in connection with their investigation of the growth of large crystals on prolonged annealing of fine-grained copper with cube-texture by "secondary recrystallization" (abnormal grain growth). The fact that, according to their experiments, this occurs only in twin-bearing material, leads KRONBERG and WILSON to the assumption that "nucleation" occurs preferentially at twin boundaries and is connected with stacking faults existing at such boundaries. Moreover, they point out that the orientation relationship existing between the new crystals and the primary texture (they are related by a rotation about either a [111]- or a [100]-axis over approximately definite angles) is such that the atoms in the (111)-resp. (100)-planes show definite coincidences or near-coincidences in both orientations, so that the atoms of one net can be brought into the sites of the new net by simple movements. If, reasoning along the lines set forth above, two such lattice regions were adjacent in the deformed state, it might be envisaged that such movements, bringing about better fit, were apt to give a stress release and to initiate growth.

3. A more defined conception of the nucleation process has been given in a paper by CAHN (1950). This paper, which starts from the assumption given above that the growth nuclei are actually formed in the most distorted parts of the lattice, i.e. in the "local curvatures", postulates (as is also done in a note by BECK, 1949) that the process which transforms the potential nuclei in growth nuclei, is essentially the recently much discussed process of "polygonization" (OROWAN, 1947; CAHN, 1949). This process, taking place in curved lattice regions, is considered to consist of a diffusion of dislocations parallel to the slip planes, thus producing a redistribution of dislocations causing a change of a continuously bent lattice into a number of polygon elements, each keeping the orientation of that part of the bent lattice from which it is formed but free of elastic strain [1]). (Cf. CAHN, 1950 p. 326). This is illustrated in a schematic way in

[1]) It was pointed out to us by Dr W. SHOCKLEY that also local redistributions of dislocations in a somewhat different way may involve a release of strain energy. For example under special conditions a displacement of two sets of dislocations along intersecting glide-planes can built up a boundary between lattice elements including a definite angle under release of strain.

figure 2, which is taken from CAHN's paper (1950). Figure 3, also taken from this paper, illustrates the effect of polygonization of a local curvature, representing a potential nucleus (PP in figure 3*a*) and suggests (figure 3*b*) that a strain-free element formed in this way is able to grow at the expense of the surrounding lattice. For details of CAHN's conception, in particular his assumption that the "incubation period" for

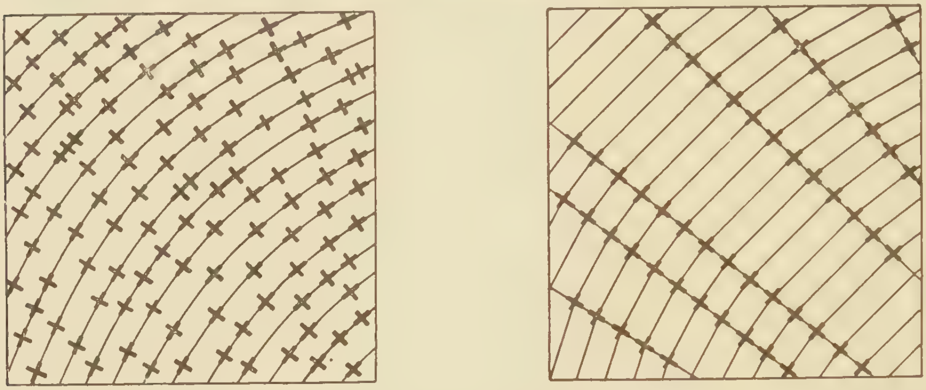

Fig. 2. Nature of polygonization in a bent crystal lattice: *a*) as bent; *b*) annealed. The crosses represent excess positive dislocations remaining on the glide-planes after bending. [After CAHN (1950)].

activation is inversely proportional to the radius of a local curvature, and the possibilities of his theory to explain a.o. ANDERSON and MEHL's observations on the kinetics of nucleation we must refer to his paper.

4. The conceptions of the process of nucleation, discussed in the foregoing paragraphs, are in our view supported by X-ray diffraction results recently obtained by TIEDEMA (1950). These results show that LAUE photographs of the "center" (the nuclear region) of aluminium crystals formed by recrystallization have a peculiar striated appearance, as if they are accompanied by satillite spots, these pecularities being absent on photographs of parts of the crystal *outside* the nuclear spot region. This fact points to the presence, in the nuclear region, of lattice elements differing in orientation of the order of a degree of arc from the main body of the crystal, which have been left unconsumed by the growing nucleus.

As it is well established that a growing crystal cannot, or in any case only very reluctantly, consume lattice regions of approximately parallel orientation (TIEDEMA, MAY and BURGERS, 1949; LACOMBE and BERGHEZAN, 1949), the above result is in excellent agreement with the idea that a crystal grows from a lattice element, which forms part of a local curvature, as schematically shown in CAHN's paper (see figure 3), leaving unabsorbed some neighbouring elements of approximately the same orientation and growing at the expense of the deformed matrix outside the local curvature, which differs from it far more in lattice orientation.

Moreover, the non-focussed LAUE photographs taken by TIEDEMA

according to GUINIER and TENNEVIN's method (1949) show that *not one but a few elements* of the local curvature, with *slightly* deviating orientation (of the order of minutes of arc), may function simultaneously as actual growth nuclei, growing as it where side by side and producing a crystal consisting of as many parts with the same slight orientation differences [2]).

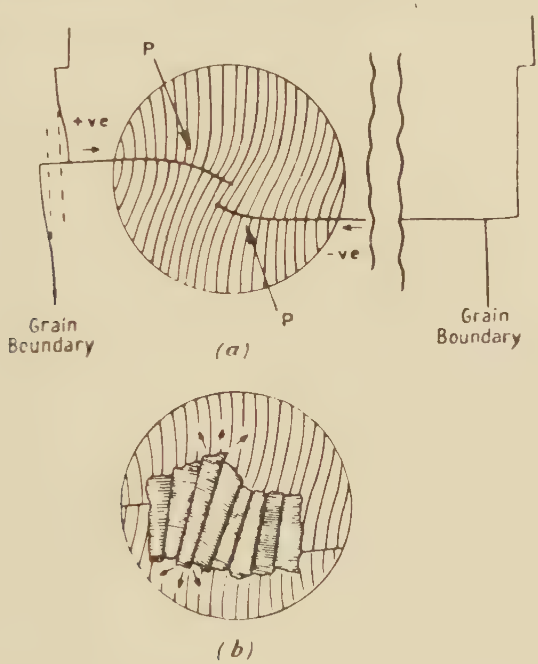

Fig. 3. Local curvature in a deformed crystal: *a*) as deformed; *b*) after annealing. [After CAHN (1950)].

Such a behaviour seems compatible with either of the two conceptions of nucleation discussed above, (1) by elimination or (2) by redistribution of dislocations. In fact, it seems a priori very well possible that in both cases more than one element of the original local curvature becomes sufficiently free of strain to obtain the faculty to grow. These, due to their approximately coinciding orientations, can grow at equal rate at the expense of the surrounding matrix, thus forming a straight "boundary" between them as observed in TIEDEMA's photographs, while leaving unabsorbed those elements of the original local curvature, which differ from them in orientation to a larger degree.

In this connection it seems of interest to remark that in rapidly heated samples KRONBERG and WILSON (see paragraph 2) often observed roughly

[2]) These regions of *macroscopic* size, dividing the final crystal in a few parts only, must not be confused with the very much smaller lattice regions (dimensions $\sim$ 0.1 mm) with still smaller (less than one minute of arc) orientation differences existing over the whole extension of the crystal and therefore inside each of the larger blocks, as discussed by LACOMBE and GUINIER and co-workers (see f.e. references in CHAUDRON, 1949).

elliptical grains containing a twin boundary near the center and along the major axis, the occurrence of which they take as an indication that in that case growth develops a twinned crystal as the first unit of growth, which continues to grow most rapidly parallel to the twin boundary. This may perhaps be compared with the occurrence of two or three side-by-side growing parts of slightly different orientation observed by TIEDEMA in aluminium crystals. We are, however, aware that this comparison is highly speculative. For instance Dr SHOCKLEY raised the question whether in our case such a process will give a sufficient release in strain energy (angle between two lattice elements about 1/200 radians corresponding to about 1 dislocation per 200 interatomic distances), and if not the origin of the crystal parts must be explained in quite a different way.

On the origin of recrystallization textures:

5. The nucleation theories discussed in the foregoing paragraphs are denoted by BECK and co-workers (1949, 1950) as "oriented nucleation", in so far they imply the supposition that the occurrence of potential nuclei in definite orientations, as refound later after completed recrystallization in the orientation of the new crystals (recrystallization texture), is a direct consequence of the foregoing deformation process. To state an extreme case: BURGERS and LOUWERSE (1931), when recrystallizing aluminium single crystal discs, which had been subjected to homogeneous compression between flat discs, thus causing almost pure shear parallel to definite glide-combinations (glide-plane (111), glide-direction in this plane [110]), found crystal orientations, which could be deduced from those of the deformed crystals by a rotation about the normal to the glide-direction (a [112]-direction).

The conception of "oriented nucleation" has been disputed by BECK c.s. in favour of a theory of "oriented growth". These authors, in an interesting series of papers (1949, 1950) found that in aluminium a pronounced orientation relationship exists between grains growing in a matrix with a strong single orientation texture and the matrix itself, namely a rotation of 30—40° around a [111]-axis. This relationship exists as well when the matrix is a cold worked single crystal as in coarsening when the matrix is an annealed primarily recrystallized fine-grained material with a pronounced preferential orientation. Similar orientation relationships have been found (in some cases together with other ones) in other cubic face-centred metals: in copper by BOWLES and BOAS (1948) and by KRONBERG and WILSON (1949) (see paragraph 2), and in nickel-iron alloys by RATHENAU and CUSTERS (1949).

The occurrence of the same relationship in all these cases is considered by BECK c.s. as an indication that grains with definite orientations with respect to the matrix grow much faster than others. Taking this as starting point, and assuming that in a deformed matrix, and also in a recrystallized material, even when a strong texture is present, there are always some

lattice elements in practically any orientation [3]), it is supposed that only those domains can serve as actual growth nuclei, which are favorably oriented with regard to the matrix for their growth. The resulting texture would thus be caused by "selective growth" and not by "selective nucleation" of domains in special positions only. A similar view was tentatively advanced by BARRETT (1940), and discussed by DUNN (1948).

6. When comparing these two conceivable theories (see also DUNN, 1948), it must be said first of all that also in the opinion of the present authors it is certain that a selectivity of the growth process exists, in that sense that the ease with which a crystal can grow in a matrix depends very much on the mutual orientation of growing and disappearing lattice domain. This was already evident from early researches by VAN ARKEL (1932) on "secondary recrystallization" of aluminium, in which it was observed that a large crystal, growing at the expense of a fine-grained matrix, often stops its growth on attaining a part of the matrix, which had a texture different from that present on the part where the crystal was growing. Also experiments on crystal growth in locally deformed single crystals of aluminium (by scratching: BECK, SPERRY and HSUN HU, 1950) or in pseudo-unicrystalline nickel-iron foil (by pinprick: RATHENAU and CUSTERS, 1949) show that the largest crystals developed occupy special positions with regard to the matrix texture. Finally the non-consumability by a growing crystal of lattice domains in approximately identical or twin position (TIEDEMA, MAY and BURGERS, 1949; BURGERS and DALITZ, 1949; LACOMBE and BERGHEZAN, 1949) is an exponent of this phenomenon (cf. also BOWLES and BOAS, 1948).

A quantitative measure of the variation of rate of growth with orientation of large crystals growing in a given texture can be deduced from experiments by DUNN (1948) with silicon iron. A variation up to about 30 % was observed. From these investigations it may be concluded that a high rate of growth is connected with a considerable difference in orientation between growing and disappearing lattice domains. The question remains, however, whether the selectivity is itself sufficiently pronounced to explain the observed orientations after recrystallization. In considering this question it seems to us important to take into account the following experimental facts:

a. A consideration of the results obtained with aluminium by BECK c.s., according to which a 30—40° rotation about an [111]-axis between growing grain and matrix is particularly favorable for growth, shows that deviations from such positions up to at least 15° in some direction are present. This is a considerable amount in a lattice with cubic symmetry

[3]) There is no doubt that this is certainly true in most cases (cf., however, paragraph 6, under (*b*)), as X-ray photographs apart from the interference spots due to the prefered orientation, practically always show at least some intensity along the Debye-Scherrer rings outside the intense reflection region due to the texture.

and means that crystals with a quite different crystallographic orientation with regard to the matrix can also grow with a comparable rate [4]).

That this is actually true can be demonstrated in a direct way by forcing a crystal with a prechosen orientation to grow in a matrix with a pronounced texture, using the method of "growing round the corner", as realized by TIEDEMA (1949) with aluminium and by DUNN (1949) with silicon-iron. For aluminium, such experiments (TIEDEMA, unpublished results) show that in a matrix, with a sharp texture, obtained by stretching a single crystal, apart from crystals deviating from this texture by a rotation about [111] as considered by BECK, also crystals related to the matrix by a rotation about for example an [110]-axis can grow over large distances. This agrees to some extent with the fact that in some cases (for example in RATHENAU and CUSTERS' work on nickel-iron and in that of

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4) The above can be illustrated in another way. As set forth in paragraph 5, the recrystallization texture of homogeneously compressed aluminium single crystals was interpreted by BURGERS and LOUWERSE on the basis of oriented nucleation in lattice regions rotated about an axis [112], perpendicular to the glide-direction [110]. In the original paper it is shown that the observed orientations could approximately be ascribed to rotations around the perpendiculars to the various active glide-directions. BECK and HSUN HU (1949), in an effort to fit these results into the theory of selective growth, show decidedly that the same orientations, at least the prominent groups, can also be described, perhaps even somewhat better, by a rotation around [111]-direction.*) As far as this statement goes, the argument may be taken as an example that also in this case the scattering of the observed orientations is so considerable that they can approximately be described by rotations about different sets of axes. One may say also that a description of a scattered texture by a rotation about a definite axis or set of axes has in itself not much value. It derives this value from the interpretation: either (BURGERS and LOUWERSE) on the basis of "oriented nucleation" in local curvatures, or (BECK and HSUN HU) in terms of growth selectivity.

*) In this connection it seems of interest to remark that, as set forth by KOCHENDÖRFER (1950) (cf. also BILBY, 1950), shear along a [111]-plane may be produced as well by propagation in a direction parallel to the glide-direction of a "Taylor" or "line" dislocation, as by a sideways displacement of a screw-dislocation. Whereas the holding-up of line-dislocations gives rise to local curvatures about the normal to the glide-direction [112], considered in paragraph 5, the holding-up of screw-dislocations will presumably cause local lattice rotations about the normal to the glide-plane [111] (cf. J. M. BURGERS, 1940). Only the first type of curvatures was considered at the time of the BURGERS and LOUWERSE 1931-paper. If gliding in these experiments was also produced by the second mechanism, rotations about the [111]-axis normal to the active glide-planes might be expected. They were apparently observed by HEIDENREICH and SHOCKLEY (1948) (cf. also FRANK, 1948). If such local rotations actually existed, the presence of [111]-related deformation and recrystallization textures could even be expected on the ground of an oriented nucleation theory.

It must be mentioned, however, that the [111]-axes applied by BECK and HSUN HU (1950) (cf. also BARRETT, 1940) to explain the recrystallization textures in compressed aluminium single crystals, were *not* the [111]-axes perpendicular to the prominent glide-planes.

KRONBERG and WILSON on copper), besides the "[111]"-rotated crystals, occasionally also large crystals with other orientations with respect to the matrix, grew by "secondary recrystallization".

b. A second point to be considered is the following: a deformation process apparently not always produces growth nuclei in every possible orientation, from which the recrystallizing matrix may "chose" those best fitted to grow. This is clear from the following (unpublished) experiments: a drawn and annealed aluminium wire, on prolonged heating, often shows the formation of large crystals with a [210]-direction parallel to the wire axis (BURGERS and SANDEE, 1942). This fact itself appears to fit in BECK's et al. selective growth theory, as the [210]-orientation can be deduced from the [111]-texture by a 40°-rotation about one of the [111]-axes, which is not parallel to the wire axis.

If, however, a *single* crystal wire, with a [111]-direction parallel to the wire axis, is extended circa 8 %, and then subjected to prolonged annealing, then among the large crystals developed by "secondary recrystallization", the formation of a [210]-crystal was in no case observed. Yet, by the method of "growth round the corner", it was found that such a crystal, if presented to the matrix, could consume the deformed [111]-crystal readily. This can be interpreted that in this case 8 % extension of the single crystal did not produce potential nuclei in the [210]-orientation.

An analogous conclusion can be drawn from the experiments of RATHENAU and CUSTERS with nickel-iron, mentioned in paragraph 5. Although in their experiments with locally deformed sharp-texture matrixes, the largest crystals developed possessed the same orientation with respect to the matrix as those formed "spontaneously" on prolonged annealing, a fact apparently pointing to a dominant influence of growth selectivity in chosing from the available nuclei, yet it must be mentioned that the local deformation after subsequent annealing not always gave rise to the growth of large crystals (loc. cit. fig. 5): this again might be interpreted in the sense that the local deformation (pin-prick) had not produced growth nuclei in all possible orientations, so that the matrix could not "chose" the right one.

c. Thirdly, there is the phenomenon of "stimulated crystal growth", discussed in paragraph 2. Here we find an example of two crystals, growing both at the cost of the same fine-grained matrix, of which the one with the faster rate of growth, viz. the "stimulated" crystal, starts to grow at a *later* moment than the one with the slower rate of growth, viz. the "stimulating" crystal. Therefore, notwithstanding its faster rate, which undoubtedly means that its orientation with respect to the matrix texture is more favorable for consuming this texture, the "stimulated" crystal would perhaps not have developed at all, if not, according to our view, the *establishing of contact* with the approaching "stimulating" crystal had "activated" its "potential" growth nucleus to a centre actually capable to grow.

7. The points raised in the foregoing section make us ask whether the theory of "selective growth" is capable of explaining the observed recrystallization textures without taking into account the part played by the preceding deformation process in producing specially oriented lattice elements, which we may call "potential nuclei", which have to undergo some "activation process" before they can start to grow, their growth then being subjected to the laws of "growth selectivity".

To state our point more precisely, we think that for growth of a crystal a combination of factors is required:

1) there must be, in the deformed matrix, a lattice region (lattice element) in the orientation, later found again in the resulting crystal;
2) in order to be able to grow, this element must be essentially "strain-free" and in contact with strained (c.q. higher strained) lattice regions;
3) the "strain-free" state is brought about by some activation process, which probably consists of a proper redistribution of dislocations (as in "polygonization") or perhaps of a mutual dissolution of dislocations at the boundary of two adjacent lattice elements, which transforms a "potential" growth nucleus into an "actual" growth nucleus;
4) the capacity of actual growth nuclei to grow at the expense of the surrounding matrix depends on their lattice orientation with respect to the matrix texture ("growth selectivity factor").

Of these four points, 1) and 2) are probably widely accepted and considered inherent to both the "oriented nucleation" and the "selective growth" theories. With regard to 3), however, the conceptions differ, apparently not so much as to the idea that some "activation" process in the nuclear region has to precede actual growth (cf. BECK, 1949), but in the assumption in the former theory, that the orientations of the lattice-elements, where such an activation can take place, are in first instance determined by the deformation process, whereas the latter theory supposes that in general activation may take place in elements with all kinds of orientation, leaving it to the selective character of the growth process to chose those properly oriented with regard to the matrix for growth.

We think that, generally speaking, neither of the two factors 3) or 4) can be considered to be the exclusively active one in determining the orientation of the crystals observed after recrystallization is complete; but that the final texture has to be considered to be the combined effect of both, it depending on the experimental conditions of deformation and annealing, which one has the dominating influence.

If actually the deformation process produces potential nuclei in all possible orientations and with nearly equal "incubation periods" (cf. the end of paragraph 3), then growth selectivity may be expected to be the dominant factor, as all nuclei start their growth at the same moment. Therefore the faster growing ones may impede to a large extent the growth of the slower ones, and thus determine the final texture.

If, however, the incubation periods are widely varying, then the

potential nuclei with the shortest periods become actual growth nuclei and start their growth before nuclei with longer periods. If the "short period nuclei" happen to possess an orientation different from that most favourable for growth, then, due to the fact (cf. point *a* of paragraph 6) that growth selectivity is not so very pronounced, such nuclei may consume a considerable part of the matrix before nuclei with a longer incubation period (and perhaps more favourable orientation with respect to the matrix) may start their growth, or such nuclei may even be consumed by the already growing crystals. In such a case the final texture is not that to be expected according to the growth selectivity conception. The occurrence of a large scattering in the observed textures as well as that of large crystals with orientations different from that of the main group (cf. point *a* of paragraph 6) can be understood in this way.

In other cases, however, the deformation process may have a less all-round character (cf. point *b* of paragraph 6) and produce "local curvatures" or adjacent lattice elements with "unstable" boundaries only in special positions with regard to the matrix, either because the "curvatures" are directly correlated to the orientation of the glide-planes, or because unstable boundaries, corresponding to "cusp positions" in SHOCKLEY and READ's picture, require quite definite orientation relationships between two adjacent lattice elements, one of which may belong to the main orientation of the deformed matrix. In such cases the orientation of the new crystals, according to our view, is determined by the orientation of the potential nuclei, produced by the deformation process, i.c. by oriented nucleation. If these orientations do not conform to that which is most favourably oriented with respect to the matrix from the point of view of growth selectivity, then such "favorable" orientations cannot develop and (to quote CAHN, 1950, footnote p. 333) growth must necessarily occur from the available nuclei, whatever their orientations. Again the selectivity of the growth process seems to us to be sufficiently weak to allow such growth, if only the orientation of the available nuclei is sufficiently different from that of the matrix.

In conclusion, the considerations given above do not pretend in any way to give a definite solution to the problem of the origin of recrystallization textures. We think it impossible at the present state of our knowledge to decide with certainty between the various possibilities. However, we thought it opportune to draw attention to some experimental facts, which, at least in our opinion, are not readily explained on the basis of the selective growth theory *alone*.

Summary:

The first part of this paper intends to show that X-ray results obtained by TIEDEMA (1950) on the structural state of the nuclear region of aluminium crystals grown by recrystallization, support the idea that new crystals originate in "local curvatures" of the deformed matrix.

The second part discusses the question to what extent the orientations of crystals grown by recrystallization of a given matrix can be understood on the assumption of what may be called "oriented nucleation" or "selective growth".

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MATHEMATICS

REALISATIONS UNDER CONTINUOUS MAPPINGS

BY

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1. *Introduction and summary*

To what extent is a continuous mapping topological on a well chosen subset?

Given a continuous mapping $f(M) = M'$ of a space M on a space M', one may ask for the existence of a subset $M^* \subset M$ such that

$$f(M^*) = M'$$

and f even *topological* (on M^*). If this is possible, we call M^* a homeomorphic *realisation* of M' in M under f.

Furthermore, f defined on M^* is called a (topological) realisation of f (defined on M). Since we may remark, that in this case

$$f^{-1} f(M) = M^* \tag{1}$$

(f^{-1} now denoting the topological inverse mapping of M'). M^* is a retract of M, the retraction given by $f^{-1} f$.

Conversely, any retraction

$$f(M) = A \subset M$$

is determining a trivial realisation A of A.

It will be shown in 4. by some examples, that very strong conditions are required, imposed on M and f, if a realisation will be possible.

But even then I cannot obtain general conclusions.

The mapping $A \times B \to B$ (defined by $A \times b \to b$, $b \in B$) of a topological product on one of its factors, gives a simple example of a mapping by which a realisation is possible.

One might consider realisations M^* in connection with M as a certain generalisation of the topological product concept. This is one reason why realisations seem to me of some interest.

Realisations generally being impossible, one may ask how far realisations are possible with respect to certain subsets of M'. In this way we arrive at the following definition.

Be given a continuous mapping $f(M) = M'$. If it is possible to find a subset $^*M$ of M such that f is topological on $^*M$ and the image $^*M'$ of $^*M$ under f is *dense* in M', we define $^*M$ as a *weak* (*topological*) *realisation* of M' in M under f.

Shortly, $*M$ is a weak realisation, if

$$\overline{f(*M)} = M', \qquad f \textit{ topological} \text{ (on } *M).$$

Weak realisations are not always possible. If M denotes the set of a countable number of isolated points and this set is mapped one to one on the set M' of rational numbers, obviously this mapping is continuous but there does not exist any weak realisation. The situation is completely changed however, if we are considering continuous mappings of *compact* sets.

We shall prove (see theorem III and III′), that to *any continuous mapping of an arbitrary compact metric space corresponds a weak realisation.*

We obtain this result by using an important theorem of Hill [8] and Kuratowski [7], which essentially says that, given an upper semi-continuous decomposition of a compact metric space, this decomposition is continuous on a well chosen subset, while the corresponding set in the decompositionspace (hyperspace) is dense in this space [1]).

This contention may be interpreted as a weak (interior) realisation in respect to *interior* mappings instead of the formerly mentioned topological mappings (theorem II).

This "weak *interior*-realisation theorem" is established by Stoilow [10]. This theorem is moreover an almost immediate consequence of the theorem of Hill and Kuratowski.

Our weak topological realisation is somewhat strengthened in theorem III′ by extension of the topological mapping, giving us the following mainresult: *any continuous mapping f of a compactum M is topological on a G_δ-subset S of M, such that the G_δ-set $f(S)$ is dense in $f(M)$.*

This result intersects with theorems of Kuratowski, Hurewicz and Stoilow, our result being far stronger but only proved in compact spaces. Kuratowski [6], p. 227 proved that any continuous mapping of a complete separable space is topological on a certain set D being a discontinuum of Cantor, provided that the image is an uncountable set.

Now considering a compactum instead of a complete space this fact is a simple consequence of theorem III′. Indeed, the uncountable set $f(M)$ contains a compact subset K dense in itself; according to theorem III′ there exists a G_δ-subset S of the compact set $f^{-1}(K)$ such that f is topological on S and $f(S)$ is a G_δ-subset of K dense in K. $f(S)$ is therefore a G_δ-set (therefore topologically complete), dense in itself and contains a discontinuum of Cantor D' according to a theorem of Young (compare e.g. Hahn [11], p. 127).

The topological inverse of D' gives the required D. Hurewicz [12] gives generalisations of this theorem of Kuratowski; for instance: a

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1) Kuratowski's theorem is expressed as follows: "La famille des tranches de continuité est un ensemble G_δ dense dans l'hyperespace". For a partial result see also Moore [9], p. 348, theorem 24.

continuous mapping of a separable space with an uncountable image is topological on a certain perfect set.

Finally: STOILOW [10] proves a theorem of very restricted character assuming that M and M' are n-dimensional manifolds and that the inverse set $f^{-1}(m')$ of any point $m' \in M'$ is countable.

All mentioned spaces are separable metric, all mappings continuous. Commonly we are using the terminology of WHYBURN [1]. The realisation problem originates from a suggestion of J. CH. BOLAND.

2. *Theorem I.* (HILL, KURATOWSKI). If $f(M) = M'$ is a continuous mapping of the compact metric space M, the corresponding upper semi-continuous decomposition $\{f^{-1}(x)\}$ $(x \in M')$ of M contains a sub-collection being a continuous decomposition, while the corresponding image under f of this sub-collection in M' (M' is homeomorphic with the decomposition-space (hyperspace) of M) is dense in M'.

2. 1. We can express theorem I by means of interior mappings. To this end we call $M_* \subset M$ a *weak interior realisation* of M' in M under f, $f(M) = M'$ if

$$\overline{f(M_*)} = M', \qquad f \textit{ interior on } M_*.$$

As a simple result of theorem I we obtain

Theorem II. (STOILOW). *At any continuous mapping of a compact metric space a weak interior realisation may be found* [2]).

Proof. Applying theorem I to the upper semicontinuous decomposition performed by the mapping $f(M) = M'$, we obtain a subcollection $M_* = \Sigma E$, being continuous (in the limit sense), such that

$$\overline{f(M_*)} = M'$$

$M_* = \Sigma E$ is a total inverse set under f. Hence we have only to prove that f is interior on M_* (if M_* would be compact, this fact is expressed by a well known theorem, but M_* is not compact in general).

If a neighbourhood $U(p \mid M_*)$ of a point

$$p \in E \subset M_*$$

is not transformed on a neighbourhood of $p' = f(p)$ in $f(M_*)$, there may be found a sequence of points $p'_i \in f(M_*)$ converging to p' such that the intersection of $f^{-1}(p'_i)$ and $U(p \mid M_*)$ is vacuous. A subsequence of points $\{p_l\}$

$$p_l \in f^{-1}(p'_{i_l}) \subset M_*$$

is converging to a point $q \in M$; q is a point of E, since f is continuous, and $f(q) = \lim f(p_l) = f(p) = p'$ gives us

$$q \in f^{-1}(p') = E$$

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2) *Note.* We shall prove even more. *In fact M_* is a total inverse set under f:*

$$M_* = f^{-1} f(M_*)$$

Taking this into consideration we see that theorem II is not an immediate consequence of theorem III.

Therefore

$$\lim\ \inf f^{-1}(p'_{i_l}) \cdot E \neq 0.$$

This leads to (compare note 1))

$$\lim f^{-1}(p'_{i_l}) = E.$$

Applying this result we can find in our neighbourhood $U(p|M_*)$ a point p'_k contrary to the fact that the intersection of $f^{-1}(p'_i)$ and $U(p|M_*)$ is vacuous. The transformed set of $U(p|M_*)$ is therefore a neighbourhood of $f(p)$ in $f(M_*)$; thus is f interior on M_*.

3. *Theorem III. At any continuous mapping of a compact metric space a weak (topological) realisation may be found.*

It is worth noting, that the points of the realisation set $*M \subset M$ must be carefully chosen.

We are giving a simple example for illustration. Let M be the plane set

$$\begin{aligned} -1 \leqslant x \leqslant 0 \quad &, \quad y = 1 \\ x = 0 \quad &, \quad 1 \leqslant y \leqslant 2 \\ 0 \leqslant x \leqslant 1 \quad &, \quad y = 2 \end{aligned}$$

Let f be the projection of M on the X-axis, such that M' is the set

$$-1 \leqslant x \leqslant 1 \quad , \quad y = 0$$

It is clear that there does not exist any (topological) realisation. A weak realisation $*M$ however is given e.g. by the set

$$\begin{aligned} -1 \leqslant x < 0 \quad &, \quad y = 1 \\ 0 < x \leqslant 1 \quad &, \quad y = 2 \end{aligned}$$

On the other hand it is apparently impossible to find a weak realisation $*M$ which contains a point of the total inverse set

$$x = 0 \quad , \quad 1 \leqslant y \leqslant 2$$

of

$$x = 0,\ y = 0.$$

Proof of theorem III. We put as before,

$$f(M) = M'$$

M and M' compact metric with metrics ϱ and ϱ'.

$$M_* = \Sigma E_*$$

is a continuous subcollection of the corresponding upper semi-continuous decomposition $M = \Sigma E$, while $M'_* = f(M_*)$ is dense in $f(M)$. All this is possible according to theorem I. We start changing the given metric ϱ' in M'_*.

To obtain this metric we are using a well known distance function α

of the compact subsets of M. Given two compact subsets C_1 and C_2 of M, we define

$$\alpha(C_1, C_2) = \inf \beta$$

if β is a real number such that

$$U_\beta(C_1) \supset C_2 \text{ and } U_\beta(C_2) \supset C_1.$$

It is known (comp. [5], p. 115) that in a compact space "metric" convergence of compact subsets imposed by this distance function is identical with the "topological" convergence.

Now we contend, that

$$\tilde{\varrho}(x', y') = \varrho'(x', y') + \alpha(f^{-1}(x'), f^{-1}(y')), \qquad x', y' \in M'_*$$

is a metric in M'_* equivalent to the given metric ϱ' (it is worth noting that generally this is not true if we define $\tilde{\varrho}$ on M'!). We see at once that $\tilde{\varrho}$ is a distance function since ϱ' and α are distance functions. Further, if

$$\lim_i \varrho'(x'_i, x') = 0 \quad , \quad x'_i, x' \in M'_* \tag{1}$$

we have

$$\lim_i f^{-1}(x'_i) = f^{-1}(x'),$$

since the decomposition of M_* is continuous. This convergence being identical with metric convergence, we obtain

$$\lim_i \alpha(f^{-1}(x'_i), f^{-1}(x')) = 0$$

and therefore

$$\lim_i \tilde{\varrho}(x'_i, x') = 0.$$

Conversely this relation obviously implies the relation (1).

M'_* (the imposed metric is from now on $\tilde{\varrho}$) has *finite* ε-coverings with mesh $\leqslant \varepsilon$ for any $\varepsilon > 0$. This will be proved, if any sequence $\{p'_i\}$ of points $p'_i \in M'_*$ has a fundamental subsequence, according to a known theorem (comp. [5], p. 104).

To prove this contention we select from $\{p'_i\}$ a subsequence converging at $q' \in M'$, say $\{q'_k\}$. Thus for any $\varepsilon > 0$ we may find a natural number N_1 such that

$$\varrho'(q'_l, q'_m) < {}^1/_2\, \varepsilon \text{ if } l, m > N_1. \tag{2}$$

The collection

$$\{f^{-1}(q'_k)\} = \{{}_kE_*\}$$

is a collection of E_*— sets and has a convergent subsequence $\{{}_lE_*\}$, its limit being contained in $f^{-1}(q')$.

Obviously

$$\alpha({}_{l_1}E_*, {}_{l_2}E_*) < {}^1/_2\, \varepsilon \text{ if } l_1, l_2 > N_2. \tag{3}$$

Now from (2) and (3) we immediately obtain

$$\tilde{\varrho}\,(q'_{l_1}, q'_{l_2}) < \varepsilon \text{ if } l_1, l_2 > N_3.$$

Hence $\{q'_i\}$ is a fundamental subsequence of $\{p'_i\}$.

Let A'_* be a countable subset of M'_* dense in M'_* and therefore dense in M'.

There exists a real number $\gamma_1 < 1$ such that the neighbourhood $U_{1/2\gamma_1}(p'|A'_*)$ of any point $p' \in A'_*$ is open and closed in A'_*. This results from the countability of A'_* and the non-countability of the set of real numbers between 0 and 1.

Therefore we may determine a finite open and closed γ_1-covering (with mesh $\leqslant \gamma_1$) $\{U_i\}$ $(i = 1, 2, \ldots, k)$ of A'_*.

This is possible since $A'_* \subset M'_*$. Putting

$$V_i = U_i - \sum_{j=1}^{i-1} U_j \quad (i = 1, 2, \ldots, k),$$

we obtain a finite γ_1-covering $O_1 = \{V_i\}$ of A'_* of disjoint sets both open and closed in A'_* with mesh $\leqslant \gamma_1$.

Similarly it is apparently possible to determine a refinement of this covering being a finite γ_2-covering $O_2 = \{V_{ij}\}$ consisting of disjoint sets both open and closed in A'_* such that

$$\gamma_2 < {}^1\!/_2\,\gamma_1 \quad , \quad V_{ij} \subset V_i.$$

Continuing this process we obtain a sequence of subsequently refined coverings $\{O_i\}$, each O_i being a finite γ_i-covering

$$\gamma_i < {}^1\!/_2\,\gamma_{i-1}$$

consisting of disjoint sets both open and closed in A'_*. Now we construct a sequence of points $\{a'_i\}$, $a'_i \in A'_*$ as follows; in each V_k of O_1 we select one point a'_k $(k = 1, 2, \ldots, k_1)$; in each V_{rj} of O_2 we select one point a'_{k_1+l} $(l = 1, 2, \ldots, l_1)$, except for a V_{rj} which already contains a point a'_k. In each V_{rjs}, not containing a point a'_k or a_{k_1+l} we select a point

$$a'_{k_1+l_1+m} \qquad (m = 1, 2, \ldots, m_1)$$

and so on ad infinitum.

The countable set

$$A' = \Sigma\, a'_i \subset A'_* \subset M'_* \subset M'$$

is obviously dense in M'.

Now we start transforming A' one to one on a set $A \subset M$, such that

$$f(A) = A'.$$

Beginning with the points $a'_k (k = 1, 2, \ldots, k_1)$ we select in each E_*-set $f^{-1}(a'_k) = E^k_*$ an arbitrary chosen point a_k. Proceeding with the points $a'_{k_1+l}\,(l = 1, 2, \ldots, l_1)$, obtained at the covering O_2, we select in each

$$f^{-1}\,(a'_{k_1+l}) = E^{k_1+l}_*$$

a point a_{k_1+l} such that

$$(4) \qquad \varrho\,(a_{k_1+l}, a_k) = \min \varrho\,(E_*^{k_1+l}, a_k),$$

k being determined by the conditions

$$a'_{k_1+l} \in V_k\,,\ f\,(a_k) = a'_k \in V_k \in O_1.$$

In other words: a'_{k_1+l} is contained in one and only one V_k; select the corresponding a'_k in V_k and determine $a_{k_1+l} \in E_*^{k_1+l}$ according to (4).

Continuing this process we select in each

$$f^{-1}\,(a'_{k_1+l_1+m}) = E_*^{k_1+l_1+m}$$

a point $a_{k_1+l_1+m}$ such that

$$(5) \qquad \varrho\,(a_{k_1+l_1+m}, a_\pi) = \min \varrho\,(E_*^{k_1+l_1+m}, a_\pi),$$

π fixed, being exactly one of the numbers $1, 2, \ldots, k_1 + l_1$, determined by the conditions

$$a'_{k_1+l_1+m} \in V_{rj}\,,\ a'_\pi \in V_{rj} \in O_2.$$

Indefinitely continuing this process defined by means of induction, we arrive at our one to one mapping of A' on A. We assert, that the mapping

$$f(A) = A'$$

is topological (on A), and that A gives us the required weak realisation. f is continuous and apparently one to one on A; hence we have only to show the continuity of the mapping f^{-1}, if this time by f^{-1} is indicated the mapping of A' on A. Let

$$U_\varepsilon = U_\varepsilon\,(a_i\,|\,A)$$

be an ε-neighbourhood in A of a fixed point $a_i \in A$. We shall determine a neighbourhood

$$V' = V'(a'_i|A')$$

which is mapped under the restricted f^{-1} on a subset of U.

a'_i originates from the selection of a point in an open set being an element of a certain covering O_s.

Let

$$V' = V_{a_1a_2\ldots a_s a_{s+1}\ldots a_{s+l}} \in O_{s+l}$$

be an open set of the finite covering O_{s+l} such that

$$a'_i \in V',\ \gamma_{s+l} < {}^1/_2\,\varepsilon.$$

Suppose a'_k is some point of $V' \cdot A'$. a'_k originates from the selection of a point in an open set

$$V'' = V_{a_1a_2\ldots a_{s+l}\ldots a_{s+l+t}} \in O_{s+l+t}.$$

One may find one and only one finite sequence of V-sets between V' and V'' such that

$$a'_i \in V' = V_{a_1a_2\ldots a_{s+l}} \supset V_{a_1a_2\ldots a_{s+l+1}} \supset \ldots \supset V_{a_1a_2\ldots a_{s+l+t}} = V'' \ni a'_k.$$

This sequence corresponds to a fixed finite sequence of a'-points contained in the corresponding V-sets

$$a'_i, a'_{i_1}, a'_{i_2}, \ldots, a'_{i_n} = a'_k$$

such that $a'_{i_j} \in A'$ and

$$i \leqslant i_1 \leqslant i_2 \ldots \leqslant i_n = k.$$

Apparently

$$\varrho\,(a'_i, a'_{i_1}) < {}^1/_2\,\varepsilon, \varrho\,(a'_{i_1}, a'_{i_2}) < {}^1/_4\,\varepsilon, \ldots, \varrho\,(a'_{i_{n-1}}, a'_{i_n}) < \frac{\varepsilon}{2^n}.$$

From this it is clear that

$$\alpha\,(E^i_*, E^{i_1}_*) < {}^1/_2\,\varepsilon, \alpha\,(E^{i_1}_*, E^{i_2}_*) < {}^1/_4\,\varepsilon, \ldots, \alpha\,(E^{i_{n-1}}_*, E^{i_n}_*) < \frac{\varepsilon}{2^n}.$$

From the obvious inequality

$$\min \varrho(E_*, p) \leqslant \alpha(E_*, {}_1E_*) \text{ if } p \in {}_1E_*$$

and (4), (5), ..., we obtain

$$\varrho\,(a_i, a_{i_1}) < {}^1/_2\,\varepsilon, \varrho\,(a_{i_1}, a_{i_2}) < {}^1/_4\,\varepsilon, \ldots, \varrho\,(a_{i_{n-1}}, a_k) < \frac{\varepsilon}{2^n}$$

such that

$$\varrho\,(a_i, a_k) \leqslant \varrho\,(a_i, a_{i_1}) + \varrho\,(a_{i_1}, a_{i_2}) + \ldots + \varrho\,(a_{i_{n-1}}, a_k) <$$
$$< \sum_{p=1}^{n} \frac{\varepsilon}{2^p} < \sum_{p=1}^{\infty} \frac{\varepsilon}{2^p} = \varepsilon.$$

$\varrho(a_i, a_k) < \varepsilon$ means however that $a_k \in U_\varepsilon(a_i|A)$. V' is therefore mapped under (our new restricted) f^{-1} in U_ε, which we had to prove.

3. 1. In our proof we established a weak *countable* realisation-set A. How far is it possible to extend the topological mapping $f(A) = A'$ under f to a topological mapping

$$f(S) = S', \quad A \subset S \subset M, \quad A' \subset S' \subset M' \;?$$

In general $S' \neq M'$, for $S' = M'$ would produce a realisation S of M' in M under f. This problem however is immediately solved by means of known theorems.

Indeed, according to a well-known theorem of LAVRENTIEFF (comp. [6], p. 214) any homeomorphism $f(A) = A'$ may be extended to a homeomorphism $g(S) = S'$

$$A \subset S \subset M = \overline{A}, \; A' \subset S' \subset M' = \overline{A'}$$

where S and S' are G_δ-sets in M and M'.

Any continuous extension however of $f(A) = A'$ on a subset of M must coincide with f defined on M.

Therefore $g \equiv f$ on S. At last we observe, that the G_δ-sets S and S' are topologically-complete sets (comp. [6], p. 215).

Thus we obtain

Theorem III'. Weak realisation theorem for continuous mappings of compacta.

Any continuous mapping $f(M) = M'$ of a compactum M is topological on a suitably selected subset S of M, such that S (and therefore $f(S) = S'$) are topologically-complete sets and S' is dense in M'.

4. In 3, we have given a first simple example of a continuous mapping $f(M) = M'$ of a compactum for which no realisation (of M' in M) is possible. At first sight however one might expect that this situation is altered, if we consider a (continuous and) interior mapping, the decomposition of which is continuous. The well-known existence of dimension-raising interior mappings however makes it clear that in this case too realisations are generally impossible.

But the interior mapping $w = z^2$ of the circle $|z| = 1$ on $|w| = 1$ in the complex domain has no realisation either (although M and M', being circles, are homeomorphic), as turns out by a slight examination. In this case however the reason for the impossibility of a realisation might originate from the fact that the inverse set of an image point is not connected (consists in fact of exactly two points). Thus we arrive at continuous interior monotone mappings. Here again realisations are not possible in general, as may be shown by examples of different kind.

We give a simple but rather typical example.

Well-known is the example of Brouwer (comp. [2], [3], or [5], p. 118—120) of three simply-connected disjoint regions R_1, R_2 and R_3 (any of them therefore homeomorphic with a circle region), the boundaries of which are identical, while the sum of regions and boundary B fills up a square S.

R_1, R_2 and R_3 are mapped topologically on three circle-regions R_1', R_2' and R_3'. The collection of concentric circles filling up R_1', resp. R_2', resp. R_3' are mapped continuously on the plane sets $y = 0$, $0 \leqslant x < {}^1/_2$, resp. $y = 0$, ${}^1/_2 < x \leqslant 1$ resp. $x = {}^1/_2$. $0 < y \leqslant {}^1/_2$, each circle corresponding exactly to one point. The productmapping of R_1, resp. R_2, resp. R_3 in this triod together with the mapping of B on $x = {}^1/_2$, $y = 0$, gives us the required mapping of S on M. This produces *an example of a continuous interior monotone mapping of a square on a triod at which no realisation is possible.*

To prove this last statement we only have to recall to mind the fact that any point of B is not accessible from R_1, R_2, and R_3. The possibility of realisation therefore breaks down at $x = {}^1/_2$, $y = 0$. It is however worth nothing that the inverse set B of $x = {}^1/_2$, $y = 0$, is not locally connected.

In (13) Knaster gives an example of a plane irreducible continuum C which yields an interior monotone mapping of C on an interval. A realisation is apparently impossible, C being irreducible. In this example however the inverse sets of the image points are rather pathological continua. In our previous example there is one pathological inverse set B. For this reason we might not consider monotone but

Peano-monotone, interior mappings. We call a mapping Peano-monotone, if the total inverse set of any image point is a locally connected continuum. But even for Peano-monotone interior mappings realisations are generally impossible, as D. van Dantzig has pointed out; this may be proved by the mapping of the space of tangential line elements on a 2-sphere on this sphere by identifying the line elements of a point with this point. The impossibility of a realisation follows from the theorem of Poincaré–Brouwer on the impossibility of a continuous field of tangential line elements on a 2-sphere.

On the other hand it seems probable to me that Peano-monotone interior mappings of *plane* sets — at least of Peano-continua — always have topological realisations. In the general case however there may arise great difficulties. Certain retraction-properties of the original sets are required. These problems are all combinatorial.

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MATHEMATICS

ESPACES LINÉAIRES À UNE INFINITÉ DÉNOMBRABLE DE COORDONNÉES

PAR

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Introduction

Dans une série d'articles [1]) KÖTHE — en partie en collaboration avec TOEPLITZ — a développé une théorie concernant les espaces linéaires dont les points sont des suites de nombres complexes $\mathfrak{x} = (x_1, x_2, \ldots)$. Cette théorie, qui ne fait pas usage d'une métrique, a un caractère tout à fait différent de celui de la théorie abstraite des espaces de BANACH qui est d'ailleurs métrique. Récemment j'ai donnée une extension de cette dernière théorie - qui n'était développée que pour le cas où les coëfficients sont des nombres réels ou complexes ou, ce qui revient au même, appartiennent à un corps muni d'une valuation archimédienne — pour le cas où les coëfficients appartiennent à un corps muni d'une valuation non-archimédienne [2]). Il me semble donc qu'il n'est pas sans intéresse à rechercher ce que devient de la théorie de KÖTHE et TOEPLITZ si les coordonnées des points sont pris dans un corps muni d'une valuation non-archimédienne. Remarquons en outre que, dans le cas des nombres réels, l'espace dual au sens de la théorie de K. et T. aussi bien que l'espace conjugué de la théorie des espaces de BANACH apparaîssent comme des cas particuliers d'une notion d'espace conjugué, introduit par BIRKHOFF dans la théorie des structures vectorielles (vector-lattices) [3]). Il semble donc utile d'étudier la théorie de K. et T. avant d'aborder l'étude de la théorie de BIRKHOFF par rapport à des corps plus généraux que celui des nombres réels.

Dans ce qui suit nous ne donnerons que les traits fondamentaux de la théorie et nous n'insisterons pas sur le développement de la théorie tel qu'il a été donné par KÖTHE. En conséquence le nombre des conditions,

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[1]) G. KÖTHE und O. TOEPLITZ, Lineare Räume mit unendlichvielen Koordinaten und Ringe unendlicher Matrizen. Journal f. reine u. angew. Math. 171, 193—226 (1934). Cité par K. et T.

G. KÖTHE, Die Teilräume eines linearen Koordinatenraumes. Math. Ann. 114, 99—125 (1937).

G. KÖTHE, Lösbarkeitsbedingungen für Gleichungen mit unendlich vielen Unbekannten. Journal f. reine u. angew. Math. 178, 193—213 (1938). Cité par K 2.

[2]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 1045—1055, 1056—1062, 1134—1141, 1142—1152 (1946); 51, 197—210 (1948); 52, 151—160 (1949).

[3]) G. BIRKHOFF, Lattice theory, p. 246, New York (1948).

imposées au corps K où appartiennent les coëfficients, sera aussi peu que possible. En particulier on ne suppose pas que K soit localement-compact.

On verra qu'un nombre de résultats prennent une forme plus simple. Les démonstrations ne seront pas données si on peut les prendre de K. et T. sans modifications essentielles.

I. *Propriétés topologiques*

1. 1. Soit K un corps muni d'une valuation non-archimédienne. Bien qu'il ne soit pas nécessaire pour tout ce qui suit, nous supposons que K est complet.

Définition 1. *Les espaces linéaires λ que nous allons considérer sont des ensembles d'éléments $\mathfrak{x} = (x_1, x_2, \ldots)$, où $x_i \in K$, tel que si $\mathfrak{x} \in \lambda$, $\mathfrak{y} \in \lambda$, on a $\mathfrak{x} + \mathfrak{y} = (x_1 + y_1, \ldots) \in \lambda$ et $a\mathfrak{x} = (ax_1, ax_2, \ldots) \in \lambda$ pour tout $a \in K$.*

Définition 2. *Appelons l'espace dual λ^* de λ l'espace linéaire dont les points sont les suites $\mathfrak{u} = (u_1, u_2, \ldots)$ $(u_i \in K)$ telles que la série $\mathfrak{u}\mathfrak{x} = \sum u_i x_i$ converge pour tout $\mathfrak{x} \in \lambda$.*

Ici Köthe et Toeplitz, donc dans le cas des nombres complexes, exigent la convergence absolue de la série $\sum u_i x_i$, puisque le cas de convergence ordinaire conduit à une théorie moins satisfaisante. Ces objections ne se présentent pas dans le cas non-archimédien. La condition de la convergence de $\sum u_i x_i$ est équivalente avec la condition $u_i x_i \to 0$ si $i \to \infty$. La convergence de $\sum |u_i x_i|$ implique donc la convergence de $\sum u_i x_i$. La théorie fondée sur la définition 2 renferme donc la théorie fondée sur la convergence absolue.

Supposons que la valuation de K soit triviale. La condition $u_i x_i \to 0$ se réduit alors à $u_i x_i = 0$ pour toutes les valeurs suffisamment grandes de i. Seulement des séries limitées doivent donc être considéré comme convergente et ils ne se présentent plus des raisonnements de convergence proprement dits. Les topologies que nous introduisons plus tard se réduisent dans ce cas à une topologie discrète. C'est pourquoi que nous supposons dans tout ce qui suit que la valuation de K ne soit pas triviale (voir une remarque dans IV concernant les "espaces dépourvus de convergence" (konvergenzfreie Räume)).

1. 2. $\lambda \subset \lambda^{**}$.

1. 3. $\lambda^{***} = \lambda^*$.

2. 1. Définition 3. *L'espace λ est appelé parfait ("vollkommen") si*

$$\lambda^{**} = \lambda$$

Il n'y a pas de danger de confondre cette dénomination avec la notion d'ensemble parfait qui ne paraît pas dans tout ce qui suit.

2. 2. φ étant l'espace dont les points sont les suites n'ayant qu'un nombre fini de coordonnées $\neq 0$, on a $\varphi \subset \lambda$ si λ est parfait.

3. 1. Définition 4. *L'espace λ est appelé normal si $\mathfrak{x} \in \lambda$ et $|y_i| \leqq |x_i|$ entraînent $\mathfrak{y} = (y_1, y_2, \ldots) \in \lambda$.*

3. 2. Chaque espace parfait est normal.

Remarque. Cette propriété n'est pas vraie dans la théorie fondée sur les nombres complexes et la convergence ordinaire.

4. 1. Définition 5. *La suite* $\mathfrak{x}^{(1)}, \mathfrak{x}^{(2)}, \ldots$ *dans* λ *est appelée faiblement convergente si la limite*

$$\lim_{n\to\infty} \mathfrak{u}\,\mathfrak{x}^{(n)}$$

existe pour tout $\mathfrak{u} \in \lambda^*$. *Si*

$$\lim_{n\to\infty} \mathfrak{u}\,\mathfrak{x}^{(n)} = \mathfrak{u}\,\mathfrak{x}$$

pour tout $\mathfrak{u} \in \lambda^*$, *alors* $\mathfrak{x}$ *est appelé la limite faible de la suite.*

4. 2. $\mathfrak{x}^{(n)}$ étant une suite faiblement convergente, $\lim\limits_{n\to\infty} x_i^{(n)}$ existe pour tout i.

5. 1. Définition 6. *Un espace est appelé faiblement complet si chaque suite faiblement convergente dans cet espace possède une limite faible.*

5. 2. *Chaque espace parfait est faiblement complet.*

Démontrons d'abord le théorème auxiliaire suivant.

Soit $\mathfrak{x}^{(n)}$ une suite faiblement convergente. Alors pour tout $\mathfrak{u} \in \lambda^*$ et $\varepsilon > 0$ il existe un $N(\varepsilon, \mathfrak{u})$ tel que

$$\sup_{1\leqq i<\infty} |u_i|\,|x_i^{(p)} - x_i^{(q)}| \leqq \varepsilon$$

pour $p, q \geqq N(\varepsilon, \mathfrak{u})$.

Démonstration. Supposons que ceci n'est pas vrai. Il existe alors un $\varepsilon > 0$, un $\mathfrak{u} \in \lambda^*$ et une suite d'indices p_j, q_j tel que

$$\sup_{1\leqq i<\infty} |u_i|\,|x_i^{(p_j)} - x_i^{(q_j)}| > \varepsilon. \tag{1}$$

Choisissons N_1 tel que

$$\sup_{N_1+1\leqq i<\infty} |u_i|\,|x_i^{(p_1)} - x_i^{(q_1)}| \leqq \varepsilon. \tag{2}$$

Cela est possible puisque la série

$$\sum u_i\,(x_i^{(p_1)} - x_i^{(q_1)})$$

est convergente de sorte qu'on a, à partir d'une valeur suffisamment grande de i,

$$|u_i|\,|x_i^{(p_1)} - x_i^{(q_1)}| \leqq \varepsilon.$$

(1) et (2) entraînent

$$\sup_{1\leqq i\leqq N_1} |u_i|\,|x_i^{(p_1)} - x_i^{(q_1)}| > \varepsilon. \tag{3}$$

Nous allons introduire un nouveau vecteur $\mathfrak{v} = (v_1, v_2, \ldots)$ dans λ^* dont les coordonnées, satisfaisant à $|v_i| \leqq |u_i|$, seront déterminées successivement. D'abord déterminons les $v_i (1 \leqq i \leqq N_1)$ tels que

$$\Bigl|\sum_{i=1}^{N_1} v_i (x_i^{(p_1)} - x_i^{(q_1)})\Bigr| = \sup_{1\leqq i\leqq N_1} |u_i|\,|x_i^{(p_1)} - x_i^{(q_1)}|. \tag{4}$$

Remarquons que dans la formule générale

$$|\alpha+\beta+\ldots| \leqq \max\,(|\alpha|,|\beta|,\ldots)$$

on a certainement l'égalité si parmi les termes α, β, . . . la terme à valeur maximale ne paraît qu'une seule fois. On peut donc atteindre la formule (4) par un changement convenable des $u_i(1 \leqq i \leqq N_1)$, effectuant une réduction des valeurs de ces coordonnées, cependant en restant invariante la (ou une) terme à valeur maximale. On a alors

$$(5) \qquad \Big|\sum_{i=1}^{N_1} v_i\,(x_i^{(p_1)}-x_i^{(q_1)})\Big| > \varepsilon.$$

Quelque soient $v_i(N_1+1 \leqq i < \infty)$, pourvu que $|v_i| \leqq |u_i|$, on a en vertu de (2)

$$\Big|\sum_{N_1+1}^{\infty} v_i\,(x_i^{(p_1)}-x_i^{(q_1)})\Big| \leqq \sup_i |v_i|\ |x_i^{(p_1)}-x_i^{(q_1)}| \leqq$$

$$\leqq \sup_i |u_i|\ |x_i^{(p_1)}-x_i^{(q_1)}| \leqq \varepsilon.$$

On en tire, en tenant compte de (5)

$$(6) \qquad \Big|\sum_{i=1}^{\infty} v_i\,(x_i^{(p_1)}-x_i^{(q_1)})\Big| = \max\Big[\Big|\sum_{i=1}^{N_1}\ldots\Big|,\ \Big|\sum_{N_1+1}^{\infty}\ldots\Big|\Big] > \varepsilon.$$

N_1 étant choisi, puisque la suite $\mathfrak{x}^{(n)}$ converge par coordonnées (voir 4. 2) on peut déterminer $j_2 < j_1 = 1$ tel que

$$(7) \qquad \sup_{1 \leqq i \leqq N_1} |u_i|\ |x_i^{(p_{j_2})}-x^{(q_{j_2})}| \leqq \varepsilon.$$

A fortiori cette inégalité est vraie si on remplace les u_i par v_i pourvu que $|v_i| \leqq |u_i|$.

Déterminons alors $N_2 > N_1$ tel que

$$(8) \qquad \sup_{N_2+1 \leqq i < \infty} |u_i|\ |x_i^{(p_{j_2})}-x_i^{(q_{j_2})}| \leqq \varepsilon.$$

Cette inégalité reste vraie pour des $|v_i| \leqq |u_i|$. En vertu de (1) on a

$$(9) \qquad \sup_{1 \leqq i \leqq N_2} |u_i|\ |x_i^{(p_{j_2})}-x^{(q_{j_2})}| > \varepsilon,$$

et avec (7) donc

$$(10) \qquad \sup_{N_1+1 \leqq i \leqq N_2} |u_i|\ |x_i^{(p_{j_2})}-x_i^{(q_{j_2})}| > \varepsilon.$$

On peut déterminer $v_{N_1+1}, \ldots, v_{N_2}$ tel que

$$\Big|\sum_{N_1+1}^{N_2} v_i\,(x_i^{(p_{j_2})}-x_i^{(q_{j_2})})\Big| = \sup_{N_1+1 \leqq i \leqq N_2} |u_i|\ |x_i^{(p_{j_2})}-x_i^{(q_{j_2})}| > \varepsilon.$$

Il suit de (7) et (8), les $v_i(i > N_2)$ restant indéterminés pourvu que $|v_i| \leqq |u_i|$,

$$\Big|\sum_{i=1}^{\infty} v_i\,(x_i^{(p_{j_2})}-x_i^{(q_{j_2})})\Big| = \max\Big[\Big|\sum_{1}^{N_1}\Big|,\ \Big|\sum_{N_1+1}^{N_2}\Big|,\ \Big|\sum_{N_2+1}^{\infty}\Big|\Big] > \varepsilon.$$

En continuant ainsi on construit un vecteur $\mathfrak{v} = (v_1, v_2, \ldots)$ et $|v_i| \leqq |u_i|$. Il suit de 1. 3 et 3. 2 que $\mathfrak{v}$ appartient à λ^*. Pour $\alpha = 1, 2, \ldots$ on a donc

$$|\mathfrak{v}(\mathfrak{x}^{(p_{j\alpha})} - \mathfrak{x}^{(q_{j\alpha})})| > \varepsilon,$$

ce qui est en contradiction avec la convergence faible de la suite $\mathfrak{x}^{(n)}$.

Remarque. Pour ce théorème auxiliaire il suffit que l'espace λ est normal.

La démonstration que chaque espace parfait est faiblement complet va maintenant comme il suit.

Soit $\{\mathfrak{x}^{(n)}\}$ une suite faiblement convergente et soit $\mathfrak{u} \in \lambda^*$. En vertu du théorème auxiliaire on a pour chaque valeur de m

$$\sup_{1 \leqq i \leqq m} |u_i| \; |x_i^{(p)} - x_i^{(q)}| \leqq \varepsilon.$$

Puisque $\lim\limits_{p \to \infty} x_i^{(p)} = x_i$ existe, on a $|x_i^{(p)}| \to |x_i|$ et, si q est fixé,

$$|x_i^{(p)} - x_i^{(q)}| \to |x_i - x_i^{(q)}|.$$

Si $x_i - x_i^{(q)} \neq 0$ on a même à partir d'une certaine valeur de p

$$|x_i^{(p)} - x_i^{(q)}| = |x_i - x_i^{(q)}|.$$

Donc

$$\sup_{1 \leqq i \leqq m} |u_i| \; |x_i - x_i^{(q)}| \leqq \varepsilon$$

et si $m \to \infty$

$$\sup_{1 \leqq i < \infty} |u_i| \; |x_i - x_i^{(q)}| \leqq \varepsilon.$$

Ensuite

$$|u_i x_i| = |u_i| \; |x_i - x_i^{(q)} + x_i^{(q)}| \leqq \max[|u_i| \; |x_i - x_i^{(q)}|, \; |u_i| \; |x_i^{(q)}|] \leqq$$
$$\leqq \max[\varepsilon, \; |u_i| \; |x_i^{(q)}|].$$

Puisque la série

$$\sum_{i=1}^{\infty} u_i x_i^{(q)}$$

converge, $u_i x_i^{(q)}$ tend vers 0, donc aussi $u_i x_i \to 0$ de sorte que $\sum u_i x_i$ converge pour tout $\mathfrak{u} \in \lambda^*$. Donc $\mathfrak{x} = (x_1, x_2, \ldots) \in \lambda^{**} = \lambda$. On montre alors d'une façon analogue que $\mathfrak{x}$ est la limite faible de $\mathfrak{x}^{(n)}$:

$$\lim_{n \to \infty} \mathfrak{x}^{(n)} = \mathfrak{x}.$$

Remarque. La démonstration ci-dessus est construite en analogie avec la démonstration de Köthe et Toeplitz. Ces auteurs montrent par un exemple (qu'on ne peut pas transposer dans notre cas), en restant dans le cas des nombres complexes, que la propriété n'est plus vraie si on remplace la convergence absolue par la convergence ordinaire. La propriété 5. 2 est essentielle pour une grande partie de la théorie (voir la partie III).

6. 1. L'introduction d'une topologie faible, dont la notion correspondante de limite coïncide avec la limite faible, peut se faire comme chez Köthe (K 2).

Définition 7. *Un ensemble $N \subset \lambda$ est appelé borné si pour tout $\mathfrak{u} \in \lambda^*$ il existe un nombre réel $k(\mathfrak{u})$ tel que*

$$|\mathfrak{u}\mathfrak{x}| \leqq k(\mathfrak{u})$$

pour tout $\mathfrak{x} \in N$.

Définition 8. *Soit M un ensemble borné dans λ^*. Posons pour $\mathfrak{x} \in \lambda$*

$$\sup_{\mathfrak{v} \in M} |\mathfrak{v}\mathfrak{x}| = (\mathfrak{x})_M .$$

L'ensemble M est appelé fortement borné si

$$\lim_{n \to \infty} (\mathfrak{x}^{(n)})_M = 0$$

pour chaque suite dans λ qui converge faiblement vers 0.

Définition 9. *Soient M un ensemble fortement borné dans λ^*, $\mathfrak{x} \in \lambda$ et $\varepsilon > 0$. Alors, par définition, l'ensemble des points $\mathfrak{y} \in \lambda$ tels que*

$$(\mathfrak{x} - \mathfrak{y})_M < \varepsilon$$

est un voisinage faible de $\mathfrak{x}$.

On montre que, par cette topologie, λ devient un espace de Hausdorff.

En posant

$$\sup_{1 \leqq i < \infty} |u_i x_i| = [\mathfrak{u}\mathfrak{x}],$$

$$\sup_{\mathfrak{v} \in M} [\mathfrak{u}\mathfrak{x}] = [\mathfrak{x}]_M ,$$

les ensembles, définis par les inégalités

$$[\mathfrak{x} - \mathfrak{y}]_M < \varepsilon$$

constituent un système de voisinages qui est équivalent à la topologie faible.

Un ensemble dans λ est borné si et seulement s'il existe un nombre $k(\mathfrak{u})$ tel que

$$[\mathfrak{u}\mathfrak{x}] \leqq k(\mathfrak{u})$$

pour tout $\mathfrak{u} \in \lambda^*$.

6. 2. Un espace λ, muni de la topologie faible, est 0-dimensionnel. Il suffit de montrer que le voisinage U de 0

$$(\mathfrak{x})_M \leqq \varepsilon,$$

qui est, comme on peut montrer, fermé, est aussi ouvert. Soit $\mathfrak{y}^0 \in U$. Le voisinage

$$(\mathfrak{y} - \mathfrak{y}^0)_M \leqq \varepsilon$$

est tout entier dans U. En effet

$$(\mathfrak{y})_M = \sup_{\mathfrak{u} \in M} |\mathfrak{u}\mathfrak{y}| = \sup_{\mathfrak{u} \in M} |\mathfrak{u}(\mathfrak{y} - \mathfrak{y}^0) + \mathfrak{u}\mathfrak{y}^0| \leqq$$

$$\leqq \sup_{\mathfrak{u} \in M} [\max(|\mathfrak{u}(\mathfrak{y} - \mathfrak{y}^0)|, |\mathfrak{u}\mathfrak{y}^0|)] \leqq \varepsilon.$$

6. 3. Définition 10. *Soient N un ensemble borné dans λ^*, $\mathfrak{x} \in \lambda$ et $\varepsilon > 0$. Alors, par définition, l'ensemble des points $\mathfrak{y} \in \lambda$ tels que*

$$(\mathfrak{x} - \mathfrak{y})_N < \varepsilon$$

est un voisinage fort de $\mathfrak{x}$. La topologie ainsi déterminée est la topologie forte.

La limite forte qui correspond à cette topologie peut être définie comme il suit: la suite $\mathfrak{x}^{(n)}$ converge fortement vers $\mathfrak{x}$, si on a

$$\lim_{n \to \infty} (\mathfrak{x} - \mathfrak{x}^{(n)})_N = 0$$

pour chaque ensemble borné $N \subset \lambda^*$.

Un espace muni de la topologie forte est 0-dimensionnel. Chaque espace parfait est fortement complet.

7. Soit X un ensemble borné dans λ. Alors pour chaque ensemble borné U dans λ^* il existe un nombre $r(U)$ tel que

$$[\mathfrak{u}\mathfrak{x}] \leqq r(U)$$

pour $\mathfrak{x} \in X$, $\mathfrak{u} \in U$.

Avec quelques modifications on peut suivre en principe la démonstration de K. et T.

8. Dans ce qui précède nous avons donné les propriétés topologiques principales des espaces linéaires de la forme considérée qu'on peut déduire sans aucune condition spéciale concernant le corps complet K muni d'une valuation non-archimédienne; en particulier on n'a pas supposé que K est localement-compact. Dans les travaux cités on trouve encore d'autres propriétés topologiques — mentionnons une topologie qui diffère de la topologie faible et de la topologie forte et qui est fondée sur les ensembles bornés faiblement compacts de l'espace dual [4]) — mais la théorie ainsi construite suppose que K est localement compact. C'est par ces méthodes que Köthe atteint une théorie satisfaisante. Il n'y a pas de difficultés essentielles dans la transposition de cette théorie dans notre cas. Restant dans le cadre d'une revue générale, nous n'entrons pas dans ces questions. On pourrait rechercher ce que devient de cette théorie plus développée si on fait tomber la supposition que K est localement-compact.

La théorie est indépendant de toute métrique. Remarquons ici que, selon la définition de Köthe, un espace de la forme considérée est métrique si 1⁰ la métrique se déduit d'une norme et 2⁰ la convergence métrique est équivalente à la convergence forte (comme dans les espaces de Banach).

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[4]) G. Köthe, Erweiterung von Linearfunktionen in linearen Räumen. Math. Ann. 116, 719—732 (1939).

Il y a des espaces qui ne sont pas métriques en ce sens. Il faut bien distinguer cette méthode d'introduire une métrique de la définition d'une métrique dans le produit topologique d'une infinité dénombrable d'espaces métriques. Un tel produit peut toujours être métrisé, cependant en un tout autre sens et en général pas par une norme. En effet, une suite de points de l'espace produit est alors appelée convergente si les coordonnées du même indice forment des suites convergentes. La convergence forte et faible de K. possèdent aussi cette propriété, mais elle ne suffit ni pour la convergence forte ni pour la convergence faible; ces dernières imposent des conditions plus graves (comme dans les espaces de BANACH). Au sens d'un produit topologique les espaces λ sont tous métrisables.

II. *Fonctions linéaires*

1. Une fonction $u(\mathfrak{x})$, prenant ses valeurs dans K sera appelée linéaire si $u(\mathfrak{x} + \mathfrak{y}) = u(\mathfrak{x}) + u(\mathfrak{y})$ et $u(r\mathfrak{x}) = ru(\mathfrak{x})$ pour tout $r \in K$. La fonction $u(\mathfrak{x})$ est appelée faiblement, respectivement fortement continue si $u(\mathfrak{x}^{(n)}) \to u(\mathfrak{x})$ pour chaque suite $\mathfrak{x}^{(n)}$ qui converge faiblement, resp. fortement vers $\mathfrak{x}$. KÖTHE introduit encore deux autres notions de continuité, à savoir la continuité faiblement topologique et la continuité fortement topologique. On les définit d'une façon connue au moyen des voisinages faibles respectivement forts. En général les quatres notions ne coïncident pas. Tout ceci ne change pas dans notre cas.

2. Pour tout $\mathfrak{u} \in \lambda^*$ la fonction $u(\mathfrak{x}) = \mathfrak{u}\mathfrak{x}$ est une fonction linéaire dans λ qui est continue au sens de chacune des quatres définitions. La question de l'existence de fonctions linéaires continues qui ne sont pas identiquement égale à 0 est donc ici bien plus simple que pour les espaces de BANACH: dans les espaces de BANACH réels ou complexes ils existent de telles fonctions. Cependant pour les espaces de BANACH non-archimédiens le problème n'a pas encore atteint sa résolution définitive; si la valuation de K est discrète, la réponse est affirmative.

Il y a maintenant deux problèmes. 1⁰ la question si l'ensemble des fonctions linéaires continues est épuisé par les fonctions $\mathfrak{u}\mathfrak{x}$. 2⁰ le problème de l'extension d'une fonction linéaire continue, qui est définie sur un sous-espace linéaire, sur l'espace tout entier sous la condition que la relation qui exprime que la fonction est bornée doit être gardée.

3. Chaque fonction linéaire faiblement continue ou faiblement topologique est de la forme $\mathfrak{u}\mathfrak{x}$ si $\lambda \supset \varphi$. Remarquons qu'on ne suppose pas, comme le fait K. dans le cas complexe, que λ soit normal. Ceci donne une réponse partielle au problème 1 du numéro précédent: l'espace conjugué faible au sens de la théorie des espaces de BANACH coïncide avec l'espace dual.

Ce théorème n'est plus vrai pour les fonctions fortement continues [5]).

[5]) Il est par exemple en défaut dans l'espace σ_∞ (voir IV). Voir aussi: A. F. MONNA. Over niet-archimedische lineaire ruimten. Versl. Ned. Akad. v. Wetensch. **52**, 308—321 (1943), p. 316, où on a déterminé la forme des fonctionelles linéaires dans l'espace des suites convergentes.

Les démonstrations de K. qui lui permettent de résoudre le premier problème dans ce cas, supposent cependant que K est localement compact (comparer I. 8).

4. Une fonction linéaire $u(\mathfrak{x})$, définie sur un sous-ensemble linéaire $\mu \subset \lambda$ est faiblement (fortement) continue topologique si et seulement s'il existe un ensemble fortement borné (borné) M respectivement N dans λ^* tel que

$$|u(\mathfrak{x})| \leqq (\mathfrak{x})_M \qquad (\mathfrak{x} \in \mu)$$

respectivement

$$|u(\mathfrak{x})| \leqq (\mathfrak{x})_N \qquad (\mathfrak{x} \in \mu).$$

5. Le problème 2 du numéro 2 est ramené au problème correspondant pour les espaces de Banach non-archimédiens. En ce dernier cas l'extension est possible si la valuation de K est discrète. Cependant, si la valuation est partout dense, l'extension n'est pas toujours possible (des critères permettant de décider si ou non l'extension est possible dans ce cas manquent encore). On a alors le théorème suivant.

La valuation de K soit discrète. Soit $u(\mathfrak{x})$ une fonction linéaire, continue faiblement topologique, définie sur un sous-espace linéaire $\mu \subset \lambda$; $\lambda \supset \varphi$. Alors on peut extendre $u(\mathfrak{x})$ à λ tout entier et il existe un $\mathfrak{u} \in \lambda^*$ tel que $u(\mathfrak{x}) = \mathfrak{u}\mathfrak{x}$. Une relation

$$|u(\mathfrak{x})| \leqq (\mathfrak{x})_M \qquad (M \subset \lambda^* \text{ fortement borné})$$

valable sur μ, reste vraie sur λ.

Les symboles $(\mathfrak{x})_N$ etc. puissent être remplacés par $[\mathfrak{x}]_N$ etc.

III. *Applications*

1. Suivant K. et T., on a les définitions suivantes.

Définition 1. *La matrice $A = (a_{ik})$ ou la transformation linéaire*

$$y_i = \sum_{k=1}^{\infty} a_{ik}\, x_k$$

est appelée associée à l'espace λ si la série au membre à droite converge pour tout $\mathfrak{x} \in \lambda$ et tout i et si $\mathfrak{y} = A\mathfrak{x}$ appartient à λ. Le système de toutes les matrices associées à λ sera désigné par $\sum(\lambda)$.

Définition 2. *Un ensemble composé de matrices $A = (a_{pq})$ $(p, q = 1, 2, \ldots)$ est appelé un anneau de matrices si les axiomes connues des anneaux sont vérifiées et si pour chaque couple de matrices A et B de cet ensemble toutes les séries $\sum_{\alpha=1}^{\infty} a_{p\alpha}\, b_{\alpha q}$ convergent.*

Définition 3. *Un anneau M de matrices est appelé maximal s'il n'existe aucun anneau de matrices qui contient M comme vrai sous-anneau.*

On a alors:

Si l'espace λ est parfait, alors $\sum(\lambda)$ est un anneau de matrices maximal.

Remarques. *a.* Les définitions analogues de K. et T. supposent la

convergence absolue et dans le cas complexe cela est essentielle pour la validité de cette propriété: par un contre-exemple ils font voir qu'elle est fausse si on ne suppose que la convergence ordinaire. Ici il y a donc une différence essentielle avec le cas non-archimédien. I. 5. 2 est essentiel pour la démonstration

b. Ce théorème exprime pour les espaces parfaits une propriété qui correspond aux "Faltungssätze" de HILBERT dans l'espace de HILBERT (les matrices bornées constituent un anneau).

2. La théorie permet de donner des critères concernant les systèmes d'équations linéaires à une infinité d'inconnues. La transposition de ces critères va sans difficultés.

IV. *Espaces spéciaux*

1. Désignons par σ_∞ l'espace des suites $\mathfrak{x} = (x_1, x_2, \ldots)$ telles que $|x_i| \leqq M(\mathfrak{x})$ et par σ l'espace des suites telles que $x_i \to 0$.

On montre facilement les relations

$$\sigma_\infty^* = \sigma$$

$$\sigma^* = \sigma_\infty .$$

Dans σ_∞ et dans σ on peut définir une métrique par

$$\|\mathfrak{x}\| = \sup_i |x_i| .$$

En appliquant I. 7 on montre que dans σ_∞ et σ la convergence forte est bien identique avec la convergence selon cette métrique.

Enfin la convergence forte et la convergence faible coïncident dans σ.

La dernière propriété n'est plus vraie dans σ_∞. Par exemple la suite $\{\mathfrak{e}_n\}$, où $\mathfrak{e}_n = (0, 0, \ldots, 1, 0, \ldots)$, considéré comme ensemble dans σ_∞ est faiblement convergente vers $(0, 0, \ldots)$. Cependant, cette suite ne converge pas au sens métrique et n'est donc pas fortement convergente.

On peut considérér la suite $\{\mathfrak{e}_n\}$ comme un ensemble dans σ. En faisant ainsi, cette suite n'est pas faiblement convergente. En effet, la limite faible ne pourrait être autre que $(0, 0, \ldots)$. Cependant $\mathfrak{u}\mathfrak{e}_n = u_n$ ne tend pas vers 0 pour chaque $\mathfrak{u} \in \sigma_\infty = \sigma^*$, de sorte que I, définition 5 n'est pas satisfaite. Puisque $\sigma \subset \sigma_\infty$, on voit donc que la convergence faible est une propriété relative: elle dépend de l'espace dont on considère l'ensemble donné comme sous-ensemble.

2. Il n'y a pas d'espaces qui sont duals avec soi-même. Supposons en effet $\lambda = \lambda^*$. Si $\mathfrak{x} \in \lambda$, on tire de I, définition 2, que $x_i^2 \to 0$, donc $x_i \to 0$. Donc $\lambda \subset \sigma$, d'où $\lambda^* \supset \sigma^*$ et donc $\lambda \supset \sigma_\infty$, en contradiction avec $\lambda \subset \sigma$. Dans cette théorie il n'existe donc pas un espace qui est analogue à l'espace de HILBERT.

3. En rapport avec la propriété précédente on peut étudier l'espace σ_r dont les éléments sont les suites $x_1, x_2, \ldots$ telles que la série

$$\sum_{i=1}^{\infty} |x_i|^r \quad (r \geqq 1)$$

converge [6]). En suivant K. et T. appelons un tel espace symmétrique, ce qui veut dire que, si $\mathfrak{x} = (x_1, x_2, \ldots)$ appartient à l'espace, toutes les suites qu'on obtient de $\mathfrak{x}$ par une permutation arbitraire des coordonnées, appartiennent à l'espace.

Montrons d'abord:

L'espace dual λ^* d'un espace symmétrique λ est symmétrique.

Soit $u_1, u_2, \ldots \in \lambda^*$ et soit $v_1, v_2, \ldots$ une suite obtenue par une permutation des coordonnées de $\mathfrak{u}$. Il faut montrer $\mathfrak{v} \in \lambda^*$. Considérons donc la série $v_1x_1 + v_2x_2 + \ldots$. On ne peut pas obtenir cette série par une permutation des termes de $u_1x_1 + u_2x_2 + \ldots$. Cependant on peut l'obtenir par une permutation de la série $u_1y_1 + u_2y_2 + \ldots$, où $y_1, y_2, \ldots$ est une suite obtenue par une permutation convenable de $x_1, x_2, \ldots$. Puisque λ est symmétrique, on a $\mathfrak{y} \in \lambda$, de sorte que la série $u_1y_1 + u_1y_2 + \ldots$ est convergente (I, définition 2). Après la permutation on a encore $v_i x_i \to 0$ (démonstration par l'absurde). La série $\sum v_i x_i$ est donc convergente et $\mathfrak{v} \in \lambda^*$.

Montrons ensuite que $\sigma_r^* = \sigma_\infty$.

D'abord on a $\sigma_r \subset \sigma$, donc $\sigma_r^* \supset \sigma^* = \sigma_\infty$. Supposons que σ_r^* a comme élément une suite non-bornée $u_1, u_2, \ldots$. Il y a alors une suite $u_{i_1}, u_{i_2}, \ldots$ telle que

$$|u_{i_n}| > M_n, \quad M_n \to \infty.$$

Cherchons le plus petit indice n tel que $M_n > 1$. Appelons, après une rénumération, u_1 la coordonnée correspondante. Prenons alors la coordonnée d'indice le plus petit tel que $M_n > 2^2$ et appelons la, en rénumérant, u_2. En continuant ainsi on obtient une suite $u_1, u_2, \ldots$ telle que $|u_n| > n^2$. Remplaçons les coordonnées de la suite donnée, qui ne sont pas encore utilisées ainsi, par 0 et posons les entre les coordonnées de la suite qu'on vient de construire de façon qu'il n'y en a qu'un nombre fini entre chaque couple de coordonnées. On trouve

$$\mathfrak{u} = u_1, 0, \ldots, u_2, 0, \ldots, \ldots.$$

σ_r^* étant symmétrique et normal (puisque parfait) on a $\mathfrak{u} \in \sigma_r^*$.

Considérons alors le point

$$\mathfrak{x} = \frac{1}{u_1}, 0, \ldots, \frac{1}{u_2}, 0, \ldots.$$

On a

$$\sum |x_i|^r = \sum \frac{1}{|u_n|^r} < \sum \frac{1}{n^{2r}},$$

d'où on tire $\mathfrak{x} \in \sigma_r$. Puis on a $\mathfrak{u}\mathfrak{x} = 1 + 1 + \ldots$, donc une série divergente, en contradiction avec $\mathfrak{u} \in \sigma_r^*$; on a donc bien $\sigma_r^* = \sigma_\infty$.

Remarquons que par

$$\left\{\sum |x_i|^r\right\}^{\frac{1}{r}}$$

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[6]) On trouve une autre étude de ces espaces dans A. F. Monna, Over een lineaire P-adische ruimte. Versl. Ned. Akad. v. Wetensch. 52, 74—82 (1943).

on ne peut pas définir dans σ_r une métrique au sens de la théorie précédente. La raison en est que la convergence forte — avec laquelle doit coïncider, d'après la définition, la convergence métrique — se définit au moyen des ensembles bornés de l'espace dual, en ce cas l'espace σ, le même que l'espace dual de σ_∞. On peut métriser σ_r et σ_∞ par la même métrique, à savoir

$$\sup |x_i|.$$

4. K. et T. introduisent les espaces dépourvus de convergence. Ce sont des espaces λ tels que, si $\mathfrak{x} \in \lambda$, on a $\mathfrak{y} = (y_1, y_2, \ldots) \in \lambda$, où y_i est un élément arbitraire de K si $x_i \neq 0$ et $y_i = 0$ si $x_i = 0$.

K. et T. montrent qu'il est impossible de métriser ces espaces au sens de notre théorie.

Si la valuation de K est triviale, λ^{**} est dépourvu de convergence pour chaque espace λ. Danc ce cas chaque espace parfait est donc dépourvu de convergence. On a dans ce cas $\sigma = \varphi$.

MATHEMATICS

EINSTEIN SPACES AND CONNECTIONS. I

BY

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In this paper we use, following CH. EHRESMANN [10, 11, 12], the terminology of the theory of fibre spaces, while dealing with spaces with a connection as defined by E. CARTAN [5, 6, 7]. This is important in view of the study of spaces in the large, but here it gives us also in the small a satisfactory view on several spaces with a connection uniquely definable with a given Riemannian, in particular Einstein-, space. Some important notions are: the fixed oblique cross section; torsion of a figure in a fibre with respect to a connection; covariant constant figures. We present two main theorems (3 and 11) on the normal conformal and the normal projective connection of an Einstein space, and several corollaries.

S. SASAKI [15, 16] kindly drew my attention to his interesting papers on Einstein spaces. I think that my presentation of the relation between the connections of an Einstein space has advantages over his method, in particular with respect to his "Poincaré's representation in the underlying manifold" (which I did not understand).

1. *The conformal group G*

Let S^n be a quadratic hyper surface of the signature of a hypersphere in a real projective $n+1$-dimensional space P^{n+1}. The projective transformations of P^{n+1} that leave S^n invariant correspond (isomorphy) with the same transformations restricted to S^n. These are if $n > 2$ the (only) twice differentiable conformal transformations of S^n.

For suitable wellknown coordinates ($-\infty < y^a < +\infty$; $a = 1, \ldots, n$), called conformally preferred (G-preferred) coordinates, covering S^n with the exception of one point, a set of operators Y_σ called infinitesimal conformal transformations are:

$$(1)\qquad Y_a = \frac{\partial}{\partial y^a}\,; \quad Y_{a\beta} = Y^a{}_\beta = -Y^\beta{}_a = y^a \frac{\partial}{\partial y^\beta} - y^\beta \frac{\partial}{\partial y^a}\,;$$

$$(2)\qquad Y_0 = y^a \frac{\partial}{\partial y^a}\,; \qquad (3)\qquad Y_{*a} = \textstyle\sum_{\beta=1}^{n} (y^\beta)^2 \dfrac{\partial}{\partial y^a} - 2 y^a y^\beta \dfrac{\partial}{\partial y^\beta}.$$

("translation; rotation; multiplication; inversion-translation-inversion"; $a, \beta = 1, \ldots, n$).

If the mapping f: part of $S^n \to n$-dimensional number space, is one

G-preferred coordinate system, then the other G-preferred coordinate systems are $f \cdot g$; $g \in G$ i.e. the group of conformal transformations of S^n.

The structure equations of the basis (1, 2, 3) of the infinitesimal conformal transformations are:

(4) $[Y_\lambda, Y_\mu] = Y_\lambda Y_\mu - Y_\mu Y_\lambda = c^\tau_{\lambda\mu} Y_\tau$ (λ, μ, τ have the same range as σ).

The constants $c^\tau_{\lambda\mu}$ are zero except those in the following equations and trivially related ones:

$$(4) \quad \begin{cases} [Y_\beta, Y_{\alpha\beta}] = [Y_\alpha, Y_0] = Y_\alpha; \quad [Y_{*\beta}, Y_{\beta\alpha}] = [Y_0, Y_{*\alpha}] = Y_{*\alpha}; \\ [Y_{\alpha\gamma}, Y_{\gamma\beta}] = Y_{\alpha\beta}; \quad [Y_\alpha, Y_{*\beta}] = 2 Y_{\alpha\beta}; \quad \alpha \neq \beta \neq \gamma \neq \alpha. \end{cases}$$

A subgroup of the conformal group G operating in S^n, or of the corresponding group operating in P^{n+1}, consists for example of those group elements that leave a fixed point ψ in P^{n+1} invariant. Three possibilities may occur:

If ψ is a point of S^n, then we choose G-preferred coordinates such that ψ is the excluded or "infinite" point. The infinitesimal transformations of the subgroup has the base (1) (2). This "similarity group" has the Euclidean group G_0 with infinitesimal generators (1) as a subgroup, a subgroup, under which there exists an invariant Euclidean metric in S^n minus ψ. Those G-preferred coordinate systems for which this metric has the expression $ds^2 = \sum_{\alpha=1}^n (dy^\alpha)^2$ are called Euclidean-preferred or G_0-preferred. If f is one of them, then the others are $f \cdot g$, $g \in G_0$. Note that the metric is not (completely) determined by ψ.

If ψ in P^{n+1} is in the interior bounded by S^n (= no tangent line of S^n passes through ψ) then the polar hyper plane of ψ with respect to S^n does not intersect S^n. P^{n+1} with the exception of this hyper plane can then be covered, with preservation of topology and of straight lines, by a Euclidean space E^{n+1} such that S^n is covered by a Euclidean hypersphere of radius $\sqrt{K}$ $(K > 0)$. Note that K is not determined by ψ. The subgroup (G_K) is represented by the group of motions of the space of constant curvature K introduced in S^n. For suitable G-preferred coordinates the infinitesimal transformations of G_K are those that leave invariant (the family of polarities with respect to)

$$(5) \qquad K \cdot \sum_{\alpha=1}^n (y^\alpha)^2 + 4 = 0.$$

These coordinate systems, and also those of the next paragraph, are called non-Euclidean preferred, or more specific G_K-preferred. If f is one of these, then the others are $f \cdot g$, $g \in G_K$.

A base of the infinitesimal transformations of G_K is

$$(6) \qquad Y_{\alpha\beta}(K) = Y_{\alpha\beta}; \quad Y_\alpha(K) = Y_\alpha - K' \cdot Y_{*\alpha} \; (K' = K/4).$$

The invariant metric, which osculates at $y^\alpha = 0$ with the Euclidean metric $ds^2 = \sum_{\alpha=1}^n (dy^\alpha)^2$, is (E. Cartan [4] p. 164):

$$(7) \qquad ds^2 = \sum_{\alpha=1}^n (dy^\alpha)^2 \cdot [1 + K' \cdot \sum_{\beta=1}^n (y^\beta)^2]^{-2}.$$

If ψ is exterior of S^n, then the polar hyperplane of ψ does intersect S^n in a S^{n-1} which is invariant under the subgroup. For suitable G-preferred coordinates this S^{n-1} has the equation (5) with $K < 0$. The infinitesimal transformations of the subgroup have the base (6). There is an invariant metric (7) defined in the pointset $1 + K' \cdot \sum^a (y^a)^2 > 0$. The other points of S^n are called "infinite" points.

From (4) (6) the non trivial structure equations of the Euclidean and non Euclidean groups are found to be ($K \gtreqless 0$)

$$(8) \quad [Y_a(K), Y_\beta(K)] = -K \cdot Y_{a\beta}\,;\; [Y_a(K), Y_{a\beta}] = Y_\beta(K)\,;\; [Y_{a\gamma}, Y_{\gamma\beta}] = Y_{a\beta}.$$

An infinitesimal transformation of G has an expression in any particular (G- or G_K-) preferred coordinate system. If we take another preferred coordinate system, then the same transformation will in general get another expression. We are in particular interested in the influence of those preferred coordinate transformations that yield one and the same expression for the point which has under one choosen preferred coordinate system the coordinates $y^a = 0$ (the "origin").

In the case of the G_K-preferred coordinate systems, only those coordinate transformations are then possible, that are obtained from rotations about the "origin" of the Euclidean or non-Euclidean space under consideration. Under such a coordinate transformation the operators $Y_{a\beta}$ and $Y_a(K)$ transform as a tensor and a vector of the kind indicated by the indices.

In the case of the G-preferred coordinate systems, all G-preferred coordinate transformations, also those that leave the expression of a choosen "origin" fixed, are generated by a few, the influence of which on the expression for the infinitesimal transformations is as follows:

(9)
Multiplication $\bar{y}^a = \sigma y^a$: $Y_a = \sigma \overline{Y}_a,\; Y_0 = \overline{Y}_0,\; Y_{a\beta} = \overline{Y}_{a\beta},\; \sigma Y_{*a} = \overline{Y}_{*a}$.

Rotation: The operators transform as vectors etc. of the kind indicated by their indices.

Inversion $\bar{y}^a = y^a / \sum_{\gamma=1}^n (y^\gamma)^2$: $\quad Y_0 = -\overline{Y}_0,\quad Y_{*a} = \overline{Y}_a,\quad Y_a = \overline{Y}_{*a},\; Y_{a\beta} = \overline{Y}_{a\beta}$.

Translation $\bar{y}^a = y^a - \delta^a_\beta \cdot p$: $\; Y_0 = \overline{Y}_0 + p \cdot \overline{Y}_\beta,\; Y_{\eta\beta} = \overline{Y}_{\eta\beta} - p \cdot \overline{Y}_\eta,\; Y_{\beta\eta} = \overline{Y}_{\beta\eta} + p \cdot \overline{Y}_\eta,\quad Y_{*a} = \overline{Y}_{*a} - 2p \cdot \overline{Y}_{a\beta} + p^2 \cdot \overline{Y}_a\,(a \neq \beta \neq \eta),\; Y_{*\beta} = \overline{Y}_{*\beta} - 2p \cdot \overline{Y}_0 - p^2 \cdot \overline{Y}_\beta$, *others invariant.*

2. *Spaces with a conformal connection*

All spaces and mappings to be considered from now on, are of a sufficiently high differentiability class, say ≥ 3. Let B and X be manifolds of dimension $2n + 1$ and n respectively. X is a neighborhood as small as we please at any moment. A mapping called *projection* is given, $p : B \to X$, such that if $x \in X$ then $p^{-1}(x)$ is an $n + 1$-dimensional manifold called fibre at x and denoted by Y_x.

Definition: A *reference system* is a mapping $b : B \to P^{n+1}$ such that $b \mid Y_x$ (restricted to Y_x) is topological onto P^{n+1}.

Let H ($= G$ or G_K, compare section 1) be a subgroup of the group of projective transformations in P^{n+1}. P^{n+1} assigned with the group H is called *the reference space.* Two reference systems b_1 and b_2 are called H-aequivalent, if for any $x \in X$:

$$b_2 (b_1^{-1} Y_x) = h_x \in H.$$

(h_x has by assumption three derivatives with respect to x!).

We now moreover assume that B has a complete ($=$ not contained in a larger class) H-aequivalence class of reference systems. Then B is called an *H—P^{n+1}-fibre bundle* with base space X. The reference systems of the class are called *preferred.*

Remark: If X is not restricted to be a small neighborhood, then a fibre bundle is defined by B, X, p and aequivalence classes of reference systems for neighborhoods in X.

From the definition it follows that each fibre has the structure of the reference space. The complete class of G_K-preferred reference systems of a G_K—P^{n+1}-fibre bundle is contained in one unique complete class of G-preferred reference systems. The last determines a G—P^{n+1}-fibre bundle, the *G-abstractum* of the first.

Restricting to $H = G$ or $H = G_K$ we observe that in each fibre Y_x the invariant image (under the reference system) of the quadric $S^n \subset P^{n+1}$ occurs. These images together are the pointset of an H—S^n-fibre bundle $B(H, S^n, X)$, with reference space S^n assigned with H. Vice versa $B(H, S^n, X)$ also determines $B(H, P^{n+1}, X)$.

A coordinate system for X (differentiability class > 3), a reference system of the H-aequivalence class of $B(H, S^n, X)$, and an H-preferred coordinate system for S^n (section 1), yield in a natural way a coordinate system for $B(H, S^n, X)$. A coordinate system obtained in this way will be called H-preferred.

Definition: A space X is said to possess an *a conformal connection* (or displacement) if: A): X is base space of a fibre bundle $B(G, S^n_z, X)$ · B): To every curve segment in X a conformal mapping is assigned of the fibre at the initial point onto the fibre at the end point of the segment. This is a continuous function of curve segments with the same initial and end point in X. C): The pseudo group of curve segments in X (addition of two curves is possible if the initial point of the second is the end point of the first curve) is homomorph onto the pseudo group of displacements. This homomorphism is obtained from the function mentioned in B) · D): the expression of a displacement along a differentiable curve segment in X, with respect to a preferred coordinate system for the fibre bundle, is obtained by integration from equations of the kind:

(10) $$dy^a + \omega_i^\sigma \, dx^i \, Y_\sigma \, y^a = 0.$$

The range of the indices is as before. $\omega_i^\sigma = \omega_i^\sigma(x^1, \ldots, x^n)$ is a vector of the space X in the index i.

Let $x^i = x^i(u, v)$ be a differentiable surface with differentiable parameters u, v in X. The displacement along a small closed curve in X consisting of segments of parameter curves: $(u, v) = (0, 0)$, $(0, v)$, $(0, b)$, (u, b), (a, b), (a, v), $(a, 0)$, $(u, 0)$, $(0, 0)$ is approximated by

$$(11) \quad \left\{ \begin{array}{l} \triangle y^a = (\delta d - d\delta) y^a = -2\,\Omega^\sigma_{ki}\, dx^k\, \delta x^i\, Y_\sigma\, y^a \\ \qquad (2\,\Omega^\sigma_{ki} = \partial_i \omega^\sigma_k - \partial_k \omega^\sigma_i + \omega^\lambda_k\, \omega^\mu_i\, c^\sigma_{\lambda\mu}\,;\; dx^i = \dfrac{\partial x^i}{\partial u} \cdot a\,;\; \delta x^i = \dfrac{\partial x^i}{\partial v} \cdot b). \end{array} \right.$$

If $\Omega^\sigma_{ki} = 0$ then the space is called (locally) flat. Then the displacement along any (contractible) closed curve in X, is the identity mapping of the fibre at the initial = end point. Ω^σ_{ki} is a tensor in X with respect to the indices k, i.

A change of reference system for the fibre bundle carries with it self a change of coordinates for each fibre. In one fibre, but not in all at the same time, this change of coordinates by virtue of the reference transformation. can be annihilated by a preferred coordinate transformation of the reference space S^n. The influence of some preferred coordinate transformations of S^n on $\Omega^\sigma\,(= \Omega^\sigma_{ki})$, is as follows:

$$(12) \quad \left\{ \begin{array}{l} \textit{Multiplication } \bar{y}^a = \sigma \cdot y^a\colon\ \bar{\Omega}^a = \sigma\,\Omega^a,\ \sigma\,\bar{\Omega}_{*a} = \Omega_{*a},\ \bar{\Omega}^0 = \Omega^0,\ \bar{\Omega}^{a\beta} = \Omega^{a\beta}. \\ \textit{Rotation: } \Omega^a, \Omega^{*a}, \Omega^{a\beta} \textit{ transform as vectors of the kind indicated by} \\ \textit{the indices} \cdot \bar{\Omega}^0 = \Omega^0. \\ \textit{Inversion: } \bar{\Omega}^0 = -\,\Omega^0,\ \bar{\Omega}^a = \Omega^{*a},\ \bar{\Omega}^{*a} = \Omega^a,\ \bar{\Omega}^{a\beta} = \Omega^{a\beta}. \\ \textit{Translation: see } (9).\quad \bar{\Omega}^0 = \Omega^0 - 2\,p \cdot \Omega^{*\beta},\ \bar{\Omega}^{*a} = \Omega^{*a}, \\ \qquad \bar{\Omega}^{a\eta} = \Omega^{a\eta} - p \cdot \delta^\eta_\beta\, \Omega^{*\eta} + p\, \delta^a_\beta\, \Omega^{*a}, \\ \qquad \bar{\Omega}^a = \Omega^a - 2\,p\,\Omega^{a\beta} + p^2\,(1 - 2\,\delta^a_\beta)\,\Omega^{*a} + p\,\delta^a_\beta\,\Omega^0. \\ \textit{Inversion-translation-inversion:} \\ \qquad \bar{\Omega}^0 = \Omega^0 + 2\,p\,\Omega^\beta,\ \bar{\Omega}^a = \Omega^a, \\ \qquad \bar{\Omega}^{a\eta} \doteq \Omega^{a\eta} - p\,\delta^\eta_\beta\,\Omega^\eta + p\,\delta^a_\beta\,\Omega^a, \\ \qquad \bar{\Omega}^{*a} = \Omega^{*a} - 2\,p\,\Omega^{a\beta} + p^2\,(1 - 2\,\delta^a_\beta)\,\Omega^a - p\,\delta^a_\beta\,\Omega^0. \end{array} \right.$$

The space with a conformal (G-) connection which we defined sofar, needs much additional structure, before a space with a conformal connection in the customary sense (CARTAN E.) is obtained.

First Assumption. With each fibre is associated one fixed point in that fibre, such that these points form a n-dimensional sub manifold of the fibre bundle. We call this pointset *the fixed cross section*. The point in the fibre represents the corresponding point in the base space X. We choose the preferred reference system of the fibre bundle and the preferred coordinate system of the reference space such, that the point of the fixed cross section in each fibre has coordinates $y^a = 0$ ("origin").

Definition: A point with coordinates say y^a in the fibre Y_x at the point x of X is said to be without torsion if equation (11) yields

$$\text{at } x:\ \triangle y^a = 0. \tag{13}$$

Under a displacement along a small circuit in the base space X, such a point returns in first approximation to its original place. It is easily seen that the origin ($y^a = 0$) of the fibre Y_x is without torsion, if and only if Ω^a (at x) $= 0$ ($a = 1, \ldots, n$).

Second Assumption. All points of *the fixed cross section* are *without torsion* with respect to the given connection:

$$\Omega^a(x) = 0 \qquad (a = 1, \ldots, n\ ;\ x \in X). \tag{14}$$

(This is usually expressed by the words: the connection is without torsion. Note that also the vanishing of the torsion of any figure, instead of a point, in a fibre can be defined in a similar way.)

The influence of reference transformations which leave the expression of the "origin" invariant, on the numbers Ω^σ_{ki} of the space with a conformal connection without torsion is as follows (Compare (12)):

(15) $\begin{cases} \textit{Similarity}: \overline{\Omega}^a = \Omega^a = 0,\ \overline{\Omega}^{*a} = \sigma^{-1} \cdot \Omega^{*a}, \textit{ others invariant.} \\ \textit{Rotation}: \textit{as before } (12). \\ \textit{Inversion-translation-inversion } (\textit{Compare } (9)): \\ \qquad \overline{\Omega}^{*a} = \Omega^{*a} - 2\, p\, \Omega^{a\beta} - p\, \delta^a_\beta\, \Omega^0, \textit{ others invariant.} \end{cases}$

Consider the points of a curve segment in X, and also the points of the fibres which by virtue of the fixed cross section represent these points. Displacement along any segment with end point x_0 of the curve, yields a representation of the initial point of that segment in the fibre at x_0. These points in the fibre at x_0 which correspond with points of the curve form by definition *the development of the curve in that fibre* with respect to the connection and the fixed cross section. This development is a curve in the fibre at x_0 which passes through the "origin".

If the curve $x^i = x^i(t)$ in X is differentiable with parameter t, then the tangent to the development in the fibre at $x^i = x^i(0)$, at the origin, has the direction $\omega^a_i.\ dx^i/dt$ if this is different from 0. If this direction exists ($\neq 0$) for every choice of $dx^i/dt \neq 0$ at the point x_0 to be considered, then the fixed cross section is called oblique with respect to the connection at the point $x_0 \in X$. A necessary and sufficient condition is: determinant $\omega^a_i \neq 0$.

Third Assumption. The fixed *cross section is oblique* with respect to the connection at all points of X.

Let Y_x be a fibre of the fibre bundle $B(G, S^n, X)$ of a space with a G-connection, and let χ be a point of Y_x. Consider the n-dimensional space

$T(\chi)$ of tangent vectors at χ with respect to the fibre. The angle between two of these vectors is defined by virtue of the reference system, which is a mapping onto S^n. From $T(\chi)$ a Euclidean n-dimensional space can be obtained, by assigning a number (length) to one of the vectors. The other vectors then also have a natural length. Such an "imbedding" of a Euclidean space in $T(\chi)$ is called a *gauge* at χ.

Now let χ be without torsion. Let the fibre Y_x be displaced along a small closed parallellogram as above. Considering first order terms only, let all gauges at χ be invariant under this displacement. If the same is true for any small closed parallellogram with vertix x in X, then the gauges at the point χ of the fibre Y_x are said to be without torsion.

Fourth Assumption. The gauges at every point of the fixed cross section are without torsion. This condition is expressed by the invariant equations

(16) $$\Omega^0_{ki} = 0.$$

The first three assumptions are also made in the cases of spaces with a G_K-connection, and they have the same expression (Compare section 1). The fourth assumtion need not be stated because it is true anyhow in this case.

$\omega^a_i \dfrac{dx^i}{dt}$ $(x = x_0)$ is a vector of the fibre at x_0. It is a vector at the point of the fixed cross section of this fibre. If we define $\Omega^a_{\beta ij} = \Omega^{a\beta}{}_i$ then $\Omega^a_{\beta ij}$ is a tensor in the indices a, β at the same point of the fibre (Compare (14)). We now define numbers ω^i_a by

(17) $$\omega^i_a \omega^\beta_i = A^\beta_a \quad \text{hence} \quad \omega^i_a \omega^a_j = A^i_j \quad (a, \beta, i, j = 1, \ldots n).$$

$A^a_\beta = 0$ if $a \neq \beta = 1$ if $a = \beta$; A^i_j analogous.
ω^i_a is a covariant vector with respect to the fibre in the index a and a contravariant vector with respect to the base space in the index i.

From the influence of preferred reference transformations (with a fixed expression $y^a = 0$, for the fixed point in each fibre) on $\Omega^a_{\beta ij}$, ω^a_i, ω^i_a, it now follows, that the following tensors with respect to the base space, are invariant under these reference transformations of the fibres, and therefore these entities are just tensors with respect to the base space:

The conformal curvature tensor

(18) $$\Omega^i_{jkm} = \Omega^a_{\beta km}\, \omega^i_a \omega^\beta_j .$$

The conformal Ricci tensor

(19) $$\Omega^i_{jim} .$$

If we consider instead of the conformal group G, the subgroup G_K, then we get by definition a space with a Euclidean or Non-Euclidean connection. Instead of (11) we then have:

(20) $$\triangle y^a = -2\,\Omega^\sigma_{ki}(K)\, dx^k\, \delta x^i\, Y_\sigma(K)\, y^a$$

where σ has now only the range $\sigma = a, a\beta$ $(a, \beta = 1, \ldots n; a \neq \beta)$.

A G_K-connection in a $G_K - S^n$-fibre bundle is carried over in a natural way to the G-abstractum of this fibre bundle. The space with a G-connection obtained in this way is called the *G-abstractum (conformal abstractum) of the given space with a G_K-connection.* The conformal curvature Ω^{σ}_{ki} of this abstractum of the space with curvature $\Omega^{\sigma}_{ki}(K)$ obeys:

$$\left\{ \begin{array}{l} \Omega^{a}_{ki} = \Omega^{a}_{ki}(K) = 0, \ \ \Omega^{*a}_{ki} = -K\, \Omega^{a}_{ki}(K) = 0 \\ \Omega^{0}_{ki} = 0\,; \ \ \Omega^{a}_{\beta ki} = \Omega^{a}_{\beta ki}(K). \end{array} \right. \tag{21}$$

Hence:

$$\Omega^{i}_{jkm} = \Omega^{i}_{jkm}(K)\,; \ \ \Omega^{i}_{jik} = \Omega^{i}_{jik}(K) \tag{22}$$

EINSTEIN SPACES AND CONNECTIONS. II

BY

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3. *Einstein spaces*

Let a Riemannian metric be given in the space X: $ds^2 = g_{ij} dx^i dx^j$. Then a unique space with a Euclidean connection and with base space X is determined by, apart from the assumptions of section 2, the properties: The development of any differentiable curve $x_i = x_i(t)$ in the fibre at one of its points, has the same length as the curve itself; the development of a broken curve the two parts of which meet under an angle α, consists of two parts that meet under the same angle α (LEVI–CIVITA [14]; SCHOUTEN [17]: parallellism). We recall the equation

$$\sum_{\alpha=1}^{n} \omega_i^\alpha \omega_j^\alpha = g_{ij}. \tag{23}$$

Let us now consider at each point of X a space of constant curvature K, osculating at the fixed point of the Euclidean fibre with this Euclidean space. The G_K-connection defined by the same numbers ω_i^α, $\omega_i^{\alpha\beta}$ as those to be found in the unique way mentioned above (with respect to given preferred coordinates) has the same properties that we mentioned for the Euclidean connection. It is therefore the unique G_K-connection (non-Euclidean connection) with these properties. The curvature of the two connections are related as follows (Compare (8) and (11)).

$$\Omega^\alpha_{\beta km}(K) = \Omega^\alpha_{\beta km}(0) - K(\omega_k^\alpha \omega_m^\beta - \omega_m^\alpha \omega_k^\beta). \tag{24}$$

Also, after multiplication with $\omega_\alpha^i \omega_j^\beta$ and summation over $\alpha, \beta = 1, \ldots n$.

$$\Omega^i_{jkm}(K) = \Omega^i_{jkm}(0) - K(\delta_k^i g_{jm} - \delta_m^i g_{jk}). \tag{25}$$

Hence:

$$\Omega^i_{jik}(K) = \Omega^i_{jik}(0) - K(n-1) g_{jk}. \tag{26}$$

From (25) follows that the tensor of the non-Euclidean connection of a Riemannian space obeys identities analogous to the well known identities of *the* Riemann tensor:

$$\Omega_{ijkm}(K) \stackrel{\text{def}}{=} g_{ip} \Omega^p_{jkm}(K); \; \Omega_{ijkm}(K) = -\Omega_{jikm}(K) = \Omega_{kmij}(K). \tag{27}$$

A Riemannian space is called an Einstein space of scalar curvature K if the right hand side of (26) vanishes. Hence it is *characterised by*

Theorem 1. The G_K-connection of an Einstein space of scalar curvature K has a vanishing G_K-Ricci tensor.

A space of constant curvature K has the characteristic property that the right hand side of (25) vanishes. It is *characterised by*

Theorem 2. The G_K-connection of a space of constant curvature K is (locally) flat.

The conformal structure of a Riemannian space determines a unique conformal connection (CARTAN [5]), the so called *normal conformal connection*, determined by, apart from the assumptions of section 2, the property: The conformal Ricci-tensor vanishes

$$\Omega^i_{jim} = 0. \tag{28}$$

It seems to be difficult to understand geometrically the contents of the equations (28). The normal conformal connection of an Einstein space however can be characterised in a satisfactory geometrical way as follows: Theorem 1, (21), (22) and (27) yield

Main theorem 3. *The normal conformal connection of an Einstein space of scalar curvature K is the conformal abstractum of the G_K-connection of this space.*

Theorem 4 (SCHOUTEN, STRUIK [18]). *A conformally flat Einstein space is a space of constant curvature.*

Proof: Let the scalar curvature of the Einstein space be K. Then the G_K-connection is flat. (Th. 3).

Theorem 5 (SCHOUTEN, STRUIK [18]). *A three dimensional Einstein space is of constant curvature.*

Proof: The conformal curvature tensor of the normal conformal connection of a three dimensional Riemannian space vanishes. Let the scalar curvature of the given Einstein space be K. Then the G_K-connection of this space is (locally) flat (Th. 2 and 3).

In section 1 we introduced the conformal group of transformations in $S^n \subset P^{n+1}$. All conformal, Euclidean and non-Euclidean fibres can be assumed to be imbedded in the way of section 1 in P^{n+1}'s. Also: the spaces with a conformal etc. connection determine (extend) in a unique way (to) spaces with a connection and as fibres P^{n+1}'s. We call those connections *conformal P^{n+1}-connections.* If the connection is obtained from a normal conformal connection, then we call it also *normal.*

In the case of a Euclidean or non-Euclidean connection, the in this way constructed space with a P^{n+1}-connection, has in each fibre one point of particular interest: the point ψ of section 1. These points map onto each other under displacements along arbitrary curves in the base space.

They form a cross section, which for these reasons is called *a covariant constant point* of the conformal P^{n+1}-connection. From th. 1 and 3, we now have:

Theorem 6 (SASAKI [15]). *The normal conformal P^{n+1}-connection of an Einstein space has a covariant constant point, which lies outside, on, inside the covariant constant S^n in case the scalar curvature K is $<, =, > 0$ respectively.*

Example: DEBEVER [9] proved, without knowing this theorem of SASAKI, its application to the group space of a semi simple group which has an Einstein metric according to CARTAN and SCHOUTEN [8].

Conversily, if a normal (!) conformal P^{n+1}-connection has a covariant constant point, then this point can be used to introduce a covariant constant metric in the covariant constant S^n or part of it. The metric and the connection, determine a G_K-connection of which the G_K-Ricci tensor vanishes. This connection introduces a metric in the base space, except in those points in the fibres of which the fixed cross section intersects the "infinite" point(s) of the fibre, infinite with respect to the introduced metric in the fibre S^n. Those points of the base space have to be excluded, because their distance to ordinary points cannot be defined properly. We have:

Theorem 7 (SASAKI [15]). *If the normal conformal P^{n+1}-connection of a Riemannian n-dimensional space V has a covariant constant point in the interior, on, in the exterior of the covariant constant S^n, then V is conformal to an Einstein space with (constant) positive, zero and negative scalar curvature respectively. In the last two cases it may happen that some "infinite" points of the base space have to be excluded.*

In the case of theorem 7 it is obvious that the given Riemannian space is conformal to a set of Einstein spaces each of which is obtained from one of them by a multiplication of all distances with a positive factor. This corresponds with different choices of the metric in the $S^n \subset P^{n+1}$. Compare section 1; with a point ψ in P^{n+1} correspond several metrics in S^n.

V is said to admit a conformal mapping onto k *conformally independant Einstein spaces*, if its normal conformal P^{n+1} connection has k covariant constant points not contained in a (covariant constant) projective subspace of dimension $< k-1$.

A set of covariant constant figures, e.g. points, S^n, a non-Euclidean metric in S^n, determines the subgroup of those projective transformations in one fibre (P^{n+1}) that leave the representative figures invariant. All figures in this fibre, invariant under this subgroup, are then also representative for covariant constant figures. If the subgroup is the identity then the connection is flat.

Suppose for example that a conformal P^{n+1} connection has two covariant constant points in the interior of S^n. Then all points of the line passing

through these two points are also covariant constant (consider representative figures in one fibre).

A covariant constant figure is in each fibre represented by a figure which is a fortiori without torsion. Sometimes it is also known that in each fibre another figure is without torsion e.g. the point of the fixed cross section, or the gauges there. Other figures without torsion may then be found, eventually all points of each fibre in which case the connection is flat.

These are the ways to prove theorems like:

Theorem 8. *If a Riemannian space is conformal to two conformally independant Einstein spaces, then the space is certainly conformal to an Einstein space of negative scalar curvature* (Eventually some "infinite" points excluded); *If one of the given Einstein spaces has positive scalar curvature, then the given space is also conformal to an Einstein space with vanishing scalar curvature.* (Compare SASAKI [15], BRINKMAN [1, 2, 3]. The proof is left to the reader.

Theorem 9 (SASAKI [15]). *If an Einstein space V of dimension n and scalar curvature K admits a conformal mapping onto $n-2$ conformally independant Einstein spaces* (itself included), *then it is a space of constant curvature K.*

Proof: Consider the G_K-connection of the given space. It is easy to check that the Euclidean ($K=0$) or non-Euclidean ($K \neq 0$) fibre contains at least $n-3$ independant covariant constant points ("infinite" points are not counted). Suppose that in a fibre to be considered the point of the fixed cross section is independant of the points representing the covariant constant points *. Then at the point of the fixed cross section we find $n-3$ independant directions without torsion. If we choose favourable preferred coordinates for that fibre, and coordinates in the base space such that ω_i^a (at that point of the base space) equals δ_i^{a}', then

$$\text{(29)} \quad \text{at } x=x_0,\ \Omega^i_{jkm}=\Omega^i_{jkm}(K)=0 \text{ if at least one of the indices} > 3.$$

(27), (28) and (29) give $\Omega^i_{jkm}(K)=0$ at $x=x_0$, $i, j, k, m=1, \dots n$. The same holds for all points x for which * holds, and these points are every where dense in V, so that continuity of $\Omega^i_{jkm}(K)$ leads to the theorem.

We conclude this section with a theorem in the large. A Riemannian space is called (metrically-)*complete*, if any segment of a geodesic is contained in a segment with the same begin point and one unit longer. The G_K-connection of the Riemannian space has the characteristic property, that any curve segment in one fibre, passing through the point of the fixed cross section, which does not contain "infinite" points of the fibre, *is* the development of some curve in the base space. Suppose the given space is an Einstein space of non-positive scalar curvature. Consider the normal conformal P^{n+1} connection. From the completeness of the given

space and the theorem 3 follows that the "infinite" covariant constant point(s) in the covariant constant S^n are determined by the normal conformal connection and the base space as point set. Hence the point ψ of section 1 is determined by the conformal structure. The metric in each fibre is determined by ψ, but for the choice of a unit of distance. The same holds true for the metric in the base space. Therefore:

Theorem 10. *A conformal mapping of a metrically complete Einstein space of non-positive scalar curvature onto another metrically complete Einstein space is the product of a (locally-)congruent mapping and a multiplication of the metric (ds^2) with a constant positive factor.*

Example: A conformal mapping of the Euclidean space onto itself, or onto a locally Euclidean n-dimensional torus.

4. *Einstein spaces and projective connections*

If: the reference space for the fibres of a fibre bundle with an n-dimensional base space X, is an n-dimensional projective space; the group analogous to H in section 2 is the group of projective transformations; and a displacement in the fibre bundle is defined as in section 2; then a space with a projective connection is defined. The existence of a fixed oblique cross section without torsion is assumed.

Let us consider the reference space P^n. The Euclidean space is obtained from P^n by a choice of: a hyperplane P^{n-1} of excluded "infinite" points in P^n; a polarity with respect to an imaginary quadric in P^{n-1}; a unit of distance in P^n minus P^{n-1}.

The non-Euclidean space of constant negative curvature is obtained by a choice of: a real quadratic hypersurface S^{n-1} of the signature of a sphere in P^n; a unit of distance for the interior of this sphere. (The other points of P^n are called "infinite" points.)

The non-Euclidean space of constant positive curvature is obtained by a choice of: an imaginary quadratic hypersurface in P^n; a unit of distance.

Vice versa these spaces can always be considered to be imbedded in the described way in a uniquely determined P^n. Summarising, in all three cases there is an (eventually degenerated) hypersurface of the second class, and a unit of distance.

Because the Euclidean and non-Euclidean groups (G_K) are subgroups of the group P of projective transformations operating in the same space or a uniquely determined space obtained by addition of some "infinite" points, a definition of the *projective abstractum of a Euclidean or non-Euclidean connection* can be given, analogous to the conformal abstractum of section 3.

The system of geodesics of a Riemannian space determines one unique "normal" projective connection. This connection obeys apart from the conditions allready mentioned: a condition which has analogy with the

fourth assumption (16) of section 2; and a condition of vanishing of a projective Ricci-tensor analogous to (28). (CARTAN [7]). It is difficult to understand geometrically this normal projective connection. For Einstein spaces of scalar curvature K however a satisfactory geometrical characterisation can be given:

Main theorem 11. *The normal projective connection of an Einstein space of scalar curvature K is the projective abstractum of the G_K-connection.* The proof is analogous to the proof of main theorem 3.

The equality of the so called (normal) projective curvature tensor P^i_{jkm}, the (normal) conformal curvature tensor C^i_{jkm} and the G_K-curvature tensor (25) of an Einstein space (K), can easily be checked from the formulas for P^i_{jkm} and C^i_{jkm}. (EISENHART [13], SCHOUTEN-STRUIK [19])

As a corollary of theorem 11 we have for example:

Theorem 12 (SASAKI–YANO [16]. *The normal projective connection of an Einstein space has a covariant constant hypersurface of the second class.* If the scalar curvature is K, then the hypersurface has the normalised equation in homogeneous hyperplane coordinates for one fibre P^n

(30) $$\sum_{i=1}^{n} \xi_i^2 + K\, \xi_{n+1}^2 = 0.$$

Vice versa: if the normal projective connection of a Riemannian space V does have a covariant constant figure of the kind (30), then the connection has a covariant constant Euclidean or non-Euclidean metric in the fibres, with the help of which an agreeing Einstein metric in the base space can be introduced. V then admits a mapping with preservation of geodesics (a projective mapping) onto that Einstein space (SASAKI [16]). As with the analogous theorem on conformal connections, it may happen that some points, the "infinite" points, have to be excluded from the new space. Here however worse may happen: In case $K < 0$, the theorem does not hold, if all points of the fixed cross section are exterior of the covariant constant (infinite) S^n.

The proof of the next theorem is representative for the proofs of a class of theorems. If the Einstein spaces which are given have non-vanishing scalar curvature, then a fairly simple proof can be given. The "exceptional" cases when some of the Einstein spaces have vanishing scalar curvature make the proofs lengthy.

Theorem 13. *If a fourdimensional Einstein space admits an essentially projective mapping onto another four dimensional Einstein space, then it is a space of constant curvature.*

A mapping is here called essentially projective, if it is not the product of a congruent mapping and a multiplication of all distances with a constant factor; or, what amounts to the same, if the two metrics of the

space with the common normal projective connection realise by different covariant constant hypersurfaces of class two. (Note that we deal with local properties, that is with a small neighborhood of the base space, so that topological difficulties do not occur here).

Proof: Let the scalar curvatures of the two Einstein spaces be K and K'.

Case A: $K \neq 0$, $K' \neq 0$. The normal projective connection has two covariant constant non-degenerated hyperquadrics. Also the pencil of hyperquadrics with these two as basis, is covariant constant. Consider a fibre in which the point of the fixed cross section is not contained in any of the two hyperquadrics *. By assumption this point is without torsion. Also the hyperquadric of the pencil which passes through this point (in the fibre under consideration) is then without torsion, and the same is true for the hyperplane (of dimension 3) tangent at this point to the hyperquadric. We assume the existence of this hyperplane *. In view of main theorem 11, the G_K-connection of the first Einstein space has at the point of the base space under consideration a threedimensional hyperplane through the point of the fixed cross section *without torsion*. The direction perpendicular to this hyperplane at the same point is also without torsion. The rest of the proof is the same as in the proof of theorem 9.

Case B: $K \neq 0$, $K' = 0$. The normal projective connection has a covariant constant non degenerated hyper quadric and a covariant constant hyperplane. The hyperplane counted twice may serve as a second hyperquadric. The rest of the proof is as in case A.

Case C: $K = 0$, $K' = 0$. If the two covariant constant hyperplanes do not coincide, then they are the base of a pencil of covariant constant hyperplanes. The rest of the proof is then as before. Now suppose that the covariant constant hyperplanes coincide. The covariant constant imaginary quadrics in this hyperplane have for suitable homogeneous coordinates the equations:

$$(31) \qquad \begin{cases} x^2 + y^2 + z^2 + t^2 = 0, \quad ax^2 + by^2 + cz^2 + dt^2 = 0, \\ a, b, c, d > 0, \quad a + b + c + d = 4. \end{cases}$$

They determine invariantly (hence also covariant constant) a point in case the four numbers a, b, c, d are not equal in pairs. The covariant constant point determines in almost every fibre a line without torsion: the line through this point and the point of the fixed cross section. The rest of the proof is as before.

In case the numbers a, b, c, d are equal in pairs (they are not all mutually equal, because then the given mapping would not be essential) the normal projective connection has two covariant constant lines in the covariant constant ("infinite") hyperplane. The Euclidean (G_0) connection of say the first space ($K = 0$) has two perpendicular covariant constant two-directions. For suitable coordinates, for which $g_{ij} = \delta_{ij}$, $\partial_i g_{jk} = 0$ at

$x = x_0$, in a neighborhood of a point x_0 of the base space, the (ordinary !) Riemann tensor then obeys, apart from

$$\sum_{j=1}^{4} \Omega_{ijjk} = \Omega_{1234} = 0 \quad \text{also} \quad \Omega_{31\,km} = \Omega_{41\,km} = \Omega_{32\,km} = \Omega_{42\,km} = 0,$$

hence this tensor vanishes. q.e.d.

The generalisation to higher dimensions (compare theorem 9) of theorem 13 holds true and is easy to prove if the Einstein spaces under consideration have non-vanishing scalar curvature. Otherwise the proofs get complicated by the large number of details.

We conclude section 4 with the statement of a theorem analogous to theorem 10

Theorem 14. *A projective mapping of a metrically complete Einstein space of negative (vanishing) scalar curvature onto another such Einstein space is the product of a (locally-)congruent mapping and a multiplication of all distances with a constant factor (is an affine mapping, that is: it preserves ratios of lengths of segments of any geodesic).*

July 1950. *Technische Hogeschool, Delft*

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ASTRONOMY

THE PROFILES OF THE LINES OF THE PASCHEN AND BRACKETT SERIES OF HYDROGEN IN THE SOLAR SPECTRUM

BY

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(Communicated by Prof. M. G. J. MINNAERT at the meeting of Sept. 30, 1950)

Resumo. La profilojn de la hidrogen-linioj en la sunspektro, kiuj apartenas al la transiroj 3—5; 3—6; 3—7; 3—9; 3—10; 3—11; 3—12; 4—7; 4—10; 4—11; 4—12; 4—13; kaj 4—14 oni determinis ĉe la scienca esplorejo de Jungfraujoch, Svisujo, pere de la foto-elektra infraruĝo-spektografo de la astrofizika Instituto de la Universitato en Lieĝo, Belgujo.

La linioj, precipe tiuj de la Brackett-serio, estas tre larĝaj kaj mal-profundaj: ĉe tiuj de la lasta serio oni povas ankoraŭ percepti la enpro-fundiĝon je distancoj ĝis 50, ja eĉ ĝis 100 A de la liniocentro.

La ekivalenta larĝo de la linioj montriĝas granda; tiu de la linioj el la Brackett-serio ekzemple estas pli granda ol tiu de la analogiaj linioj el la Paschen-serio kaj laŭ grandeco samklasa kiel tiu de la analogiaj linioj el la Balmerserio.

La mezkvadrata ekarto de la mezuritaj punktoj de la linio-profilo estas granda je $\pm 6\,^0/_{00}$ escepte ĉe la altnombraj membroj el la serioj, kiuj ofte estas grave perturbitaj.

Summary: The profiles of the hydrogen lines in the solar spectrum corresponding with the transitions 3—5, 3—6, 3—7, 3—9, 3—10, 3—11, 3—12, 4—7, 4—10, 4—11, 4—12, 4—13, and 4—14 have been determined at the Scientific Station of the Jungfraujoch, Switzerland, with the photo-electric infrared spectrograph of the "Institut d'Astrophysique" of the University of Liège, Belgium.

The lines, especially those of the Brackett series appear to be extremely broad and shallow; the wings of the latter may be followed even to distances of 50 to 100 A from the line centre. The equivalent widths of the lines of the Brackett series are remarkably great and of the same order as those of the corresponding Balmer lines.

The mean error of the mean profile points is of the order of $\pm 6\,^0/_{00}$; it is greater for the higher series members, which are often seriously blended.

1. *Introduction.* The higher series of hydrogen in the near infrared spectral regions between 0,8 and 2,2 μ are interesting. Their broad profiles,

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completely differing from the normal solar and atmospheric lines, have already been noted by some observers. From the theoretical point of view their investigation seems to be important in several respects.

Their study makes possible to check the theory of the Stark-widening of the high hydrogen levels. Their profiles may be compared with those of the Balmer lines in the photographic spectral region.

At the same time it is possible to compare the line excitation coefficients with the continuous absorption coefficient in the solar atmosphere. This is an independent method in order to compare the continuous absorption coefficient, determined observationally, with the theoretical values, calculated for the H^--ion by Chandrasekhar and Breen [1]). Chalonge and Kourganoff [2]) found that important differences between the theoretical data and those, derived from the observations do still occur.

For the construction of a model of the solar atmosphere these lines are especially useful. The problem of the abundance of the hydrogen atom does not play a serious role, since both the continuous and the selective absorption are due to the same atom.

Several authors have tried to explain the behaviour of the hydrogen lines by superexcitation [3]) [4]), assuming deviations from Boltzmann's law. This phenomenon can only be investigated properly if one possesses measurements concerning transitions originating from different atomic levels.

For these reasons these lines have been put on our observational program. A special observational difficulty is produced by the fact that the spectral regions considered here are often heavely blended by atmospheric lines and especially by lines of the water vapor molecule. They can only be studied with success in a high altitude station. We are extremely obliged to Dr M. Migeotte from the "Institut d'Astrophysique" at Liège, who generously put the infrared spectrograph from this institute at our disposal. This spectrograph is temporarily mounted at the Sphynx observatory of the Jungfraujoch Scientific Station in Switzerland (Altitude 3580 m). At this height the blending by atmospheric lines is greatly reduced as compared with low-altitude observations.

2. *Photometric properties of the apparatus.* The apparatus has been described by Migeotte [5]). Its present mounting at the Sphynx observatory and its current program have been described by Migeotte and Neven [6]).

During our observations we always used a solar image with a diameter

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[1]) Ap. J. **104**, 446 (1946).

[2]) Ann. d'Astr. **9**, 69 (1946); see also R. Peyturaux, Contr. I.A.P. série A, 56 (1950).

[3]) A. Rosa, Zs. für Ap. **24**, 38 (1947).

[4]) P. ten Bruggencate, H. Gollnow, S. Günther, W. Strohmeier, Zs. für Ap. **26**, 51 (1949).

[5]) Thesis, Liège, 1945.

[6]) Congres National des Sciences, Bruxelles, 1950 (in press).

of 15 mm. The monochromator slit of the spectrograph had a height of 7,5 mm. A Wood echelette grating was used with 600 lines per mm.

The light was received by a Cashman PbS cell. The light beam was chopped 1080 times per second; the chopper was driven by a synchronous motor at a frequency of 50 c/s, stabilized to 10^{-4} by a tuning fork controlled drive amplifier. The photocurrent was amplified by a WILSON amplifier [7]) and recorded on a Speedomax recorder (Leeds and Northrup).

The mean linear dispersion on the plane of the second slit of the spectrograph was 7,7 A/mm for the first order spectrum (which was always used by us). On the tracing one A corresponds with 2,5 mm at 1,5 μ.

Special attention was paid to the photometric properties of the apparatus: (*a*) the apparatus profile, (*b*) the intensity of the ghosts and (*c*) the response curve.

The apparatus profile: The optimal slitwidth, calculated according to the theory of VAN CITTERT [8]), is 1,04 $\lambda(\mu) \cdot 10^{-2}$ cm. As we sometimes worked with a slit width 2 to 2,5 times its optimal value (in that wavelength regions where little energy was available), the apparatus profile has been determined for 4 different ratios between slit width and theoretical width, namely resp. 1,1; 1,4; 2,5 and 3,7.

The atmospheric CO_2 lines at 1,57 μ are appropriate for the determination of the apparatus profile: these lines, as observed by us, are weak and their theoretical width is small, compared with their observed

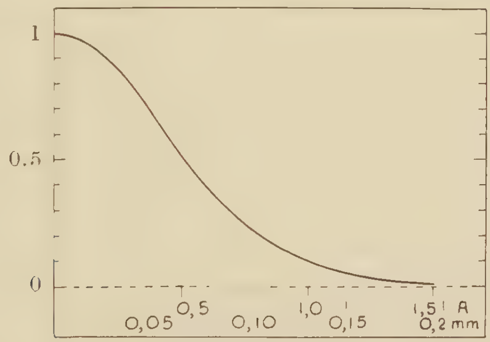

Fig. 1. Apparatus curve of the infrared spectrograph at 1,57 μ.

TABLE 1. Mean apparatus profile for 1,57 μ

$\Delta\lambda$ (A)	$\Delta\lambda$ (mm)	intensity
0	0	1.00
0.25	0.032	0.85
0.50	0.065	0.52
0.75	0.098	0.25
1.00	0.130	0.10
1.25	0.163	0.03
1.50	0.195	0.01

[7]) KUIPER, WILSON and CASHMAN, Ap. J. **106**, 246 (1947).

[8]) Z. Phys. **65**, 547, (1930); **69**, 298 (1931).

width. For a temperature of 253° K, which may be the mean temperature of the layers where these lines originate, one finds a Doppler width of 0,03 A, which is negligible compared with the observed width of 1,03 A.

It was found that for the four slit width ratios given above, no systematic differences occurred between the profiles for the first three values; so a mean profile was determined for the ratios 1,1, 1,4, and 2,5. This profile is given in table 1 and in figure 1, both in mm on the second slit as well as in Angstrom units. It corresponds with a wavelength of 1,57 μ. The profile width depends linearly on the wavelength.

As we never worked with a width exceeding 2,5 times its theoretical value, the profile for width-ratio 3,7 is not included here.

The half width of the profile, found here is of the same order as other half widths determined for spectrographs with gratings of the same dimensions and numbers of lines pro unit of length.

We compare: Liège: $10,5 \times 10,5$ [cm^2]; 600 lines/mm; halfwidth 0,13 mm
Utrecht 5×8 [cm^2]; 568 lines/mm; halfwidth 0,079 mm
Meudon 6×8 [cm^2]; 600 lines/mm; halfwidth 0,101 mm

All halfwidths have been reduced to 1,57 μ and to the same focal length.

The fact that the half width of the Liège apparatus is still somewhat greater than both other values can be explained simply by the fact that in Liège *two* slits are used, both with the same width. The resolving power is generally greater in photographic than in photoelectric spectro-photometry. One only arrives at the same value for both methods if the width of the second slit is small as compared with the width of the instrumental profile.

The intensities of the ghosts have been determined with the aid of a mercury vapor lamp. The green ray at 5460 A has been recorded together with the adjacent wavelength region in the first, second and third order. It turned out, that even in the third order no ghosts could be detected; the ghosts in the first order spectrum are a fortiori completely negligible.

The response curve of the apparatus has been found to be linear only in its first part; we practically always worked in this range. The shape of the curve has been determined in the laboratory by recording a small wavelength region of H_2O-lines several times with different amplification factors. This procedure is justified if one may assume that the Cashman cell has indeed a linear response curve [9]). The reduction of the material showed that this assumption was right.

The linear part of the curve was determined, assuming that it could be represented by a curve of the form $D = a + b\,I$, where D is the deflection of the recorder and I the intensity of the light.

D is expressed in mV; the whole scale has a range of 10 mV.

For a a least squares determination yielded $0,025 \pm 0,052$ mV; hence a

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[9]) R. R. McMath and O. C. Mohler, Journal Opt. Soc. Amer. 39, 903 (1949).

has been assumed to be equal to zero. The curved part of the response curve was then easily determined; the whole curve is given in figure 2.

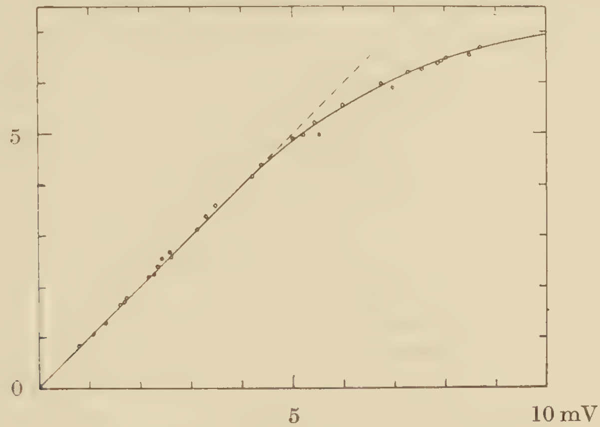

Fig. 2. Response curve

3. *Observed profiles of lines of the Paschen series.* Profiles were obtained of the Paschen lines corresponding with the transitions 3—5, 3—6, 3—7, 3—9, 3—10, 3—11, 3—12. The lines 3—4 and 3—8 are too heavily blended by lines of water vapor. The line 3—13 lies in a wing of one of the infrared Ca^{-} lines and cannot be detected; the other Paschen lines are too weak. Figure 3 contains the records of all Paschen lines given above. This figure may already give an impression of the difficulties encountered when drawing the line profiles. These difficulties are small for the lines 3—5; 3—6 and 3—7; but for the higher members, which are already very weak and often more or less blended, they are sometimes very great. In drawing the line profiles several questions had to be considered:

a. The extension of the wings: Generally in the literature the wings of the hydrogen lines are underestimated. In order to have an impression of the extension of the wings we always made at least two different records of the same line: one on a large scale, covering only the central regions up to about 15—25 A from the line centre; and another, on a smaller scale, covering a region of in all 150 to 200 A. This latter record makes it possible to find the wing profile exactly; after that the central profile can be connected to this wing profile. Another difficulty which was also met in the construction of the wing profiles was that of the fluctuations in the height of the continuous background, caused either by wavelength-dependent variations of the sensitivity of the PbS-cell, or by variations of the continuous radiation of the sun as a function of the wavelength. However since these latter variations are rather smooth, the principal cause must be the first one.

We could overcome this difficulty by recording — in otherwise similar circumstances — the continuous radiation of a Nernst filament in the same

wavelength region. An analogous procedure was already applied by EVANS [10]) when studying the profile of H_{2-3} (= H_α).

In order to be mutually comparable, the records have to be corrected for the differences in colour temperature between the Nernst filament and

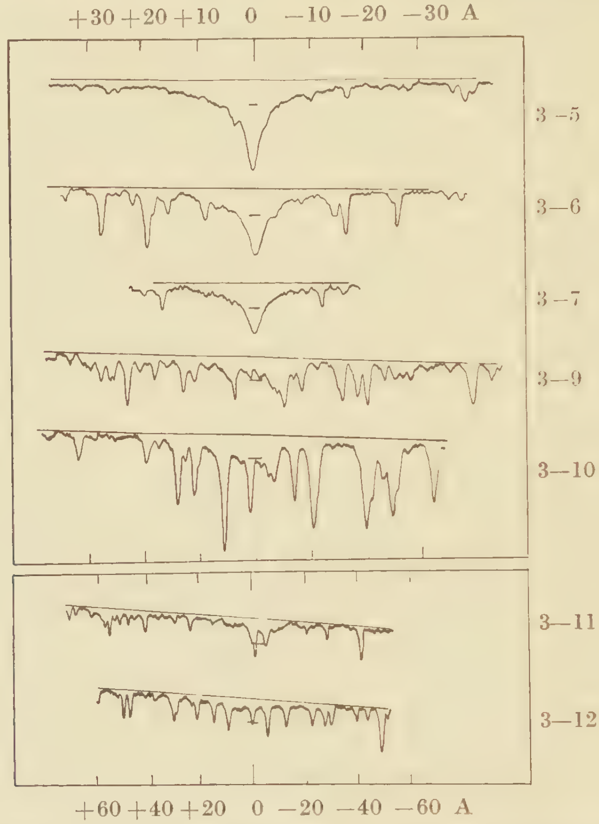

Fig. 3. Records of infrared PASCHEN lines at the centre of the sun's disc. A short horizontal line gives the intensity at 90 %.

the sun. There must exist (at least in the small wavelength regions around every hydrogen line) a constant difference between the gradients $d \log I / d\,(1/\lambda)$ for the filament and the sun.

This difference has been determined empirically. With the aid of it the true background could be determined without difficulties. In the region of the Paschen lines this procedure has been followed for the line 3—5. For the other lines no obvious irregularities in the continuous background could be found.

b. Elimination of blends. For this purpose it is necessary to estimate the influence on the profile of each individual blend line, to separate these lines and to estimate their influence on the line contour. This work was always done by one of the authors, controlled by the other

[10]) M. N. 100, 156 (1940).

(or done again independently) and afterwards discussed. It might seem that the process is rather subjective, leaving the possibility of great errors; but this is not in fact the case. Fairly exact objective standards are also available: the profiles of the blend lines must be symmetric; moreover, since the profiles of the faint blend lines can nearly always be considered as only determined by the apparatus profile, the whole blend line profile can be constructed, even when only the top part of it is given.

It indeed occurred very seldom that differences between line contours, thus corrected, as determined by one author or the other amounted to 1 % of the continuous background. Nearly always the difference between both authors or the agreement between left and right wing of the lines or between two different records was perfect. Of course this does not exclude the possibility of greater systematic errors of both authors.

A discussion of the data showed that the mean error of one determined profile point is $\pm$ 6 ‰. In this figure are included both the recording (instrumental) errors as well as the errors made in correcting the profiles. It evidences the accuracy of photo-electric photometry as compared with photographic photometry, where one generally cannot avoid errors four times greater.

In determining profiles we used one record for the high Paschen members 3—9 .... 3—12 and at least two for the other lines. As we did not find any systematic difference between the left and the right wings of the lines, the data have always been combined to one set of mean values.

c. Correction for the apparatus profile. This correction was only necessary for the lines 3—5; 3—6 and 3—7. It was applied numerically with BURGER and VAN CITTERT's method[11]). For 3—5 and 3—6 two approximations were necessary, for 3—7 one was sufficient. For the cores of the lines this correction amounted to:

for 3—5	...	16 ‰
3—6	...	12 ‰
3—7	...	7 ‰

Corrections were zero up from a wavelength distance of 3 A from the centre of the lines.

The mean ultimate profiles are given in table 2 and in figure 4. The fact that the profile of H_{3-9} intersects some of the other profiles may be ascribed to errors in the construction of the line profiles.

4. *A comparison with other observations of the Paschen lines.*

Equivalent widths of some Paschen lines are given by ROSENTHAL[12]), by DAHME[13]) and by ALLEN[14]). Moreover Mr E. W. DENNISON put at our disposal unpublished results on the profile of the H_{3-12} line as measured

[11]) Zs. Phys. 79, 722, (1932); 81, 428 (1933).
[12]) Nature 134, 533 (1934).
[13]) Zs. f. Ap. 11, 93, (1935).
[14]) Ap. J. 88, 125 (1938).

TABLE 2

Mean profiles and equivalent widths of Paschen lines in the central parts of the solar disc

(The table gives the residual intensities multiplied by 1000)

line	3−5	3−6	3−7	3−9	3−10	3−11	3−12
λ (A)	12818	10938	10049	9292	9015	8863	8750
E.W. (A)	4.388	2.426	1.602	0.748	0.736	0.560	0.486
$\Delta\lambda$ (A)							
0	627	742	788	956	961	969	969
0.5	657	768	807				
1.0	739	806	831				
1.5	787	851	874				
2.0	819	864	893	962	967	975	976
4	884	925	935	972	971	980	982
7	925	962	970	979	976	984	986
10	944	976	984	987	986	986	988
15	967	985	991	992	989	992	995
20	976	990		996	995	995	998
30	981	992					

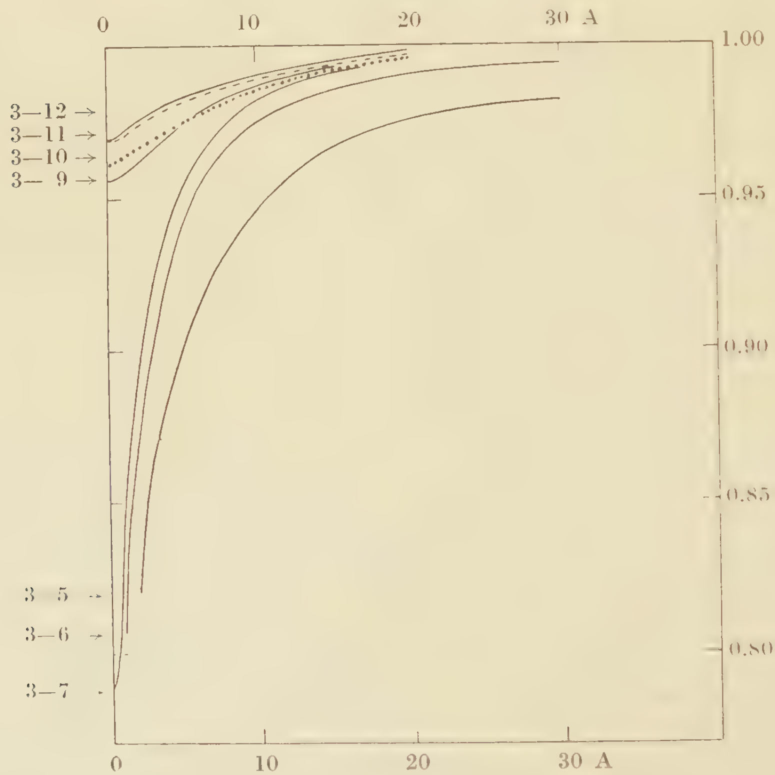

Fig. 4. Mean profiles of solar PASCHEN lines.

on Mount Wilson plates, previously used for the Utrecht Solar Atlas [15]).

[15]) MINNAERT, MULDERS and HOUTGAST, Photometric Atlas of the Solar Spectrum, Amsterdam 1940.

We are much indebted to Mr DENNISON for his permission to use his data and to publish them here. Finally, still unpublished results of photographic photometry performed with the Utrecht solar telescope by one of the authors, are available. Firstly we compare the aequivalent widths (expressed in Angström units):

Line:	3–5	3–6	3–7	3–8	3–12
ROSENTHAL		1.66	1.57		
DAHME	3.27	2.19	1.81	2.67	
ALLEN		1.32	1.39		
DENNISON					0.22
DE JAGER		1,38	1.40		
this work	4.39	2.43	1.60		0.49

The comparison shows that generally our values are greater than those given by the other authors. In similar cases nearly always the explanation has to be sought in the height of the continuous spectrum. We think that in this case too, such an explanation applies. We note further that DAHME's value for 3—8 must be suspected, since even at the Jungfraujoch it was impossible for us to find the profile of this line, which is completely blended by water vapor.

The same fact is shown when comparing our profile of H_{3-12} with that derived by DENNISON. Below we give DENNISON's profile points and ours; the latter are found by interpolation. (All residual intensities are multiplied by 1000).

$\Delta\lambda (A)$	0	1.46	2.93	4.40	5.86	7.32	8.79	10.26
DENNISON	981	981	984	990	995	994	998	1000
This note	969	974	979	983	985	986	987	988

It is seen that DENNISON's profile reaches a depression of 2 $^0/_{00}$ at 8,8 A from the centre of the line while in our profile this is the case at a distance of 20 A. This is almost certainly caused by the dispersion of the plates used for the Utrecht Solar Atlas, which is too great for the detection of lines as broad as the hydrogen lines.

5. *The lines of the Brackett series*

We give here the profiles of the Brackett lines 4—7; 4—10; 4—11; 4—12; 4—13 and 4—14. The tracings are reproduced in figure 5.

The lines 4—5 and 4—6 are too far in the infrared to be recorded with a non-cooled PbS cell as was used by us; moreover they are blended. The lines 4—8, 4—9 and 4—15 are also too heavily blended. Around 4—16 the continuous spectrum is not disturbed by other lines; a small depression of about 1 % could be detected there, but this is too small for the derivation of a reliable profile. The line 4—17 could not be found, although the continuous spectrum was undisturbed.

The method that was followed was essentially the same as that followed for the Paschen lines. We only note the following facts:

a) The extension of the wings. Small scale tracings were made for all lines over a region of at least 300 Å, since it could be expected that these lines are still much broader than those of the Paschen series. The

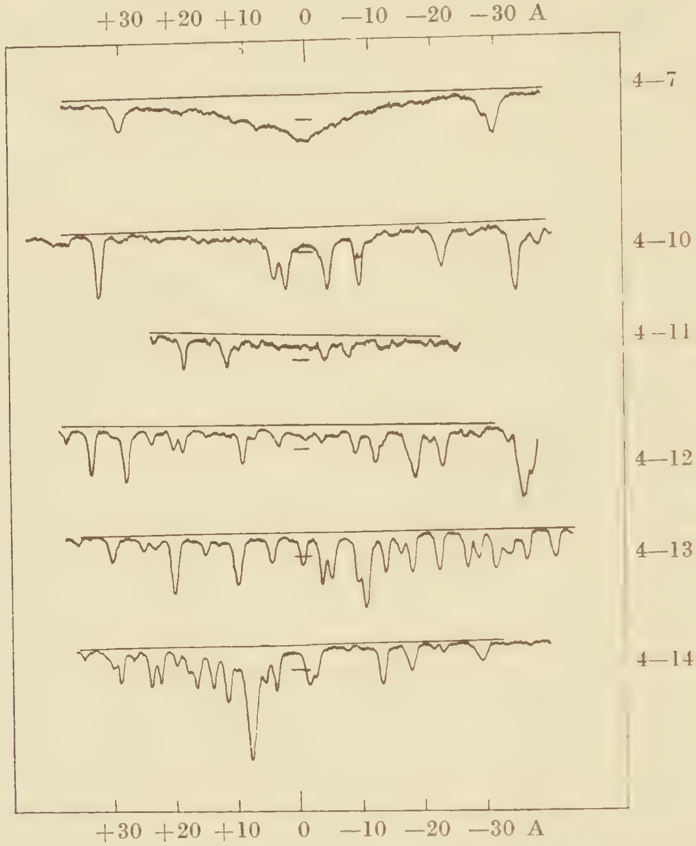

Fig. 5. Infrared Brackett lines at the centre of the sun's disc. A horizontal line gives the intensity at 90 %

continuous spectrum was always very regular, so that no additional measurements with the Nernst-filament were necessary. We made a laboratory comparison only in the case of 4—13 and 4—14, where the depressions are already very small and the lines still broad: this showed that the tracings are indeed reliable.

b) The apparatus profile correction was only applied to the 4—7 line (one approximation). In the centre of the line this correction amounts to 5 $^0/_{00}$. In table 3 and figure 6 the mean corrected profiles are given.

6. *A comparison with other observations of the Brackett lines*

Only very few data concerning these lines are known as yet. Goldberg,

TABLE 3

Mean profiles and equivalent widths of BRACKETT lines in the central part of the solar disc.

(The table gives the residual intensities multiplied by 1000)

line	4—7	4—10	4—11	4—12	4—13	4—14
λ (A)	21655	17362	16806	16407	16109	15880
E.W. (A)	7.368	3.182	2.534	1.850	1.334	0.832
$\Delta\lambda$ (A)						
0	794	925	940	958	963	972
1	809	930	945	962	967	
2	826	936	952	966	970	975
4	852	944	957	968	974	979
7	885	957	964	973	978	985
10	912	964	968	977	984	986
15	938	970	976	980	986	990
20	950	974	982	984	988	994
30	966	983	987	988	992	997
50	979	992	992	996	996	998
100	990					

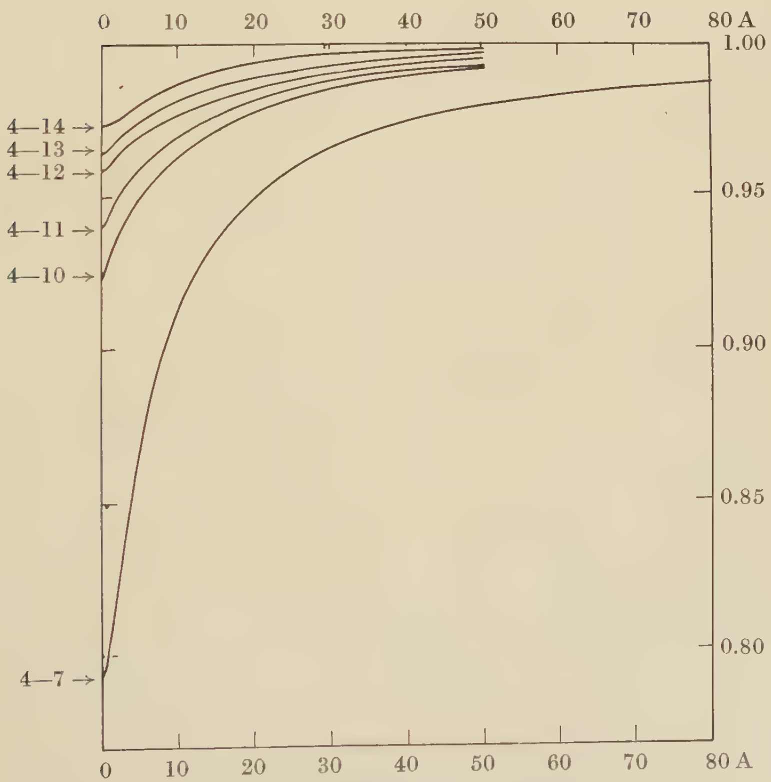

Fig. 6. Mean profiles of solar BRACKETT lines

Mohler and McMath [16]) published observations of 4—7; 4—11 and 4—12. A comparison of their figure 1 with our figure 5 shows the following facts:

1) the depressions derived at Michigan are too small. Two causes can be found for this error: a general psychological effect of underestimating the width of these extremely broad lines; and a physical effect: the noise on the Michigan tracings is greater than on our tracings. This is caused due to the ten times greater diameter of their solar image, giving much less energy on the cell. From the tracings we estimate that the Michigan noise is 4 to 5 times greater than our noise.

The influence on the widths of the lines is important: for 4—7 a width of 40 A is given by the Michigan workers while we find about 200 A. This causes at the same time a great reduction of the equivalent width: we found the value 7,37 A, while the Michigan tracing should give about 2,4 A, showing the very important contribution of the far wings of the lines to the equivalent widths.

2) The influence of *superposed atmospheric lines* is greater at Michigan, caused by the thicker atmosphere containing more water vapor. This makes it difficult to find accurate profiles in general; particularly the *higher members* can only be detected with great difficulty and with uncertainty.

The case of the central depression of 4—7 is interesting. We found a value of 20,6 % in good agreement with the Michigan depth of 20 %, notwithstanding the fact that the continuous spectrum is drawn too low in Michigan by more than 4 %. This error is however compensated by a small water vapor line, which is situated exactly in the centre of the hydrogen line, giving an extra depression. The influence of this line, as well as the influence of other water vapor lines on the profile is shown by the following figure 7, where the 4—7 region is reproduced four times:

1. as observed in Liège with low sun [17])
2. as observed in Liège with high sun [17])
3. as observed in Michigan
4. as observed at the Jungfraujoch.

One should note the intensity of the strong water vapor lines at 25 A on both sides of the line centre, as well as the effect produced by the weak line on the centre of 4—7.

7. *A comparison of the Balmer, Paschen and Brackett series of hydrogen in the solar spectrum.*

There are some interesting facts that should be noted:

a) The number of lines in the series. The last detectable Balmer line corresponds with the transition 2—16 or 2—17 [18]); for the Paschen

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[16]) Ap. J. 109, 28 (1949).

[17]) These data are extracted from the "Memoire de Licence" of J. Ottelet from Liège. We are obliged to mr. Ottelet for his permission to publish these data.

[18]) A. Unsöld, Zs. f. Physik 59, 253 (1930).

and Brackett series we find respectively the transitions 3—12 and 4—14 or 4—16. It is striking that in the Paschen series less lines are detectable than in the Brackett series, although the widths of the Brackett lines are greater than those of the Paschen lines.

b) The same effect is noticed when one considers the equivalent widths of the lines. Here too the greater equivalent widths of the Brackett lines

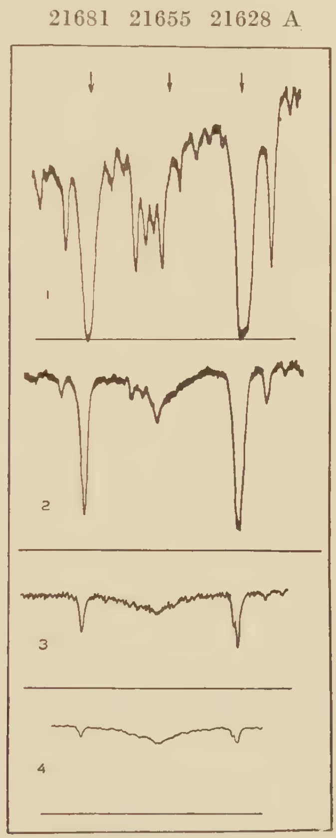


Fig. 7. A comparison of H_{4-7}. 1) Low sun at Liège. 2) High sun at Liège. 3) McMath Observatory. 4) Jungfraujoch.

are obvious. In table 4 we give the equivalent widths for the different lines of the three series [19]). The table shows the very remarkable fact that the lines of the Brackett series are always much stronger than those of the Paschen series and are sometimes even stronger than those of the Balmer series. Of course the data are not directly comparable; questions of saturation of the lines, of continuous absorption coefficient and of the depth of formation in the solar atmosphere play a role. A quantitative comparison is necessary to ascertain if we are really concerned with super-excitation here, or if this is another effect.

[19]) Data on the Balmer lines are extracted from unpublished results of one of the authors.

c) A further comparison of the lines of the three series is given in tables 5, 6 and 7, where we give respectively the half widths of the lines, the total width at a depression of 2 % and the central depression. These tables show again how the widths of the lines increase as n (number of the

TABLE 4

Equivalent widths of the lines of the Balmer, Paschen and Brackett series of hydrogen. (Angström units) (centre of solar disc)

Lowest level (n)	2	3	4
Upper level (m) = 3	4.03		
4	3.80		
5	2.76	4.39	
6	3.37	2.43	
7		1.60	7.37
8	2.46		
9		0.75	
10	3.59	0.74	3.18
11	2.42	0.56	2.53
12	1.44	0.49	1.85
13			1.33
14			0.83
15	0.62		
16	0.48		

lowest level) increases; how, for constant n, the widths increase firstly when m increases, but fall off for the highest numbers. The far wings of the Paschen lines extend to wavelength distances which are of the same order as at the corresponding lines in the Balmer series; both extend less far than the lines of the Brackett series. The central depressions again

TABLE 5. Total half-widths of Hydrogen lines (A)(centre of solar disc)

n	2	3	4
$m =$ 3	1.66		
4	1.36	3.9	
5	1.70	3.9	
6	1.60	4.0	
7		4.0	16.6
8	1.96		
9		12.6	
10	3.21	14.2	19.2
11	2.61	12	22
12	3.0	10	25
13			21
14			18
15	1.6		
16	1.6		

TABLE 6. Total widths of hydrogen lines at 2 % depression (A) (centre of solar disc)

$n =$	2	3	4
$m =$ 3	34		
4	29		
5	14	46	
6	20	24	
7		19	103
8			
9		15	
10		14	51
11	17	9	37
12	8	7	27
13			17
14			10
15	6		
16	5		

show the remarkable position of the Paschen series between the Balmer and Brackett series; especially for the highest series members the central depressions of the Paschen lines are less than those of the corresponding Balmer and Brackett lines.

We wish to express our sincere thanks to Dr M. Migeotte and to Prof. P. Swings, who permitted us to work with the infrared spectrograph of the Institut d'Astrophysique of Liège, and to intercalate our investigation between the current observational programs.

TABLE 7
Central depression of hydrogen lines (× 1000)
(centre of solar disc)

n	2	3	4
$m =$ 3	828		
4	824		
5	815	373	
6	774	258	
7		212	206
8	737		
9		44	
10	649	39	75
11	545	31	60
12	421	31	42
13			37
14			28
15	287		
16	220		

Besides we are very obliged to the board of the Jungfraujoch Scientific Station, who most kindly made it possible for us to work at this Station. We are glad to thank Mr H. Wiederkehr, housekeeper of the Station, who showed us much hospitality and a never failing helpfulness in the solution of many technical problems.

This investigation has been made possible by a grant of UNESCO through the intermediary of Prof. F. J. M. Stratton, president of the commission for the exchange of astronomers of the International Astronomical Union.

UN CEPHALOPODE NOUVEAU: PHOLIDOTEUTHIS BOSCHMAI GEN. ET SP. NOV.

PAR

W. ADAM

(Communicated by Prof. H. BOSCHMA at the meeting of Sept. 30, 1950)

Parmi les Céphalopodes, récoltés par l'Expédition du Snellius dans la partie orientale de l'archipel indien (1929—1930), se trouve une espèce nouvelle, intéressante, dont je donne ici la description.

Je me fais un agréable devoir de remercier sincèrement Monsieur le Professeur H. BOSCHMA, biologiste de l'Expédition du Snellius et directeur du Musée d'Histoire naturelle de Leyde (Pays-Bas), qui a eu l'amabilité de me confier l'étude des Céphalopodes de cette expédition, dont les résultats complets seront publiés ultérieurement.

Les photographies ont étés réalisées par Monsieur H. F. ROMAN; les figures 5 et 6 du texte ont été exécutées par Monsieur G. W. KURPERSHOEK.

Pholidoteuthis, gen. nov.

Type du genre: *Pholidoteuthis boschmai* sp. nov.

Comme le nouveau genre ne comprend pour le moment qu'une seule espèce, il n'est pas possible de faire ressortir nettement les caractères génériques de l'ensemble des caractères spécifiques. Provisoirement, la diagnose du genre coïncide avec celle de l'espèce.

Pholidoteuthis boschmai, sp. nov.

Holotype: "Rijksmuseum van Natuurlijke Historie" à Leyde (Pays-Bas).

Localité du type: Station du Snellius, 192: 5° 58'.0 S., 121° 32'.0 E., straminpose employé en filet vertical, de 2000 à 0 m.

Description:

Le seul spécimen est une femelle, dont la longueur dorsale du manteau est de 273 mm. L'exemplaire ayant été plié en deux dans un bocal, il n'est malheureusement pas possible de donner sa largeur exacte. Les figures 1 et 2 (pl. I) montrent cependant suffisamment la forme générale de l'exemplaire.

Le corps est allongé, la moitié antérieure du manteau, plus ou moins cylindrique, se prolonge postérieurement en un cône allongé. Le bord palléal se montre très peu saillant dorsalement, légèrement concave du côté ventral.

Les nageoires sont rhomboïdes; leur longueur atteint 45 % de la longueur

dorsale du manteau, leur largeur totale, 52 %. Leur bord antérieur est très légèrement convexe, le bord postérieur presque droit. La plus grande largeur des nageoires se situe un peu en avant du milieu de leur longueur.

A l'exception des nageoires et de l'extrémité postérieure du corps, toute la peau du sac palléal est couverte de papilles très serrées et aplaties, à contour arrondi, polygonal, ou parfois vaguement stellaire. Sur la face dorsale, les papilles s'étendent en arrière jusqu'à la ligne concave limitant l'insertion des nageoires. Sur la face ventrale, la limite de la partie ornée de papilles est constituée par une ligne fortement incurvée en arrière des points d'attache antérieurs des nageoires et s'étendant sur la ligne médio-ventrale jusqu'à 75 mm de l'extrémité postérieure. Le diamètre des papilles dorsales varie de 0.75 à 1.5 mm; de 0.5 à 1.5 mm sur la face ventrale. Parfois leur bord est pourvu d'un cercle de chromatophores.

L'état de conservation de l'exemplaire ne permet pas une description histologique détaillée de la peau. Sur des coupes transversales, la forme des papilles rappelle celle des papilles de *Moroteuthis ingens* (Smith), figurées par E. Lönnberg (1898, pl. V fig. 1) et celle des papilles de *Lepidoteuthis grimaldii* Joubin, figurées par L. Joubin (1900, pl. X fig. 5). L'intérieur des papilles montre une structure réticulaire. Extérieurement, l'épiderme suit le contour des papilles et ne les recouvre pas comme celles représentées par la figure (pl. V fig. 1) d'E. Lönnberg (1898), où il s'agit de papilles sous-cutanées. Les papilles de *Pholidoteuthis* ressemblent plutôt à celles de *Lepidoteuthis*, sans toutefois se recouvrir partiellement comme des écailles (pl. II).

La tête semble avoir eu à peu près la même largeur que le corps; les yeux ne sont pas saillants. L'ouverture oculaire est plus ou moins réniforme-verticale; le sinus antéro-ventral est très marqué (pl. III fig. 1). Postérieurement, la tête (pl. III fig. 1—2) montre deux plis nuchaux transversaux et, de chaque côté, quatre plis lamelliformes longitudinaux, dont le quatrième, situé à côté du siphon, est peu marqué. Derrière le troisième pli se trouve le tubercule olfactif. La peau entre les plis longitudinaux est nettement ridée (pl. III fig. 2) dans le sens longitudinal, sans cependant former des lamelles. Il s'agirait plutôt de faisceaux de muscles causant un relief par suite de la fixation.

Le siphon est très robuste. L'appareil de connection est simple: la partie palléale se compose d'une paire de crêtes cartilagineuses, d'une longueur d'environ 4 cm, qui atteignent le bord palléal. Les cartilages du siphon sont ovalaires, allongés (30 × 8 mm) et pourvus d'une simple rainure médiane. Le siphon possède à l'intérieur une grande valve et l'organe siphonal en forme de $|\wedge|$.

Sur le bulbe oculaire on distingue un anneau clair, entourant la pupille. Cet organe, qui est peut-être lumineux, constitue un cercle complet, large de 3 mm, sauf ventralement, où, sur une longueur de 5 mm, il devient très mince.

L'état de conservation de la face dorsale de la tête et des bras n'est pas

parfait. La face extérieure des bras dorsaux, dorso-latéraux et ventraux est légèrement concave, peut-être suite à la fixation.

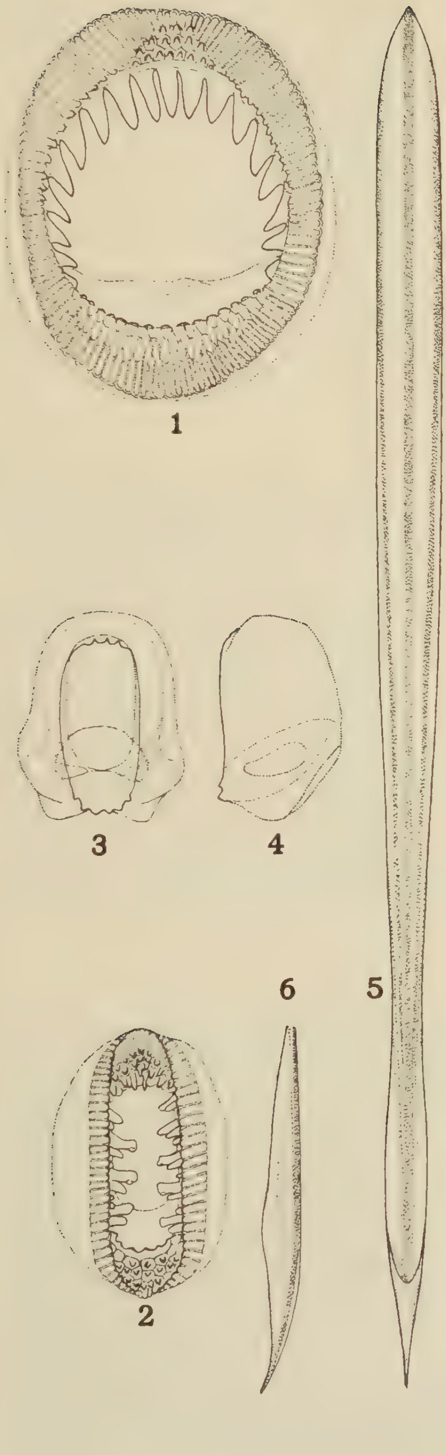
Les bras sont sub-égaux, les latéraux légèrement plus longs, atteignant 36 % de la longueur dorsale du manteau. L'état de conservation ne permet pas de décider si les bras dorsaux possédaient des membranes natatoires. Les bras dorso-latéraux sont pourvus du côté ventral d'une mince membrane natatoire, large de 4 mm sur une grande partie de sa longueur. Les bras ventro-latéraux ont une grande membrane natatoire, très large dans la partie basale du bras, où elle forme un triangle. Les bras ventraux sont aplatis extérieurement et pourvus de larges membranes natatoires, dont la dorsale entoure la base du tentacule.

Les membranes protectrices sont nettement développées dans tous les bras et renforcées de fortes brides musculaires, alternant avec les ventouses (pl. III fig. 4).

Tous les bras portent 55—60 paires de ventouses dont les plus grandes ont un diamètre de 2.5 mm (0.9 % de la longueur dorsale du manteau). Chaque ventouse se compose d'une cupule oblique sur un pédoncule dont la longueur ne dépasse pas la hauteur de la cupule. Le cercle corné (fig. 1) est armé dans les trois quarts distaux d'environ 18 grandes dents, pointues, contigues. Le quart proximal du cercle est irrégulièrement denticulé. Le cercle corné est entouré d'un large anneau d'attache, composé de bâtonnets radiaires, bruns, prolongés en papilles autour du cercle corné. On distingue également un groupe de papilles du côté distal.

Les tentacules sont comprimés latéralement et pourvus d'une faible crête membraneuse sur leur face extérieure, constituant une membrane natatoire peu marquée dans le tiers distal de la massue tentaculaire (pl. I fig. 3). Les membranes protectrices sont peu développées, la ventrale un peu plus large que la dorsale, avec des brides musculaires à peine indiquées. La massue tentaculaire, dont la longueur atteint 24 % de la longueur du manteau, est très étroite. A sa base se trouvent deux paires de petites ventouses sessiles, arrondies, d'un diamètre de 0.3—0.5 mm, puis cinq ventouses longuement pédonculées et latéralement comprimées comme les autres ventouses qui recouvrent la massue, disposées en près de cinquante rangées transversales de quatre. Les plus grandes ventouses tentaculaires mesurent 1.0 × 1.5 mm.

A la base externe de chaque ventouse latérale se trouve une petite lamelle membraneuse, libre, généralement un peu plus courte que le pédoncule de la ventouse. Les ventouses tentaculaires dont le pédoncule mince est généralement plus long que la cupule, ont un cercle corné assez compliqué (figs. 3—4). Le bord supérieur de l'ouverture du cercle porte quatre petites dents, arrondies; le bord inférieur six petites dents un peu pointues. Les bords latéraux de l'ouverture sont lisses. L'ouverture ovalaire est entouré d'un large anneau d'attache (fig. 2) dont la partie extérieure est composée de petits bâtonnets, tandis que la partie intérieure porte des papilles, dont les latérales sont tellement grandes qu'elles obstruent une partie de l'ouverture.



Pholidoteuthis boschmai gen. et sp. nov.

Fig. 1. Ventouse d'un bras dorso-latéral. $\times$ 22.
Fig. 2. Ventouse tentaculaire. $\times$ 22.
Fig. 3. Cercle corné d'une ventouse tentaculaire, vue frontale. $\times$ 22.
Fig. 4. Idem, vue latérale. $\times$ 22.
Fig. 5. Gladius, face ventrale. $\times \frac{2}{3}$.
Fig. 6. Idem, vue latérale. $\times \frac{2}{3}$.

La membrane buccale est bien développée, mais très mince. Sa face interne est fortement plissée, avec sept pointes, attachées aux bras par sept attaches, dont la dorsale est bifurquée et attachée aux bras dorsaux. Les attaches dorso-latérales sont fixées à la face dorsale des bras dorso-latéraux; les attaches ventro-latérales, à la face ventrale des bras ventro-latéraux et les attaches ventrales à la face ventrale des bras ventraux.

L'état de conservation ne permet pas la description de l'anatomie interne.

Le gladius (figs. 5—6) est svelte, sa largeur n'atteignant pas 5 % de sa longueur. Il est acuminé aux deux extrémités et muni d'un renforcement médian, arrondi et de deux épaississements latéraux qui rejoignent le médian vers l'extrémité distale. Dans le tiers postérieur il y a d'autres épaississements moins marqués. La moitié antérieure est la plus large, aux bords d'abord presque parallèles, puis convergeant vers la partie la plus étroite qui se situe près des deux tiers de la longueur totale. Dans le tiers postérieur, les parties membraneuses de la plume s'élargissent un peu, puis se rejoignent en formant un cornet. Dorsalement, l'épaississement médian porte une faible crête longitudinale le long de la partie postérieure du cornet (fig. 6).

Diagnose:

L'espèce se caractérise par: 1 — sa forme générale; 2 — la peau de son sac palléal pourvu de papilles caractéristiques; 3 — la forme et l'armement des ventouses des bras sessiles; 4 — la disposition et la structure très caractéristique des ventouses tentaculaires; 5 — la disposition des attaches de la membrane buccale; 6 — l'appareil de connection palléo-siphonal simple; 7 — son gladius.

Rapports et différences:

En 1839 (—1848) A. D'ORBIGNY décrivait sous le nom d'*Onychoteuthis Dussumieri* (p. 335, pl. 13) un Céphalopode, provenant de "200 lieues au nord de l'île Maurice". L'animal se caractérise surtout par la peau du manteau "finement chagrinée", par les "bras sessiles, pourvus d'un sillon creux sur toute leur longueur, ce qui les rend canaliculés en dehors" et par les ventouses des bras sessiles qui rappellent celles d'*Onychoteuthis banksii* par leur forme générale et par l'absence de dents au cercle corné. Quant aux tentacules, l'auteur fait remarquer qu'ils se distinguent par l'absence d'une massue tentaculaire et qu'ils paraissent avoir été couverts d'au moins trente crochets, disposés en deux lignes alternes. A. D'ORBIGNY est d'avis que l'espèce est très voisine des Ommastrephidae.

En 1900, G. PFEFFER (p. 161) a créé le genre *Tetronychoteuthis* pour l'espèce de D'ORBIGNY. Sous le nom de *Tetronychoteuthis dussumieri*, l'auteur signale un spécimen, trouvé dans l'estomac d'un dauphin de provenance inconnue. Il fait remarquer que tout comme l'original, ce spécimen avait perdu tous les crochets, mais que les parties molles démontraient qu'il y avait eu des crochets et non pas des ventouses et que ceux-ci étaient disposés en quatre rangées d'environ cinquante.

PLANCHE I

Pholidoteuthis boschmai gen. et sp. nov.

Fig. 1. Face dorsale. × $\frac{1}{3}$.

Fig. 2. Face ventrale. × $\frac{1}{3}$.

Fig. 3. Massue tentaculaire. × $1\frac{1}{2}$.

PLANCHE II

Pholidoteuthis boschmai gen. et sp. nov.

Fig. 1. Papilles du sac palléal, face dorsale près du bord palléal. × 6.

Fig. 2. Idem, face dorsale, au milieu. × 6.

Fig. 3. Idem, face dorsale, près de l'insertion des nageoires. × 6.

Fig. 4. Idem, face ventrale, au bord palléal. × 6.

Fig. 5. Idem, face ventrale, au milieu. × 6.

Fig. 6. Idem, face ventrale, à la limite postérieure. × 6.

PLANCHE III

Pholidoteuthis boschmai gen. et sp. nov.

Fig. 1. Tête, vue latérale. × $1\frac{1}{2}$.

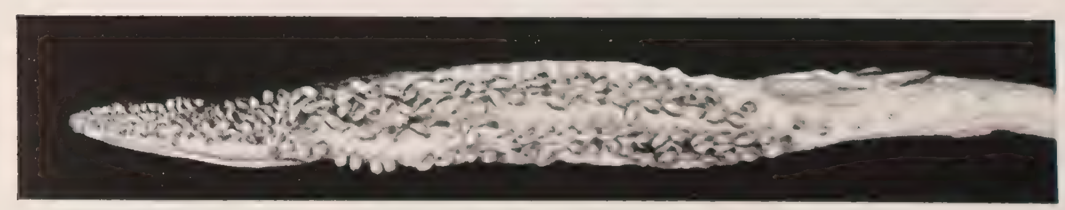
Fig. 2. Tête, face dorsale. × $1\frac{1}{2}$.

Fig. 3. Massue tentaculaire. × 6.

Fig. 4. Bras dorso-latéral droit, près de sa base. × 6.

1

2



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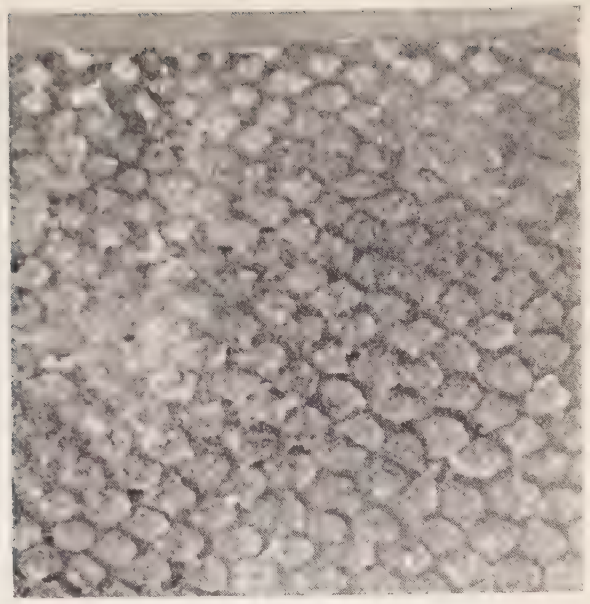
PL. II

1



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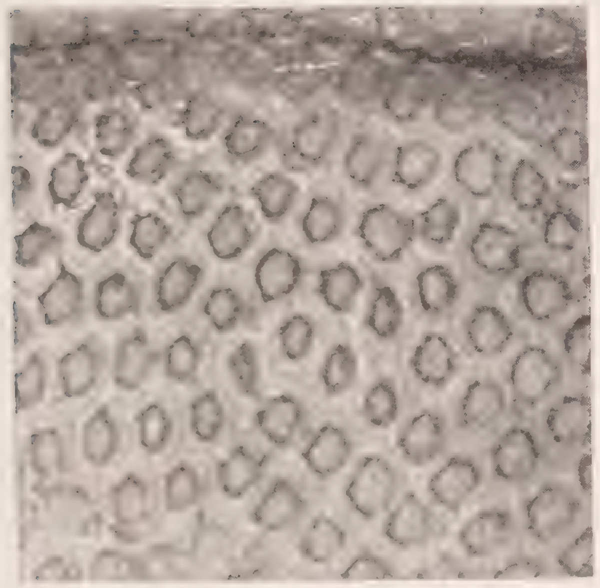

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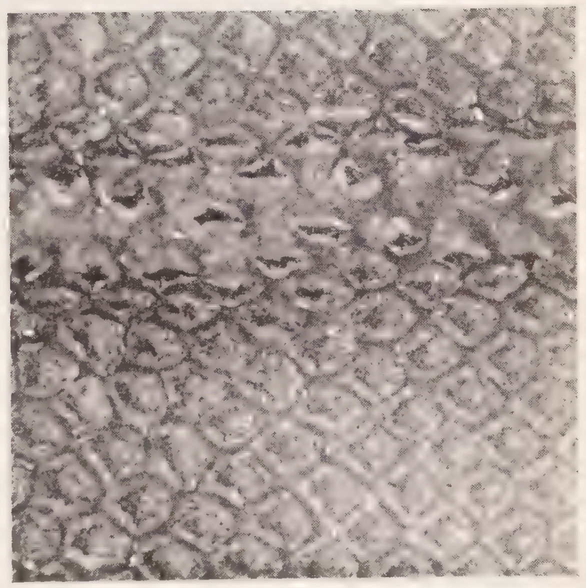

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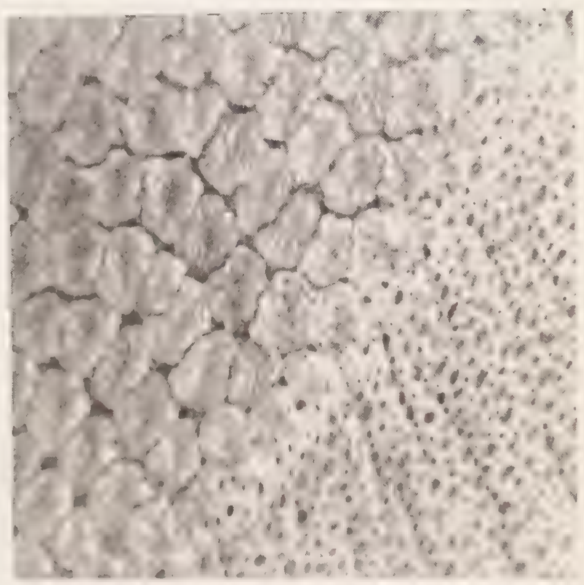

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PL. III

1

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2

D'après G. PFEFFER (p. 160), la structure "chagrinée" remarquable de la peau, que l'on trouve également chez *Onychoteuthis ingens* Smith et chez *Lepidoteuthis grimaldii* Joubin, serait due entièrement ou partiellement à l'action des sucs digestifs de l'estomac des cétacés dans lesquels les spécimens récoltés avaient séjournés.

En 1912, G. PFEFFER (p. 98, pl. 13; pl. 14 fig. 10—14) a décrit son matériel en détail et a donné une description du genre *Tetronychoteuthis*. A son avis, les tentacules avaient été probablement couverts de ventouses et non pas de crochets, comme il le signalait dans sa publication antérieure. Une seconde espèce, *Tetronychoteuthis Massyae* Pfeffer, dont l'auteur n'avait qu'un jeune spécimen, pourrait appartenir, à son avis, à la même espèce. G. PFEFFER a laissé le genre dans la famille des Onychoteuthidae, malgré l'absence supposée de crochets et d'autres caractères bien différents.

Selon A. NAEF (1923, p. 301), l'absence de crochets dans ce genre serait un caractère larvaire et il considère le genre comme un cas de néoténie.

A mon avis l'explication se trouve facilement dans l'erreur que G. PFEFFER a commis de considérer son spécimen comme appartenant à l'*Onychoteuthis dussumieri* d'Orbigny. La remarque d'A. D'ORBIGNY que le tentacule paraissait avoir été couvert d'au moins trente crochets sur deux lignes alternes, fait supposer que l'*Onychoteuthis dussumieri* se rapproche peut-être d'*Onychoteuthis raptor* Owen, 1881, dont les bras dorsaux montrent la même rainure (voir R. OWEN, pl. 29 fig. 1). G. PFEFFER (1912, p. 102) explique cette ressemblance en supposant que le dessinateur d'OWEN se serait inspiré de la figure d'*Onychoteuthis dussumieri* pour dessiner l'*Onychoteuthis raptor*!

Comme, d'autre part, une peau "chagrinée" se trouve chez *Moroteuthis ingens* (Smith), il n'y a rien qui s'oppose à considérer l'espèce de D'ORBIGNY comme un véritable Onychoteuthide dont les tentacules auraient porté des crochets. Sa plume ressemble également à celle des Onychoteuthidae.

Par contre, j'ai l'impression que le spécimen de G. PFEFFER, qui ressemble beaucoup à l'exemplaire du Snellius, n'a rien à voir avec l'*Onychoteuthis dussumieri*. S'il en était ainsi, nous nous trouvons devant un cas de nomenclature très compliqué. En effet, G. PFEFFER a décrit le genre *Tetronychoteuthis*, basé sur une seule espèce, *T. dussumieri* (d'Orbigny) [1]). Seulement, pour la description du genre et de l'espèce, l'auteur ne s'est pas basé sur l'original, mais sur un autre spécimen qui, à mon avis, n'y est pas identique. Il est possible que ce spécimen de G. PFEFFER appartienne à la même espèce que l'exemplaire du Snellius, mais le pauvre état du premier spécimen ne permet pas une décision.

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[1]) Monsieur G. CHERBONNIER, assistant au Muséum National d'Histoire Naturelle de Paris, a bien voulu m'informer, qu'il n'a pas su retrouver le type d'*Onychoteuthis dussumieri* dans les collections du Muséum.

Je me vois donc obligé de créer une nouvelle espèce et un nouveau genre pour le spécimen du Snellius, que j'appelle *Pholidoteuthis boschmai* en honneur de son récolteur.

Par les attaches ventrales de la membrane buccale, se fixant du côté ventral des bras ventraux et par l'appareil de connection palléo-siphonal simple, l'espèce ressemble aux Onychoteuthidae. Cependant, son gladius et les ventouses de ses bras sessiles sont bien différents et ressemblent plutôt à ceux des Ommastrephidae. La massue tentaculaire, très caractéristique, diffère complètement de celle de ces deux familles.

Le genre *Lepidoteuthis* Joubin, 1895, dont la tête et les bras sont inconnus, diffère de *Pholidoteuthis* par les papilles beaucoup plus grandes, se recouvrant comme des écailles et par les nageoires d'une forme différente.

Actuellement on ne connait pas de famille de Céphalopodes dans laquelle on saurait placer le genre *Pholidoteuthis*. Je considère donc ce genre comme le type d'une nouvelle famille de Céphalopodes décapodes oegopsides: les **Pholidoteuthidae**, caractérisé provisoirement par son seul représentant: *Pholidoteuthis boschmai* gen. et sp. nov.

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BOTANY

ACCUMULATION OF CYTOPLASM IN ALLIUM CELLS AS A CONSEQUENCE OF EXPOSURE TO ORGANIC SUBSTANCES. I

1) *Description of the phenomenon of accumulation in the cell corners,*
2) *Method to compare the action of organic substances quantitatively,*
3) *Results with iso-alcohols and urethanes.*

BY

W. A. LOEVEN

(Communicated by Prof. H. G. BUNGENBERG DE JONG at the meeting of Oct. 28, 1950)

1. *Introduction*

In all the theories on the permeability, which have been built up in the course of the years, the explanation has been searched in a regulating protoplasmic membrane, a part of the protoplasm, which only owes its particular properties to its position (border cell/medium or protoplasm/vacuole).

The structure of this membrane, which contains lipoids, has been conceived in many of the theories as built up from mono-, di- or pauci-molecular layers of lipoid molecules. Besides these lipoids proteins and ions can be considered as a main building material. BUNGENBERG DE JONG has developed the complex theory of permeability, in which the protoplasmic membrane is considered to be a tri-complex system between protein, lipoid (in many cases phosphatides) and a cation. As the same bio-colloids are found in the protoplasm itself, it is perhaps possible, that the protoplasm may be considered as such a system too.

For the greater part this theory has been developed by studying model experiments with coacervates of lecithin and in the last years with coacervates and elastic systems of soap solutions (mainly Na-oleate). Investigations about the influence of various substances on the structure of these lecithin and soap systems may be used to realize some details of the structure of the protoplasm.

However, it is very desirable to make experiments about the influence of various substances on the protoplasm itself besides experiments with models such as these oleate coacervates and elastic systems, because in this way it is possible to test the usefulness of the hypotheses obtained from these model experiments for biological material.

The influence of salts on the structure of the protoplasmic membrane (for instance through the exchangeability of the cations) seen from the point of view of the complex theory, is already investigated in some details by DE HAAN (permeability of Allium cells) and BOOIJ (fermentation of sugar with yeast and germination of pollen of sweet pea). Besides these

investigations experiments with organic electrolytes and non-electrolytes would be valuable.

It is of great importance to have at our disposal a method, which enables us to investigate the influence of organic substances on the structure of the protoplasm as a whole systematically.

During his experiments about the influence of organic substances on coacervates BUNGENBERG DE JONG had also made some preliminary experiments on the influence of some of these substances on the living cell itself.

A strip of epidermis of the bulb scales of Allium cepa was laid in for instance n-heptane and then already after a few minutes a change in the cell was observed under the microscope.

Especially in the corners of the cells a "swelling" of the protoplasm appeared, in many cases stretching out itself to an image resembling the phenomenon being known in the literature as "vacuole contraction". Also with alcohols it appears to be possible, for instance with a 0,5 M solution of n-butylalcohol. After drawing my attention to this phenomenon BUNGENBERG DE JONG proposed to investigate if perhaps it was possible to make an useful method out of it.

In this communication first the morphological changes in the protoplasm through the action of organic substances will be described (section 2), after which the quantitative method developed by means of this phenomenon and some results already obtained will be discussed (sections 4 and 5) and some short conclusions resulting from these experiments will be given (section 6).

2. *Description of morphological changes of the protoplasm resulting from the action of organic substances on epidermic cells of Allium-bulbs*

As already described in the Introduction BUNGENBERG DE JONG found that many organic substances have a "swelling" influence on the protoplasm. At closer investigation, however, it appears, that this "swelling" must not be understood in the common sense of the word alone. We will use it as a short denomination for the phenomenon.

If we bring cells in an alcohol solution of sufficient concentration and study them under the microscope, we see, that the protoplasmic streaming shifts towards the corners and edges of the cell and soon (usually within a quarter of an hour) there is no more question of a real protoplasmic streaming. Usually it can then be seen very weakly in the "swollen" protoplasmic caps and along the edges of the cell.

It seems therefore, as if at the same time there is a flow of protoplasm towards the corners (in section 6 we shall further return to it). This seems to be no phenomenon of dying of the protoplasm, caused by the action of these organic substances, because the cells may still be plasmolysed very well with e.g. 1 N KNO_3 and de-plasmolysed with water. Also the cell can still be properly stained with vital dyes such as neutral red.

Sometimes during plasmolysis the protoplasmic caps flow off along the vacuole, in other cases they stay visible as falcate layers against the vacuole. In the last case they return partly to their old corners at deplasmolysis.

With many substances it is possible to obtain this change of the protoplasm. We have tried, for instance, aromatic and aliphatic hydrocarbons, alcohols and urethanes. We observe a similar effect in solutions of some so-called vital dyes. With erythrosin and chrysoidine the "swelling" can be observed very well as the protoplasm will be coloured red, respectively yellow.

If the concentrations of the added substances are too high, secondary effects may also be obtained sometimes: appearance of a strong granulation and or vacuolisation of the protoplasm and decrease of the protoplasmic streaming. Often the nucleus is also strongly swollen and the two nucleoli are clearly visible.

That these substances influence the structure of the protoplasm appears from the fact that at plasmolysis the separation of the protoplast from the cell wall takes place much better in many cases (especially at larger concentration of the substances).

Phenomena of cramp-plasmolysis often fail to come to the fore and the plasmolysis will be convex already from the beginning. Threads of Hecht are no longer present in most cases.

Finally we will mention a particular case.

Cells in pure cyclohexane show excellent protoplasmic caps very quickly (often within two minutes). If plasmolysed in 1 N KNO_3 the protoplast shows one or more hyaline caps at the border, which appear to be the primary "swollen" plasmic corners. During plasmolysis we sometimes see two swollen corners turn around together, so that two hyaline caps come up or against together.

In the photographs *A—G* some of these morphological changes are shown.

3. *Method to express the influence of organic substances quantitatively*

The question, which arises at once, is: Is it possible to express the influence of organic substances on the phenomenon, described in the section above, in a quantitative sense? In this case the possibility may be created to coordinate the influence of various substances on the structure of the protoplasm, so that the results of the experiments with a living biological object can be compared with those obtained from model experiments. At the same time the usefulness of the hypotheses about the structure of the protoplasm can be tested.

The experiments showed, that the appearance of the phenomenon depended on the concentration of the organic substance in the solution, in which the strip of epidermic tissue of a bulb scale of Allium was placed. The phenomenon appears in a relatively small area of concentrations,

after which it is maintained at an almost maximal value over a large area. If we make the concentration still higher, secondary effects appear (as described in section 2) and the real "swelling" effect is lost.

In all experiments we used the cells of the inner-epidermis of the second, third or fourth bulb scale. This epidermis can be stripped off very easily in contrast with the outer-epidermis, so that damages may be limited to a minimum. The further advantage is, that this epidermis is only one cell layer thick, thus the permeation of the used substances via other cell layers is avoided.

The strips of epidermis are torn from the middle of the bulb scale in the manner shown in fig. 1. In this way we use identic cells of practically the same age. Differences of age between top cells and basic cells of the bulb scale might cause troubles as will be proved in the next sections.

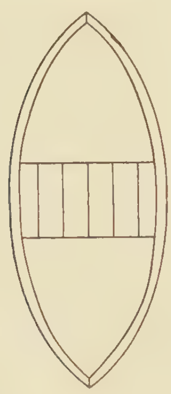


Fig. 1. Sector of an Allium bulb scale to show the making of incisions in the epidermis.

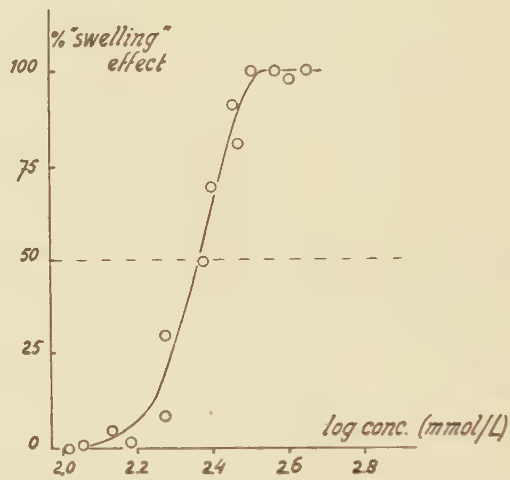


Fig. 2. Curve, showing the "swelling" of the protoplasm as a function of the concentration of an added substance.

Strips of epidermis of the second, third and fourth bulb scale of a single onion are brought in a series small bottles (with grounded glass stoppers when using volatile substances) with increasing concentrations of the substance to be investigated.

We took care that the strips of epidermis were completely immersed in the solution. After lying half an hour in the solution the strip was fished out and laid under the microscope. Then we determined what percentage of ± 100 cells showed the "swelling" effect (counting only those cells which were otherwise completely intact).

We called the action positive, when the "swelling" effect was distinctly visible in at least two corners of the cell.

After having investigated a whole series of concentrations of a substance in this way, we may plot the logarithms of the concentrations on

the abscissa and the percentages of the "swelling" effect on the ordinate, and we get S-curves as shown in fig. 2.

In most cases the spreading of the experimentally determined percentages is small (as shown in the graphs of the iso-alcohols and urethanes in the next sections).

The influence of the substance investigated in this way can be expressed quantitatively by reading off the diagram at what concentration of the substance the "swelling" effect is 50 % (in other words when 50 % of the cells show swollen protoplasmic caps in at least two corners).

These values may be compared to the activities which the same substances exert in model experiments. Perhaps it will be possible in future to build up a picture of a possible structure of the protoplasm from the data obtained in this way.

4. *Influence of iso-alcohols on the protoplasm of epidermic cells*

To compare the influence of three iso-alcohols (iso-propylalcohol, iso-butylalcohol and iso-amylalcohol) we used the method, described in section 3. In table I and fig. 3 we have given the results of this investigation (using the cells of the inner-epidermis of the second, third and fourth bulb scale of only one single bulb of the so-called "winter proof" Dutch Onion).

In the first place we see that the influence of the three alcohols is very different and that the concentrations needed increase in the order

iso-amylalcohol < iso-butylalcohol < iso-propylalcohol.

Further we observe that the succeeding bulb scales give small differences mutually. The concentrations needed increase for each of the three alcohols in the order

second < third < fourth bulb scale.

This order denotes the "age" of the bulb scales and indicates that the

TABLE I: Dutch Onion

iso-propylalcohol				iso-butylalcohol				iso-amylalcohol			
log conc. (mmol/l)	% effect on bulb scale no.			log conc. (mmol/l)	% effect on bulb scale no.			log conc. (mmol/l)	% effect on bulb scale no.		
	2	3	4		2	3	4		2	3	4
2,602	—	—	—	2,000	—	--	—	1,602	—	—	—
2,653	13	—	—	2,097	4	2	—	1,699	4	—	—
2,699	38	8	—	2,176	17	8	3	1,778	12	5	2
2,740	55	10	2	2,243	31	19	9	1,845	14	12	6
2,778	49	14	9	2,301	62	39	21	1,903	49	38	25
2,813	84	72	41	2,352	100	57	34	1,954	40	31	18
2,845	69	54	32	2,398	96	84	68	2,000	84	67	51
2,875	96	80	48	2,439	100	91	79	2,041	97	78	66
2,903	100	91	65	2,477	100	99	94	2,079	100	92	89
2,929	100	98	76	2,512	100	100	100	2,114	100	100	90
2,954	100	100	100					2,146	100	100	100
2,978	100	100	100								

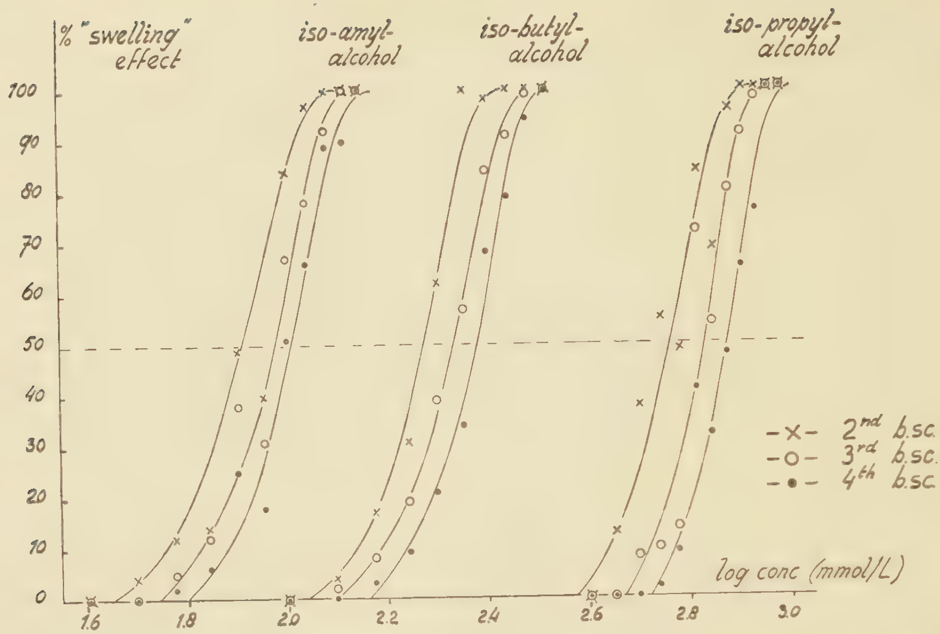


Fig. 3. Influence of iso-alcohols on the epidermic cells of the Allium race "Dutch Onion".

structure of the protoplasm is modified when the cells grow older. Then the influence of an alcohol added shows itself at a lower concentration.

That there are indeed differences in the influence of substances on the succeeding bulb scales appears from the fact that the curves drawn are wholly separated one from each other.

If we compare the influence of these alcohols closer, it is seen that the concentrations needed (compared for each single bulb scale) give approximately a relation, which can be recognized as the Traube's rule: in a homologous series the concentration needed to give a certain effect is three times smaller at lengthening of the carbon chain with one C atom.

In survey I we have given the logarithms of the concentration of the alcohol (these concentrations as mmol l.) needed to give 50 % effect for each alcohol and for each bulb scale and their differences for each bulb scale.

In fig. 5 this influence of the alcohols at 50 % effect is once again plotted for each bulb scale by setting the number of C atoms of the alcohol on the abscissa and the logarithm of the concentration of the alcohol at 50 % effect given in the survey I on the ordinate. Here Traube's rule finds expression in a straight line for each bulb scale.

If we repeat these experiments with an other onion of the same race, the results are almost similar to the ones given above. There are but relatively small differences in the concentration of the alcohols needed. Their order of magnitude is almost similar to those of the differences in concentration between the bulb scales of one onion.

It was considered worth while to perform the same experiments with yet another race of onion (the "Red Onion", a race of which the cells of the outer-epidermis of the bulb scales contain anthocyane in the vacuoles). The results obtained with an onion of this race are given in table II and fig. 4.

TABLE II: Red Onion

iso-propylalcohol				iso-butylalcohol				iso-amylalcohol			
log conc. (mmol/l)	% effect on bulb scale no.			log conc. (mmol/l)	% effect on bulb scale no.			log conc. (mmol/l)	% effect on bulb scale no.		
	2	3	4		2	3	4		2	3	4
2,699	7	6	—	2,097	2	—	—	1,699	3	—	—
2,740	12	7	1	2,176	5	2	—	1,778	7	4	—
2,778	21	18	6	2,243	37	15	4	1,845	9	12	5
2,813	59	52	39	2,301	62	48	27	1,903	35	11	4
2,845	70	61	50	2,352	88	76	55	1,954	46	25	14
2,875	97	84	72	2,398	97	89	79	2,000	58	52	25
2,903	100	99	90	2,439	100	95	85	2,041	74	64	48
2,929	98	100	93	2,447	97	100	98	2,079	80	75	59
2,954	97	98	100	2,512	100	100	100	2,114	91	88	78
2,978	100	100	100					2,146	100	98	95
								2,176	100	100	100

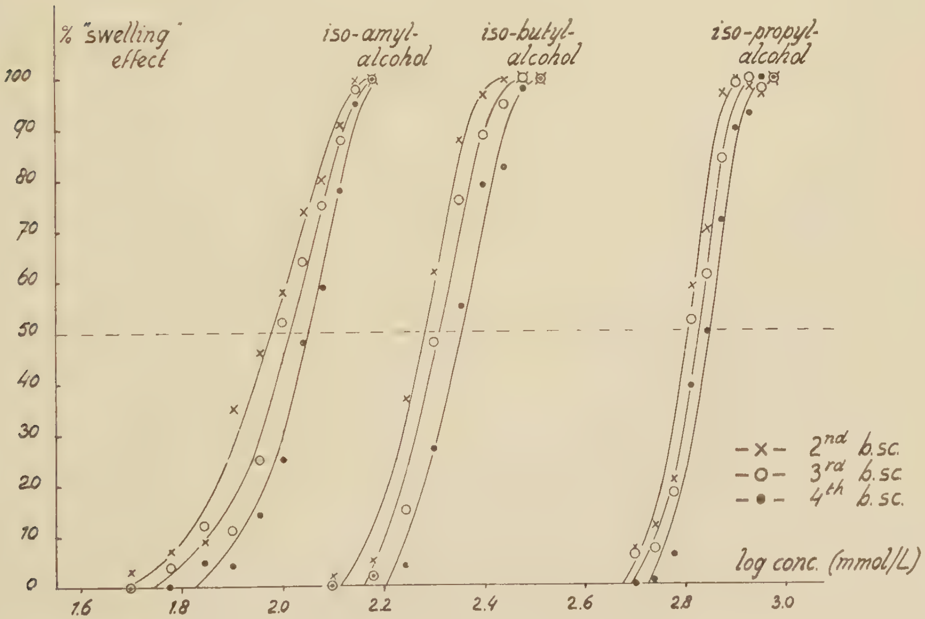


Fig. 4. Influence of iso-alcohols on the epidermic cells of the Allium race "Red Onion".

We see that the differences between these two races of onions are very small and agree in the order of magnitude with those between two individuals of a single race.

SURVEY I: Dutch Onion

bulb scale	iso-propylalcohol		iso-butylalcohol		iso-amylalcohol
second	2,757	0,482	2,275	0,368	1,907
third	2,827	0,497	2,330	0,360	1,970
fourth	2,880	0,500	2,380	0,373	2,007

SURVEY II: Red Onion

bulb scale	iso-propylalcohol		iso-butylalcohol		iso-amylalcohol
second	2,805	0,528	2,277	0,302	1,975
third	2,825	0,515	2,310	0,300	2,010
fourth	2,847	0,497	2,350	0,300	2,050

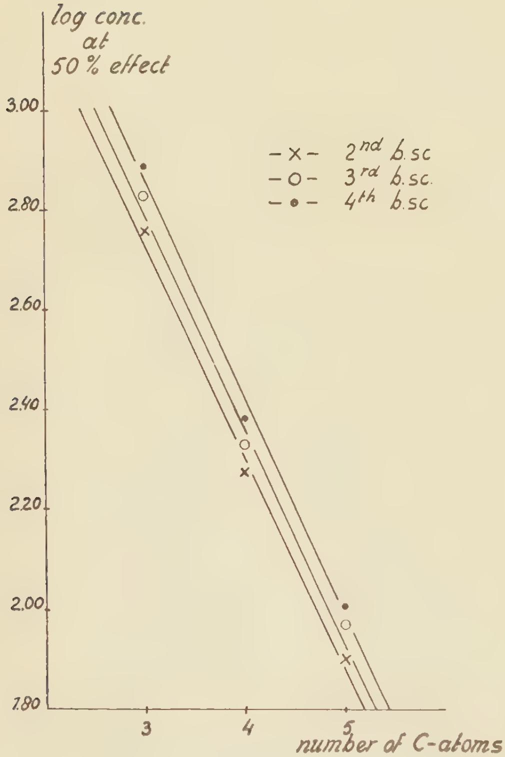


Fig. 5. Logarithms of the concentration of iso-alcohols needed for a 50 % "swelling" effect as a function of the number of C-atoms at the race "Dutch Onion".

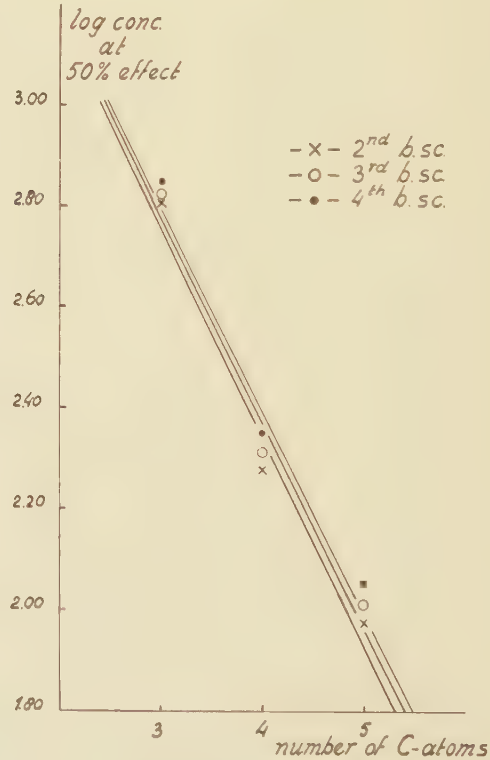


Fig. 6. Logarithms of the concentration of iso-alcohols needed for a 50 % "swelling" effect as a function of the number of C-atoms at the race "Red Onion".

The concentrations of the alcohol needed to get a 50 % effect for each bulb scale are given in survey II, from which it appears that the differences with those in survey I are very small too.

Traube's rule is also found here and the differences of slope of the straight line are very small (compare figs 5 and 6).

5. *Influence of some urethanes*

The influence of the urethanes were examined on cells of the Red Onion. We used: ethylurethane, propylurethane, butylurethane (in these three esters the alcohol group is varied) and the methyl derivative of ethylurethane (in which the free NH_2-group is replaced by a $HN\text{-}CH_3$ group).

The results obtained are given in fig. 7. Here it is seen that lengthening

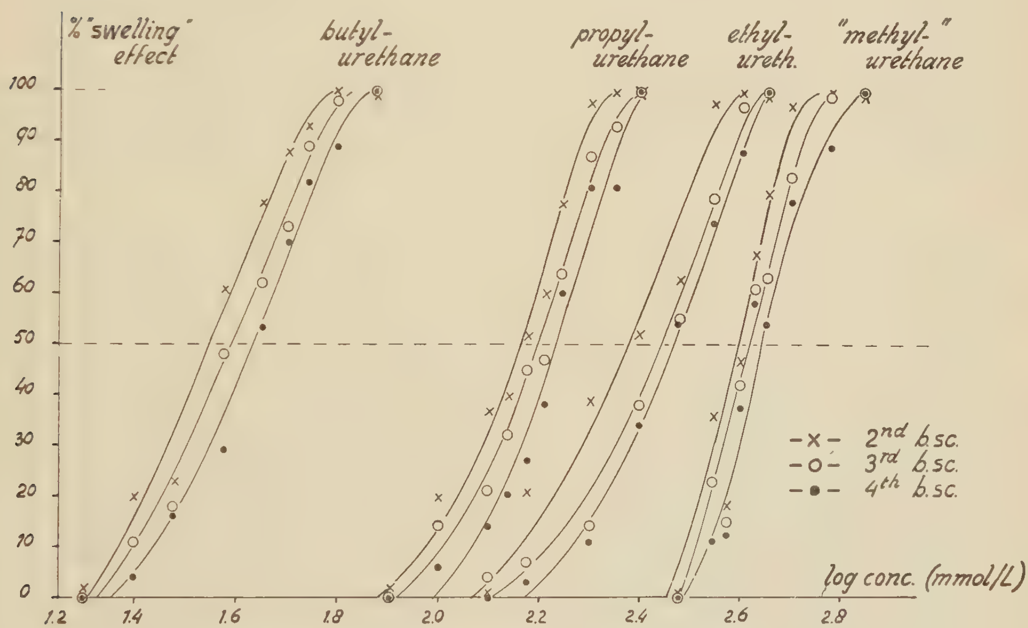


Fig. 7. Influence of urethanes on the epidermic cells of the Allium race "Red Onion".

SURVEY III

bulb scale	butylurethane		propylurethane		ethylurethane		methylder. ethylurethane
second	1,545	0,617	2,162	0,218	2,380	0,212	2,592
third	1,577	0,618	2,195	0,245	2,440	0,177	2,617
fourth	1,625	0,610	2,235	0,240	2,475	0,165	2,640

of the carbon chain of the alcohol group gives an effect at lower concentration, but replacing of the NH_2-group in ethylurethane by the $HN\text{-}CH_3$-group brings with it that the concentration lies higher for each bulb scale. In these experiments we also observe that the concentration of each urethane needed for the appearance of the effect increases when going in the direction: second, third and fourth bulb scale.

In survey III we have given the logarithms of the concentration needed for a 50 % effect for each urethane and for each bulb scale.

6. *Conclusions*

Evidently, it is not possible to explain the structure of the protoplasm of the Allium cells from a few data. However, some conclusions may be drawn from these very first experiments. First of all we will consider the slight difference between the consecutive bulb scales of one onion and between the corresponding bulb scales of onions of one or two races.

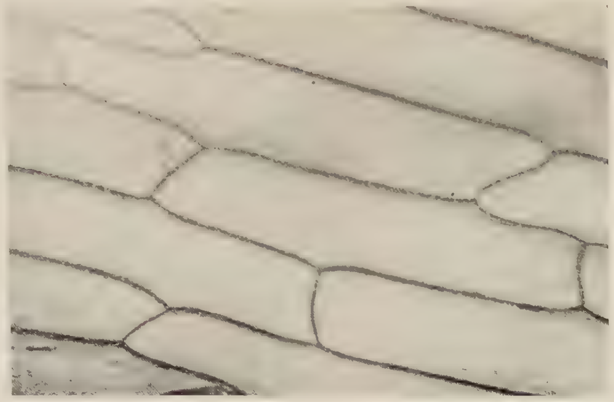
The explanation of it may be, that at maturing of the cells (cells of the second bulb scale are older than those of the third and fourth ones, provided that they are taken from equal heights of the bulb scales) the structure of the protoplasm undergoes a change. Then the influence of these compounds will assert itself at another concentration.

Also in the case of experiments on the influence of salts on the permeability of these Allium cells (not yet published) we found slightly different qualities of the cells depending on the position from which the cells are taken. Degree of plasmolysis and time of de-plasmolysis of the cells varied when using basic and top cells of one bulb scale and when using cells of different bulb scales.

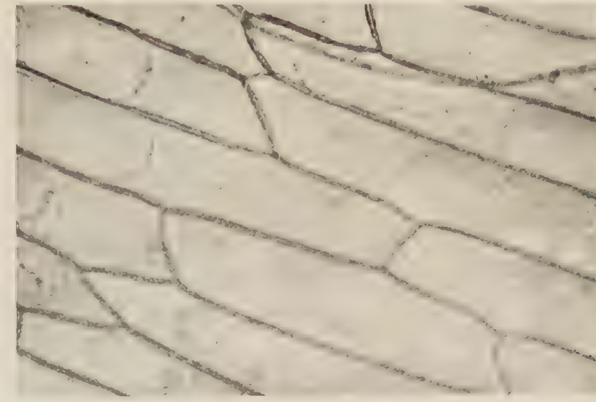
If we compare the influence of the investigated terms of the homologous series of the iso-alcohols and urethanes, it appears, that in both cases their action increases very considerably when the carbon chain is lengthened by one carbon atom. The differences in the logarithms of the concentration needed to give 50 % "swelling" effect correspond (when going from iso-propyl- to iso-amylalcohol and from ethyl- to butylurethane) to ratios approximately corresponding to those of Traube's rule.

As already mentioned in section 2 the "swelling" of the protoplasm caused by the action of organic substances is not only a swelling without more. It is evident that the phenomenon is mainly caused by a displacement of protoplasm from the longitudinal walls towards the corners of the cell. This may be explained by a decrease of protoplasmic viscosity and a loss of its elastic properties, resulting in its redistribution in the cell by the surface tension of the vacuolar membrane.

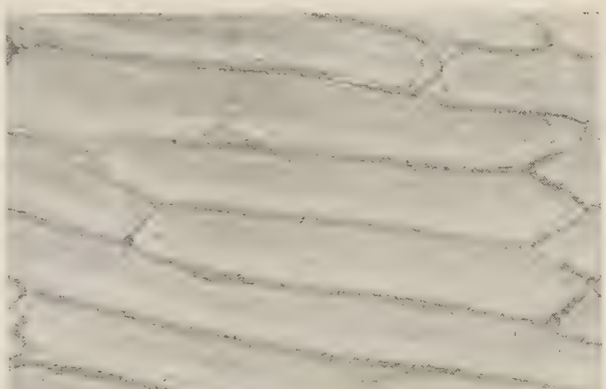
A more detailed discussion, however, will be better postponed to a later communication of this series, after having investigated the actions of more organic substances.



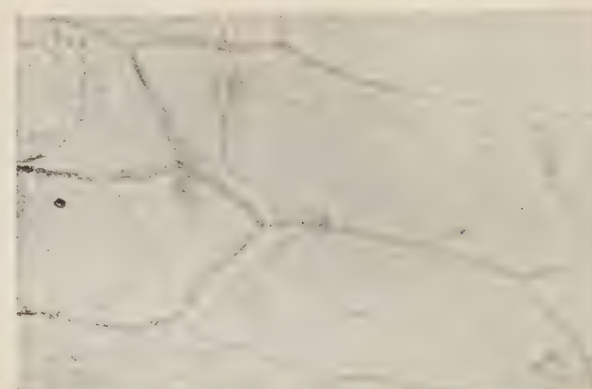
A. Cells of the inner-epidermis of Allium cepa coloured with toluidinblau, which gives no "swelling" effect.



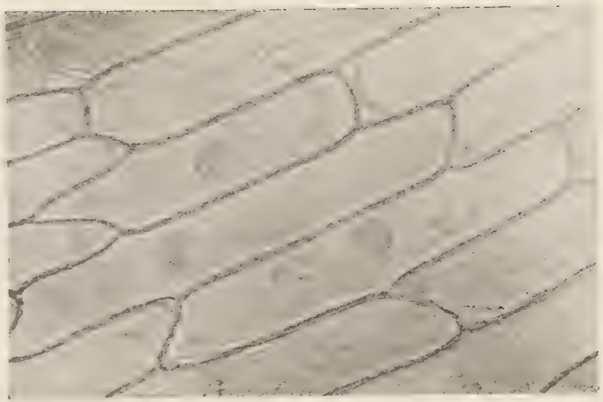
B. Cells of the inner-epidermis plasmolysed with 1 N KNO_3. Cells also coloured with toluidinblau, which gives a normal plasmolysis.



C. "Swelling" effect with cyclohexane. Swollen caps often with hyaline structure.



D. Cells, plasmolysed with 1 N KNO_3 after the action of cyclohexane. Swollen caps remain and do not flow along the protoplast.



E. "Swelling" effect with erythrosin.

F. Begin of plasmolysis of the cells after the action of erythrosin.

G. Plasmolysis of the cells after the action of erythrosin.

I am greatly indebted to Professor Dr T. H. v. d. Honert, who has permitted me to carry out these experiments under the direction of Professor Dr H. G. Bungenberg de Jong, who has given me the help of his expert advice and valuable criticisms, for which I thank him very much.

7. *Summary*

1. A description is given of morphological changes in the cell under the influence of organic substances. Epidermic cells of the bulb scales of Allium cepa have been used.

2. The phenomenon of "swelling" of the protoplasm may be composed of two effects:

a swelling of the protoplasm in the proper sense of the word and a rounding off of the protoplasm (perhaps resulting from a decrease of the viscosity and/or the elasticity).

3. A method is developed to measure quantitatively the influence of organic substances on cells by means of a morphological change of the protoplasm.

4. Results of preliminary experiments with iso-alcohols and urethanes are given.

5. Slight differences have been found between the influences of an organic substance on different bulb scales of one race of Allium and on corresponding bulb scales of two different races.

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INFLUENCE OF DAYLIGHT ON THE FRUITING OF TWO ORANGE-YELLOW PIGMENTED MYXOMYCETE PLASMODIA

BY

JOHANNA C. SOBELS AND HENDERICA F. J. VAN DER BRUGGE

(Communicated by Prof. V. J. KONINGSBERGER at the meeting of Oct. 28, 1950)

Introduction

In 1938 GRAY established that the yellow pigmented plasmodia of several myxomycetes: *Physarum polycephalum*, *Physarum tenerum*, *Didymium difforme* and *Fuligo septica*, are not able to fruit in absence of light. He obtained sporangia after treatment with diffuse daylight and with artificial light of different intensity.

This investigation deals again with the influence of diffuse daylight on the fruiting of two orange-yellow plasmodia: *Badhamia utricularis* (strain I) and *Physarum polycephalum*. We made use of the flowerpot-method of Miss SCHURE (1949), already mentioned briefly by SOBELS (1950): "Une culture associée de *Badhamia utricularis* (souche I) cultivée à la lumière du jour dans un pot de fleur, suivant la méthode de M.elle SCHURE, développe son plasmode dans les parties ombrées du récipient. Lorsque le plasmode veut fructifier, il se déplace pour former ses sporanges vers les parties plus éclairées."

SCHURE (1949) describes the following method to obtain sporangia of myxomycetes: "The culturing of the plasmodia to obtain fructifications succeeded best when red earthenware flowerpots were used as a basis for the plasmodium instead of a petri dish with agar. For this purpose flowerpots already used for cultivation of plants in the garden were scrubbed clean with a brush, the hole in the bottem was stopped and the pots were kept in water until the pores were entirely filled. The pots were kept moist by placing them in basins containing tapwater which were covered with a glass plate."

The above mentioned method has been elaborated for fructification of two-membered cultures and pure cultures of plasmodia.

Materials and method

New red earthenware flowerpots, about 8 cm high, were washed during one night with running tapwater. The hole in the bottom was stopped with a cork. The dried pots were wrapped up in paper and sterilized (1 hour at 140° C). In our first experiments we used 500 ml beakers of pyrex glass covered with a petri dish, but stray infections were then very frequent (fig. 1). In later experiments each flowerpot was placed in a high

petri dish fitted with a round piece of cellulose, which stabilized the pot and kept it moist. The dimensions of the petri dish were 8.5 cm high and 14.5 cm diameter (fig. 2).

The flower-pots were provided with half a teaspoon of oatmeal and 10 ml unwashed agar (buffered with phosphates on pH 6), or with a teaspoon of dry sterilized oatmeal. Sixty ml of tapwater or double dist. water or diluted phosphate buffer pH 6, (total phosphate M/75 in 1 liter) was poured into each petri dish. Lime containing tapwater was preferred, because these species of myxomycetes produce lime crystals on sporangia and capillitium.

The dishes, closed and wrapped in paper, tied over the top, were autoclaved (30 min. at 120° C).

This method is advantageous, because humidity in the earthenware pot decreases gradually toward the upper edge. The fruiting plasmodium will develop its sporangia wherever optimum moisture conditions for fruiting on the pot exist. Liquid on the bottom of the petri dish kept the pot moist.

The flowerpots were inoculated with a small part of a healthy plasmodium (about 1 cm^2) deposited near the bottom of the pot. The plasmodium spread quickly and moved toward the oatmeal-agar. The cultures were kept one week in darkness. The plasmodia grew abundantly, partly covering the flowerpot and in some cases spreading on the cellulose. The cultures were then placed in diffuse daylight, direct irradiation by the sun was avoided.

Growth and movement of the plasmodia could be watched through the cover of the petri dish, but soon the top became steamy. To prevent condensation a small thermo-element with a circular opening of 10 cm diameter was placed on each cover. Several elements were linked up and fitted with weak current. In this way each cover was heated and remained clear. The heating was so slight that it had no harmful influence on the growth of the plasmodium, but by using the thermo-elements we had to supply the petri dishes with more water (80 ml).

This method was also used with pure cultures; the flowerpots were provided with 10—15 ml unwashed agar, pH 6, and with a sterilized yeast suspension, deposited near the plasmodia.

For the media used, the way of preparing a sterilized yeast suspension and the growth in stock cultures of myxomycete plasmodia on oatmeal-agar we refer to Cohen (1939) and Sobels (1950).

Experiments and results

Experiment 1, carried out with a wild culture of *B. utricularis* (strain I).

Six flowerpots were provided with oatmeal-agar and placed in double dist. water. They were inoculated on June 23, 1949 with a small piece of plasmodium. The cultures were kept in darkness 12 days; they developed into large plasmodia which spread on the flowerpots.

Then the cultures were brought into diffuse daylight entering from a window. Two of the 4 petri dishes were supplied with 20 ml non-diluted phosphate buffer pH 6. All the plasmodia showed the same reaction, they moved away from the daylight toward the shady side of the flowerpot (both inside and outside).

Fruiting occurred in the cultures buffered on pH 6 after 5 days in daylight, one day later in the dishes with double dist. water. A day before sporangia appeared, the plasmodia changed their reaction toward daylight: they moved to the light exposed sides of the flowerpots, contracted and started fruiting. In most cases the sporangia appeared in a well defined zone just on the border of shade and light in a sickle shaped figure. Sometimes we observed sporangia on the moist cellulose.

At first the sporangia were yellow similar to the orange yellow pigment of the plasmodium. A few hours later the colour darkened into light brown and finally into brown-black. One or two days later mature sporangia were often covered with a white layer of lime crystals. The two plasmodia kept in darkness showed no sign of fruiting.

Experiment 2, carried out with a wild culture of *B. utricularis* (strain I).

This experiment showed similar results as experiment one. The plasmodia were inoculated on May 24, 1949 in flowerpots provided with oatmeal-agar, this time placed in pyrex beakers. The cultures were kept 6 days in darkness and then moved in daylight. After a rather long time sporangia appeared.

Fig. 1 shows the dark sporangia arranged in a sickle shaped figure just on the border of shade and light in the flowerpot. The plasmodium fruited after being in daylight 19 days. The arrow shows the direction of the light.

Experiment 3, carried out with a two-membered culture of *B. utricularis* (strain I), with *Rhodotorula minuta*.

Eight flowerpots were provided with oatmeal-agar and placed in lime containing tapwater. They were inoculated on November 24, 1949 and kept in darkness for 8 days. The plasmodia developed abundantly and were then placed in daylight.

Two cultures fruited in 5 weeks, sporangia appeared on the illuminated sides of the flowerpots. One plasmodium changed into a sclerotium and 3 cultures died. The 2 cultures kept in darkness died without fruiting.

Experiment 4, carried out with a pure culture of *B. utricularis* (strain I) feeding upon agar supplied with a sterilized suspension of *Torulopsis laurentii*.

Six flowerpots were provided with 15 ml unwashed agar pH 6, and placed in 60 ml double dist. water. They were inoculated on July 12, 1949. The plasmodia were supplied with a sterilized suspension of *T. laurentii*, during the first month weekly and then every fortnight.

The first week the cultures were kept in darkness, then they were placed in diffuse daylight. All cultures were affected by an excess of moisture and appeared poor in comparision with two-membered cultures. After 3 months one culture produced a few dark sporangia of normal shape. The spores when inoculated on unwashed agar, developed into a small orange-yellow plasmodium. The remaining 6 cultures died without fruiting.

Experiment 5, carried out with a two-membered culture of *P. polycephalum* with *Saccharomyces cerevisiae* var. *ellipsoideus*. This culture was obtained through the kindness of Dr W. Seifriz, Philadelphia.

Eleven flowerpots were provided with dry sterilized oatmeal, each placed in 80 ml double dist. water and inoculated on July 27, 1950. After one week in darkness the cultures had developed into abundant growing plasmodia, covering the inside and the outside of the pots.

After 7, 8, 9 and 10 days cultures, two at a time, were transferred into daylight at one o'clock. In each case the cultures fruited about 20 hours later. Next morning we could observe the formation of yellow sporangia scattered all over the flowerpots, showing no preference for light or shady side, similar to the spreading of the network of the plasmodial veins. The maturing sporangia darkened slowly and about 7 hours later their colour became brown-black. After 2—3 days of drying some of the sporangia were covered with a white layer of lime crystals. The 3 cultures kept in darkness 16 days did not show any sign of fruiting.

Experiment 6, carried out with a two-membered culture of *P. polycephalum* with *S. cerevisiae* var. *ellipsoideus*.

Six flowerpots were supplied with dry sterilized oatmeal, each placed in 80 ml double dist. water and inoculated on January 9, 1950. After one week in darkness the plasmodia showed good development and were transferred into daylight. In each case fruiting occurred in a few days; sporangia were scattered over the flowerpots on the light as well as on the shady side and on the moist cellulose.

2 cultures fruited after 1 day in the light,
3 cultures fruited after 2 days in the light,
1 culture fruited after 3 days in the light.

Fig. 2 shows one of the cultures which fruited after 2 days in the daylight. Note that some of the mature sporangia are covered with lime crystals.

Experiment 7, carried out with a two-membered culture of *P. polycephalum* with *S. cerevisiae* var. *ellipsoideus*.

Two erlemeyer flasks of 100 ml, the bottom covered with one teaspoon dry oatmeal and about 15 ml tapwater, were autoclaved (30 min. at 120° C). They were inoculated on July 23, 1950. After 16 days in darkness the plasmodia spread upon the glass wall with a dense network of orange-yellow veins.

One culture was transferred into daylight and within 24 hours fruiting occurred. The plasmodium had changed into a dark network, speckled with brownish-black sporangia. The second culture kept in darkness, remained vegetative.

Fig. 3 shows the sporangia of the above mentioned culture. Only very rarely sporangia of *B. utricularis* have been obtained in this way.

Discussion of the results

The orange-yellow plasmodia of *B. utricularis* (strain I) and of *P. polycephalum* form sporangia in diffuse daylight only. The two species react, however, quite differently with respect to daylight.

B. utricularis moves away from the light, toward the shady side of the flowerpot. Just before fruiting the reaction of the plasmodium with regard to daylight changes and it moves to the illuminated side of the pot, (exp. 1—4 and fig. 1). The plasmodium shows a contraction, followed by the formation of sporangia.

The time between the moment of the transfer of the flowerpots into daylight and the fruiting of *B. utricularis* is fairly long and varied. In summer (May, June) it takes from 19—5 days, (exp. 2, fig. 1 and exp. 1), in winter (November) it takes up to 35 days, (exp. 3). In the pure culture which fruited (exp. 4) the time was abnormally long.

Fruiting of *P. polycephalum* occurs almost immediately after exposure to daylight. There is no question of movement of the plasmodium to the shady side of the flowerpot, sporangia appear anywhere on the pot. Apparently the light-intensity is of importance, as is shown by variation in time between transfer to daylight and fruiting. This is shown clearly in the case of *P. polycephalum*. In winter (January) it takes from 1—3 days (exp. 6, fig. 2), in summer (July) it takes about 20 hours (exp. 5).

Gray (1938) has already established for *P. polycephalum* a correlation between the light intensity and the rhythm of fruiting. With more light fruiting proceeds earlier; this is in agreement with our own results. In July the days are longer, the light intensity is higher and fruiting proceeds faster than in January.

The statement of Gray (1938): "that myxomycetes, like other organisms, have the power of changing their phototropic responses", agrees with our own observations on *B. utricularis* (strain I). In 1949 Gray established that the fruiting of yellow plasmodia is more frequent in light than in darkness, whereas light is without any influence on the fruiting of colourless plasmodia. Gray points out the direct connection between the yellow pigment and the influence of light on the fruiting process, or as expressed in another way, the yellow pigment of the plasmodium is the receptive element to light.

The proper nature of the yellow pigment is hardly known and several authors do not agree in their conceptions (Sobels (1950)). We can ask what happens in the plasmodium of *B. utricularis* just before fruiting

and also whether the change in phototropic responses has its reflection upon the light sensitivity of the pigment?

Another problem of great importance, independent from the above mentioned questions, is the origin of the process of slow darkening in the maturing sporangia. SOBELS (1950) suggests that the lethal reddening of the dying plasmodium caused by an oxidation under influence of the enzyme tyrosinase and the darkening of the maturing sporangia, are based on analogous processes.

The shape of the sporangia of *P. polycephalum* varies with the humidity of the medium. On the flowerpot the sporangia have distinct stalks. When fruiting takes place on the very wet cellulose, the stalks are much shorter or wanting. The plasmodia are sometimes combined into plasmodiocarp-like clusters. For *B. utricularis* we did not notice the exact shape of the sporangia, often they were close together, only sometimes separate (fig. 1).

These observations agree with COHEN (1942). He studied the shape of the sporangia with regard to the humidity of the medium and observed that with decreasing moisture sporangia develop separately with stalks. With increasing moisture they show transitions to sessile forms and plasmodiocarp-like forms. Perhaps the flowerpot method opens new possibilities for work in this direction.

The writers wish to express their appreciation to Dr J. B. THOMAS, Dr H. P. BOTTELIER and L. ANKER for their interest and constructive criticism throughout the course of the work. They adress to Mrs E. OLMSTED their sincere gratitude for the correction of the english text.

Summary

The method of Miss SCHURE in cultivating plasmodia on flowerpots, has been worked out for two-membered and pure cultures.

Light is necessary for the fruiting of two orange-yellow pigmented plasmodia: *Badhamia utricularis* (strain I) and *Physarum polycephalum*. They react however differently upon daylight.

In agreement with GRAY we assume that the yellow pigment of the plasmodium is the receptive agent to light.

Botanical Laboratory of the University Utrecht, October 1950.

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Legend

Fig. 1. A wild culture of *Badhamia utricularis* (strain I) on oatmeal-agar inoculated on May 24, 1949, was kept 6 days in darkness and then moved into diffuse daylight. The plasmodium fruited after being in daylight 19 days. The dark sporangia are close together arranged in a sickle shaped figure just on the border of shade and light in the flowerpot. The arrow shows the direction of the light.

Fig. 2. A two-membered culture of *Physarum polycephalum* with *Saccharomyces cerevisiae* var. *ellipsoideus* inoculated on January 9, 1950, was kept in darkness one week and then brought into diffuse daylight. The plasmodium fruited two days later, sporangia are scattered over the flowerpot on the light as well as on the shady side. Some of the mature sporangia are covered with lime crystals.

Fig. 3. A two-membered culture of *Physarum polycephalum* with *Saccharomyces cerevisiae* var. *ellipsoideus* inoculated on July 23, 1950. The culture was kept in darkness 16 days and then transferred into daylight. The yellow plasmodium fruited within 24 hours and changed into a dark network speckled with brownish-black sporangia.

JOHANNA C. SOBELS AND HENDERICA F. J. VAN DER BRUGGE: *Influence of daylight on the fruiting of two orange-yellow pigmented myxomycete plasmodia.*

BOTANY

DEVELOPMENTAL PROCESSES OF THE RICE PLANT IN RELATION TO PHOTOPERIODISM

II

BY

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(Communicated by Prof. S. J. WELLENSIEK at the meeting of Oct. 28, 1950)

5.5. *Stem elongation.* Before germination the insertion of the coleoptile, the leaf primordia and the growing point are at the same level. Two weeks after germination the horny, curved, pointed stem base has been formed, showing no signs of nodes or internodes. It lasts throughout the life-cycle. At the base of the 6th leaf the lowest internode is formed, which reaches its final length at the 16th day. After a leaf has reached maturity, the underlying internode extends. The successive internodes reach increasing lengths. The rate of the elongation depends on the variety and may also be influenced by external conditions. Fig. 10 and fig. 11 show the elongation of the main stem and the laterals respectively of the July-series of the Untung-variety and of the Baok-variety. In the latter the rate of elongation is much lower.

The laterals show no extended internode below the prophyll. As soon as a leaf of a lateral has reached maturity, the internode below it enters the grand period of growth. It is remarkable that such an internode, what even its place in the lateral, elongates to nearly the same extent as that internode of the main stem, that enters the grand period of growth at the same moment. It happened for instance that the 4th internode of the 9th lateral reached the same length as the 15th internode of the main stem at the 97th day.

Even young laterals with a small number of leaves react in the same way. Finally the peduncles elongate almost simultaneously: they are the top-most internodes between the bract at the base of the panicle and the insertion of the flag leaf. Earing of the main stem occurs first, immediately followed by earing of the laterals.

5.6 *Initiation and development of the inflorescence.* The first sign of the reproductive stage is an elongation of the growing-point from 0,080 mm to about 0,200 mm. It also increases in diameter. The highest primordium stops elongating and remains as a bract of about 0,250 mm under the inflorescence. The growing point is soon divided into lobes (fig. 12 and 13), which form the future flower-primordia. Each one splits off the 3 glumes, the palea, the lodicules and the 6 anthers (fig. 14 and 15). Finally the

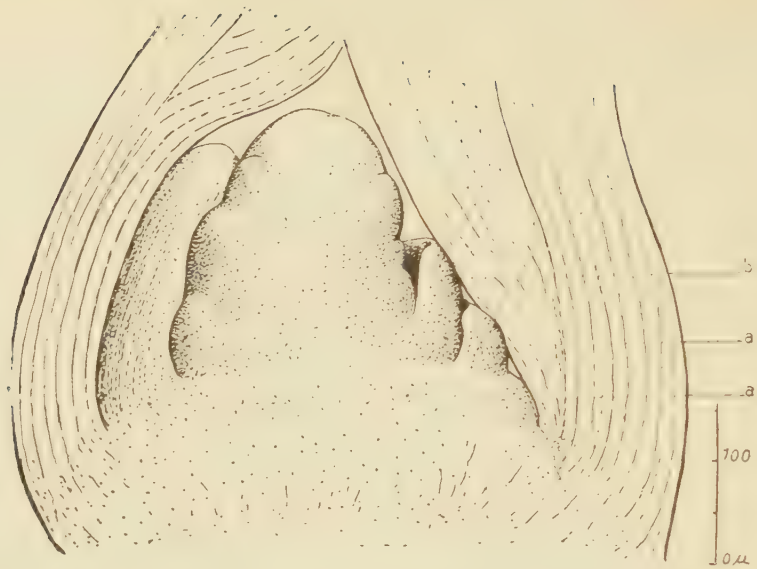

Fig. 12. Growing point of the main stem of a 51 days old plant of the Untung-variety (July-series), entered into the reproductive stage showing flower primordia. *a:* flag leaf; *b:* bract.

Fig. 13. Growing point of the main axis of a 52 days old plant of the Untung-variety (July-series), in the reproductive stage, surrounded by the 15th and the 16th leaf.

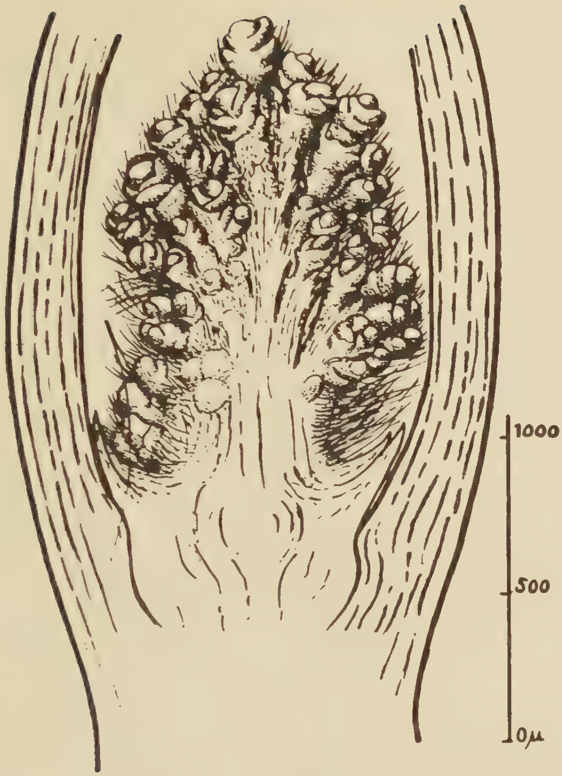

Fig. 14. Plumule of the main axis of a 66 days old plant of the Untung-variety. The topmost flowers with paleas and glumes developed.

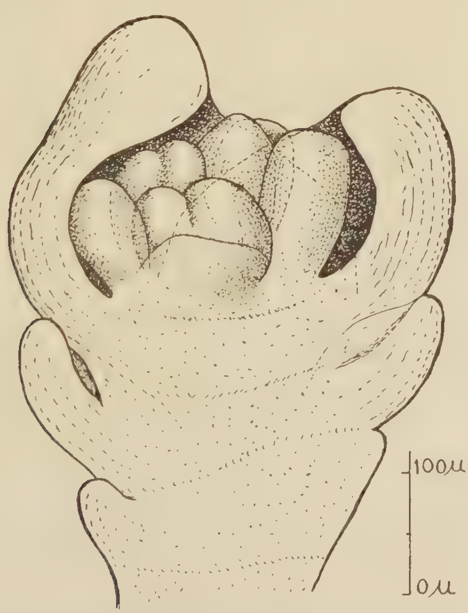

Fig. 15. Flower primordium of a 69 days old plant of the Untung-variety with paleas and glumes. The primordium of the 5th anther is just visible.

pistils are formed and the rest of the growing point changes into the ovary (fig. 16). At this stage the flower has a length of 1,5 mm to 2 mm. Maturity is reached after 24 to 30 days. The main stem reaches the reproductive

Fig. 16. Flower with anthers, carpels and lodicules of a 83 days old plant of the Untung-variety. The growing point has changed into ovary.

stage first, followed in turn by the biggest laterals of the first order and within a week by the smaller ones of the second and the third order. The same order of succession occurs in the elongating of the peduncles and in flowering.

5.7. *The period of inhibition.* After the described period of vigorous growth, a period of inhibition sets in. Especially bud development is retarded. No correlation could be found between the beginning of the period of inhibition and the time of stem elongation and earing. In the Untung-variety (July-series) the beginning of the inhibition period could be detected after stem elongation had started and after the flower-primordia were initiated. In the Baok-variety, however, the first signs of inhibition were visible long before (fig. 17). Only in one case a correlation was found between the time of earing and another phenomenon of inhibition. It concerns the elongation of the sheaths of the successive leaves of the main stem which are of increasing lengths until the growing point changes into an ear primordium. The sheaths which are immature at that time, elongate to a lesser extent than could be expected (fig. 18).

Concerning bud initiation, it was observed that the rate of initiation decreased in the Untung-variety (July-series) from the 55th day on.

Between the 75th and the 95th day, however, a period of a somewhat increased bud initiation and bud development could be observed in the midst of the inhibition period which lasted until the ears had ripened. Thus the observation of KUILMAN [21 and 22] was confirmed.

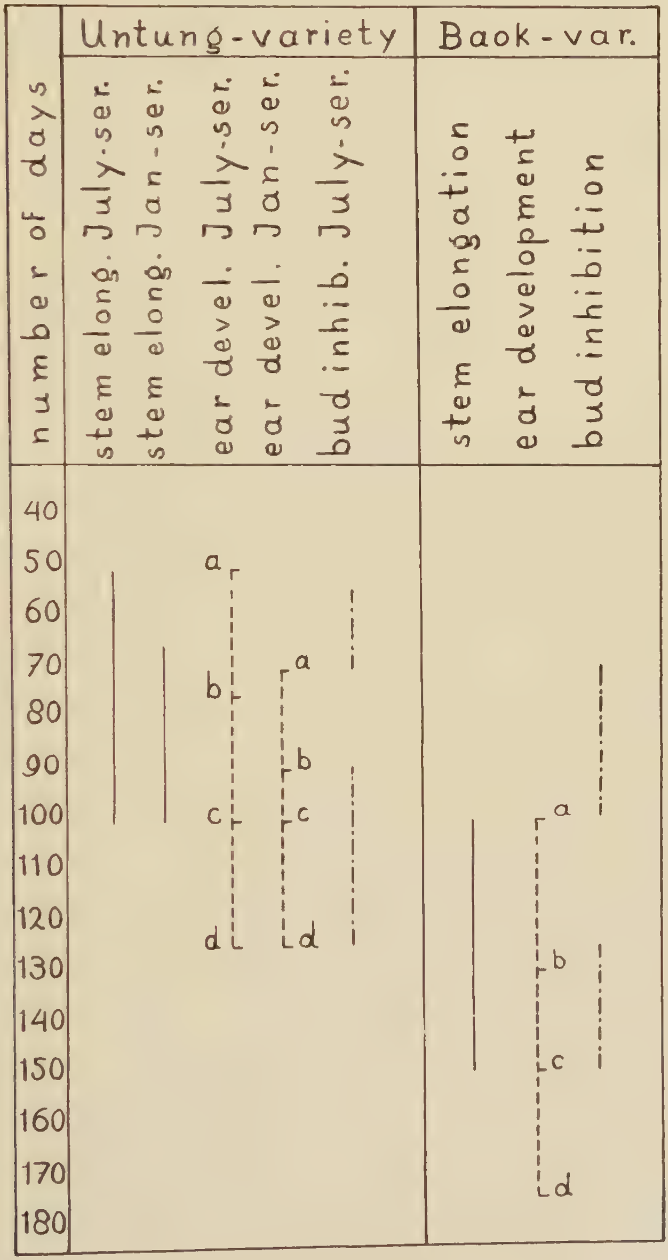

Fig. 17. Periods of stem elongation, ear development and bud inhibition in the 3 series. *a:* ear initiation; *b:* the peduncles start elongating; *c:* earing; *d:* panicles are mature.

During the period of inhibition the buds of about 4 mm do not enter the grand period of growth as readily as before, though the growing point does not stop initiating new leaf primordia. Thus thickened buds of a spherical shape can be found with a great number of short leaves, sur-

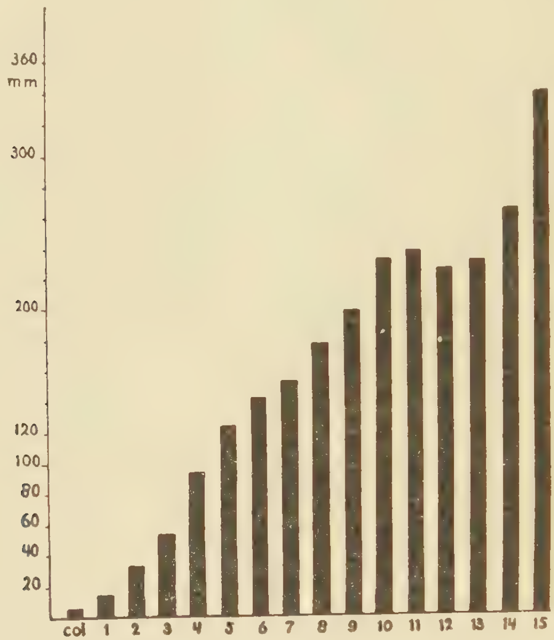

Fig. 18. Average lengths of the sheaths of 195 plants of the Untung-variety sown in July. Abcissa: the numbers indicate the place of the sheaths and the coleoptile at the main stem; ordinate: length in mm

rounding each other (fig. 19 and 20; table 1, 76 and 84 days). Here again the rate of leaf production seems to be rather independent of the factors causing inhibition of the further development.

The rate of elongation of those buds already in the grand period of growth decreases at the time inhibition sets in. As they do not reach the stage of a "shoot" as readily as before, the result is an increasing number of buds between 4 mm and 60 mm. They are flat, pressed between stem and leaf sheath (fig. 19).

These buds die off after the 70th day (Untung-variety, July-series), becoming brown, decayed or shrivelled. The thickened buds die off as well, though less readily than the flat ones. The outer leaves are first in decaying. The growing point may finally die as well, or it may still be living at the time the plant resumes vegetative growth after the ripening of the ears. In that case it may play a rôle in the secondary vegetative development (p. 1625). The flat and the thickened types of bud are found at the base of culms with a developing ear, just above the insertion of the highest lateral in the generative stage. The processes of decaying and

Fig. 19. An elongated and a thickened axillary bud of the axis 1/5/2 of a 80 days old plant of the Untung-variety. 2 ×.

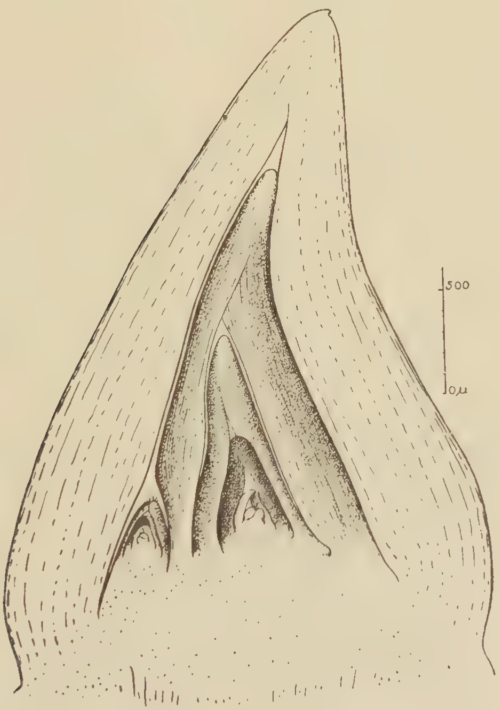

Fig. 20. The thickened bud 1/6/3 of a 84 days old Untung-plant with 5 primordia under the prophyll.

thickening advance in an apical direction. Fig. 21 shows the schemes of 3 of the dissected plants. In the 51 days old plant the buds are still "normal", in the 84 days old plant dead buds are found for instance at the culm 1/5/1, above the insertion of the earing shoot 1/5/1/3. If the lateral 1/11

Fig. 21. Schemes of 3 dissected plants (Untung, July-series). The leaves are omitted and the internodes are lengthened at random, to show the place and the types of the buds.

is the highest earing culm, the buds 1/12 and 1/13 may die off which often happens as in the case of the 104 days old plant. At the time the ears reach maturity only a few normal buds are left at the top of the culms. Though the number of "normal" buds decreases, the total number of buds increases during the period of inhibition, as well as the number of decayed flat and thickened buds.

5. 8. *Which shoots are "ripe to flower"?* Ripeness to flower is reached if the minimal leaf number necessary for ear initiation has been laid down. Probably this number depends on the variety and may differ for the main stem and for the laterals which may need fewer leaves. Also day-

length may be of influence. In the Untung-variety a shoot with a mature prophyll, one mature leaf at the top of an elongated internode, one leaf in the grand period of growth and 2 leaf primordia were ripe to flower in August. The internode may have a length of 20 mm or as little as 5 mm provided it has a cylindrical shape. Shoots never initiate a flower-primordium while they still have immature flat internodes. In most cases 6 leaves is the minimum number for an earing lateral, though culms with 4, even with 3 leaves were recorded at the end of the generative period. On the other hand, after the ripening of the ears of the main stem and the biggest laterals, shoots with 8 or more leaves might still be vegetative, a difference which could not be explained. May be the plant is not able to initiate an ear primordium just after having entered a second vegetative period under all circumstances, may be the shoots developed during September and October were influenced by the increasing daylength, which inhibits ear initiation.

At the time the ears are initiated, about 88 % of the shoots is not yet ripe to flower. It might be expected that they would gradually reach this stage by a continuous process of leaf initiation and elongation. By the inhibition process, however, they stop developing and only a few buds at the top of the main stem and earing laterals, having escaped inhibition, reach a stage of ripeness to flower and join the first group in flower initiating later on. In each dissected plant a few ear primordia were found much younger than the others. They flower and mature 2 to 3 weeks after the first group (fig. 10 and 11, the laterals 1/15, 1/16 and the younger ones). These culms are cut 2 to 3 weeks after the harvest.

5. 9. *The second vegetative stage.* After the ripening of the ears, the inhibition stops rather suddenly and a vigourous vegetative development of all living buds and meristematic tissues occurs. In a short time the apparently dead culms are covered with green needle shaped shoots, the internodes of which at once show an increased elongation. A culm partly developed during the generative and partly during the second vegetative period, may show short internodes, originating from a thickened bud with many leaves at the base, sharply separated from the longer topmost internodes, which bear longer leaves. Flower initiation of these shoots is delayed. The moment of ear initiation could not be determined, as the old decaying culms could not bear the heavy load of new shoots and succumbed. The behaviour of the buds and shoots was further studied by examining cuttings.

5. 10. *The behaviour of rice cuttings.* Pieces of old stems each with a thickened bud or a young shoot rooted easily helped by the high number of adventitious roots. Cuttings of the Untung-variety were taken from the field directly after harvesting in February. Cuttings were taken weekly up to 7 weeks after harvesting. The first set tillered abundantly and flowered after 84 days. Every plant developed about 8 ears, each with

about 170 spikelets. The other groups showed less tillering: the longer the buds remain on the mother plant, the less they tiller after transplanting, which might be due to the lack of nutrients in the early stages. All cuttings flowered, however, at the same time, the ears having been initiated simultaneously, independent of the time of transplanting and probably influenced by daylength. The shoots cut 7 weeks after harvesting only showed 1 to 3 ears per plant, each with about 20 to 60 spikelets. From all sets the number of leaves of a culm varied between 3 and 9: most culms showed 5 leaves, independent of the date of transplanting. Such a small number of leaves necessary for earing is characteristic for laterals of a high order. All shoots developed by the cuttings seem to retain their original character as laterals of high order, forming only a small number of leaves before ear initiation. None of the shoots behaved as a main stem, forming 15 or more leaves before entering the generative period. Yield was reduced by this readiness to flower.

From the varieties: Skrivimankoti, Bayang, Brondol poetih, Tjina and Baok cuttings were taken directly after harvesting. From Skrivimankoti and Bayang respectively 66 % and 43 % failed, a fact probably due to the coarse unbranched root system. Cuttings of these varieties, as well as those of Brondol poetih eared before tillering. After 75 days Skrivimankoti flowered irregularly and did not produce tillers afterwards. Bayang produced only a few tillers after earing, but Brondol poetih produced such a great amount of tillers after earing at the 40th day, that the plants showed an even more vigourous growth than seedlings of the same age. Of the Tjina- and Baok-variety 100 % of the cuttings succeeded: the plants tillered abundantly before earing and the growth was as vigourous as of seedlings. Probably the long growing period and a high number of leaves required before earing, are favourable factors.

The circumstances of the war stopped the experiments. For a rapid propagation of new valuable varieties further study of the behaviour of cutting may be of interest.

5. 11. *Developmental differences between the January- and the July-series of the Untung-variety.* Fig. 1 shows, that the duration of the growth-periods of Bayang and of Skrivimankoti are nearly equal, when sown in January and when sown in July. Earing occurs after the same number of days. However, it is not surprising that processes preceding a phenomenon of such a complex character as earing: e.g. leaf and ear initiation, leaf and stem elongation, may proceed in different ways. The plants of the January-series were exposed to decreasing daylengths, those of the July-series to increasing daylengths. Both flowered after 100 days though the plants of the July-series were more irregular. Concerning development the following facts were established:

a. The number of leaves on the main stem. In the July-series ear initiation took place at the 50th day, after 15 leaves and leaf primordia

had in most cases been formed. The 16th primordium is the bract under the panicle. The number of leaves in the January-series increased until the 62nd day, when the 19th leaf primordium was developed. At the 70th day the ear primordium was differentiated.

b. The process of inhibition. As mentioned before the sheaths of the leaves, which are still immature at the time of ear initiation remain shorter than might be expected. In the July-series those of the 12th and of the 13th leaf (fig. 16), in the January-series those of the 16th and of the 17th leaf were thus shortened. The delay in the process of bud inhibition enabled the plants of the January-series to tiller longer and more abundantly than those of the July-series.

c. The rate of leaf initiation and leaf development. As mentioned before the rate of leaf initiation is independent of daylength. In both series, 15 leaves were initiated before the 45th day. Between the 45th and the 70th day the January-series initiated 5 leaves more before ear initiation. Did elongation of a leaf continue to take 3 to 5 days, the January-series might be expected to flower 15 to 25 days later than the July-series. Flowering occurred, however, in both cases after about 100 days. This is due to the fact that after the 45th day leaf elongation proceeds with increasing speed in the January-series and is retarded in the July-series.

d. Elongation of the internodes and the peduncle. Till the 50th day in the July-series the stem consisted of about 10 small internodes, the 5 lower ones each with a length of 0,5—1 mm, the higher ones not surpassing 4 or 5 mm. When earing occurred, total stemlength was 20 to 25 mm. After the 50th day it was the internode under the 11th leaf that suddenly showed an increased elongation. It reached a length of 40 mm, in some plants even of 50 mm. Each of the following internodes elongated to a higher extent: the stem had entered the grand period of growth simultaneously with ear initiation. Finally, at the 100th day, when earing occurred, total stemlength was about 600—700 mm (fig. 17). In the January-series, the 10 lowest internodes formed a stem of only about 10 or 11 mm, the biggest internodes not surpassing 5 mm. Here it was the 14th internode that suddenly elongated to an extent of 30 mm at the 64th day. In this series the stem had entered the grand period of growth before ear initiation, which occurred at the 70th day, when stemlength was already 175 mm. After 100 days, at the time of earing the total stemlength was about 1200 mm. The youngest internodes of the January-series, below the leaves 15 to 18, together with the peduncle, needed less time for elongation than those of the July-series under the leaves 11 to 14. From the moment the flag leaf became visible till ear emergence the former needed only 8 to 10 days, the latter about 15 to 16 days.

e. The number of grains in the ear. The ears of the main stem of 13 plants of the July-series had an average of 135 grains, of the January-series over 200 grains. Ears of the laterals had averages respectively of 100 and of 150 grains.

6. *Discussion*

6. 1. *The interrelation between the developmental processes.* Fig. 17 shows, that in the July-series of the Untung-variety the process of stem elongation was immediately followed by ear initiation and inhibition. In the Baok-series inhibition occurred first, followed by the entering of the stem in the grand period of growth and finally by ear initiation. A correlation was found between ear initiation and inhibition of the rate of elongation of those leaf sheaths, which were still immature at that time. Between inhibition of bud development and earing, however, no correlation could be found. On the contrary, after the first period of bud inhibition in the Baok-series bud development increased again when stem elongation and ear differentiation were observed. The only coincidence between these processes can be observed at the beginning of the second vegetative period when maturity has been reached and bud inhibition stops suddenly.

Concerning bud inhibition, RAMIAH and NARASIMHAM [36] consider a lack of food as the causal factor of the abortion of tillers, the mother shoots taking so much of the available nutrients that no adventious roots can be formed. "All the late and undesirable tillers are thus eliminated by the time the plant passes into the reproductive phase". KUILMAN [21, 22] considered his second interval in the process of tillering to be correlated with formation and development of the ear; later he saw in the elongation of the internodes the causal factor. None of these opinions has been proved exactly.

Concerning the coincidence of the developmental processes, the 3 series behaved in different ways, no causal connection could be established. They may be caused independent of each other by different factors. The influence of growth substances as auxin or of the still hypothetic caulocaline formed in the roots, causing stem elongation under certain conditions, (WENT [42]), will not be discussed here. Before a theory of flowering and flowering hormones can be formulated for rice, as has been done by PURVIS and GREGORY [32] for rye and by VAN DE SANDE BAKHUYZEN [37] for wheat, a considerable body of experimental work still has to be completed.

6. 2. *The influence of daylength; earlines and lateness.* Though in the vegetative period of the January-series the average number of hours of insolation was far less than in the July-series, the plants tillered to a greater extent. The long days of January and February may delay bud inhibition and therefore promote tillering. They delay ear initiation, inhibit leaf- and stem elongation and promote lateness. The shorter days of March and April, however, induce flowering and provoke a rapid leaf- and stem elongation and a rapid earing; just as in winter rye elongation and ear-initiation are influenced by the same factors (PURVIS [31]). The reverse can be said of the conditions and their effect on the development

of the plant of the July-series. From the graphs given by VAN DER MEULEN ([27], fig. 1), it is clear that the life cycle is shortest under daylength conditions as short as possible at Buitenzorg. Earliness is promoted, but yield is lowest. It must be stated that the Untung-variety is better adapted to the light conditions prevailing during the first half than during the second half of the year. Possibly the decreasing or the increasing daylengths are decisive, more likely the absolute daylength determines development. In that case the number of hours of daylength necessary to obtain flower initiation as early as possible is still an unknown factor. On the other hand it is known that continuous light suppresses heading (KONDO [18]) or heading may occur at some delayed moment (FUKE [13]).

The influence of daylength on the different developmental processes has to be studied separately as far as possible. Long days may have an after-effect on stem elongation during a following period of decreasing daylength (PURVIS [31]).

Whether there is an obligate minimum number of leaves which has to be initiated before the ear can be differentiated, is another question to be elucidated. In winter rye this number is 7; for different rice varieties this number is still unknown. That a certain stage has to be reached before flowering can be induced has been reported by ALAM: "all varieties require a minimum period of 30 days for vegetative growth" (WHYTE [43], p. 329), and by FUKE [13], who discriminates two tillering stages. The effect of short day-conditions on plants in the first and on plants in the later part of the tillering stage was similar. Probably the minimum number of leaves required for flower initiation had not yet been reached in the earlier stages. Thus short day conditions had no effect on the growing point, as the stems were not yet "ripe to flower".

6. 3. *Breeding of rice varieties based on developmental characters.* In the Untung-variety factors may be distinguished independent of the date of sowing, such as the rate of leaf-initiation before the 45th day, the relation of the lengths of the mature sheaths and the increasing lengths of the successive internodes. Others are dependent on the date of sowing, such as the number of leaves and internodes initiated in the first vegetative period, the rate of leaf- and stem-elongation, the moment of inhibition of bud development and the number of spikelets per ear.

In a variety sensitive to daylength these factors could be recognized under different lengths of day; in varieties indifferent to daylength, factors for earliness and for a high yield can be determined as well: the tendency to develop a small number of leaves or the tendency to rapid leaf- and stem-elongation. In breeding for earliness in combination with a high yield, knowledge of the physiology of the parent plants is of highest importance. The program proposed for the wheat plant by McKINNEY and SANDO [26] can be applied to rice hybrids as well: "populations which are segregating for earliness and lateness should be tested and

classified as far as practicable under several suitable conditions of temperature and daylength. This should facilitate the selection of genotypes homozygous for the several characters influencing earliness and lateness". Lines still heterozygous to a small degree may segregate for some developmental factors. By selection of types with the most favourable factors more homogeneous material may be obtained. The factors discussed here have not to be considered as units of heredity. They have to be studied first before the physiological factors depending on a single gene can be analyzed, in the way it is proposed by BOONSTRA [6].

Summary

The development of 3 series of rice plants was studied by the dissection method:

1. Two series of the variety "Untung", one sown in January, the other in July. Untung is an unawned pure line, grown at Buitenzorg, Java. The growing period depends on the date of sowing.

2. One series of the variety "Baok", sown in January, an awned pure line with a growing period of 130 days, independent of the date of sowing.

Both varieties were grown on nutrient solution.

The following facts were established:

The growing point of the embryo is surrounded by a coleoptile, two young leaves, closely packed together and a third primordium. After germination a fourth primordium is formed exactly when the coleoptile starts elongation. The same principle is maintained during the vegetative stage, as every growing point during that period is always surrounded by: a young primordium of about 0,200 mm; a primordium of about 0,200 mm to 1 or 2 mm; a leaf in the "grand period of growth"; one or more mature leaves. A primordium only elongates when the preceding leaf has reached maturity and a new primordium has been formed.

When a leaf has reached maturity the internode below starts elongating, the peduncle being last of all. The immature internodes of the main stem and of all flowering laterals elongate to nearly the same extent at the same moment. The growing point of the sufficiently developed laterals change almost simultaneously into ear-primordia which develop in about 25 to 30 days. The simultaneous elongation of the peduncles synchronises flowering. In the Untung-variety ripeness to flower may be reached if a lateral has a minimum number of 5 leaves. At a certain stage, depending on the variety, an inhibition of the process of tillering prevents the younger shoots from reaching this stage. Buds, apparently in the grand period of growth, are found to be dead, pressed between stem and leaf sheath. In younger buds leaf primordia are still initiated but as no elongation occurs, the buds become spherical in shape. These two types of inhibition continue from the base of a flowering shoot in an

apical direction. At the time of maturity only the uppermost buds are still in a normal condition.

At that time the plant enters a second vegetative stage: the buds at the top and the thickened ones develop quickly into shoots, which may flower some 3 weeks after the first set.

The behaviour of the shoots was studied further with cuttings.

The period in which stem elongation and in which bud inhibition takes place, as well as the date of ear initiation depend on the variety and on the environmental conditions. An interrelation between these processes has not yet been determined.

Daylength at Buitenzorg (7° Lat. S.) is 12 h. + 31′ in December and 12 h. — 17′ in June. This difference of 48′ throughout the year may induce a difference of 50 or 60 days in growth period in some varieties, depending on the date of sowing. When sown in January and July an equal number of days is required for earing.

Both the series of Untung examined reached maturity after 120 days, though the developmental processes were different:

The vegetative period of the July-series lasted 50 days, that of the January-series 70 days, with 15 and with 20 leaves initiated respectively. In the July-series the elongation of the uppermost leaves and the peduncle took 50 days, in the January-series these processes took only 30 days. Flowering occurred in both cases at the 100th day, though in the July-series the ears were smaller and flowering occurred more irregularly. Probably in the July-series short days promote flower initiation, resulting in earliness, but the longer days of September delay leaf and stem elongation. In the January-series the long days delay flower initiation but promote vegetative development. In March and April the shorter daylength promotes elongation and earing, resulting in earliness.

The influence of a constant length of day on the developmental processes requires investigation, varietal differences need to be recorded. Attention should be paid to those simple developmental factors which control earliness. New varieties may be obtained by crossings of varieties combining factors favourable for earliness and for a high yield under certain conditions.

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DETERMINATION OF THE DIAMETER, THICKNESS, VOLUME AND SURFACE OF ERYTHROCYTES BY MEANS OF THE DIFFRACTION METHOD. I

BY

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I. DIFFRACTOMETRIC DETERMINATION OF THE DIAMETER OF RED BLOODCORPUSCLES

In a series of three articles the results will be communicated here of a study about the diffraction method of YOUNG–PIJPER.

This first article will only deal with the measurement of the diameter of red bloodcells.

In a following article the method will be extended to the determination of the thickness, the volume and the surface, while in a concluding article a mathematical-physical theory will be developed which makes it possible to compute all sorts of diffraction patterns.

1. *Considerations on the diffraction method*

When comparing determinations of the diameter of red bloodcells according to PIJPER's diffraction method with the results obtained by measurements according to PRICE JONES I decided, about 1939, to make an experimental and theoretical study of the problem of diffraction. In the beginning, I only knew PIJPER's first formula, published in 1919 and as I was then working in the Dutch East Indies, at a spot far removed from medical and physical centres, I had to tackle the problem without the help of others. It was only after my liberation from imprisonment by the Japanese and when I had returned to the Netherlands that I had an opportunity to study the literature. I had then already made up my mind as to the direction in which the problem should be solved.

In the first place, the art of being able to make good preparations is indispensable in order to observe the phenomenon of diffraction with blood-preparations. A drop of blood (which at the skin has about the size of a pin's head) should be slided very regularly on a clean object-glass. The last third part of the film then usually has a soft, silky, shining aspect. Here the erythrocytes lie distributed separately. If one looks through this part at a small, clear source of light, which has been placed at some metres' distance, while the preparation is held very close in front of the eye, one sees round the lamp beautiful, coloured rings. Immediately round the source of light there is a round zone of the colour

of the lamplight, with a narrow orange and red edge only at the circumference. Round this aureole lies a spectrum-like ring with the colours violet, blue, green, yellow, orange and red, from within. Round this, analogous systems again group themselves, the colours of which, however, are more blended and in which green and red predominate.

This effect was already known to YOUNG in 1807, but PIJPER discovered it again in 1919 and developed a method which is of clinical importance for quickly diagnosing macrocytic anemias.

PIJPER's apparatus projects the rings on a screen, so that they can be gauged. He improved his apparatus in 1929. This, beside measurement of the diameters of the rings, also makes it possible to compare two preparations. PIJPER's explanation of the effect is that a blood-preparation behaves like a diffraction grate, the diameter of the cells which lie themselves irregularly with respect to one another acting as a grate-constant. The coloured rings form spectra, i.e. a certain wavelength can be ascribed to certain colours. So they form real maxima. The rings with the most brilliant colours correspond to a spectrum of the second order, so that in his formula a factor 2 occurs. In 1925 he reduced this to 1,7, as by this more satisfactory results were obtained. A theoretical foundation of this change was not given.

PIJPER's theory has not been able to stand the test of criticism. Already in 1923 BERGANSIUS wrote that a constellation of irregularly distributed erythrocytes cannot be a grate in a physical sense and that the diameter of the cells cannot have the function of a grate-constant. If the cells are to be considered as dark disks and lie irregularly, very definite formulas for this are known in physics. The constant 2 in a second maximum should then be 2,67. PIJPER's factor 2, however, gives much better values. BERGANSIUS thought he could show that in suitable places the erythrocytes lie regularly with respect to one another. Through lens-action each cell has above it an image of the source of light. Thus a grate effect can occur. Not the diameter, however, is measured, but the average distance between the cell-centres, so a somewhat greater value, as the elements do not lie exactly against one another.

In 1925, however, SIEGENBEEK VAN HEUKELOM pointed out that spores of lycopodium, though opaque, also give beautiful diffraction rings.

In 1936 MILLAR applied the classical theory of irregularly distributed dark round disks. His results are also too large by 10 to 20 %. He suggests a possible influence of the thickness of the erythrocytes, which has made it considerably more difficult to solve the problem physically.

In 1928 ALLAN and PONDER and in 1935 HARNAPP and MÖBIUS in principle used monochromatic light. They found that the classical physical disk-theory yielded good results.

In 1941, however, COX and PONDER establish that a satisfactory theory has not yet been drawn up. Also with the disk-theory the constant in the classical formula should be reduced by about 10 %.

I should like to begin, so as to be able to compare a grate- and a disk-theory, by describing a theoretical experiment. As a grate we take a sheet of glass, on which very many parallel grooves have been made at equal distances d. As a "disk" object a sheet of glass on which there are many lines with a diameter d, but which, though parallel to one another, are grouped in the most disorderly manner with regard to their distances from one another.

A parallel monochromatic beam of light with a wavelength λ falls perpendicularly on both sheets. This light is at the same time coherent, i.e. in a plane perpendicular to the beam the light is everywhere in the same phase. After passing the elements causing the diffraction phenomenon the light appears to be scattered. By means of a lens with a focus distance f we collect the light on a screen placed in the focal plane of the lens. If the source of light had the form of a slit and was parallel to the positions of the elements which cause the diffraction phenomenon, we see on the screen light and dark diffraction bands on either side of the image of the source of light.

If the elements causing the diffraction phenomenon as well as the source of light are round, we see light and dark rings (plate I, fig. 1).

If we call the distance of any point in the diffraction pattern to the centre r, we can write down a general formula for small scattering angles:

$$r = n \cdot \lambda \cdot \frac{f}{d}$$

and from the formula $d = n \cdot \lambda \cdot f/r$ the sought grate-constant or diameter can be computed.

It is clear that, with fixed values of λ, f and d, the places in the diffraction pattern where a maximum or a minimum of light is present (also the value r for them) are dependent on n.

In the following table, for the 1st to the 3rd maximum and minimum the corresponding values of n have been indicated, viz. for a grate in row I, for irregularly distributed parallel lines in row III and for irregularly distributed round dark disks in row IV.

For blood the corresponding factor-corrections have also been given, viz. those of Pijper in row II (correction to row I), those of Cox and Ponder in row V (correction to row IV). Those corrections were established empirically. In row VI my own correction has been given, computed on the erythrocytes model to be mentioned in these articles according to a theory to be developed in the concluding article.

In this table it has also been indicated that with grates the maxima are narrow, while with disks the minima are narrow, whereas inversely with grates the minima are broad, while with disks the maxima are broad.

It can also be seen from rows I and III that with grates the maxima lie in exactly the same places as the corresponding minima with irregularly distributed parallel lines and in approximately corresponding places of the minima in rows V and VI.

TABLE I

		Values of n							
		narrow maxima				broad minima			
		0	1st	2nd	3rd		1st	2nd	3rd
I	Physical grate; linedistance d is grateconstant	0	1	2	3		0,5	1,5	2,5
II	Pijper's constant (erythr. diameter d is grate-constant)	0	—	1,7	—		—	—	—
			narrow minima			broad maxima			
			1st	2nd	3rd	0	1st	2nd	3rd
III	Irregularly distributed parallel lines with thickness d . .		1	2	3	0	1,43	2,46	
IV	Opaque round disks, distributed irregularly, with diameter d		1,22	2,22	3,23	0	1,65	2,67	
V	Constant of Allan and Ponder (empiric)		1,095	—	—	0	—	—	
VI	Constant of Verveen, computed on erythr. model . .		1,11	—	—	0	—	—	

When blended light is used, with erythrocytes the maxima of the various wavelengths will fall broadly across one another. In the places of the minima a certain colour will drop out, so that in the blending of colours the complementary colour will make the impression of a (seeming) maximum. When monochromatic light is used the real character of the phenomenon appears. Plate I clearly shows that with blood the pattern corresponds to a disk-configuration. For these photographs monochromatic light ($\lambda = 0{,}546\ \mu$) was used.

Pijper has stuck to his theory. For the classical theory also needed a factor-correction of about 10 %, there being no foundation for this that could be understood theoretically. As Pijper also derives from his theory the possibility to establish an anisocytosis quantitatively and in 1947 tries to show the clinical practicability from comparative measurements according to Price Jones, it is of great importance that quantitatively there is a foundation of the effect, as regards the positions of the minima and maxima respectively.

According to Pijper the yellow ring measures the average diameter, the utmost red the microcytes and the inmost violet the macrocytes.

Harnapp and Möbius could not discover a correspondence between the positions of the colours red and violet and the sizes of the micro- and macrocytes.

De Monijé mixed erythrocytes of varying diameters, originating from various animals and could not confirm Pijper's opinion.

Knoche finds that with macrocytes these should amount to 8 % of

the population, if the violet ring is to give a measure for this, with microcytes (measurement of the red ring) all values should be reduced by 0,6 to 0,7 μ.

It can be shown that with blood the diffraction pattern is a disk-effect. One can also make beautiful grate preparations artificially.

For this I made the following experiments:

1. A blood-preparation is semifixed by drawing it a few times through a blue flame (at first with the side of the preparation from the flame so as to prevent haemolysis through the condensing vapour and then towards it) and then a part is immersed in water after cooling. It appears that here the rings have become much smaller. As the positions of the erythrocytes with respect to one another have remained identical and the macrocytosis shows itself at once microscopically, it is clear that only the increase in diameter can be held responsible for the diminution of the rings. By this BERGANSIUS' theory has been refuted and it has been proved experimentally that the disk-theory is correct.

2. If one smears out blood with a finely notched object-glass, in the extreme end of the film the bloodcells will lie in regular chains. One sees here now a fine grate pattern, i.e. broad minima and narrow clear maxima.

So experimentally the two forms that are theoretically possible can be realized.

With less successful preparations there sometimes occurs a mixed form of disk- and grate pattern. The aureole of the disk pattern has a zone of greater intensity at its border. For good measurements these places should not be used.

We regret to say that no foundation can be given to PIJPER's deduction, unless in principle the preparations are made with notched glasses. Then, however, the distortions of the erythrocytes in the chains are so great that they are useless for delicate measurements for the discrimination of micro- and macrocytes.

The classical theory fails owing to the fact that the red bloodcells were wrongly considered as opaque. The light that is transmitted should also be taken into account. If one does so (for this one should, therefore, use a model that approaches the vertical section over a diameter, i.e. profile of the bloodcorpuscles as well as possible) one does get good values for the places of the minima.

I found the mathematical basis for these kind of computations in VAN DER HULST, Optics of spherical particles, *thesis* Utrecht 1946.

The curious thing about the diffraction pattern of blood is that the distances from each other of the minima are nearly equally large everywhere and are equal to the radius of the aureole (Plate I fig. 1, 2). This may have been the cause of the wrong interpretation of the polychromatic effect, which owing to this shows a strong resemblance to a grate pattern indeed.

The constant I computed corresponds well to the empiric one of Cox and Ponder (see table). The difference is only 1,5 %.

2. *Methodology and measurements on normal blood*

For the measurements of rings, when monochromatic light is used, the projection-method of Pijper is less suitable. Monochromatic light, as it is obtained by sifting all undesired wavelengths by means of filters, from a strong source of light which sends out certain spectral lines, has undergone such a weakening in intensity that the projected rings are no longer clear enough and have become unsuitable for measurement. I therefore applied two other methods, viz. the photographic and the direct visual one; with the latter the retina serves as a screen.

A. Photographic method (see fig. 1).

With two cameras *A* and *B*, both with an *f* of 13,5 cm and suitable for 9 × 12 cm plates, the ground-glass of *A* was replaced by an opaque plate, which exactly in the middle at 0 had a small round opening with a diameter of 0,75 mm. The cameras were placed in such a position that the lenses

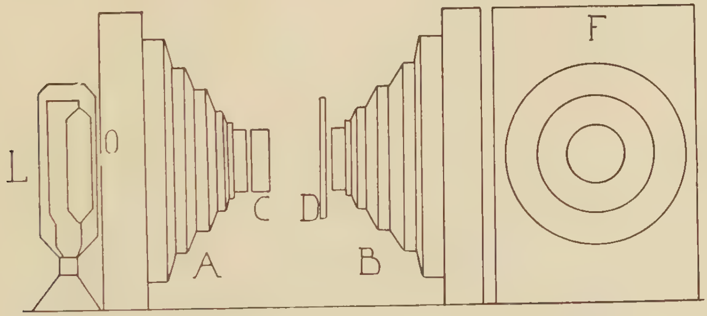

Fig. 1. Photographic position to obtain photographs of diffration.

faced each other. Before the lens of *A* coloured filters were placed at *C* (Wratten numbers 27 and 70) which of the light furnished by a low pressure mercury lamp *L* (Philips Philora type 5381 E) only transmitted the green mercury line ($\lambda = 0{,}546$ micr).

As both cameras were adjusted to infinity, a beam of parallel rays come out of *A*. So on the ground-glass of *B* there arose a sharp image of the opening 0 with a diameter of 0,75 mm.

Now before the lens of *B* the blood-preparation (uncoloured) *D* was placed, by means of a clip, in such a way that the beam of rays from *A* fell perpendicularly on it. The diffraction effect which occurred was now photographed at *F* (plate I, 2*a*).

The whole was protected from undesired entrance of light by a dark cloth.

When the filters C were removed, a red-coloured filter was put before C and the mercury lamp L was replaced by an ordinary strong bulb, beautiful photographs were also taken of the diffraction rings (plate I, $2b$).

B. Visual method (fig. 2 and plate I, fig. 3).

The direct visual method has the advantage of great strength of light and makes it possible to read quickly. Fig. 2 represents its build.

A rectangular little box of 8×10 cm and of about $2\frac{1}{2}$ cm thickness has been blackened on the inside and has a small round opening at 0. On the outside at F a red-coloured filter has been placed. On the inside, quite close to 0 a curved graduated scale has been placed, the distance of the calibration-lines of which is $= \frac{1}{2}$ mm. The zero of the scale lies against the

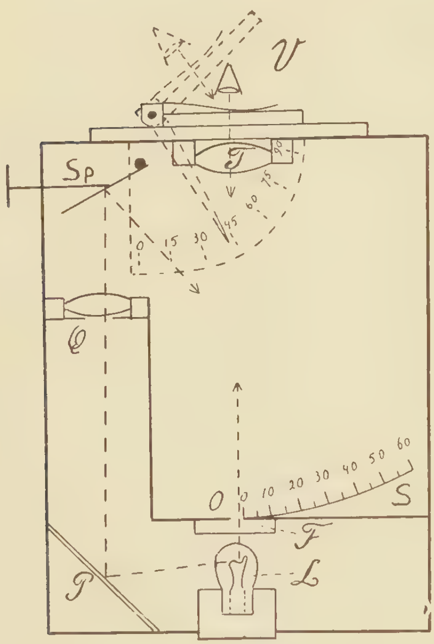

Fig. 2. Apparatus for the measurement of the diameter and thickness of red bloodcorpuscles.

edge of 0 and not, as one would expect, in the middle of 0. For up to a considerable increase of the opening the distances of the minima remain constant as far as the edge. At P a part has been left open at the back, so that the scale can be illuminated via an adjustable mirror Sp. At T there is a lens of 13.5 diopters ($f = 74$ mm).

At V the blood-preparation is fixed in an object-glass carrier. This has been constructed in a special way. For it revolves round a little axis mounted on a slide which can be shifted to and fro along the upper surface of the apparatus. Attached to this slide is a calibrated grade arc. The carrier has a pointer. Thus the position of the preparation with respect to the lens of the apparatus can be read at all times on the graduated arc, which runs from 0 to 90 degrees.

In order to determine the diameter the preparation is placed under an angle of 0 degrees with the plane of the lens.

The intention of the revolving object-glass carrier will be discussed extensively in a following article. It can already be communicated here that it makes the apparatus suitable for measuring the thickness of the E (erythrocytes).

The electric lighting-apparatus L consists of a bulb of $3\frac{1}{2}$ volts, which has been put immediately before F, further of a small metal mirror, which leads the light of this bulb inside at P and finally of a little lens with a short focusdistance and a rectangular fixed diaphragm at Q, so that on scale S a small rectangular part is clearly illuminated. This spot of light can be moved along the whole scale by means of an adjustable screw Sp. The bulb is fed either by a battery or via a transformator by the lighting-mains of the town.

Now if from L light is cast on 0, red light enters into the apparatus through 0 and 0 being in the focus of T, a parallel beam of light falls on the preparation from below at V. As L at the same time lights the scale and the latter is in the focal plane of the lens, the eye, which looks at the scale through the preparation, sees both the diffraction rings and the graduated scale and can read the first minimum.

This reading can be difficult with the uncoloured preparation, as it is less transparent, though with very thin places (and these are exactly the fields which offer the best conditions for a reliable measurement) it is always possible.

A so-called *absorption-preparation* is very transparent, however. It is made in this way: An ordinary bloodslide is fixed for half a minute with methanol and then coloured for 3—5 minutes with leicht-grün S. F. from Sandoz, Basel, 1,2 %. After washing and drying, a drop of cedar-oil is laid on the thin part of the preparation and this is provided with a coverslip. It is now put on the apparatus and the first dark ring is read on the scale. We want the middle of this ring. The borders are not completely sharp, but can be well determined by contrast-effect. Both the inner border and the outer border of the ring are noted down and the average of this is the value sought.

If one sticks to the rule of only using those parts of the preparation where it is thin, which can be judged best on the still uncoloured preparation and if, moreover, one makes the measurements in places not close to the edges, one always gets constant results for different preparations of one and the same person, made at the same hour.

It is easy to see that near the edges influences may occur during sliding which distort the E. in a certain way. They are here sometimes lying so close against one another that they become somewhat squashed, so smaller, with the consequence: larger rings and sometimes grate effect (see p. 1638).

One can also very well make measurements on the still moist prepar-

ation by covering the last third part with a cover-slip and hermetically closing the edges with vaseline, directly after sliding, before the film has dried. One should then make the measurement immediately, for if one waits too long, agglutination sets in and the effect disappears.

Sometimes smaller rings are found than with the dry preparation, which points to the fact that the *E.* are somewhat larger (about 0,3 μ).

A fixed uncoloured preparation can also be made transparent by covering it with a drop of water and a cover-slip.

The red-coloured filter I used in my apparatus was gauged visually by means of the green mercury line. I found its λ to be 0, 64 microns.

It should also be observed that with the direct visual method the eye of the investigator should be corrected for anomalies caused by anomalous eye refraction. Only young hypermetropic persons will read the scale without difficulty. Myopic as well as hypermetropic presbyopic persons should use spectacles; the latter, as is perhaps superfluous to add, should use the spectacles which they need for seeing distant objects.

In order to prevent unnecessary calculations and consequent loss of time the diameter d corresponding to a measured scale value s can be read from a curve at once.

I made 100 measurements on normal blood. As average diameter-sizes I found $7{,}6 \pm 0{,}34\ \mu$, varying from 6,8 to 8,4 μ.

3. *Comparison with measurements according to* PRICE JONES

If one measures the diameter of the *E.* diffractometrically, it appears on comparison with an analysis according to PRICE JONES that the value found lies very close to the top or modus of the diameter distribution curve (see fig. 3). Even when many diameters occur, smaller than that of the majority of macrocytes, they have no influence as it only the top-diameter which is measured (fig. 3*c* and *d*). Only with symmetrically distributed diameters will the average be found diffracto metrically. So one should never speak of an average size of the *E.*, if it is not known whether the distribution of the diameter is symmetrical with regard to the modus. In fig. 3*a* and *b* the curve is symmetrical, so that the value for M, i.e. the average diameter, does indeed correspond well to the value given by the diffraction method.

From a theoretical-physical viewpoint it is also to be expected that the macrocytes come to the fore. The intensity of a certain place in the diffraction pattern is proportional to the square of the resultant amplitude of the rays of light in that direction. This resultant is proportional to the volume of an *E.* and so to the third power of the radius or diameter (see formula 13,2 in the concluding article). So the intensity is proportional to the sixth power of the diameter.

Consequently, when there is an equal number of two different sizes, that group always predominates which has the largest diameter.

Even with a considerable degree of anisocytosis we can yet expect

D. VERVEEN: *Determination of the diameter, thickness, volume and surface of erythrocytes by means of the diffraction method.* I.

PLATE I

Fig. 1. Diffraction pattern of normal blood.

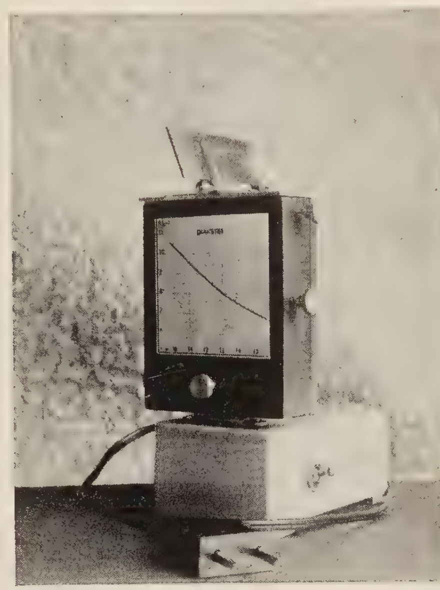

Fig. 3. Apparatus for diffractometric determination of the diameter and thickness of erythrocytes.

Fig. 2*a*. Normal blood in monochromatic green light. ($\lambda = 0.546\ \mu$).

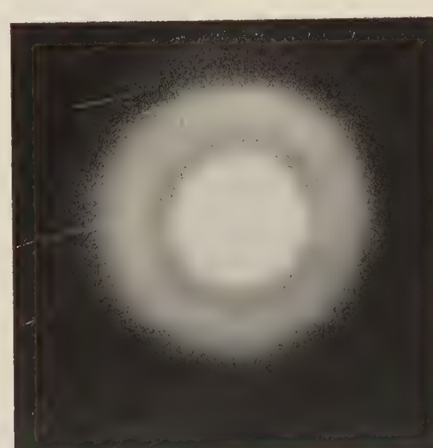

Fig. 2*b*. The same blood as in fig. 2*a* but in ordinary light and using a red glass filter.

clear diffraction rings. It is wrong to assume that there is anisocytosis on a blood-preparation when the rings are blurred. With good sliding technics there may be several causes for the rings being blurred:

1. Marked poikylocytosis. The irregular forms of the *E.* have an

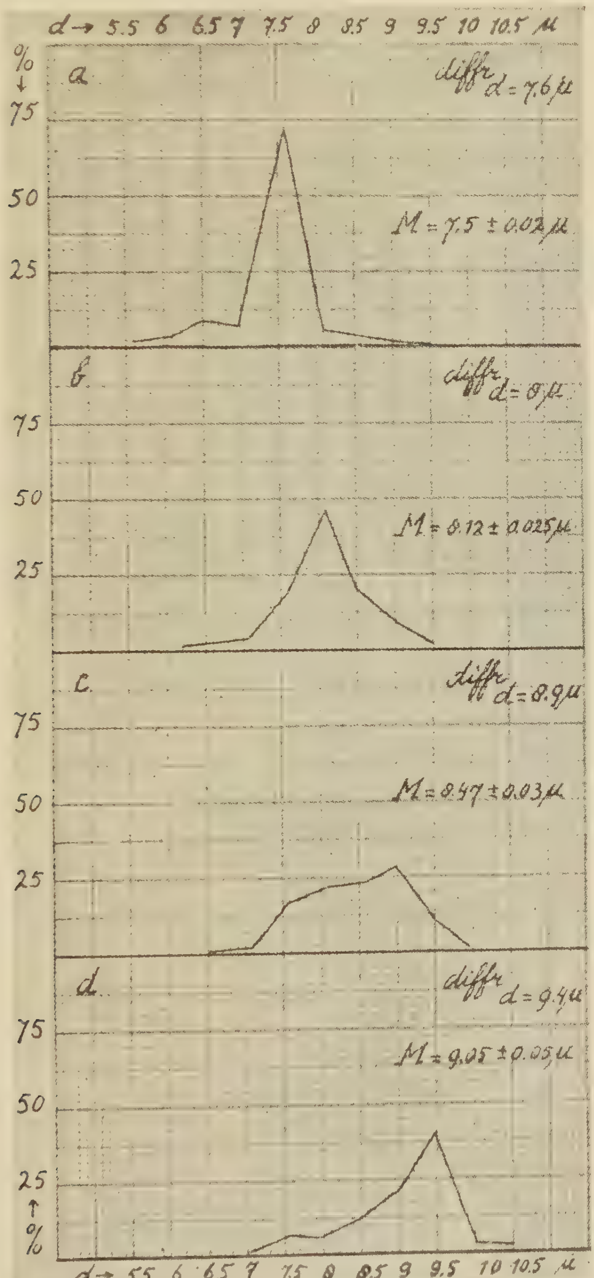

Fig. 3. Curves according to PRICE JONES
a normal blood; *b* light macrocytosis; *c* and *d* blood in sprue.

unfavourable influence on the clearness of the contrasts, because all kinds of irregular diffractions occur.

2. Strong leucocytosis.

3. Ovalocytosis. The *E.* have all about the same surface, but the long and short axes lie distributed in all directions in the same quantities.

With very strong ovalocytosis, as appears in chicken's blood, no diffraction effect at all should occur, theoretically speaking. During sliding, however, there is a current of the blood in the longitudinal direction of the preparation. By this a tendency of the *E.* arises to lie with their long axes in the direction of the sliding. This tendency appears clearly in the diffraction effect. The (blurred) rings are short in the longitudinal direction and long in the transverse direction of the object-glass.

In a following article I hope to show that it is also possible to measure the greatest thickness of the edge of *E.* by means of the diffraction method.

Summary:

The diffraction effect occurring in blood preparations was discussed. It was proved experimentally that it occurs on account of the erythrocytes behaving like irregularly distributed disks. If the transmitted light is taken into account a quantitative relation can be computed between the diameters and the sizes of the diffraction patterns.

The degree of anisocytosis cannot be established by means of this method, as only the macrocytes determine the effect.

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AN ARITHMETICAL THEOREM CONCERNING LINEAR DIFFERENTIAL EQUATIONS OF INFINITE ORDER

BY

J. POPKEN

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It is a matter of common knowledge that analysis often plays an important part in deriving purely arithmetical results. However, by combining in the same manner analytical methods and ideas from the theory of numbers, one is often led to theorems of mixed arithmetical and analytical character. The theorems derived in this paper are of this type.

In order to state our principal result, it is convenient to introduce first for a given integral function $y(z)$ the notion of an "*exceptional point*"; we shall call a complex number ζ an exceptional point for the function $y(z)$ if both values ζ and $y(\zeta)$ are algebraic numbers. If μ is a positive integer, such that ζ, $y(\zeta)$, $y'(\zeta), \ldots, y^{(\mu-1)}(\zeta)$ all are algebraic, but $y^{(\mu)}(\zeta)$ transcendental, then μ will be called the "*multiplicity*" of the exceptional point ζ. If possibly ζ and all values $y(\zeta)$, $y'(\zeta), \ldots$ are algebraic, then the multiplicity of μ will be infinite by definition.

Theorem I. *Let the integral function*

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!} \quad , \quad \limsup_{h\to\infty} \sqrt[h]{|c_h|} \leqq q,$$

where q denotes an arbitrary positive number and all coefficients $c_0, c_1, c_2, \ldots$ are algebraic, satisfy a linear differential equation of infinite order

$$a_0\, y(z) + a_1\, y'(z) + a_2\, y''(z) + \ldots = 0,$$

with constant coefficients $a_0, a_1, a_2, \ldots$ not vanishing simultaneously. Let the corresponding characteristic function

$$a_0 + a_1\, t + a_2\, t^2 + \ldots$$

be regular in the circle $|t| \leqq q$ and let ν denote the maximum of the multiplicities of its zeros in the region $0 < |t| \leqq q$.

Then the following two assertions are true:

1. *If $y(z)$ has ν or more exceptional points different from zero* (counting a point of multiplicity μ also μ-times), *then $y(z)$ necessarily is a polynomial with algebraic coefficients.*

2. *If $a_0 \neq 0$, then every exceptional point different from zero and with multiplicity μ necessarily is a zero of $y(z)$ with the same multiplicity μ.* (This assertion still holds if μ is infinite.)

Remark. It follows from assertion 1: If the *transcendental* function $y(z)$ fulfills the conditions of our theorem and if moreover the zeros of the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$ are *simple* (hence $\nu = 1$ in the preceding theorem), then $y(\zeta)$ is transcendental for every algebraic value of $\zeta \neq 0$.

We give a proof of this theorem in § 1; some of the ideas we use are due to McMillan, who stated a theorem closely related to ours in an earlier paper [1]). However the result obtained by McMillan is erroneous as will be shown in § 2. In § 3 we give some interesting applications of our theorem I. In this manner we obtain three known theorems, respectively due to Itihara, to Dietrich and Rosenthal and to R. Rado. Moreover we find a new result concerning linear differential-difference equations (theorem II). The paper closes with some references.

§ 1. The principal tools we need for the proof of theorem I are:

a) The Lindemann–Weierstrass theorem: *Let $\alpha_1 \alpha_2, \ldots, \alpha_n$ denote different algebraic numbers, let $\beta_1, \beta_2, \ldots, \beta_n$ denote arbitrary algebraic numbers. If*

$$\beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \ldots + \beta_n e^{\alpha_n} = 0,$$

then necessarily $\beta_1 = \beta_2 = \ldots = \beta_n = 0$.

b) The analogous but elementary theorem: *Let $\varrho_1, \varrho_2, \ldots, \varrho_n$ denote different numbers, let $P_1(z), P_2(z), \ldots, P_n(z)$ denote arbitrary polynomials. If*

$$P_1(z)\, e^{\varrho_1 z} + P_2(z)\, e^{\varrho_2 z} + \ldots + P_n(z)\, e^{\varrho_n z} \equiv 0,$$

then necessarily $P_1(z) \equiv P_2(z) \equiv \ldots \equiv P_n(z) \equiv 0$.

c) A theorem essentially due to Schürer [2]): *Let the integral function*

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z_h}{h!} \quad , \quad \limsup_{h \to \infty} \sqrt[h]{|c_h|} \leqq q,$$

satisfy a linear differential equation of infinite order

$$a_0\, y(z) + a_1\, y'(z) + a_2\, y''(z) + \ldots = 0,$$

with constant coefficients not vanishing simultaneously. Let the characteristic function

$$A(t) = a_0 + a_1 t + a_2 t^2 + \ldots$$

be regular for $|t| \leqq q$.

[1]) For references see the list at the end of this paper.

[2]) See also the papers of Perron and Sheffer.

If $A(t)$ has no zeros in the circle $|t| \leqq q$, then necessarily $y(z) \equiv 0$.

In all other cases there exists a polynomial $b_0 + b_1 t + \ldots + b_k t^k$ with zeros (also with respect to their multiplicities) *identical with those of $A(t)$ in the circle $|t| \leqq q$. Then $y(z)$ satisfies the linear differential equation of finite order*

$$b_0\, y(z) + b_1\, y'(z) + \ldots + b_k\, y^{(k)}(z) = 0. \tag{1}$$

From the theorems *b*) and *c*) we deduce the following lemma:

Lemma: *Let the integral function*

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!} \not\equiv 0 \quad , \quad \limsup_{h\to\infty} \sqrt[h]{|c_h|} \leqq q,$$

with algebraic coefficients $c_0, c_1, c_2, \ldots$ satisfy a linear differential equation of infinite order

$$a_0\, y(z) + a_1\, y'(z) + a_2\, y''(z) + \ldots = 0,$$

with constant coefficients not vanishing simultaneously and such, that the characteristic function

$$a_0 + a_1 t + a_2 t^2 + \ldots$$

is regular for $|t| \leqq q$.

Then $y(z)$ can be written

$$y(z) = \sum_{i=1}^{j} P_i(z)\, e^{\varrho_i z}.$$

Here $\varrho_1, \varrho_2, \ldots, \varrho_j$ represent different algebraic numbers, zeros of the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$ in the circle $|t| \leqq q$; moreover every $P_i(z)$ is a polynomial with algebraic coefficients and of degree $\nu_i - 1$ at most, ν_i denoting the multiplicity of the zero ϱ_i $(i = 1, 2, \ldots, j)$.

Proof. 1. The function $y(z)$ considered here satisfies all the hypotheses of Schürer's theorem *c*). Now $y(z) \not\equiv 0$, hence the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$ must have zeros in the circle $|t| \leqq q$. Let $\varrho_1, \varrho_2, \ldots, \varrho_s$ represent these zeros and let $\nu_1, \nu_2, \ldots, \nu_s$ denote their respective multiplicities. Let $b_0 + b_1 t + \ldots + b_k t^k$ be a polynomial with zeros $\varrho_1, \varrho_2, \ldots, \varrho_s$ of multiplicities $\nu_1, \nu_2, \ldots, \nu_s$ $(b_k \neq 0)$. Then, on account of Schürer's theorem, $y(z)$ satisfies

$$b_0\, y(z) + b_1\, y'(z) + \ldots + b_k\, y^{(k)}(z) = 0 \qquad (b_k \neq 0). \tag{2}$$

Hence $y(z)$ can be written

$$y(z) = \sum_{\sigma=1}^{s} P_\sigma(z)\, e^{\varrho_\sigma z}, \tag{3}$$

where every $P_\sigma(z)$ represents a polynomial of degree $\nu_\sigma - 1$ at most $(\sigma = 1, 2, \ldots, s)$.

2. Now we shall use the condition, that all coefficients $c_0, c_1, c_2, \ldots$ of $y(z)$ are algebraic. In stead of (2) we may write

$$L\,[y(z)] \equiv 0, \tag{4}$$

if we introduce the linear differential operator

$$L = b_0 + b_1 D + \dots + b_k D^k.$$

The $k+1$ numbers $b_0, b_1, \dots, b_k$ have a linear independent basis $\tau_1, \tau_2, \dots, \tau_r$ with respect to the field of algebraic numbers; hence

(5) $$b_\varkappa = b_{\varkappa 1}\tau_1 + b_{\varkappa 2}\tau_2 + \dots + b_{\varkappa r}\tau_r \qquad (\varkappa = 0, 1, 2, \dots, k)$$

with algebraic $b_{\varkappa 1}, b_{\varkappa 2}, \dots, b_{\varkappa r}$. It follows

(6) $$L = \tau_1 L_1 + \tau_2 L_2 + \dots + \tau_r L_r,$$

if we put

$$L_\varrho = b_{0\varrho} + b_{1\varrho} D + \dots + b_{k\varrho} D^k \qquad (\varrho = 1, 2, \dots, r).$$

Now all coefficients in $y(z) = c_0 + c_1 \frac{z}{1!} + c_2 \frac{z^2}{2!} + \dots$ are algebraic; also the coefficients $b_{0\varrho}, b_{1\varrho}, \dots, b_{k\varrho}$ in the above operator L_ϱ are algebraic. It follows easily

$$L_\varrho [y(z)] = c_{0\varrho} + c_{1\varrho}\frac{z}{1!} + c_{2\varrho}\frac{z^2}{2!} + \dots$$

with algebraic coefficients $c_{0\varrho}, c_{1\varrho}, c_{2\varrho}, \dots$ Hence, taking account of (4) and (6),

$$\begin{gathered} c_{01}\tau_1 + c_{02}\tau_2 + \dots + c_{0r}\tau_r = 0, \\ c_{11}\tau_1 + c_{12}\tau_2 + \dots + c_{1r}\tau_r = 0, \\ \dots\dots\dots\dots\dots\dots \end{gathered}$$

Here $\tau_1, \tau_2, \dots, \tau_r$ are linearly independent; it follows therefore

$$\begin{gathered} c_{01} = c_{02} = \dots = c_{0r} = 0, \\ c_{11} = c_{12} = \dots = c_{1r} = 0, \\ \dots\dots\dots\dots\dots \end{gathered}$$

or

$$L_\varrho [y(z)] \equiv 0 \ \textit{for} \ \varrho = 1, 2, \dots, r.$$

Every linear differential operator $L_\varrho = b_{0\varrho} + b_{1\varrho} D + \dots + b_{k\varrho} D^k$ has algebraic coefficients; there is at least one whose coefficient $b_{k\varrho}$ does not vanish (on account of (5) and $b_k \neq 0$). Let $\bar{b}_0 + \bar{b}_1 D + \dots + \bar{b}_k D^k$ denote this operator. *Hence $y(z)$ satisfies a linear differential equation*

(7) $$\bar{b}_0 y(z) + \bar{b}_1 y'(z) + \dots + \bar{b}_k y^{(k)}(z) = 0$$

with algebraic coefficients $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_k$ and $\bar{b}_k \neq 0$.

3. The auxiliary equation

$$\bar{b}_0 + \bar{b}_1 t + \dots + \bar{b}_k t^k = 0$$

of (7) clearly has algebraic roots, say $\bar{\varrho}_1, \bar{\varrho}_2, \dots, \bar{\varrho}_l$; let $\mu_1, \mu_2, \dots, \mu_l$ denote their respective multiplicities. Hence

$$\mu_1 + \mu_2 + \dots + \mu_l = k,$$

and we can write

$$y(z) = \sum_{\lambda=1}^{l} \overline{P}_\lambda(z)\, e^{\overline{\varrho}_\lambda z}, \tag{8}$$

where $\overline{P}_1(z), \overline{P}_2(z), \ldots, \overline{P}_l(z)$ are polynomials. We shall prove, that these polynomials have algebraic coefficients.

The general solution

$$\eta(z) = \gamma_0 + \gamma_1 \frac{z}{1!} + \gamma_2 \frac{z^2}{2!} + \cdots \tag{9}$$

of the equation (7) is given by

$$\eta(z) = \sum_{\lambda=1}^{l} \Pi_\lambda(z)\, e^{\overline{\varrho}_\lambda z} = \sum_{\lambda=1}^{l} \Pi_\lambda(z) \left(1 + \overline{\varrho}_\lambda \frac{z}{1!} + \overline{\varrho}_\lambda^{2} \frac{z^2}{2!} + \cdots\right), \tag{10}$$

where

$$\Pi_\lambda(z) = p_{\lambda 0} + p_{\lambda 1} z + \cdots + p_{\lambda,\mu_\lambda - 1} z^{\mu_\lambda - 1}$$

are polynomials.

Equating corresponding coefficients in (9) and (10) we obtain

$$\left\{\begin{aligned} \gamma_0 &= \sum_{\lambda=1}^{l} p_{\lambda 0} \\ \frac{\gamma_1}{1!} &= \sum_{\lambda=1}^{l} \left(p_{\lambda 0} \frac{\overline{\varrho}_\lambda}{1!} + p_{\lambda 1}\right) \\ &\cdots\cdots\cdots\cdots \\ \frac{\gamma_{k-1}}{(k-1)!} &= \sum_{\lambda=1}^{l} \left(p_{\lambda 0} \frac{\overline{\varrho}_\lambda^{k-1}}{(k-1)!} + p_{\lambda 1} \frac{\overline{\varrho}_\lambda^{k-2}}{(k-2)!} + \cdots\right). \end{aligned}\right. \tag{11}$$

This constitutes for given $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ a system of k linear equations in the $\mu_1 + \mu_2 + \ldots + \mu_l = k$ quantities $p_{\lambda 0}, p_{\lambda 1}, \ldots, p_{\lambda,\mu_\lambda - 1}$ $(\lambda = 1, 2, \ldots, l)$.

For $\gamma_0 = \gamma_1 = \ldots = \gamma_{k-1} = 0$ the differential equation (7) has only the solution $\eta(z) \equiv 0$; hence in this case the (homogeneous) system (11) has only the trivial solution

$$p_{\lambda 0} = p_{\lambda 1} = \cdots = p_{\lambda,\mu_\lambda - 1} = 0 \qquad (\lambda = 1, 2, \ldots, l).$$

It follows, that the determinant of the system (11)

$$\begin{vmatrix} 1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0 & \ldots \\ \frac{\overline{\varrho}_1}{1!} & 1 & \ldots & \frac{\overline{\varrho}_2}{1!} & 1 & \ldots & \frac{\overline{\varrho}_l}{1!} & 1 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\overline{\varrho}_1^{k-1}}{(k-1)!} & \frac{\overline{\varrho}_1^{k-2}}{(k-2)!} & \ldots & \frac{\overline{\varrho}_2^{k-1}}{(k-1)!} & \frac{\overline{\varrho}_2^{k-2}}{(k-2)!} & \ldots & \frac{\overline{\varrho}_l^{k-1}}{(k-1)!} & \frac{\overline{\varrho}_l^{k-2}}{(k-2)!} & \ldots \end{vmatrix}$$

does not vanish. Moreover all elements of this determinant are algebraic numbers. In the special case, that $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ are equally algebraic, the solution $p_{\lambda 0}, p_{\lambda 1}, \ldots, p_{\lambda, \mu_\lambda - 1}$ of (11) necessarily must consist of algebraic numbers. Taking for $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ the algebraic coefficients $c_0, c_1, \ldots, c_{k-1}$ of $y(z)$ we obtain as the solution exactly the coefficients of the polynomials $\overline{P}_1(z), \overline{P}_2(z), \ldots, \overline{P}_l(z)$ from (8). Hence: *In*

$$y(z) = \sum_{\lambda=1}^{l} \overline{P}_\lambda(z)\, e^{\overline{\varrho}_\lambda z} \tag{8}$$

$\overline{\varrho}_1, \overline{\varrho}_2, \ldots, \overline{\varrho}_l$ are different algebraic numbers and the polynomials

$$\overline{P}_1(z), \overline{P}_2(z), \ldots, \overline{P}_l(z)$$

have algebraic coefficients.

4. We observe that the righthand-sides of (3) and (8) are identical functions of z. Using the elementary theorem b) we may suppose, without loss of generality,

$$\varrho_i = \overline{\varrho}_i, \qquad P_i(z) \equiv \overline{P}_i(z) \not\equiv 0 \qquad (i = 1, 2, \ldots, j) \tag{12}$$

and

$$P_\sigma(z) \equiv 0 \;(\sigma = j+1, j+2, \ldots, s), \quad \overline{P}_\lambda(z) \equiv 0 \;\;(\lambda = j+1, j+2, \ldots, l)$$

for a positive integer $j \leqq \mathrm{Min}\,(s, l)$. Hence

$$y(z) = \sum_{i=1}^{j} P_i(z)\, e^{\varrho_i z}.$$

By definition $\varrho_1, \varrho_2, \ldots, \varrho_j$ represent zeros of multiplicities $\nu_1, \nu_2, \ldots, \nu_j$ of the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$ in the circle $|t| \leqq q$. Moreover every $P_i(z)$ is a polynomial of degree $\nu_i - 1$ at most (see section 1 of this proof). On the other hand by (12) and section 3 the numbers $\varrho_1, \varrho_2, \ldots, \varrho_j$ and the coefficients of $P_1(z), P_2(z), \ldots, P_j(z)$ are all algebraic. This completes the proof of our lemma.

Proof of theorem I. Without loss of generality we may assume $y(z) \not\equiv 0$. Obviously the function $y(z)$ considered here fulfills all conditions of the preceding lemma. Hence

$$y(z) \equiv \sum_{i=1}^{j} P_i(z)\, e^{\varrho_i z}, \tag{13}$$

where $\varrho_1, \varrho_2, \ldots, \varrho_j$ denote different algebraic zeros of the characteristic function in the circle $|t| \leq q$; moreover every $P_i(z)$ represents a polynomial with algebraic coefficients of degree $\nu_i - 1$ at most, ν_i denoting the multiplicity of the zero ϱ_i.

Let $\zeta \neq 0$ be an exceptional point of multiplicity μ; hence

$$\zeta, y(\zeta), y'(\zeta), \ldots, y^{(\mu-1)}(\zeta)$$

are algebraic simultaneously. From (13) it follows

$$(14)\quad\begin{cases} -y(\zeta)\,e^0 + \sum\limits_{i=1}^{j} P_i(\zeta)\,e^{\varrho_i\zeta} = 0 \\ -y'(\zeta)\,e^0 + \sum\limits_{i=1}^{j} (P_i'(\zeta) + \varrho_i\,P_i(\zeta))\,e^{\varrho_i\zeta} = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ -y^{(\mu-1)}(\zeta)\,e^0 + \sum\limits_{i=1}^{j}\Big(P_i^{(\mu-1)}(\zeta) + \binom{\mu-1}{1}\varrho_i\,P_i^{(\mu-2)}(\zeta) + \ldots + \\ \qquad\qquad + \binom{\mu-1}{\mu-1}\varrho_i^{\mu-1}\,P_i(\zeta)\Big)\,e^{\varrho_i\zeta} = 0. \end{cases}$$

Now we are in a position to apply the LINDEMANN–WEIERSTRASS theorem on each of the μ equations of (14). For in these equations the "exponents" $\varrho_1\zeta, \varrho_2\zeta, \ldots, \varrho_j\zeta$ are different algebraic numbers and the "coefficients"

$$-y(\zeta), P_i(\zeta)\ ;\ -y'(\zeta), (P_i'(\zeta) + \varrho_i\,P_i(\zeta))\ ;\ \ldots ;$$
$$-y^{(\mu-1)}(\zeta), \Big(P_i^{(\mu-1)}(\zeta) + \binom{\mu-1}{1}\varrho_i\,P_i^{(\mu-2)}(\zeta) + \ldots + \binom{\mu-1}{\mu-1}\varrho_i^{\mu-1}\,P_i(\zeta)\Big)$$

all are algebraic.

For the application of the said theorem it is convenient to consider two special cases separately:

Case I. *Let all numbers $\varrho_1, \varrho_2, \ldots, \varrho_j$ be different from zero.* Then it follows from (14)

$$y(\zeta) = 0,\ P_i(\zeta) = 0$$
$$y'(\zeta) = 0,\ P_i'(\zeta) + \varrho_i\,P_i(\zeta) = 0$$
$$\cdots\cdots\cdots\cdots$$
$$y^{(\mu-1)}(\zeta) = 0,\ P_i^{(\mu-1)}(\zeta) + \binom{\mu-1}{1}\varrho_i\,P_i^{(\mu-2)}(\zeta) + \ldots + \binom{\mu-1}{\mu-1}\varrho_i^{\mu-1}\,P_i(\zeta) = 0$$

for $i = 1, 2, \ldots, j$, hence

$$(15a)\qquad y(\zeta) = y'(\zeta) = \ldots = y^{(\mu-1)}(\zeta) = 0$$

and

$$(15b)\qquad P_i(\zeta) = P_i'(\zeta) = \ldots = P_i^{(\mu-1)}(\zeta) = 0 \qquad (i = 1, 2, \ldots, j).$$

Case II. *Let one of the numbers $\varrho_1, \varrho_2, \ldots, \varrho_j$ vanish; say $\varrho_1 = 0$.* Then we obtain from (14) in the same manner

$$-y(\zeta) + P_1(\zeta) = 0,\ P_i(\zeta) = 0$$
$$-y'(\zeta) + P_1'(\zeta) = 0,\ P_i'(\zeta) + \varrho_i\,P_i(\zeta) = 0$$
$$\cdots\cdots\cdots\cdots$$
$$-y^{(\mu-1)}(\zeta) + P_1^{(\mu-1)}(\zeta) = 0,\ P_i^{(\mu-1)}(\zeta) + \binom{\mu-1}{1}\varrho_i\,P_i^{(\mu-2)}(\zeta) + \ldots +$$
$$+ \binom{\mu-1}{\mu-1}\varrho_i^{\mu-1}\,P_i(\zeta) = 0$$

for $i = 2, 3, \ldots, j$, hence

$$(16) \qquad P_i(\zeta) = P_i'(\zeta) = \ldots = P_i^{(\mu-1)}(\zeta) = 0 \qquad (i = 2, 3, \ldots, j).$$

By hypothesis ν is the largest multiplicity of the zeros of the characteristic function in the region $0 < |t| \leqq q$; hence in case I every polynomial $P_i(z)$ is at most of degree $\nu_i - 1 \leqq \nu - 1$ for $i = 1, 2, \ldots, j$ and in case II the same assertion holds for $i = 2, 3, \ldots, j$.

Now we prove the two assertions of our theorem:

1. Suppose there exist exceptional points $\zeta_1, \zeta_2, \ldots, \zeta_k$ different from zero and with multiplicities $\mu_1, \mu_2, \ldots, \mu_k$, such that

$$\mu_1 + \mu_2 + \ldots + \mu_k \geqq \nu.$$

We have to show, that $y(z)$ is a polynomial with algebraic coefficients.

In case I every polynomial $P_i(z)$ is at most of degree $\nu - 1$ $(i = 1, 2, \ldots, j)$; on the other hand applying (15*b*) with $\zeta = \zeta_\varkappa$ and $\mu = \mu_\varkappa$ $(\varkappa = 1, 2, \ldots, k)$ we see, that every polynomial $P_i(z)$ at least has ν zeros, hence $P_i(z) \equiv 0$ and therefore $y(z) \equiv 0$, but this gives a contradiction, for we assumed $y(z) \not\equiv 0$.

In case II we similarly apply (16) in stead of (15*b*) and we obtain $P_i(z) \equiv 0$ for $i = 2, 3, \ldots, j$, hence $y(z) \equiv P_1(z)$, a polynomial with algebraic coefficients.

2. If $a_0 \neq 0$, then $t = 0$ is not a zero of the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$, hence $\varrho_1, \varrho_2, \ldots, \varrho_j$ all are different from zero and we have case I. If $\zeta \neq 0$ is an exceptional point with multiplicity μ, then we derive from (15*a*), that ζ necessarily is a zero of $y(z)$ with multiplicity μ (for $y^{(\mu)}(\zeta)$ is transcendental and therefore different from zero).

§ 2. In 1939 McMillan stated the following theorem, closely related to our theorem:

"Given the set of algebraic numbers a_n $(n = 0, 1, 2, \ldots)$ of which an infinite number are non-vanishing, and such that

$$\limsup_{n \to \infty} |a_n|^{1/n} \leqq \varrho.$$

Let there be a set of constants c_n $(n = 0, 1, 2, \ldots)$, at least two of which are non-vanishing, such that the function

$$g(t) = \sum_{n=0}^{\infty} c_n t^n$$

is analytic for $|t| < R$ where $R > \varrho$, and has zeros for $|t| \leqq \varrho$ only at the points $t = \zeta_k$ $(k = 1, 2, \ldots, N)$ respectively of multiplicities ν_k, where the ζ_k are algebraic numbers. If now

$$\sum_{n=0}^{\infty} c_n a_{n+p} = 0$$

for all $p = 0, 1, 2, \ldots$, then the function

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

takes on a transcendental value for every algebraic $z \neq 0$."

However this assertion certainly is not true, as is easily seen from the following example: Take

$a_n = n - 1$ $(n = 0, 1, 2, \ldots)$, $\varrho = 1$, $c_0 = 1$, $c_1 = -2$, $c_2 = 1$, $c_3 = c_4 = \ldots = 0$.

Then it follows

$g(t) \equiv 1 - 2t + t^2$, $N = 1$, $\zeta_1 = 1$, $c_0 a_p + c_1 a_{p+1} + c_2 a_{p+2} = 0$ $(p = 0, 1, 2, \ldots)$,

so that all hypotheses of McMillan's theorem are satisfied. But now

$$F(z) \equiv \sum_{n=0}^{\infty} (n-1) \frac{z^n}{n!},$$

hence

$$F(1) = 0,$$

not a transcendental number.

§ 3. *A*) Our theorem I clearly is a generalization of the following result of Itihara (see the joint paper of Itihara and Ôishi in the list of references): Let the transcendental function

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!}$$

with algebraic coefficients $c_0, c_1, c_2, \ldots$ satisfy a linear differential equation

$$a_0\, y(z) + a_1\, y'(z) + \ldots + a_n\, y^{(n)}(z) = 0,$$

with constant algebraic coefficients $a_0, a_1, \ldots, a_n$, then $y(z)$ is a transcendental number for every algebraic value of z with exception of n values for z at most.

B) We can apply theorem I to the differential equation

$$y(z) - y^{(n)}(z) = 0.$$

Obviously the n functions

$$y_\nu(z) = \frac{z^\nu}{\nu!} + \frac{z^{\nu+n}}{(\nu+n)!} + \frac{z^{\nu+2n}}{(\nu+2n)!} + \ldots \qquad (\nu = 0, 1, \ldots, n-1)$$

are all integrals of this equation. Hence every function

$$y(z) = c_0 + c_1 \frac{z}{1!} + c_2 \frac{z^2}{2!} + \ldots,$$

where $c_0, c_1, c_2, \ldots$ constitute a periodic sequence of complex numbers with a "length" n of the period, satisfies the equation. If moreover all coefficients $c_0, c_1, c_2, \ldots$ are algebraic and do not vanish simultaneously,

then we can apply the Remark to theorem I (the zeros of the corresponding characteristic function $1 - t^n$ being simple). It follows, that $y(z)$ is a transcendental number for every algebraic value of $z \neq 0$.

From this we easily obtain the following theorem of DIETRICH and ROSENTHAL:

If the coefficients c_h in

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!}$$

are algebraic and form a periodic sequence from some h on, then $y(z)$ is a transcendental number for every algebraic value of $z \neq 0$, except in the trivial case that $y(z)$ is a polynomial.

C) We now show the following theorem of R. RADO:

Suppose that the real functions $f_1(x), f_2(x), \ldots, f_n(x)$, not all identically zero, of the real variable x satisfy the system of differential equations

$$f'_r(x) = \sum_{s=1}^{n} c_{rs} f_s(x) \qquad (r = 1, 2, \ldots, n) \tag{17}$$

in which the coefficients c_{rs} are rational numbers satisfying

$$\begin{vmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{21} & \cdot & \cdot & \cdot & \cdot & c_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n1} & \cdot & \cdot & \cdot & \cdot & c_{nn} \end{vmatrix} \neq 0.$$

Then, for every rational number x_0 with possibly a single exception, at least one of the numbers

$$f_1(x_0), f_2(x_0), \ldots, f_n(x_0)$$

is irrational.

Proof. Let α and β be different rational numbers, and assume, that the 2n numbers

$$f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha);\ f_1(\beta), f_2(\beta), \ldots, f_n(\beta)$$

are rational. Then we will obtain a contradiction.

Without loss of generality we may suppose $\alpha = 0$ and $f_1(x) \neq 0$.

From (17) it follows for $h = 0, 1, 2, \ldots$

$$f_r^{(h+1)}(x) = \sum_{s=1}^{n} c_{rs} f_s^{(h)}(x),$$

where the coefficients c_{rs} are rationals; hence

$$\left.\begin{matrix} f_r(0) & , & f'_r(0) & , & f''_r(0) & , & \ldots \\ f_r(\beta) & , & f'_r(\beta) & , & f''_r(\beta) & , & \ldots \end{matrix}\right\} \qquad (r = 1, 2, \ldots, n)$$

all are rational numbers.

From (17) it follows on the other hand by a well-known method, that the n functions $f_1(x), f_2(x), \dots, f_n(x)$ all are integrals of a linear differential equation with constant coefficients

$$a_0\, y(x) + a_1\, y'(x) + \dots + a_n\, y^{(n)}(x) = 0,$$

where $a_0 = \det. |c_{rs}| \neq 0$ and $a_n = (-1)^n$.

Now we are in a position to apply theorem I on the integral function

$$f_1(x) = f_1(0) + f_1'(0)\frac{x}{1!} + f_1''(0)\frac{x^2}{2!} + \dots,$$

with rational coefficients and with an exceptional argument β of infinite order. By the second assertion of this theorem we obtain

$$f_1(\beta) = f_1'(\beta) = f_1''(\beta) = \dots = 0,$$

hence $f_1(x) \equiv 0$; a contradiction.

Obviously it now is easy to extend RADO's theorem.

D) It is possible to apply theorem I to the solutions of certain functional equations. As an example I consider in the next theorem a certain class of solutions of a linear differential-difference equation with constant coefficients.

Theorem II [3]). *Let the integral transcendental function*

$$y(z) = c_0 + c_1\frac{z}{1!} + c_2\frac{z^2}{2!} + \dots \quad , \quad c_h = 0(q^h), \tag{18}$$

where q is an arbitrary positive number and where all coefficients $c_0, c_1, c_2, \dots$ are algebraic, satisfy a linear differential-difference equation

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu}\, y^{(\mu)}(z+\omega_\nu) = 0, \tag{19}$$

where the constants $A_{\mu\nu}$ do not vanish simultaneously and where $\omega_0, \omega_1, \dots, \omega_n$ are different numbers.

Then $y(z)$ is a transcendental number for every algebraic value of z with exception of a finite number of values for z.

If moreover $\sum\limits_{\nu=0}^{n} A_{0\nu} \neq 0$, then these exceptional points, which differ from 0, necessarily are zeros of $y(z)$.

Proof. We have for $\mu = 0, 1, \dots, m$ and $\nu = 0, 1, \dots, n$

$$y^{(\mu)}(z+\omega_\nu) = y^{(\mu)}(z) + \frac{\omega_\nu}{1!}\, y^{(\mu+1)}(z) + \frac{\omega_\nu^2}{2!}\, y^{(\mu+2)}(z) + \dots.$$

Substitution in (19) gives a linear differential equation of infinite order

$$a_0\, y(z) + a_1\, y'(z) + a_2\, y''(z) + \dots = 0,$$

[3]) This theorem was communicated without proof on September 1, 1950, at the International Congress of Mathematicians, Cambridge (Mass.).

with characteristic function

$$a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} \left(t^\mu + \frac{\omega_\nu}{1!} t^{\mu+1} + \frac{\omega_\nu^2}{2!} t^{\mu+2} + \ldots \right)$$

$$= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} t^\mu e^{\omega_\nu t}.$$

Clearly

$$a_0 = \sum_{\nu=0}^{n} A_{0\nu}.$$

Now $y(z)$ fulfills all conditions of theorem I and the assertions of theorem II are immediate consequences of those of theorem I.

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